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ON A CLASS OF NONPARAMETRIC TWO-SAMPLE TESTS FOR CIRCULAR DISTRIBUTIONS[1]

BY SIEGFRIED SCHACH[2]

Stanford University

0. Introduction. Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two independent samples from circular distributions. A common problem consists in deciding whether the two samples have the same underlying distribution or not. In this paper we are primarily interested in non-parametric tests for the detection of rotation alternatives. Although there are "natural" isomorphisms between the circle and the interval $[0, 2\pi)$, the usual rank tests for detecting shift alternatives applied to observations in $[0, 2\pi)$ are not satisfactory, partly because they depend on an arbitrary cut-off point on the circle. Run tests can be adapted very easily to the circular two-sample problem, but their large-sample efficiency is zero for smooth families of distributions (Bahadur [1]). Kuiper [6] and Watson [12] suggested suitable modifications for the Kolmogorov-Smirnov and Cramér-von Mises tests. In this paper we use the invariance principle to derive a class of test statistics which is closely related to the class of rank tests for distributions on the real line.

1. Notation and assumptions. We define the unit circle as the set C of complex numbers of modulus 1. Then the natural isomorphism between $[0, 2\pi)$ and C is given by the mapping $x \to e^{ix}$. Under this isomorphism distributions and densities on C can be represented by cdf's and densities on $[0, 2\pi)$. For convenience we extend densities $f(\cdot)$ to all of R by the periodicity requirement $f(2k\pi + x) = f(x)$ $(k = \pm 1, \pm 2, \cdots)$.

In this paper we always assume that

(1.1) $\quad f(x) > 0$ for almost all x, and not a constant.

(1.2) $\quad f'(x)$ exists and is continuous for all x.

(1.3) $$\int_0^{2\pi} [f'(x)/f(x)]^2\, dx = \inf(f) < \infty.$$

(1.4) $$m/(m + n) = \lambda_N \to \lambda \quad \text{with} \quad 0 < \lambda < 1, \quad \text{as} \quad N = m + n \to \infty.$$

2. Transformation group and invariant tests. As our class T of transformations of the sample space we take the set of all homeomorphisms of the circle onto itself, i.e., all bicontinuous, one-to-one mappings of C onto C. Any element $t \varepsilon T$ can be written in the form: $e^{ix} \to e^{i(c+t'(x))}$, $x \varepsilon [0, 2\pi]$, where $0 \leqq c < 2\pi$ and $t'(\cdot)$ is a bicontinuous (monotone) one-to-one mapping of $[0, 2\pi]$ onto

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[2] Now at The Johns Hopkins University.

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$[0, 2\pi]$. It follows from the definition of a homeomorphism that T has a group structure if the composition of two such mappings is taken as group operation.

For the two-sample problem the set Γ of states of nature can be described as the set of pairs (F, G) of continuous strictly increasing cdf's, such that $F(0) = G(0) = 0$, $F(2\pi) = G(2\pi) = 1$. It is easy to see that each element of the induced group $\bar{T}$ is one-to-one and maps Γ onto Γ.

Let H (hypothesis) be the subset of pairs (F, G) such that $F(\cdot) = G(\cdot)$. Obviously, elements of $\bar{T}$ map H onto H, i.e., the hypothesis is invariant under the induced group.

Hence the problem of testing $H:F(\cdot) = G(\cdot)$ against $K:F(\cdot) \neq G(\cdot)$ remains invariant under transformations $t \varepsilon T$, and it is therefore natural to restrict attention to tests which are invariant under T also.

