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TESTS FOR MONOTONE FAILURE RATE II[1]

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1. Introduction. In a previous paper [2] K. Doksum and the author investigated the asymptotic power behavior of various tests in the following problem.

We observe $X_1, \cdots, X_n$ independent and identically distributed observations from a population with unknown distribution function F, such that $F(0) = 0$. We wish to test the hypothesis that F is a negative exponential distribution with (un)known scale parameter against the alternative that F has monotone increasing nonconstant failure rate. This problem arises in life testing situations.

In accordance with the notation of [2] let $X_{(1)} \leqq \cdots \leqq X_{(n)}$ be the order statistics of the sample and define, taking $X_{(0)} = 0$,

$$D_i = (n - i + 1)(X_{(i)} - X_{(i-1)}), \qquad 1 \leqq i \leqq n,$$

to be the normalized sample spacings. Finally let R_i be the rank (counting from the bottom) of D_i among $D_1, \cdots, D_n$. If the null hypothesis is that F is negative exponential with unknown scale parameter various tests of level α have been proposed based on,

(i) Statistics of the form $\sum_{i=1}^n h_n(X_i/(\sum X_i))$ ([8]),
(ii) Linear functions of $(D_1/(\sum D_i), \cdots, D_n/(\sum D_i))$ ([1], [7], [9]),
(iii) Locally most powerful tests based on $(R_1, \cdots, R_n)$ ([2]),
(iv) Linear functions of

$$[-\log(1 - R_1(n+1)^{-1}), \cdots, -\log(1 - R_n(n+1)^{-1})] \qquad ([2]).$$

All such tests are similar for the null hypothesis. In this paper we shall establish a result stated in [2] to the effect that under the regularity conditions of Theorem 4.1 of [2] each one of the classes (i), (ii), (iii) and (iv) contains a test asymptotically equivalent to the asymptotically most powerful similar test.

This is in sharp contrast to the situation when the scale parameter is known, for which it was shown in [2] that the last three classes do not contain asymptotically most powerful tests and consequently that the ranks are not asymptotically sufficient. With the slight additional uncertainty concerning the scale the ranks yield asymptotically as much information as the "natural" statistic $[X_1/(\sum X_i), \cdots, X_n/(\sum X_i)]$.

2. The main results The notation and assertions of this section are based on those of [2] Section 1. For a fuller discussion we refer to that paper.

Let $\{f_\theta\}$, $\theta \geqq 0$, be a family of densities all vanishing off the nonnegative axis

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such that $f_0(t) = e^{-t}$, $t \geqq 0$, is the standard negative exponential density. We introduce an unknown (inverse) scale parameter $\lambda > 0$ and define,

$$f_{(\theta,\lambda)}(t) = \lambda f_\theta(\lambda t). \tag{2.1}$$

In this model we are testing $H:\theta = 0$ versus $K:\theta > 0$ on the basis of independent observations $X_1, \cdots, X_n$ each distributed according to $f_{(\theta,\lambda)}$. As in[2] assume the existence and finiteness of,

$$h_\lambda(t) = \partial \log f_{(\theta,\lambda)}(t)/\partial\theta \,|_{\theta=0} \tag{2.2}$$

and let θ_n be a sequence of parameter values such that $\lim_n n^{\frac{1}{2}}\theta_n = b$ where $0 \leqq b < \infty$.

As usual, we say a (randomized) test $\varphi_n(X_1, \cdots, X_n)$ is *similar of size* α if and only if,

$$E_{(0,\lambda)}(\varphi(X_1, \cdots, X_n)) = \alpha \tag{2.3}$$

for all $\lambda > 0$.

The notation $E_{(\theta,\lambda)}$, $P_{(\theta,\lambda)}$ is used for expectations of random variables and probabilities of events defined on the sample space when (θ, λ) is the true value of the parameter. Our first theorem is,

THEOREM 2.1. *Let $\{\varphi_n(X_1, \cdots, X_n)\}$ be any sequence of (randomized) tests each similar of size α. Suppose that $\{f_{(\theta,\lambda)}\}$ satisfy conditions* (3.5) (a), (b), (c) *of* [2] *and hence that $\{f_{(\theta_n,\lambda)}\}$ are contiguous to $f_{(0,\lambda)}$ for each fixed λ in the sense of* [2]. *Then,*

$$\limsup_n E_{(\theta_n,\lambda)}[\varphi_n(X_1, \cdots, X_n)] \leqq 1 - \Phi(\Phi^{-1}(1 - \alpha) - \sigma_1(h_1)b), \tag{2.4}$$

where Φ is the standard normal cdf, *Φ^{-1} is its inverse and*

$$\sigma_1^2(h_1) = \int_0^\infty h_1^2(t)e^{-t}\,dt - [\int_0^\infty th_1(t)e^{-t}\,dt]^2. \tag{2.5}$$

