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THE ASYMMETRIC CAUCHY PROCESSES ON THE LINE

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1. Main results. The one dimensional Cauchy processes X_t are those stable processes on the line R having log characteristic functions

$$\log E(e^{i\theta(X_t - X_0)}) = -t|\theta|[1 + i \operatorname{sgn}(\theta) h \log|\theta|], \tag{1.1}$$

where $h = 2\beta/\pi$, $\beta = p - q$, $q = 1 - p$, and p is the mass put at $+1$ by the Isotropy measure of $(X_1 - X_0)$. If $\beta = 0$ the process is the usual symmetric Cauchy process. We will henceforth assume that $\beta \neq 0$. We will also assume that we have selected versions of our processes that are standard Markov processes. [See [1] for a description of a standard process.] The transition density is

$$f(t, x) = (2\pi)^{-1} \int_R e^{-i\theta x} E(e^{i\theta(X_t - X_0)})\, d\theta, \tag{1.2}$$

and the potential kernel is $g(x) = \int_0^\infty f(t, x)\, dt$. For a Borel set B, let $T_B = \inf\{t > 0 : X_t \in B\}$ ($= \infty$ if $X_t \notin B$ for all $t > 0$) be the hitting time of B. The dual process is the process $-X_t$. Quantities referring to this process will be denoted by ^ e.g. $\hat{g}(x) = g(-x)$. Let

$$H_B(x, dz) = P_x(X_{T_B} \in dz;\, T_B < \infty),$$

where $P_x(\cdot)$ and $E_x(\cdot)$ denote the conditional probability and expectation relative to $X_0 = x$. Our principle aim will be to investigate the asymptotic behavior, for large x of $H_B(x, dz)$, and of

$$P_x(t < T_B < \infty;\, X_{T_B} \in dz)$$

for large t, for bounded Borel sets B.

If $p = 1$ then the process X_t takes only positive jumps, and it easily follows from this fact that one point sets are non-polar, i.e., $P_x(T_{\{y\}} < \infty) > 0$ for some x. In [4] Orey inquires whether or not y is regular for $\{y\}$, i.e., does $P_y(T_{\{y\}} = 0) = 1$. Our original motivation was to answer this question. To our great surprise we found the following.

THEOREM 1. *Assume $\beta \neq 0$. Then for any y and all x,*

$$P_x(T_{\{y\}} < \infty) > 0 \quad \text{and} \quad P_x(T_{\{x\}} = 0) = 1.$$

This was quite unexpected since for the symmetric Cauchy process ($\beta = 0$) one point sets are polar! (For a proof see Section 5 of [5].) Theorem 1 will be an easy consequence of the following basic

PROPOSITION 1. *Assume $\beta \neq 0$. Then $g(x)$ is a continuous function on R.*

To establish our asymptotic results on $H_B(x, dz)$ we need to know the behavior of $g(x)$ for large x.

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PROPOSITION 2. *Assume $\beta \neq 0$. Then*

$$\lim_{x\to+\infty} \log |x|\, g(x) = 2p/\pi h^2 = c^+,$$
$$\lim_{x\to-\infty} \log |x|\, g(x) = 2q/\pi h^2 = c^-.$$

Using these facts we will be able to show the following

THEOREM 2. *Assume $\beta \neq 0$, and let B have compact closure $\bar{B}$. Then there are unique bounded measures π_B and $\hat{\pi}_B$ supported on $\bar{B}$ such that $\pi_B(\bar{B}) = \hat{\pi}_B(\bar{B})$;*

$$(1.3) \qquad P_x(T_B < \infty) = \int_{\bar{B}} g(y - x)\pi_B(dy);$$
$$\hat{P}_x(\hat{T}_B < \infty) = \int_{\bar{B}} g(x - y)\hat{\pi}_B(dy);$$

and for any continuous function f on $\bar{B}$,

$$(1.4) \qquad \lim_{x\to\pm\infty} \log |x|\, H_B f(x) = c^{\mp}(\hat{\pi}_B, f)$$
$$(1.5) \qquad \lim_{x\to\pm\infty} \log |x|\, \hat{H}_B f(x) = c^{\pm}(\pi_B, f),$$

where for a measure μ on $\bar{B}$, $(\mu, f) = \int_{\bar{B}} f(x)\mu(dx)$.

