The Annals of Mathematical Statistics
1969, Vol. 40, No. 1, 66–70

ON MEASURABLE, NONLEAVABLE GAMBLING HOUSES WITH A GOAL[1]

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1. Introduction. A gambler's problem, as formulated by Dubins and Savage in [1], consists of a set F of fortunes, a bounded utility function u on F to the real numbers, and, for each f in F, a set $\Gamma(f)$ of gambles (finitely additive probability measures defined on all subsets of F). A strategy σ available in the gambling house Γ at the fortune f is a sequence $\sigma_0, \sigma_1, \cdots$ where $\sigma_0 \varepsilon \Gamma(f)$ and, for $n \geqq 1$, σ_n is a gamble-valued function defined on $F \times F \times \cdots \times F$($n$-factors) such that $\sigma_n(f_1, \cdots, f_n) \varepsilon \Gamma(f_n)$ for every partial history $(f_1, \cdots, f_n)$. The strategy σ may be regarded as a probability measure defined on the finitary subsets of the infinite product $H = F \times F \times \cdots$ and $\sigma_n(f_1, \cdots, f_n)$, as the conditional σ-distribution of f_{n+1} given $(f_1, \cdots, f_n)$ (Section 2.8 of [1]). A gambler with fortune f chooses an available strategy σ and a stop rule t and gets a return $u(\sigma, t)$, the expected value of $u(f_t)$ under σ. By $U(f)$ is denoted the maximum of $u(f)$ and the sup $u(\sigma, t)$ taken over all available σ and stop rules t. Strauch has shown in [4] that if a gambling problem is assumed to have a certain natural Borel measurability structure, then U is measurable with respect to the completion of any Borel measure on the Borel subsets of F and there exist good Borel measurable strategies (See also [5] and [6]).

If a gambler using the strategy σ is not allowed to terminate play, he receives $u(\sigma) = \lim\sup_{t\to\infty} u(\sigma, t)$. $V(f)$ is the sup $u(\sigma)$ taken over all strategies σ available at f. If Γ is leavable, that is, if the one-point gamble $\delta(f)$ is in $\Gamma(f)$ for all f, then $V = U$ ([1], Corollary 3.3.2, p. 42). If Γ has the Borel measurability structure assumed by Strauch and is not leavable, it is not known whether V is absolutely measurable or if good measurable strategies exist.

In this note, I treat the special case in which the utility function u is the indicator of a single fortune g called the goal. It is seen that $u(\sigma)$ may be interpreted as the "σ-probability of visiting g infinitely often" and the questions above are settled affirmatively.

Unless otherwise indicated, the terminology and notation of this note are intended to have the same meaning as in [1].

2. Measurable strategies and probability measures on H. Assume F is a Borel subset of a complete separable metric space and let $\mathcal{B}$ denote the Borel subsets of F. Let Γ be a gambling house defined on F such that every gamble available in Γ is countably additive when restricted to $\mathcal{B}$. (Let P be the set of countably

Received 20 May 1968.
[1] Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR Grant AF-AFOSR-1312-67.

additive probability measures p on $\mathcal{B}$ and let Σ be the smallest σ-field of subsets of P which makes $p \to p(B)$ measurable for each $B \varepsilon \mathcal{B}$. If, in addition to the assumptions above, the set $\{(f, \gamma): \gamma \varepsilon \Gamma(f)\}$ is $\mathcal{B} \times \Sigma$-measurable, then Γ is called a measurable gambling house [4].)

In this setting a strategy σ is said to be measurable if, for every integer $n \geqq 1$, $\sigma_n(f_1, \cdots, f_n)$ is a regular conditional probability on $(F, \mathcal{B})$ given $(f_1, \cdots, f_n)$. Let μ denote the probability measure induced by σ on the Borel subsets of H. That is, the μ-marginal distribution of f_1 is σ_0 and the conditional μ-distribution of f_{n+1} given $(f_1, \cdots, f_n)$ is $\sigma_n(f_1, \cdots, f_n)$. (Notice that notation has been somewhat abused in the above discussion since gambles were tacitly identified with their restrictions to the Borel sets of F.)

The next theorem and its corollary establish a relation between the measure σ defined on finitary sets and the measure μ defined on Borel sets.