Consider the following similar tests;

$$\begin{aligned} \varphi_n^{(j)}(X_1, \cdots, X_n) &= 1 \quad \text{if} \quad T_n^{(j)} > c_n^{(j)} \\ &= \gamma_n^{(j)} \quad \text{if} \quad T_{n_v}^{(j)} = c_n^{(j)} \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{2.6}$$

where $1 \leqq j \leqq 4$, and the statistics $T_n^{(j)}$ are defined as follows.

$$T_n^{(1)} = n^{-\frac{1}{2}} \sum_{i=1}^n h_1(nX_i[\sum X_i]^{-1}), \tag{2.7}$$

$$T_n^{(2)} = n^{\frac{1}{2}}\{\sum_{i=1}^n a(i(n+1)^{-1}D_i(\sum_{i=1}^n D_i)^{-1} - \bar{a}\} \tag{2.8}$$

where

$$a(t) = (1-t)^{-1}\int_{-\log(1-t)}^\infty h_1'(s)e^{-s}\,ds, \tag{2.9}$$

for $0 < t < 1$ and,

$$\bar{a} = n^{-1}\sum_{i=1}^n a(i/(n+1)). \tag{2.10}$$

As usual h_1' is the derivative of h_1.

$$T_n^{(3)} = n^{-\frac{1}{2}} \sum_{i=1}^n E_{(0,1)}[h_1(X_i) | R_1, \cdots, R_n] \tag{2.11}$$
$$= n^{-\frac{1}{2}} \sum_{i=1}^n E_{(0,1)}[h_1(\sum_{j=1}^i X_{(R_j)}(n - j + 1)^{-1})].$$

The symbol $E_{(0,1)}[\cdot | R_1, \cdots, R_n]$ denotes conditional expectation given $R_1, \cdots, R_n$ and the second line in (2.11) refers to an expectation computed when $R_1, \cdots, R_n$ are treated as constants.

Finally,

$$T_n^{(4)} = -n^{-\frac{1}{2}} \sum_{i=1}^n (a(i/n + 1) - \bar{a}) \log (1 - R_i(n + 1)^{-1}). \tag{2.12}$$

The constants $c_n^{(j)}$ and $\gamma_n^{(j)}$ where $0 \leqq \gamma_n^{(j)} \leqq 1$ are chosen so that,

$$E_{(0,\lambda)}(\varphi_n^{(j)}(X_1, \cdots, X_n)) = \alpha. \tag{2.13}$$

The test proposed by Moran [8] when $\{f_{\theta,\lambda}\}$ is a family of gamma densities is of the class $\varphi_n^{(1)}$.

If $\{f_{\theta,\lambda}\}$ is a density of "generalized Makeham" type it was shown in [2] that the test $\varphi_n^{(2)}$ corresponds to the "total time on test" procedure advanced by Lewis [7], Nadler and Eilbott [9] and Barlow [1]. The whole class of tests of type $\varphi_n^{(2)}$ are locally most powerful rank tests for the given parametric family but are unfortunately quite complex. Type $\varphi_n^{(4)}$ tests are simple and asymptotically equivalent to type $\varphi_n^{(3)}$. Our main result is,

THEOREM 2.2. *If the assumptions of Theorem* 2.1 *are satisfied then,*

$$\lim_n E_{(\theta_n,\lambda)}[\varphi_n^{(1)}(X_1, \cdots, X_n)] = 1 - \Phi(\Phi^{-1}(1 - \alpha) - \sigma_1(h_1)b). \tag{2.14}$$

If in addition the assumptions of Theorem 4.1 *of* [2] *hold, then*

$$\lim_n E_{(\theta_n,\lambda)}[\varphi_n^{(j)}(X_1, \cdots, X_n)] = 1 - \Phi(\Phi^{-1}(1 - \alpha) - \sigma_1(h_1)b) \tag{2.15}$$

for $2 \leqq j \leqq 4$ *as well. In fact, all the tests* $\varphi_n^{(j)}$ *are asymptotically equivalent in the sense that the* $T_n^{(j)}$ *have a nondegenerate limiting distribution under* H *and* $T_n^{(i)} - T_n^{(j)} \to 0$ *in* $P_{(0,\lambda)}$ *and* $P_{(\theta_n,\lambda)}$ *probability whatever be* i *and* j.

COROLLARY 2.1. *If in addition to all the hypotheses of Theorem* 2.2

$$\lim_n E_{(\theta_n,\lambda)}[\varphi^{(j)}(X_1, \cdots, X_n)] = 1 \tag{2.16}$$

whenever $n^{\frac{1}{2}}\theta_n \to \infty$ *it follows that the sequence* $\{\varphi_n^{(j)}\}$ *is asymptotically most powerful among all similar level* α *tests in the sense of Wald (see* [2] *for a discussion).*

The corollary is an immediate consequence of Theorems 2.1 and 2.2 in view of the definition of asymptotically most powerful test. The proofs of the theorems as well as the statement of a generalization of a result of LeCam [5] which is central to our argument are given in the next section. The final appendix contains a sketch proof of the generalization of LeCam's theorem.[2]

[2] A result considerably more general than our Theorem 3.1 has recently been proved by M. Wichura in his thesis [11]. His result also encompasses another extension of LeCam's theorem (to functionals of the sample spacings more general than $T_{n,n}$ which may be found in Pyke's interesting paper [10].