It is convenient to introduce some terminology before we obtain a maximal invariant under T. For fixed m and n $(N = m + n)$ let $Z = \{(z_1, \cdots, z_N):z_i = 0 \text{ or } 1, \sum_{i=1}^N z_i = m\}$. Let G be the group of transformations, acting on members of Z, spanned by rotations and inversions, where a rotation is some power $(g_r)^m$ of

$$g_r:(z_1, z_2, \cdots, z_N) \rightarrow (z_2, z_3, \cdots, z_N, z_1) \tag{2.1}$$

and an inversion is defined by

$$g_i:(z_1, z_2, \cdots, z_N) \rightarrow (z_N, z_{N-1}, \cdots, z_1). \tag{2.2}$$

The transformation group G defines equivalence classes in Z in the usual way and we may say that two elements z, z' belonging to the same equivalence class are cyclically equivalent. We call these equivalence classes "arrangements"; in particular the arrangement of z is

$$a(z) = \{z':z' = gz, g \varepsilon G\}. \tag{2.3}$$

Given two samples $X_1, \cdots, X_m$; $Y_1, \cdots, Y_n$ on C and an arbitrary cut-off point and direction we can order the combined sample (linearly) and define the statistics

$$\begin{aligned} Z_i &= 1 \quad \text{if the } i\text{th smallest element is an } x, \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{2.4}$$

It is obvious that $a(z)$ does not depend on the choice of the cut-off point and direction. Hence there is a unique "arrangement of the samples $X_1, \cdots, X_m$; $Y_1, \cdots, Y_n$." Using standard arguments (see Lehmann [7], Chapter 6) it follows easily that the arrangement of two samples is a maximal invariant under T. Since all elements of H belong to the same orbit under $\bar{T}$, any invariant test is nonparametric.

3. Locally most powerful invariant tests against rotation alternatives. From now on we restrict our attention to rotation alternatives, i.e., we assume that under the natural isomorphism K consists of pairs $(F(\cdot), F_\theta(\cdot))$ such that $F(x) = \int_0^x f(t)\,dt$, $F_\theta(x) = \int_0^x f(t - \theta)\,dt$, for $0 < |\theta| \leqq \theta_0$ and $0 < \theta_0 < \pi$.

Let $V^{(1)}, V^{(2)}, \cdots, V^{(N)}$ be the order statistics of a sample of size N from $F(\cdot)$, and define $V^{(N+k)} = V^{(k)}$ $(k = 1, 2, \cdots, N)$. Then we get the following

THEOREM 3.1. *Assuming conditions* (1.1), (1.2) *and* (1.3), *there exists a locally most powerful* (LMP) *invariant test for testing H against K. It rejects H when*

$$S_N = \textstyle\sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N E[\phi(V^{(i+k)})\phi(V^{(j+k)})]z_i z_j > C_{m,n,\alpha} \tag{3.1}$$

where

$$\phi(x) = f'(x)/f(x) \tag{3.2}$$

and the expectation is with respect to $F(\cdot)$.

REMARK. The fact that the test is invariant under T implies that S_N does not depend on the cut-off point and direction used in determining the indicators z_i.

PROOF. Using Scheffe's Lemma ([4], Theorem II.4.2) it is easy to show that

$$\lim_{\theta\to 0} \textstyle\int_0^{2\pi} |\theta^{-1}[f(x+\theta) - f(x)] - f'(x)|\, dx = 0. \tag{3.3}$$

An invariant test depends on the sample outcome only through the arrangement of this outcome, since the arrangement is a maximal invariant. By the Neyman-Pearson Lemma a (non-randomized) most powerful invariant test has critical region

$$P_\theta(a)/P_0(a) > \text{const.}, \tag{3.4}$$

where $P_\theta(a)$ is the probability of arrangement a when θ is the parameter value.