The measures π_B and $\hat{\pi}_B$ are called the capacitory, respectively co-capacitory measures of B. Their common total mass is the capacity of B. The existence of such measures satisfying (1.3) is a consequence of the Hunt potential theory (see Chapter 6 of [1]). It is immediate from (1.4) that when $0 < |\beta| < 1$, then

$$(1.6) \qquad \lim_{|x|\to\infty} E_x[f(X_{T_B}) \mid T_B) < \infty] = (\hat{\pi}_B, f)/(\hat{\pi}_B, 1),$$

thus showing that the normalized co-capacitory measure is in these cases just the conditional hitting distribution at ∞.

Our final result concerns the asymptotic hitting time.

THEOREM 3. *Assume $\beta \neq 0$ and let $\bar{B}$ be compact. Then for any continuous function f on B,*

$$\lim_{t\to\infty} (\log t) E_x[f(X_{T_B}); t < T_B < \infty]$$
$$= (2q/\pi h^2)(\hat{\pi}_B, f)P_x(T_B = \infty), \quad \text{if } \beta > 0$$
$$= (2p/\pi h^2)(\hat{\pi}_B, f)P_x(T_B = \infty), \quad \text{if } \beta < 0.$$

The analogue of the results in Theorems 2 and 3 for other stable processes can be found in [5] and [6].

2. Proofs. If follows easily from (1.1) and (1.2) that the density $f(t, x)$ satisfies the scaling relation

$$(2.1) \qquad f(t, x) = f(1, xt^{-1} - h \log t)t^{-1}.$$

Since p for X_t is q for $-X_t$ it suffices to establish our results for the case $p > q$. Hence forth then we will always assume $\beta > 0$. It is well known that the distribution of $X_1 - X_0$ is unimodal, and consequently $f(1, x)$ is monotone decreasing for $|x|$ sufficiently large. Also (see e.g., Feller [3], p. 547.)

$$\lim_{x\to+\infty} P_0(X_1 > x)x = 2p\pi^{-1}, \qquad \lim_{x\to-\infty} P_0[X_1 \leqq x]|x| = 2q\pi^{-1}$$

Consequently, by a familiar Tauberian theorem

$$\lim_{x\to+\infty} f(1, x)x^2 = 2p\pi^{-1}, \qquad \lim_{x\to-\infty} f(1, x)x^2 = 2q\pi^{-1}. \tag{2.2}$$

Since $f(1, x)x^2$ is bounded on compacts we see that there is a k, $0 < k < \infty$ such that for all x, $f(1, x) \leqq k\,|x|^{-2}$. We may now establish Proposition 1. To accomplish this we will require two lemmas.

LEMMA 2.1. *Suppose* $x_0 \neq 0$. *Given* $\epsilon > 0$ *there is a neighborhood* V_{x_0} *of* x_0 *and a* $\delta > 0$ *such that*

$$\int_0^\delta f(t, x)\, dt < \epsilon$$

for $x \in V_{x_0}$.

PROOF. Suppose $x_0 > 0$. Let V_{x_0} be a compact neighborhood such that $x > 0$ for $x \in V_{x_0}$. Then for δ sufficiently small it follows from (2.1) that

$$\int_0^\delta f(t, x)\, dt \leqq k \int_0^\delta h^{-2}t^{-1} \log^{-2} t\, dt < \epsilon.$$

On the other hand if $x_0 < 0$ then choosing V_{x_0} so $x < 0$ for $x \in V_{x_0}$, and setting $a = \max \{x : x \in V_{x_0}\}$, we see that if δ is small enough so that $ht \log t \geqq -\epsilon$, $t \leqq \delta$, then

$$\int_0^\delta f(t, x)\, dt \leqq \int_0^\delta kt[x - ht \log t]^{-2}\, dt$$
$$\leqq \tfrac{1}{2}[k\delta^2(|a| - \epsilon)^{-2}].$$

This establishes the lemma.

Next we show that the same thing is true for $x_0 = 0$.

LEMMA 2.2. *Given* $\epsilon > 0$ *there is a neighborhood* V *of* 0 *and a* $\delta > 0$ *such that*

$$\int_0^\delta f(t, x)\, dt < \epsilon$$

for $x \in V$.