3. Proof of Theorems 2.1 and 2.2 and a generalization of a theorem of LeCam. The result of LeCam that we need is given by,

THEOREM (LeCam). *Suppose that the X_i are independently and identically distributed according to a standard negative exponential distribution. Let*

$$S_n = n^{-\frac{1}{2}} \textstyle\sum_{i=1}^n (X_i - 1). \tag{3.1}$$

Suppose that $\{g_n\}$ is a sequence of measurable functions such that the $g_n(X_i)$ are u.a.n.

$$\mathcal{L}(\textstyle\sum_{i=1}^n g_n(X_i), S_n) \to \mathcal{L}(T, S) \tag{3.2}$$

where $\mathcal{L}(\cdot, \cdot)$ denotes "joint law" and $\mathcal{L}(T, S)$ is some limit law.

Then $\mathcal{L}(T, S)$ has characteristic function $\varphi(t, s) = E(\exp\{isS + itT\})$ which is necessarily of the form,

$$\varphi(t, s) = \omega(t)\psi(t, s) \tag{3.3}$$

with

$$\log \psi(t, s) = -\tfrac{1}{2}\{s^2 + 2Bst + C^2t^2\}. \tag{3.4}$$

Then, the conditional law of $\sum_{i=1}^n g_n(X_i)$ given $S_n = 0$,

$$\mathcal{L}(\textstyle\sum_{i=1}^n g_n(X_i) \mid S_n = 0) \to \mathcal{L}(T_0) \tag{3.5}$$

where T_0 has characteristic function

$$E(\exp itT_0) = \omega(t) \exp - \tfrac{1}{2}(C^2 - B^2)t^2. \tag{3.6}$$

The modifications we need follow. We use the notation of the previous theorem.

THEOREM 3.1. *Under the assumptions of LeCam's theorem,*

$$\mathcal{L}(\textstyle\sum_{i=1}^n g_n(X_i) \mid S_n = \gamma) \to \mathcal{L}(T_\gamma) \tag{3.7}$$

uniformly in γ for $|\gamma| \leqq M$ for any finite M. The variable T_γ has characteristic function given by

$$E(\exp itT_\gamma) = \omega(t) \exp\{-\tfrac{1}{2}(C^2 - B^2)t^2 + itB\gamma\}. \tag{3.8}$$

From this theorem we can derive,

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1 *if $\omega(t) = \exp it\mu$ for some μ,* i.e. *if $\mathcal{L}(T, S)$ is the joint normal distribution with $E(T) = \mu$,* Var $T = C^2$ *and* cov $(S, T) = B$ *then T_γ is normal with mean $\mu + B\gamma$ and variance $(C^2 - B^2)$.*

More generally suppose we are given a family of functions $\{g_n^{(\sigma)}\}$ where σ ranges over [0, 1] such that,

$$\mathcal{L}(\textstyle\sum_{i=1}^n g_n^{(\sigma)}(X_i), S_n) \to \mathcal{L}(T^{(\sigma)}, S) \tag{3.9}$$

uniformly in σ. Let $\omega^{(\sigma)}(\cdot)$, $B^{(\sigma)}$, $C^{(\sigma)}$ be the characteristic functions and constants of the representation. Write

$$\omega^{(\sigma)}(t) = \rho^{(\sigma)}(t) \exp[i \arg(\omega^{(\sigma)}(t))] \tag{3.10}$$

where arg. $(\omega^{(\sigma)}(t))$ is chosen to be continuous in t. This is possible by analytic continuation of log z since $\omega^{(\sigma)}(\cdot)$, being the characteristic function of an infinitely divisible distribution, never vanishes. We then have,

THEOREM 3.2. *Suppose that* $\{g_n^{(\sigma)}\}$, *satisfying* (3.9), $\omega^{(\sigma)}$, *etc. are as above. Assume that* $B^{(\sigma)}$, $C^{(\sigma)}$ *and* $\omega^{(\sigma)}(t)$ *are continuous functions of* σ. *Then*

$$\mathcal{L}\left(\textstyle\sum_{i=1}^{n} g_n^{(\sigma)}(X_i) \mid S_n = \gamma\right) \to \mathcal{L}(T_\gamma^{(\sigma)}) \tag{3.11}$$

uniformly in $|\gamma| \leqq M$ *and* σ, *where* $\mathcal{L}(T_\gamma^{(\sigma)})$ *is given by* (3.8) (*with suitable superscripts*).

The proof of Theorem 3.2 which evidently implies Theorem 3.1 and its corollary is deferred to Section 4.