Since G has $2N$ members it follows easily from Hoeffding's [5] result that

$$P_\theta(a)/P_0(a) = (2N)^{-1} \textstyle\sum_{g\varepsilon G} E\{\prod_{i=1}^N [f(V^{(i)} - \theta)/f(V^{(i)})]^{(gz)_i}\}, \tag{3.5}$$

where z is any member of a, and $(gz)_i$ is the ith component of the vector gz. Using the expansion

$$f(x+\theta) = f(x) + \theta f'(x) + R(x, \theta)f(x) \tag{3.6}$$

we obtain

$$\lim_{\theta\to 0} E\,|\theta^{-1}R(x, \theta)| = 0. \tag{3.7}$$

Expanding the product in (3.5) by using (3.6) and (3.7) we get

$$2NP_\theta(a)/P_0(a) = \textstyle\sum_{g\varepsilon G} E[1 + K_1(gz)\theta + K_2(gz)\theta^2 + K_3(gz)] + o(\theta^2), \tag{3.8}$$

where

$$K_1(gz) = -\textstyle\sum_{i=1}^N (gz)_i\phi(V^{(i)}), \tag{3.9}$$

$$K_2(gz) = \textstyle\sum\sum_{i\neq j} (gz)_i(gz)_j\phi(V^{(i)})\phi(V^{(j)}), \tag{3.10}$$

$$K_3(gz) = \textstyle\sum_{i=1}^N (gz)_i R(V^{(i)}, -\theta). \tag{3.11}$$

Now it is easy to see that $\sum_{g\varepsilon G} (gz)_i = 2m$ $(i = 1, \cdots, N)$. Hence

$$\textstyle\sum_{g\varepsilon G} EK_1(gz) = \sum_{i=1}^N 2mE\phi(V^{(i)}) = 2m \sum_{i=1}^N E\phi(V_i) = 0. \tag{3.12}$$

Using (3.6) we obtain

$$\textstyle\sum_{g\varepsilon G} EK_3(gz) = 2m \sum_{i=1}^N ER(V^{(i)}, -\theta) = 0. \tag{3.13}$$

We now analyze the K_2-term. For convenience we define $z_{N+k} = z_k$, so that $\{z_k\}$ as well as $\{V^{(k)}\}$ are periodic in k (period N). The group G consists of the $2N$ members $(g_r)^k$ $(k = 1, \cdots, N)$ and $(g_r)^k g_i$ $(k = 1, \cdots, N)$, where g_r and g_i are defined by (2.1), (2.2) respectively. Hence for $1 \leqq j, j' \leqq N$ we get

$$\sum_{g \varepsilon G} (gz)_j(gz)_{j'} = \sum_{k=1}^N (g_r^k z)_j(g_r^k z)_{j'} + \sum_{k=1}^N (g_r^k g_i z)_j(g_r^k g_i z)_{j'}$$
$$= 2 \sum_{k=1}^N z_{N+j-k} z_{N+j'-k}.$$

Using this result and periodicity we obtain

$$(3.14) \quad \sum_{g \varepsilon G} \sum_{i=1}^N \sum_{j=1}^N E[\phi(V^{(i)})\phi(V^{(j)})](gz)_i(gz)_j$$
$$= 2 \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N E[\phi(V^{(i+k)})\phi(V^{(j+k)})] z_i z_j.$$

Finally we note that the "diagonal elements" $(i = j)$ add up to the constant

$$(3.15) \quad 2 \sum_{i=1}^N \sum_{k=1}^N E[\phi(V^{(i)})^2] z_{i+k}^2 = 2mN \int_0^{2\pi} [f'(x)^2/f(x)]\, dx < \infty \text{ by } (1.3).$$

Combining (3.8), (3.12), (3.13), (3.14) and (3.15) we see that there exists a neighborhood $U = \{\theta\colon |\theta| < \theta_1\}$ such that the test which maximizes the term in (3.14) has maximum power among all invariant tests for all $\theta \varepsilon U$. This ends our proof.

4. Asymptotic distribution of the LMP invariant test statistic S_N under H. In this section we will show that, asymptotically, $N^{-2}S_N - \lambda^2 \inf(f)$ has the distribution of a weighted sum of independent χ^2 random variables with 2 df. We will obtain this result from

THEOREM 4.1. *Let the following conditions be satisfied:*

(i) $\lambda_N = m/N \to \lambda$, $0 < \lambda < 1$, *as* $N \to \infty$.