THEOREM 1. *Let σ be a measurable strategy and let μ be the measure induced by σ on the Borel subsets of H. Suppose A and B are subsets of H such that A is finitary, B is Borel and $A \supseteq B$. Then $\sigma(A) \geqq \mu(B)$. (If $A \subseteq B$, then $\sigma(A) \leqq \mu(B)$.)*

PROOF. The proof is by induction on the structure of A (i.e. the structure of 1_A).

Suppose that A has structure at most 1. Then $A = A_1 \times F \times F \times \cdots$ where A_1 is a subset of F. Let $\pi: H \to F$ be the projection map of H onto its first coordinate. Define $\bar{B} = \pi(B) \times F \times F \times \cdots$. Then $\bar{B}$ is measurable with respect to the completion of the Borel sets under μ ([2], p. 391) and $B \subseteq \bar{B} \subseteq A$. Since $\bar{B}$ has structure 1 it is also finitary and $\mu(B) \leqq \mu(\bar{B}) = \sigma_0(\pi(B)) = \sigma(\bar{B}) \leqq \sigma(A)$.

Now assume inductively that the theorem is proved for finitary sets with structure less than α and suppose A has structure α. Let $\mu[f_1]$ denote the measure induced on the Borel subsets of H by the conditional strategy $\sigma[f_1]$. If A^{f_1} and B^{f_1} denote the f_1-sections of A and B, then, by induction, $\mu[f_1](B^{f_1}) \leqq \sigma[f_1](A^{f_1})$. Also, $\mu[f_1]$ is a version of the regular conditional distribution of μ given f_1. Hence,

$$\begin{aligned} \mu(B) &= \int \mu[f_1](B^{f_1})\, d\sigma_0(f_1) \\ &\leqq \int \sigma[f_1](A^{f_1})\, d\sigma_0(f_1) \\ &= \sigma(A). \end{aligned}$$ □

COROLLARY. *If A is both finitary and Borel measurable, then $\sigma(A) = \mu(A)$.*

A result similar to this corollary was proved by Raoult in [3] (Theorem 3.3).

Now suppose Γ is a measurable, gambling house with a goal (i.e. $u = 1_{\{g\}}$) and let $B_g = [f_k = g \text{ i.o.}]$ be the event that g is visited infinitely often. Clearly, B_g is not finitary. Nevertheless, as the next theorem suggests, the gambler seeks strategies which allow him to stay in B_g.

THEOREM 2. *Let σ and μ be as in Theorem 1. Then $u(\sigma) = \mu(B_g)$.*

PROOF. First we show $u(\sigma) \leqq \mu(B_g)$.

Let $\epsilon > 0$. It suffices to find an integer N such that, for every stop rule $t \geqq N$, $\sigma[f_t = g] \leqq \mu(B_g) + \epsilon$.

Let λ be an integer-valued function defined on H which indicates the last time the gambler visits g. Specifically, let

$$\lambda(f_1, f_2, \cdots) = \text{largest } k \text{ such that } f_k = g \text{ if } (f_1, f_2, \cdots) \not\varepsilon B_g$$
$$= 1 \text{ if } (f_1, f_2, \cdots) \varepsilon B_g .$$

Choose N so that $\mu[\lambda \leqq N - 1] > 1 - \epsilon$.

Suppose $t \geqq N$. Set $C = \{(f_1, f_2, \cdots): \exists k \geqq N \ni f_k = g\}$. Then $[f_t = g] \subseteq C$. So, by Theorem 1,

$$\begin{aligned}\sigma[f_t = g] &\leqq \mu(C)\\ &= \mu(C \mid B_g)\mu(B_g) + \mu(C \mid \bar{B}_g \cap [\lambda > N - 1])\mu(\bar{B}_g \cap [\lambda > N - 1])\\ &\leqq \mu(B_g) + \epsilon.\end{aligned}$$

To prove the opposite inequality, let $\epsilon > 0$ and let t_0 be an arbitrary stop rule. It suffices to find a stop rule $t \geqq t_0$ such that $\sigma[f_t = g] \geqq \mu(B_g) - \epsilon$. For $k = 1, 2, \cdots$, let $s_k(h)$ be the time of the kth visit to g along h, if g is visited k-times, and let $s_k(h) = \infty$ otherwise. Then $\mu[s_k < \infty \text{ for } k = 1, 2, \cdots] = \mu(B_g)$ and there exist integers N_k with $N_k < N_{k+1}$ and $\mu(D) > \mu(B_g) - \epsilon$, where $D = [s_k \leqq N_k \text{ for } k = 1, 2, \cdots]$. Define $t_k = s_k \wedge N_k$ and set $t(h) = \min\{t_k(h): t_k(h) \geqq t_0(h)\}$. Notice $[f_t = g] \supseteq [f_{t_k} = g \text{ for } k = 1, 2, \cdots] \supseteq D$. So, by Theorem 1, $\sigma[f_t = g] \geqq \mu(D)$. □