PROOF OF THEOREM 2.1. We begin by noting that if $\theta = 0$ the statistic S_n is complete and sufficient for λ and hence by Theorem 2 of Section 4.3 of [6] any similar test must have Neyman structure. It follows by the Neyman-Pearson lemma that if we let

$$V_n = \textstyle\sum_{i=1}^{n} [\log f_{(\theta_n,\lambda)}(X_i) - \log f_{(0,\lambda)}(X_i)] \tag{3.12}$$

then there exist random variables $c_n(S_n)$ and $\gamma_n(S_n)$ with $0 \leqq \gamma_n(S_n) \leqq 1$ such that,

$$P_{(0,\lambda)}[V_n > c_n(S_n) \mid S_n] + \gamma_n(S_n) P_{(0,\lambda)}[V_n = c_n(S_n) \mid S_n] = \alpha \quad \text{a.s.} \tag{3.13}$$

and

$$P_{(\theta_n,\lambda)}[V_n > c_n(S_n)] + \int_{[V_n = c_n(S_n)]} \gamma_n(S_n)\, dP_{(\theta_n,\lambda)} \geqq E_{(\theta_n,\lambda)}[\varphi(X_1, \cdots, X_n)] \tag{3.14}$$

for any similar level α test φ. It is clear from the structure of the problem that we can take $\lambda = 1$.

By the assumptions of Theorem 2.1 we know that (see [2]),

$$V_n - bn^{-\frac{1}{2}} \textstyle\sum_{i=1}^{n} h_1(X_i) + \frac{1}{2} b^2 E_{(0,1)}(h_1^2(X_1)) \to 0 \tag{3.15}$$

in $P_{(0,1)}$ and by contiguity in $P_{(\theta_n,1)}$ probability.

It follows that under the null hypothesis if $\lambda = 1$, (V_n, S_n) are asymptotically jointly normal with

$$\begin{aligned} E(V) &= -\tfrac{1}{2} b^2 E_{(0,1)}[h_1^2(X_1)] = -\tfrac{1}{2} b^2 \textstyle\int_0^\infty h_1^2(t) e^{-t}\, dt, \\ \operatorname{Var} V &= b^2 \textstyle\int_0^\infty h_1^2(t) e^{-t}\, dt, \\ \operatorname{cov}(V, S) &= b \textstyle\int_0^\infty t h_1(t) e^{-t}\, dt, \end{aligned} \tag{3.16}$$

where (V, S) is the limit law of (V_n, S_n).

Applying Theorem 3.1 we see that

$$\mathcal{L}(V_n \mid S_n = \gamma) \to \mathcal{L}(V_\gamma) \tag{3.17}$$

uniformly in $|\gamma| \leqq M < \infty$ where V_γ is normally distributed with mean $-\frac{1}{2} b^2 \int_0^\infty h_1^2(t) e^{-t}\, dt + \gamma b \int_0^\infty t h_1(t) e^{-t}\, dt$, and variance $\sigma_1^2(h_1)$.

Now, for every $\epsilon > 0$ and every n there exists (by the central limit theorem) $M < \infty$ such that,

$$P_{(0,1)}[|S_n| \leqq M] \geqq 1 - \epsilon. \tag{3.18}$$

From (3.13), (3.17) and (3.18) we conclude that,

$$[c_n(S_n) + \tfrac{1}{2}b^2 \textstyle\int_0^\infty h^2(t)e^{-t}\,dt - bS_n \int_0^\infty th(t)e^{-t}\,dt] \to \Phi^{-1}(1-\alpha)b\sigma_1(h_1), \tag{3.19}$$

while

$$\gamma_n(S_n) \to 0 \tag{3.20}$$

in $P_{(0,1)}$ probability and by contiguity in $P_{(\theta_n,1)}$ probability.

Finally, if

$$\begin{aligned} U_n &= V_n + \tfrac{1}{2}b^2 \textstyle\int_0^\infty h^2(t)e^{-t}\,dt - bS_n \int_0^\infty th(t)e^{-t}\,dt \\ &= bn^{-\frac{1}{2}} \textstyle\sum_{i=1}^n [h_1(X_i) - (X_{i-1}) \int_0^\infty th(t)e^{-t}\,dt], \end{aligned} \tag{3.21}$$

then under the null hypothesis with $\lambda = 1$, (U_n, V_n) tends in law to (U, V) a joint normal law with U having mean 0 variance $b^2\sigma_1^2(h)$ and with covariance $(U, V) = b^2\sigma_1^2(h)$.

By LeCam's third lemma [4], p. 208,

$$\mathcal{L}_{(\theta_n,1)}(U_n) \to \mathfrak{N}(b^2\sigma_1^2(h_1), b^2\sigma_1^2(h_1)) \tag{3.22}$$

where $\mathfrak{N}(\mu, \sigma^2)$ denotes a normal law with mean μ and variance σ^2. Therefore combining (3.19), (3.20) and (3.22),

$$\begin{aligned} &P_{(\theta_n,\lambda)}[V_n > c_n(S_n) + \textstyle\int_{[V_n = c_n(S_n)]} \gamma_n(S_n)\,dP_{(\theta_n,\lambda)} \\ &\doteq P_{(\theta_n,1)}[U_n > \Phi^{-1}(1-\alpha)bc_1(h_1)] \to [1 - \Phi(\Phi^{-1}(1-\alpha) - b\sigma_1(h_1))]. \end{aligned} \tag{3.23}$$

From (3.23) and (3.14) we obtain Theorem 2.1.