(ii) $\{h_N(\cdot)\}$ *is a sequence of step functions, defined on* $[-1,1]$, *constant on intervals* $(2k - 1)/N$, $(2k + 1)/N$, *satisfying the relations* $h_N(-x) = h_N(x) = h_N(1 - x)$ *for* $0 \leqq x \leqq 1$, *and* $\sum_{k=1}^N h_N(k/N) = 0$.

(iii) $h_N \to h$ *in* L_2 *and* $h_N(0) \to h(0)$.

(iv) $h(x) = \sum_{k=-\infty}^{\infty} d_k e^{2\pi i k x}$, *such that* $\sum_{k=-\infty}^{\infty} |d_k| < \infty$.

Define

$$(4.1) \qquad T_N = N^{-1} \sum_{i=1}^N \sum_{j=1}^N h_N((i - j)/N) Z_i Z_j.$$

Then under H the characteristic function $\gamma_{T_N}(t)$ of T_N converges to

$$(4.2) \qquad \gamma(t) = \prod_{k=1}^{\infty} (1 - 2i\lambda(1 - \lambda)\, d_k t)^{-1}.$$

PROOF. This result follows immediately from Theorems 4.7 and 4.10 of Schach [10].

In this section it is more convenient to work with uniformly distributed random variables $U_1, \cdots, U_N$, rather than with $V_1, \cdots, V_N$ (distributed according to $F(\cdot)$). We denote the corresponding order statistics by $U^{(1)}, \cdots, U^{(N)}$. Define

$$(4.3) \qquad \psi(x) = (f'/f) \circ F^{-1}(x) \qquad 0 \leqq x \leqq 1.$$

Then it is easy to see that $\psi(U)$ and $\phi(V)$ have the same distribution, and by monotonicity of F the ordered elements correspond to each other.

Throughout this section we strengthen assumption (1.3) to

$$\text{(4.4)} \qquad \psi(\cdot) \quad \text{is continuous on} \quad [0, 1].$$

Let

$$\alpha_{ij} = N^{-1} \sum_{k=1}^{N} E[\psi(U^{(i+k)})\psi(U^{(j+k)})], \qquad 1 \leqq i, j \leqq N,$$

where we use again the cyclical definition $U^{(N+k)} \equiv U^{(k)}$. Obviously $\alpha_{ij} = \alpha_{ji}$ and α_{ij} depends only on $i - j$. On $[-1, 1]$ we define

$$\text{(4.5)} \quad h_N'(x) = \alpha_{ij} \quad \text{for} \quad (2(i-j)-1)/2N < x \leqq (2(i-j)+1)/2N.$$

Then $h_N'(\cdot)$ satisfies the relations

$$\text{(4.6)} \qquad h_N'(-x) = h_N'(x) = h_N'(1 \dot{-} x).$$

It follows from the definition of $h_N'(\cdot)$ that

$$\text{(4.7)} \qquad N^{-1}S_N = \sum_{i=1}^{N} \sum_{j=1}^{N} h_N'((i-j)/N)Z_iZ_j.$$

Simple algebraic manipulations yield the result

$$\text{(4.8)} \quad \sum_{k=1}^{N} h_N'(k/N) = N^{-1}E(\sum_{k=1}^{N} \psi(U_k))^2 = E\psi(U)^2 = \inf(f) = c < \infty.$$

The function

$$\text{(4.9)} \qquad h_N(x) = h_N'(x) - c/N$$

satisfies (ii) of the theorem above.