PROOF. If $x \geqq 0$ and δ sufficiently small

$$\int_0^\delta f(t, x)\, dt \leqq k \int_0^\delta h^{-2}t^{-1} \log^{-2} t < \epsilon.$$

Now consider $x < 0$. Choose $a > 0$ so that

$$\log ((1 - a)/(1 - a)) < \epsilon$$

and set $y = -x/h$. Then

$$f(t, x) = t^{-1}f(1, -h[yt^{-1} + \log t]).$$

If $0 < y < e^{-1}$ the function $(y/t) + \log t$ has two roots ρ_1 and ρ_2. The root $\rho_1 \leqq y \log^{-1} (1/y)$ while $\rho_2 \geqq 1/e$. Set $\rho_1 = \rho$ and let $\delta < 1/e$. Decompose

$$\int_0^\delta \quad \text{as} \quad \int_0^{(1-a)\rho} + \int_{(1-a)\rho}^{(1+a)\rho} + \int_{(1+a)\rho}^{\delta}$$

If $t < (1 - a)\rho$ then

$$\begin{aligned} t^{-1}[y + t \log t] &\geqq t^{-1}[y + (1 - a)\rho \log (1 - a)\rho] \\ &= t^{-1}[y + (1 - a)\rho \log \rho + (1 - a)\rho \log (1 - a)] \\ &= t^{-1}[ay + (1 - a)\rho \log (1 - a) \\ &= ayt^{-1}[1 - c\rho y^{-1}], \end{aligned}$$

where

$$c = (a - 1)a^{-1} \log (1 - a).$$

Since $\rho(y)/y \to 0$ we see that for y sufficiently small,

$$(1 - c\rho y^{-1}) \geqq (1 - c\epsilon) > 0,$$

and thus

$$\int_0^{(1-a)\rho} f(t, x) \leqq O(\rho^2 y^{-2}) = o(1), \qquad y \to 0.$$

Next

$$\int_{(1-a)\rho}^{(1+a)\rho} f(t, x)\, dt \leqq [\sup_x f(1, x)] \log ((1 + a)/(1 - a))]$$
$$\leqq [\sup_x f(1, x)]\epsilon.$$

Finally, as to the last integral, one easily verifies that for $(1 + a)\rho(y) < t < 1/e$,

$$yt^{-1} + (1 + a)^{-1} \log t \leqq (1 + a)^{-1} \log (1 + a),$$

and thus for $t < (1 + a)^{-a^{-1}}$

$$yt^{-1} + \log t \leqq (1 + a)^{-1} \log (1 + a) + (a/(1 + a)) \log t < 0.$$

But then

$$[yt^{-1} + \log t]^{-2} \leqq (1 + a)^2 a^{-2} \log^{-2} t[1 + (c/\log t)]^{-2}$$

where $c = (\log (1 + a))/a$. Consequently if δ is sufficiently small
$\int_{(1+a)\rho}^{\delta} f(t, x)\, dt \leqq K(1 + a)^2 h^{-2} a^{-2}[1 + (c/\log \delta)]^{-2} \int_{1/\delta}^{\infty} s^{-1} \log^{-1} s\, ds < \epsilon.$

The lemma now follows from the above three estimates.

PROOF OF PROPOSITION 1. Write

$$g(x) = \int_0^{\delta} f(t, x)\, dt + \int_{\delta}^{A} f(t, x)\, dt + \int_A^{\infty} f(t, x)\, dt.$$

By Lemmas 2.1 and 2.2 for δ small enough and x in a small enough neighborhood of x_0 the first integral on the right is $<\epsilon$. The function of x defined in the second integral is clearly a continuous function of x. As to the third, given any compact set K we may choose A so that

$$|x/ht \log t| < \epsilon$$

for $x \in K$ and $t > A$. Then for $x \in K$,

$$\int_A^{\infty} f(t, x)\, dt \leqq O(\int_A^{\infty} t^{-1} \log^{-2} t).$$

The continuity of $g(x)$ now easily follows from the above facts.

We now turn our attention to the proof of Proposition 2.