Proof of Theorem 2.2 Theorem 2.2 is a consequence of

$$T_n^{(j)} - n^{-\frac{1}{2}} \textstyle\sum_{i=1}^n h_1(X_i) + S_n[\int_0^\infty th_1(t)e^{-t}\,dt] \to 0 \tag{3.24}$$

in $P_{(0,1)}$ and hence in $P_{(\theta_n,1)}$ probability.

By the usual computations involving LeCam's third lemma and Slutsky's theorem the assertion (2.14) follows if $\lambda = 1$ as does the second assertion of theorem 2.2 about the equivalence of the $T_n^{(j)}$. Since the distribution of the $T_n^{(j)}$ is independent of λ the theorem will follow. A careful examination of the proof of Theorem 2.1 and an application of (3.24) will reveal the additional interesting fact that the critical regions of the most powerful similar level α tests of the null hypothesis versus the alternatives $f_{(\theta_n,\lambda)}$ are asymptotically equivalent to the regions given by $\varphi_n^{(j)}$.

We proceed with the proof of (3.24). It was shown in [2] that

$$T_n^{(4)} - n^{-\frac{1}{2}} \textstyle\sum_{i=1}^n (a(i/(n+1)) - \bar{a})(D_i - 1) \to 0 \tag{3.25}$$

in $P_{(0,1)}$ probability. But

$$(3.26)\quad n^{-\frac{1}{2}}\sum_{i=1}^{n}(a(i/(n+1)-\bar{a})(D_i-1) = n^{-\frac{1}{2}}\sum_{i=1}^{n}a(i/(n+1))(D_i-1)-\bar{a}n^{-\frac{1}{2}}\sum_{i=1}^{n}(X_i-1).$$

Also in [2] it was shown that the first term on the right hand side of (3.26) is asymptotically equivalent to $n^{-\frac{1}{2}}\sum_{i=1}^{n}h_1(X_i)$ if $\theta = 0$ and $\lambda = 1$.

On the other hand,

$$(3.27)\quad \bar{a}\to\int_0^1 a(t)\,dt = \int_0^1(1-t)^{-1}\int_{-\log(1-t)}^{\infty}h_1'(s)e^{-s}\,ds\,dt = -\int_0^{\infty}sh_1'(s)e^{-s}\,ds = -\int_0^{\infty}sh_1(s)e^{-s}\,ds,$$

since

$$(3.28)\quad \int_0^{\infty}h_1(s)e^{-s}\,ds = 0.$$

Therefore, (3.24) holds for $T_n^{(4)}$. In [2] it is also argued that

$$(3.29)\quad T_n^{(3)} - n^{-\frac{1}{2}}\sum_{i=1}^{n}(a(i/(n+1))-\bar{a})E_{(0,1)}[X_{(R_i)}]\to 0$$

in $P_{(0,1)}$ probability.

By a result of Hájek [3],

$$(3.30)\quad n^{-\frac{1}{2}}\sum_{i=1}^{n}(a(i/(n+1)-\bar{a})E_{(0,1)}(X_{(R_i)}) - n^{-\frac{1}{2}}\sum_{i=1}^{n}(a(i/(n+1))-\bar{a})(D_i-1)\to 0$$

in $P_{(0,1)}$ probability and (3.24) follows for $T_n^{(3)}$. Similarly by Slutsky's theorem,

$$(3.31)\quad n^{\frac{1}{2}}\{\sum_{i=1}^{n}a(i/(n+1))D_i(\sum_{i=1}^{n}D_i)^{-1}-\bar{a}\} - n^{-\frac{1}{2}}\sum_{i=1}^{n}(a(i/(n+1))-\bar{a})(D_i-1)\to 0$$

in $P_{(0,1)}$ probability and (3.24) holds for $T_n^{(2)}$. For $T_n^{(1)}$ we argue as follows. It suffices to show that,

$$(3.32)\quad P_{(0,1)}[|T_n^{(1)} - n^{-\frac{1}{2}}\sum_{i=1}^{n}h_1(X_i)+\gamma[\int_0^{\infty}th_1(t)e^{-t}\,dt]\mid\geqq\epsilon|S_n=\gamma]\to 0$$

uniformly in $|\gamma|\leqq M$ for each $\epsilon>0$ and $M<\infty$. Equivalently since $n^{-1}\sum_{i=1}^{n}X_i = n^{-\frac{1}{2}}S_n+1$ we must show that

$$(3.33)\quad \mathfrak{L}_{(0,1)}\{n^{-\frac{1}{2}}[\sum_{i=1}^{n}[h_1(X_i(1+\gamma n^{-\frac{1}{2}})^{-1})-h_1(X_i)]] + \gamma\int_0^{\infty}th_1(t)e^{-t}\,dt\mid S_n=\gamma\}\to\mathfrak{L}(\{0\})$$

uniformly in $|\gamma|\leqq M<\infty$, where $\mathfrak{L}(\{0\})$ is the law of the random variable degenerate at 0.