Obviously

$$\text{(4.10)} \qquad N^{-1}S_N - m^2c/N = \sum_{i=1}^{N} \sum_{j=1}^{N} h_N((i-j)/N)Z_iZ_j.$$

We now analyze the convergence behavior of $\{h_N\}$.

$$\text{(4.11)} \qquad h_N(0) = N^{-1} \sum_{k=1}^{N} E\psi(U^{(k)})^2 - c/N$$
$$= c(1 - 1/N) \to \inf(f) \quad \text{as} \quad N \to \infty.$$

Define

$$\text{(4.12)} \quad \eta_N(x, y) = E[\psi(U^{(i)})\psi(U^{(j)})]$$
$$\text{for} \quad (i-1)/N \leqq x < i/N, \ (j-1)/N \leqq y < j/N.$$

Then we obtain

LEMMA 4.1. *$\eta_N(x, y) \to_{L_2} \psi(x)\psi(y)$, where L_2-convergence is with respect to Lebesgue measure on the unit square.*

PROOF. Let $f_N(\cdot,\cdot; i, j)$ be the joint density of $U^{(i)}$, $U^{(j)}$ $(i \neq j)$ from a sample of size N. Using $[Nx]$ for the integral part of Nx we may write

$$\text{(4.13)} \quad \eta_N(x, y) = \int_0^1 \int_0^1 \psi(u_1)\psi(u_2)f_N(u_1, u_2; 1 + [Nx], 1 + [Ny])\, du_1\, du_2$$

for (x, y) such that $[Nx] \neq [Ny]$. For $x \neq y$ the distributions corresponding to the densities $f_N(\cdot,\cdot; 1 + [Nx], 1 + [Ny])$ converge weakly to the distribution having all its mass at (x, y), as $N \to \infty$. Since $\psi(u_1)\psi(u_2)$ is continuous and bounded we have (by the Helly-Bray Theorem)

$$\eta_N(x, y) \to \psi(x)\psi(y) \quad \text{a.e.} \tag{4.14}$$

L_2-convergence follows from the Lebesgue bounded convergence theorem since $\eta_N(\cdot,\cdot)$ is uniformly bounded.

Define $\psi(1 + y) = \psi(y)$ for $0 < y \leqq 1$ and

$$h(x) = \int_0^1 \psi(t)\psi(t + x)\, dt. \tag{4.15}$$

Then it follows easily that

$$h_N \to h \quad \text{in} \quad L_2\text{-norm.} \tag{4.16}$$

THEOREM 4.2. *If the assumptions* (1.1), (1.2), (1.4) *and* (4.4) *are satisfied, the test statistic* $N^{-2}S_N$ *of the* LMP *invariant test has, under* H, *a limiting distribution which is equal to the distribution of*

$$\lambda^2 \sum_{k=-\infty}^{\infty} |c_k|^2 + \lambda(1 - \lambda) \sum_{k=1}^{\infty} |c_k|^2 \chi_{2,k}^2,$$

where $\{\chi_{2,k}^2; k = 1, 2, \cdots\}$ *is a sequence of independent* χ_2^2 *random variables and* $\{c_k\}$ *are the Fourier coefficients of* $\psi(\cdot)$.

PROOF. We use Theorem 4.1. (i), (ii), and (iii) are obviously satisfied. To obtain (iv) note that by assumption (4.4) $\psi \in L_2$. Hence it has the expansion

$$\psi(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx} \tag{4.17}$$

and by the Parseval identity

$$h(x) = \int_0^1 \psi(t)\psi(x + t)\, dt = \sum_k |c_k|^2 e^{-2\pi ikx}. \tag{4.18}$$

Hence $h(\cdot)$ has an absolutely convergent Fourier sequence. By (4.10) $N^{-2}S_N - N^{-2}m^2c$ has the form of the statistic T_N defined in (4.1). Hence the result follows, since $c = \inf(f) = \sum |c_k|^2$.

5. Efficiency of the LMP invariant test.

Before we compute the desired efficiency we need some auxiliary results. We will obtain these results under an additional assumption:

$$\psi(\cdot) \quad \text{has a bounded derivative,} \tag{5.1}$$

where $\psi(\cdot)$ is defined by (4.3).