PROOF OF PROPOSITION 2. Consider first $x \to +\infty$. If x is large and positive then $(x/t) - \log t$ has for $t > 1$ a unique root ρ, $x/(\log x) \leqq \rho$, and in fact $\rho(x) \sim x/(\log x)$. Write

$$g(hx) = \int_0^{\infty} f(1, h[xt^{-1} - \log t])\, dt$$
$$= \int_0^1 + \int_1^{(1-a)\rho} + \int_{(1-a)\rho}^{(1+a)\rho} + \int_{(1+a)\rho}^{x} + \int_x^{\infty}.$$

We now proceed to estimate these integrals.

$$\int_0^1 f(t, hx)\,dt \leqq Kx^{-2}/2h^2,$$

$$\int_1^{(1-a)\rho} \leqq Kh^{-2}\int_1^{(1-a)\rho} (x - t\log t)^{-2} t\,dt$$
$$\leqq Kh^{-2}\int_1^{(1-a)\rho} (x - t\log(1-a)\rho)^{-2} t\,dt.$$

By computing this last integral explicitly and then examining the terms we find that

$$\int_1^{(1-a)\rho} = O((\log(1-a)\rho)^{-2}) = O((\log x)^{-2}), \qquad x \to \infty.$$

It easily follows from the monotoniety of $f(1, x)$ for large x, the fact that $f(1, x)$ is a continuous positive function, and from (2.2) that $f(x + y) \sim f(x)$ uniformly in $y \varepsilon R$. Using this we see that for $a > 0$ sufficiently small,

$$\int_{(1-a)\rho}^{(1+a)\rho} f(1, h(xt^{-1} - \log t))t^{-1}\,dt$$
$$= \int_{(1-a)}^{(1+a)} f(1, h(x(\rho s)^{-1} - \log\rho - \log s)s^{-1}\,ds$$
$$\sim \int_{(1-a)}^{(1+a)} f(1, h(x(\rho s)^{-1} - x\rho^{-1}))s^{-1}\,ds = \int_{(1+a)^{-1}}^{(1-a)^{-1}} f(1, h(x\rho^{-1}(w-1))w^{-1}\,dw$$
$$\sim \rho(xh)^{-1}\int_{-ah(1+a)^{-1}x/\rho}^{ah(1-a)^{-1}x/\rho} f(t)\,dt \sim (h\log x)^{-1}.$$

Next

$$\int_{(1+a)\rho}^{x} f(1, h(xt^{-1} - \log t))t^{-1}\,dt \leqq K\int_{(1+a)\rho}^{x} [t\log(1+a)\rho - x]^{-2} t\,dt.$$

Carrying out the integration and examining the terms we find that

$$\int_{(1+a)\rho}^{x} = O(\log\log x/\log^2 x).$$

Finally

$$\int_x^\infty f(1, h(xt^{-1} - \log t))t^{-1}\,dt \sim (2q/\pi h^2)\int_x^\infty (t\log^2 t)^{-1}\,dt = (2q/\pi h^2)(\log x)^{-1}.$$

Combining these estimates we obtain

(2.3) $\quad g(x) \sim (h^{-1} + (2q/\pi h^2))\log^{-1} x = (2p/\pi h^2)(\log x)^{-1}, \quad x \to +\infty.$

We must now examine the case $x \to -\infty$. Let $y = -x$ and write

$$g(hx) = \int_0^1 + \int_1^{y/\log y} + \int_{y/\log y}^{y} + \int_y^\infty.$$

Then

$$\int_0^1 = O(y^{-2}),$$

$$\int_1^{y/\log y} \leqq Kh^{-2}\int_1^{y/\log y} (yt^{-1} + \log t)^{-2} t^{-1}\,dt$$
$$\leqq \tfrac{1}{2}y^{-2}[y^2(\log^2 y)^{-1} - 1] = O(\log^{-2} y).$$

Next

$$\int_{y/\log y}^{y} \leqq Kh^{-2}\int_{y/\log y}^{y} (y + t\log(y/\log y))^{-2} t\,dt.$$

Computing the integral and examining the terms we find that

$$\int_{y/\log y}^{y} = O(\log\log y/\log^2 y).$$

Finally

$$\int_y^\infty \sim (2q/\pi h^2) \int_y^\infty (t \log^2 t)^{-1} \, dt = (2q/\pi h^2) \log^{-1} |x|.$$

Combining, we see that

$$g(x) \sim (2q/\pi h^2) \log^{-1} |x|, \qquad x \to -\infty. \tag{2.4}$$

Proposition 2 now follows from (2.3) and (2.4).