In view of Theorem 3.2, (3.33) will follow if we can show

$$(3.34)\quad n^{-\frac{1}{2}}\sum_{i=1}^{n}[h(X_i(1+\gamma n^{-\frac{1}{2}})^{-1})-h_1(X_i)]\to -\gamma\int_0^{\infty}th(t)e^{-t}\,dt$$

in $P_{(0,1)}$ probability uniformly for $|\gamma|\leqq M<\infty$, whatever be M. But,

$$
\begin{aligned}
&E_{(0,1)}\{n^{-\frac{1}{2}}[\textstyle\sum_{i=1}^{n} h_1(X_i(1+\gamma n^{-\frac{1}{2}})^{-1}) - h_1(X_i)] + \gamma \int_0^\infty th(t)e^{-t}\,dt\}^2 \\
(3.35)\quad &= \mathrm{Var}_{(0,1)}\,[h_1(X_1(1+n^{-\frac{1}{2}})^{-1}) - h_1(X_1)] \\
&\quad + [n^{\frac{1}{2}}E_{(0,1)}[h(X_1(1+\gamma n^{-\frac{1}{2}})^{-1}) - h_1(X_1)] + \gamma \textstyle\int_0^\infty th(t)e^{-t}\,dt]^2.
\end{aligned}
$$

Now

$$
\begin{aligned}
(3.36)\quad & n^{\frac{1}{2}}E_{(0,1)}[h_1(X_1(1+\gamma n^{-\frac{1}{2}})^{-1}) - h_1(X_1)] \\
&= n^{\frac{1}{2}} \int_0^\infty h_1(t)[e^{-t(1+\lambda n^{-\frac{1}{2}})}(1+\gamma n^{-\frac{1}{2}}) - e^{-t}]\,dt \to -\gamma \int_0^\infty th(t)e^{-t}\,dt,
\end{aligned}
$$

uniformly for $|\gamma| \leqq M$ by a standard application of the dominated convergence theorem and (3.28).

Finally,

$$
\begin{aligned}
(3.37)\quad & \mathrm{Var}_{(0,1)}\,[h_1(X_1(1+\gamma n^{-\frac{1}{2}})^{-1}) - h_1(X_1)] \\
&\leqq \int_0^\infty (h_1(t(1+\gamma n^{-\frac{1}{2}})^{-1}) - h_1(t))^2 e^{-t}\,dt.
\end{aligned}
$$

Let $h_n = \log(1+\gamma n^{-\frac{1}{2}})$. Then the right hand side of (3.37) is bounded by,

$$
\begin{aligned}
& 2[\textstyle\int_0^\infty (\exp(y-h_n))\exp\frac{1}{2}\{[y-h_n] - \exp[y-h_n]\} \\
(3.38)\quad & - h_1(\exp y)\exp\tfrac{1}{2}\{y - \exp y\})^2\,dy \\
&= \textstyle\int_0^\infty h_1^2(\exp y)[\exp\frac{1}{2}\{(y+h_n) - \exp(y+h_n)\} - \exp\frac{1}{2}\{y-\exp y\}]^2\,dy]^2.
\end{aligned}
$$

The first term in (3.38) goes to 0 by the L_2 continuity theorem while the second tends to 0 by the dominated convergence theorem. That the convergence to 0 is uniform in $|\gamma| \leqq M$ is clear. Therefore (3.34) and Theorem 2.2 are established.

It is plausible to conjecture that under the assumptions of Theorem 3.1,

$$
(3.39)\qquad \textstyle\sum_{i=1}^{n} g_n(nX_i(\sum_{i=1}^{n} X_i)^{-1}) + BS_n - \sum_{i=1}^{n} g_n(X_i) \to 0
$$

in probability. L. LeCam suggested the following counterexample to this conjecture. Let h_n be the indicator of the interval $[1, 1+n^{-1}]$. It is easy to see that,

(i) $\sum_{i=1}^{n} h_n(X_i)$ and S_n are asymptotically independent.

(ii) $\sum_{i=1}^{n} h_n(X_i)$ and $\sum_{i=1}^{n} h_n(X_i(1+\gamma n^{-\frac{1}{2}})^{-1})$ are asymptotically independent and have identical Poisson limiting distributions, uniformly in $|\gamma| \leqq M$. Applying Theorem 3.2 we see that $\sum_{i=1}^{n} h_n(nX_i[\sum_{i=1}^{n} X_i]^{-1}) - \sum_{i=1}^{n} h_n(X_i)$ is asymptotically distributed as the difference of two independent Poisson variables.