LEMMA 5.1. *Under assumption* (5.1) $h(\cdot)$ *has a continuous second derivative.*

PROOF. By definition $h(x) = \int_0^1 \psi(u)\psi(u + x)\, du$. A well-known theorem (e.g. [8], page 126) allows us to differentiate under the integral sign, hence

$$h'(x) = \int_0^1 \psi(u)\psi'(u + x)\, du = \int_0^1 \psi(u - x)\psi'(u)\, du \tag{5.2}$$

by periodicity of $\psi(\cdot)$. Using the same argument again we get

$$h''(x) = -\int_0^1 \psi'(u - x)\psi'(u)\, du = -\int_0^1 \psi'(u)\psi'(u + x)\, du. \tag{5.3}$$

To show continuity of $h''(\cdot)$ we use the Schwarz inequality to get

$$\|h''(x) - h''(x + \Delta)\| \leqq [\int_0^1 \psi'(u)^2\, du \int_0^1 (\psi'(u + x) - \psi'(u + x + \Delta))^2\, du]^{\frac{1}{2}}$$
$$= \|\psi'(\cdot)\|\, \|\psi'(\cdot) - \psi_\Delta'(\cdot)\|,$$

where $\psi_\Delta'(x) = \psi'(x + \Delta)$. Since the shift operator in L_2 is continuous, this proves the lemma.

Let

$$\eta_N(x, 1 + y) = \eta_N(x, y), \qquad 0 \leqq x, y \leqq 1, \tag{5.4}$$

and

$$g_N(x) = \int_0^1 \eta_N(t, x + t)\, dt. \tag{5.5}$$

Then we get

LEMMA 5.2. *Under assumption* (5.1) $g_N(x) \to h(x)$ *uniformly in* x.

PROOF. The functions $\{g_N\}$ are continuous and piecewise linear. A straightforward calculation shows that the slopes of the pieces are uniformly bounded. Hence the sequence $\{g_N\}$ is uniformly equicontinuous. Since it is also uniformly bounded, it is conditionally compact in $C[0, 1]$ by the Arzelá-Ascoli Theorem. Since $\{g_N\}$ converges in $L_2[0, 1]$, the limit of subsequences is unique. Hence $\{g_N\}$ converges in $C[0, 1]$.

Since little seems to be known about the probabilities of large deviations for statistics of the form (4.1), we use a method proposed by Bahadur [1] to get an approximate measure of the efficiency of the LMP invariant test relative to the most powerful parametric test. This method is an approximation since it replaces the actual tail probabilitites by the tail of the limiting distribution. We use the terminology of Gleser [3], Sections 3 and 4.

Under his assumptions the (approximate) ARE for tests with critical regions $T_N^{(i)} \geqq C_{N,\alpha}^{(i)}$ is given by

$$e(T_N^{(i)}, T_N^{(2)} \mid \theta) = a_1 c_1^{t_1}(\theta)/(a_2 c_2^{t_2}(\theta)), \tag{5.6}$$

where $\theta \varepsilon K$ is the parameter value of the alternative. Since we are interested in alternatives close to the hypothesis we will restrict ourselves to evaluating the limit

$$\lim_{\theta \to 0} e(T_N^{(1)}, T_N^{(2)}/\theta) = e(T_N^{(1)}, T_N^{(2)}), \tag{5.7}$$

where θ is the rotation parameter.

Let $T_N^{(1)}$ be the sequence of test statistics $N^{-2}S_N$, where S_N is defined by (3.1). Then the tests $T_N^{(1)} \geqq C_{N,\alpha}^{(1)}$ are LMP invariant by Theorem (3.1).

LEMMA 5.3. *If* (5.1) *is satisfied, the sequence* $\{T_N^{(1)}\}$ *satisfies assumptions* 3.1, 4.1 *and* 4.2 *of* [3] *for* θ *in a neighborhood of* 0.

PROOF. Under H $\{T_N^{(1)}\}$ has a limiting distribution $F^{(1)}$ by Theorem 4.2. By taking out a χ^2-component from (4.2) it is seen that $F^{(1)}$ can be obtained as a convolution with a continuous factor. Hence $F^{(1)}$ is itself continuous. This establishes assumption 3.1.