Having Propositions 1 and 2, it is an easy matter to establish our theorems. Indeed Theorem 1 is a direct consequence of the continuity of $g(x)$ at 0 and Theorem 4.3 and Corollary 3.1 of [2]. Alternately, by an argument similar to that used to establish Proposition 2.1 in [7] we may establish this result directly without using Hunt's capacity theory. Theorem 2 follows from the asymptotic behavior of $g(x)$ given in Proposition 2 by essentially the same argument as used to establish the corresponding facts for the transient processes with $\alpha < 1$ in Theorem 2 of [6]. We will omit these details. To establish Theorem 3 we may proceed as follows. Let

$$g^\lambda(x) = \int_0^\infty f(t, x) e^{-\lambda t} \, dt,$$

$$H_B{}^\lambda(x, dy) = \int_0^\infty P_x(T_B \in ds, X(T_B) \in dy) e^{-\lambda s},$$

$$R^\lambda(x) = \int_R g^\lambda(y - x) g(y) \, dy = [g(x) - g^\lambda(x)]\lambda^{-1}.$$

Then

$$\int_0^\infty e^{-\lambda t} p_x(t < T_B < \infty) \, dt = \int_R g^\lambda(y - x) p_y(T_B < \infty) \, dy - \int_{\bar{B}} H_B{}^\lambda(x, dz) \int_R g^\lambda(y - z) p_y(T_B < \infty) \, dy,$$

and using (1.3) we find that

$$\int_0^\infty e^{-\lambda t} p_x(t < T_B < \infty) \, dt = \int_{\bar{B}} R^\lambda(y - x) \pi_B(dy) - \int_{\bar{B}} H_B{}^\lambda(x, dz) \int_{\bar{B}} R^\lambda(y - z) \pi_B(dy). \tag{2.5}$$

But it follows from (2.2) that if $\beta > 0$

$$\int_t^\infty f(s, x) \, ds \sim (2q/\pi h^2) \int_t^\infty s^{-1} \log^{-2} s \, ds = (2q/\pi h^2)(\log t)^{-1},$$

while for $\beta < 0$

$$\int_t^\infty f(s, x) \, ds \sim (2p/\pi h^2)(\log t)^{-1},$$

the limits being uniform on compacts. But then

$$R^\lambda(x) \sim (2q/\pi h^2)\lambda^{-1} \log^{-1} (\lambda^{-1}), \quad \lambda \downarrow 0, \quad \beta > 0$$

$$\sim (2p/\pi h^2)\lambda^{-1} \log^{-1} (\lambda^{-1}), \quad \lambda \downarrow 0, \quad \beta < 0.$$

Using (2.5) we then see that

$$\begin{aligned} &\int_0^\infty e^{-\lambda t} p_x(t < T_B < \infty) \, dt \\ &\sim p_x(T_B = \infty)(\pi_B, 1)(2q/\pi h^2)(\lambda^{-1})[\log (1/\lambda)]^{-1}, \quad \beta > 0, \\ &\sim p_x(T_B = \infty)(\pi_B, 1)(2p/\pi h^2)(\lambda^{-1})[\log (1/\lambda)]^{-1}, \quad \beta < 0. \end{aligned} \tag{2.6}$$

Theorem 3 for $f \equiv 1$ now follows from (2.6) by Karamata's theorem. Let $g_B(t, x, y)$ be the density of the measure $p_x(T_B > t, X_t \varepsilon\, dy)$ and let $\varphi(x) = p_x(T_B < \infty)$. Then

$$
\begin{aligned}
& E_x[f(X_{T_B}); t < T_B < \infty] \\
(2.7) \qquad & = \int_R g_B(t, x, z)\varphi(z)[H_B\varphi(z)(\varphi(z))^{-1} - (\tilde{\pi}_B, f)(\tilde{\pi}_B, 1)^{-1}]\, dz \\
& \quad + (\tilde{\pi}_B, f)(\tilde{\pi}_B, 1)^{-1} p_x(t < T_B < \infty).
\end{aligned}
$$

If $p = 1$ or 0 then

$$\log t E_x[f(X_{T_B}; t < T_B < \infty] \leqq \|f\|_\infty \log t p_x(t < T_B < \infty) \to 0,$$

so we need only consider the case when $|\beta| < 1, \beta \neq 0$. Then it is clear from (2.7) that we must show $\log t \int_R \to 0$. Now, given $\epsilon > 0$ there is an r such that

$$|H_B f(z)(\varphi(z))^{-1} - (\tilde{\pi}_B, f)(\tilde{\pi}_B, 1)^{-1}| < \epsilon$$

if $|z| > r$ and

$$\int_{|z| \leqq r} \leqq O\left(\int_{|z| \leqq r} f(t, z)\, dz\right) = O(t^{-1}).$$

The desired result now follows from these two facts.

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