3. An identity for gambling houses with a goal. In order to visit the goal infinitely often, a gambler must first reach the goal and then return infinitely often. This simple fact suggests the following result.

THEOREM 3. *Let Γ be a gambling house with a goal g. Then, for every fortune f,*

$$V(f) = U(f)V(g).$$

PROOF. Define a (possibly incomplete) stop rule t_g to be the first time the gambler reaches g.

Let $\epsilon > 0$. Choose a strategy σ at f and a stop rule t so that $\sigma[f_t = g] > U(f) - \epsilon$. Then choose a strategy σ' at g such that $u(\sigma') > V(g) - \epsilon$. Let $\bar{\sigma}$ be that strategy which uses σ until time t_g and then uses σ'. (The strategy $\bar{\sigma}$ is called the composition of σ with σ' at time t_g. See [1], p. 22 and [6], section 3.) Recall the notation for partial histories $p_t(h) = (f_1, \cdots, f_{t(h)})$. Then

$$\begin{aligned}V(f) &\geqq u(\bar{\sigma})\\ &= \int u(\bar{\sigma}[p_{t_g \wedge t}])\, d\sigma\\ &\geqq \int_{[t_g \leqq t]} u(\bar{\sigma}[p_{t_g}])\, d\sigma\\ &= u(\sigma')\sigma[t_g \leqq t]\\ &\geqq (V(g) - \epsilon)(U(f) - \epsilon).\end{aligned}$$

Since ϵ may be chosen arbitrarily small, one of the desired inequalities is proved.

To prove the other inequality, let $\epsilon > 0$ and choose $\bar{\sigma}$ at f such that $u(\bar{\sigma}) > V(f) - \epsilon$. The strategy $\bar{\sigma}$ may be chosen in such a manner that the conditional strategies $\sigma[p_{t_g}]$ are a constant strategy σ'. (If $\bar{\sigma}$ did not have this property, we could choose an ϵ-optimal strategy σ' at g and let $\hat{\sigma}$ be the composition of $\bar{\sigma}$ with σ' at time t_g. That $\hat{\sigma}$ is almost as good a strategy as $\bar{\sigma}$ can be seen directly or by an application of Lemma 3.2 of [6].)

It now suffices to find a stop rule $\bar{t}$ such that, for any stop rule $t \geqq \bar{t}$, $u(\bar{\sigma}, t) \leqq V(g)U(f) + 2\epsilon$. Choose a stop rule t_0 such that, for $t \geqq t_0$, $u(\sigma', t) \leqq V(g) + \epsilon$. Let $\alpha = \sup \bar{\sigma}[t_g \leqq t]$, where the supremum is taken over all stop rules t. Then choose a stop rule t' such that $\bar{\sigma}[t_g \leqq t'] > \alpha - \epsilon$. Now define

$$\begin{aligned} \bar{t}(h) &= t'(h) && \text{if } t_g(h) > t'(h) \\ &= t_g(h) + t_0(f_{t_g+1}, f_{t_g+2}, \cdots) && \text{if } t_g(h) \leqq t'(h), \end{aligned}$$

where $h = (f_1, f_2, \cdots)$.

If $t \geqq \bar{t}$, then

$$\begin{aligned} u(\bar{\sigma}, t) &= \int_{[t_g \leqq t]} u(\sigma', t[p_{t_g}])\, d\bar{\sigma} \\ &\leqq \int_{[t_g \leqq t']} u(\sigma', t[p_{t_g}])\, d\bar{\sigma} + \epsilon \\ &\leqq (V(g) + \epsilon)\bar{\sigma}[t_g \leqq t'] + \epsilon \\ &\leqq (V(g) + \epsilon)U(f) + \epsilon. \end{aligned}$$ □

4. Measurable houses with a goal. The results of this final section are that for any measurable house with a goal, the function V is measurable with respect to the completion of any Borel measure and there exist "good measurable strategies."