4. Proof of Theorem 3.2. For any $1 \leqq m \leqq n$ define

$$
(4.1)\qquad S_{n,m} = n^{-\frac{1}{2}} \textstyle\sum_{j=1}^{m} (X_j - 1),
$$

and similarly,

$$
(4.2)\qquad T_{n,m}^{(\sigma)} = \textstyle\sum_{i=1}^{m} g_n^{(\sigma)}(X_i).
$$

Suppose $0 < \alpha < 1$ and let $m_n/n \to \alpha$. Then, by Lemma 1 of [5] we have that

$$
(4.3)\qquad \mathfrak{L}(T_{n,m_n}^{(\sigma)}, S_{n,m_n}) \to \mathfrak{L}(T_\alpha^{(\sigma)}, S_\alpha)
$$

uniformly in σ where S_α has a normal $(0, \alpha^{\frac{1}{2}})$ distribution and the characteristic function $\varphi_\alpha^{(\sigma)}(t, s)$ of $(T_\alpha^{(\sigma)}, S_\alpha)$ is given by

$$\varphi_\alpha^{(\sigma)}(t, s) = [\varphi^{(\sigma)}(t, s)]^\alpha. \tag{4.4}$$

(Note that the complex valued characteristic function is raised to the αth power by multiplying $\arg \omega^{(\sigma)}(t)$ by α.)

Let f_{n,m_n} denote the density of S_{n,m_n}. Then Lemma 1 of [5] states,

$$\int |f_{n,m_n}(x) - f_\alpha(x)| \to 0 \tag{4.5}$$

where f_α is the density of the normal $(0, \alpha^{\frac{1}{2}})$ distribution. Define

$$q_{n,m_n}^{(\sigma)}(x) = E(\exp\{itT_{n,m_n}^{(\sigma)}\} \mid S_{n,m_n} = x) \tag{4.6}$$

and

$$q_\alpha^{(\sigma)}(x) = E(\exp\{itT_\alpha^{(\sigma)}\} \mid S_\alpha = x). \tag{4.7}$$

Since $\mathcal{L}(T_{n,m_n}^{(\sigma)}, S_{n,m_n}) \to \mathcal{L}(T_\alpha^{(\sigma)}, S_\alpha)$ we must have

$$\int_0^\infty q_{n,m_n}^{(\sigma)}(x) v(x) f_{n,m_n}(x)\, dx \to \int_0^\infty q_\alpha^{(\sigma)}(x) v(x) f_\alpha(x)\, dx \tag{4.8}$$

uniformly in σ and uniformly for $v \varepsilon V$ where V is any set of uniformly bounded, equicontinuous functions. In view of (4.5) we may argue that

$$\int_0^\infty q_{n,m_n}^{(\sigma)}(x) v(x) f_\alpha(x)\, dx \to \int_0^\infty q_\alpha^{(\sigma)}(x) v(x) f_\alpha(x)\, dx \tag{4.9}$$

uniformly in σ and $v \varepsilon V$.

The conditional density of S_{n,m_n} given that $S_{n,n} = \gamma$ which we write $p_{n,m_n}^{(\gamma)}(x)$, is given by,

$$p_{n,m_n}^{(\gamma)}(x) = (1 + \gamma n^{-\frac{1}{2}})^{-1} p_{n,m_n}^{(0)}((x - \gamma m_n n^{-1})(1 + \gamma n^{-\frac{1}{2}})^{-1}). \tag{4.10}$$

It follows from a standard calculation given in [5] that,

$$p_{n,m_n}^{(\gamma)}(x) \to f_{\alpha(1-\alpha)}^{(\gamma)}(x) \tag{4.11}$$

uniformly in $|\gamma| \leqq M$ where $f_{\alpha(1-\alpha)}^{\gamma}(x)$ is the density of the normal distribution with mean $\gamma\alpha$ and variance $\alpha(1 - \alpha)$. If $|\gamma| \leqq M$, the set $\{p_{n,m_n}^{(\gamma)}\}$ is uniformly bounded and equicontinuous. From (4.9) if we let

$$r_{n,m_n}^{(\gamma,\sigma)} = E(\exp itT_{n,m_n}^{(\sigma)} \mid S_{n,n} = \gamma)$$

we obtain,

$$r_{n,m_n}^{(\gamma,\sigma)} \to \int q_\alpha^{(\sigma)}(x) f_{\alpha(1-\alpha)}^{(\gamma)}(x)\, dx = r_\alpha^{(\gamma,\sigma)}(t) \tag{4.12}$$

uniformly in σ and $|\gamma| \leqq M$. Of course,

$$\int q_\alpha^{(\sigma)}(x) \exp[isx] f_\alpha(x)\, dx = [\varphi^{(\sigma)}(t, s)]^\alpha. \tag{4.13}$$

Arguing as in [5], p. 12, we find that

$$(4.14)\quad r_\alpha^{(\gamma,\sigma)}(t) = (\exp \tfrac{1}{2}\gamma^2)[\omega^{(\sigma)}(t)]^\alpha(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp -\tfrac{1}{2}\{s^2 + 2(B^{(\sigma)}\alpha t + i\gamma)s + [C^{(\sigma)}]^2\alpha t^2\}\, ds$$
$$= [\omega^{(\sigma)}(t)]^\alpha \exp\{it[B^{(\sigma)}\alpha\gamma] - \tfrac{1}{2}([C^{(\sigma)}]^2\alpha - [B^{(\sigma)}]^2)\}.$$