THEOREM 4. *Let Γ be a measurable house with a goal. Then V is absolutely measurable.*

PROOF. The proof is immediate from Theorem 3 and Strauch's result in [4] that U is absolutely measurable. □

As is pointed out in [5] and [6] there are no measurable strategies available in a measurable house Γ unless there is a measurable selection map $\alpha: F \to P$ such that $\alpha(f) \in \Gamma(f)$ for all f. That is, there must be a regular conditional probability distribution on F given F with the additional property of being "available." If there is no such measurable selector, we might hope for good strategies which are available with probability one. A strategy σ is *essentially available* at f if $\sigma_0 \in \Gamma(f)$ and $\sigma_n(f_1, \cdots, f_n) \in \Gamma(f_n)$ for all n along a set of histories with σ-probability one.

THEOREM 5. *Let Γ be a measurable gambling house on F with a goal g and let $\epsilon > 0$. For every f in F, there is a measurable strategy σ essentially available at f such that $u(\sigma) > V(f) - \epsilon$. If Γ has a measurable selector α, then σ can be chosen to be available.*

PROOF. Consider a slightly modified gambling problem with fortunes $F' = F \cup \{g'\}$ where $g' \notin F$. Set $\Gamma'(f) = \Gamma(f)$ for $f \in F$ and $\Gamma'(g') = \Gamma(g)$. Finally, let $u' = 1_{\{g\}}$. Clearly, $V'(g') = V'(g) = V(g)$. By Theorem 3, $V(g) = U'(g')V(g)$. Hence, $U'(g') = 0$ or 1. If $U'(g') = 0$, then $V(g) = 0$ and the theorem is trivial. So assume $U'(g') = 1$. Thus there are policies (σ, t) available at g' in Γ' with $\sigma[f_t = g]$ arbitrarily near one. But these policies are also available at g in Γ. Moreover, by a remark in [6], σ and t may be chosen to be measurable and σ to be available at g until time t (i.e. $\sigma_0 \in \Gamma(g)$ and $\sigma_n(f_1, \cdots, f_n) \in \Gamma(f_n)$ if $t(f_1, \cdots, f_n, \cdots) < n$).

Choose a sequence ϵ_n of positive numbers to satisfy $\prod_{n=1}^{\infty} (1 - \epsilon_n) > 1 - \epsilon$. Then choose a sequence of measurable policies (σ_n, t_n) so that σ_n is available at g until time t_n and $\sigma_n[f_{t_n} = g] > 1 - \epsilon_n$. Again by [6], choose a measurable policy (σ_0, t_0) such that σ_0 is available at f until time t_0 and $\sigma_0[f_{t_0} = g] > U(f) - \epsilon$.

Now we can define the strategy σ. Roughly, σ uses σ_0 until time t_0, then conditionally uses σ_1 until time t_1 and so on. More precisely, define a sequence of stop rules s_n. Let $s_0 = t_0$ and $s_{n+1}(f_1, \cdots, f_{s_n}, f_{s_n+1}, \cdots) = s_n(f_1, \cdots) + t_n(f_{s_n+1}, \cdots)$. Define σ to agree with σ_0 until time t_0. Suppose σ has been defined along each history until time s_n. If $f_{s_n} = g$, define $\sigma[p_{s_n}]$ to agree with σ_{n+1} until time t_{n+1}. If $f_{s_n} \neq g$ and Γ has no measurable selector, let $\sigma[p_{s_n}]$ be an arbitrary measurable family which is essentially available. If $f_{s_n} = g$ and Γ has a measurable selector α, let $\sigma[p_{s_n}]$ be that family of strategies which constantly uses the gamble $\alpha(f)$ whenever the current fortune is f.

Let μ be the measure induced by σ on the Borel subsets of H. By Theorem 2,

$$\begin{aligned} u(\sigma) &= \mu(B_g) \\ &\geqq \mu[f_{s_n} = g, n = 0, 1, \cdots] \\ &\geqq (U(f) - \epsilon)(1 - \epsilon) \\ &\geqq (V(f) - \epsilon)(1 - \epsilon). \end{aligned}$$

Since ϵ was arbitrary, the proof is complete. □

Acknowledgment. I am grateful to Lester Dubins, David Gilat, and Roger Purves for several stimulating discussions.

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