Evidently by the continuity conditions of the theorem

$$(4.15)\qquad r_\alpha^{(\gamma,\sigma)}(t) \to E(\exp itT_\gamma^{(\sigma)})$$

uniformly in $|\gamma| \leqq M$, c as $\alpha \to 1$ and

$$(4.16)\qquad r_\alpha^{(\gamma,\sigma)}(t) \to 1$$

uniformly in $|\gamma| \leqq M$, σ as $\alpha \to 0$. (It is easy to see that arg $\omega^{(\sigma)}(t)$ is jointly continuous in σ and t.)

Suppose $\mathcal{L}(T_{n,n}^{(\sigma)} \mid S_{n,n} = \gamma)$ does not converge to $\mathcal{L}(T_\gamma^{(\sigma)})$ uniformly in $|\gamma| \leqq M$, σ. Then there exist sequences $\{\gamma_n\}$, $\{\sigma_n\}$ which may be taken to converge to γ_0, σ_0 respectively such that, $\mathcal{L}(T_{n,n}^{(\sigma_n)} \mid S_{n,n} = \gamma_n)$ do not converge to $\mathcal{L}(T_{\gamma_0}^{(\sigma_0)})$ while $\mathcal{L}(T_{\gamma_n}^{(\sigma_n)})$ do so converge, by the continuity assumptions of the theorem. Then, there exists t_0, $\epsilon > 0$ such that,

$$(4.17)\qquad |E(\exp it_0T_{n,n}^{(\sigma_0)} \mid S_{n,n} = \gamma_n) - E(\exp it_0T_{\gamma_0}^{(\sigma_0)})| \geqq \epsilon$$

for all n. Write

$$(4.18)\qquad T_{n,n}^{(\sigma_n)} = T_{n,m_n}^{(\sigma_n)} + \hat{T}_{n,m_n}^{(\sigma_n)}.$$

Clearly $\mathcal{L}(\hat{T}_{n,m_n}^{(\sigma_n)} \mid S_{n,n} = \gamma) = \mathcal{L}(T_{n,n-m_n}^{(\sigma_n)} \mid S_{n,n} = \gamma)$ and by the continuity assumptions of the theorem,

$$\mathcal{L}(T_{n,m_n}^{(\sigma_n)} \mid S_{n,n} = \gamma_n) \to \mathcal{L}(T_\alpha^{(\gamma_0,\sigma_0)}), \qquad \mathcal{L}(\hat{T}_{n,m_n}^{(\sigma_n)} \mid S_{n,n} = \gamma_n) \to \mathcal{L}(T_{(1-\alpha)}^{(\gamma_0,\sigma_0)}),$$

where $T_\alpha^{(\gamma,\sigma)}$ is the variable with characteristic function $r_\alpha^{(\gamma,\sigma)}$. Therefore, the sequence of bivariate laws $\mathcal{L}(T_{n,m_n}^{(\sigma_n)}, \hat{T}_{n,m_n}^{(\sigma_n)} \mid S_{n,n} = \gamma_n)$ is relatively compact and we may suppose that some subsequence converges to a limit law $\mathcal{L}(T_\alpha^{(\gamma_0,\sigma_0)}, T_{(1-\alpha)}^{(\gamma_0,\sigma)})$. Hence,

$$(4.19)\quad \liminf_n |E(\exp\{it_0T_{n,n}^{(\sigma_n)}\} \mid S_{n,n} = \gamma_n) - E(\exp it_0(T_\alpha^{(\gamma_0,\sigma_0)} + T_{(1-\alpha)}^{(\gamma_0,\sigma_0)})| = 0$$

for every $0 < \alpha < 1$. But by (4.15) and (4.16),

$$(4.20)\qquad E(\exp it_0(T_\alpha^{(\gamma_0,\sigma_0)} + T_{(1-\alpha)}^{(\gamma_0,\sigma_0)})) \to E(\exp\{it_0T_{\gamma_0}^{(\sigma_0)}\})$$

as $\alpha \to 1$ whatever be the bivariate nature of $(T_\alpha^{(\gamma_0,\sigma_0)}, T_{(1-\alpha)}^{(\gamma_0,\sigma_0)})$. Expressions (4.19) and (4.20) provide a contradiction to (4.17) and the theorem follows.

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Note added in proof. I recently became aware of the fact that Theorem 2.1 and the first part of Theorem 2.2 can also be obtained (under stronger assumptions) as consequences of results of LeCam (*Proc. Third Berkeley Symposium Math. Statist. Prob.* **6** (1956) pp. 129–156) and Neyman (*Probability and Statistics, The Harold Cramér Volume,* (1959), Ulf Grenander, ed. pages 213–234).

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