It follows from Varberg [11, Corollary 3] that assumption 4.1 is satisfied with $a_1 = [\lambda(1 - \lambda) \max_k |c_k|^2]^{-1}$, $t_1 = 1$. To obtain (4.2) let $b^{(1)}(x) = x$. Using (4.7) we obtain

(5.8) $$T_N^{(1)}/b^{(1)}(N) = N^{-2} \sum_{i=1}^N \sum_{j=1}^N h_N((i-j)/N)Z_iZ_j .$$

Let $F_m(\cdot)$, $G_n(\cdot)$, $H_N(\cdot)$ be the empirical cdf of the X-sample, Y-sample, and combined sample, respectively.
Obviously

(5.9) $$N^{-2} \sum_{i=1}^N \sum_{j=1}^N h_N'((i-j)/N)Z_iZ_j = m^2N^{-2} \int_0^{2\pi} \int_0^{2\pi} h_N'(H_N(x) - H_N(y))\, dF_m(x)\, dF_m(y) = U_N, \quad \text{say}.$$

$h_N'(\cdot)$ can be replaced by g_N in (5.9) since $h_N'(l/N) = g_N(l/N)$ and hence U_N can be split up into three parts, U_{N1}, U_{N2}, U_{N3},
where

(5.10) $$U_{N1} = \lambda_N^2 \int_0^{2\pi} \int_0^{2\pi} h(H(x) - H(y))\, dF_m(x)\, dF_m(y),$$

(5.11) $$U_{N2} = \lambda_N^2 \int_0^{2\pi} \int_0^{2\pi} \{g_N(H(x) - H(y)) - h(H(x) - H(y))\} \cdot dF_m(x)\, dF_m(y),$$

and

(5.12) $$U_{N3} = \lambda_N^2 \int_0^{2\pi} \int_0^{2\pi} \{g_N(H_N(x) - H_N(y)) - g_N(H(x) - H(y))\} \cdot dF_m(x)\, dF_m(y),$$

and where

(5.13) $$H(x) = \lambda F(x) + (1 - \lambda)G(x).$$

By the Glivenko-Cantelli Lemma and the Helly-Bray Theorem

(5.14) $$U_{N1} \to \lambda^2 \int_0^{2\pi} \int_0^{2\pi} h(H(x) - H(y))\, dF(x)\, dF(y) \quad \text{a.s.}$$

By Lemma 5.3 the integrand in U_{N2} is uniformly small for N sufficiently large. Hence

(5.15) $$U_{N2} \to 0 \quad \text{a.s.}$$

Finally, using the Glivenko-Cantelli Lemma again, $\sup_x |H_N(x) - H(x)| \to 0$ a.s. and hence the integrand in U_{N3} is a.s. uniformly small for N sufficiently large on account of Lemma 5.2.

Combining these results we get

(5.16) $$T_N^{(1)}/b^{(1)}(N) \to c_1(\theta) = \lambda^2 \int_0^{2\pi} \int_0^{2\pi} h(H(x) - H(y))\, dF(x)\, dF(y) \text{ a.s.}$$

All that remains to be shown is that $c_1(\theta)$ is positive for θ close to 0, and this will follow from the expansion

(5.17) $$c_1(\theta) = b\theta^2 + o(\theta^2), \qquad b > 0.$$

We have

$$(5.18)\qquad H(x) = \lambda F(x) + (1 - \lambda)(F(x - \theta) - F(-\theta)).$$

Inserting this expression in (5.16), a straightforward calculation, using Lemma 5.1, shows that $c_1''(\theta)$ exists and is continuous, that $c_1(0) = c_1'(0) = 0$ and that

$$(5.19)\qquad 2b = c_1''(0) = 2\lambda^2(1 - \lambda)^2 \int_0^1 h(x - y)\psi(x)\psi(y)\, dx\, dy$$
$$= 2\lambda^2(1 - \lambda)^2 \textstyle\sum_{k=-\infty} |c_k|^4 > 0.$$

This ends the proof of the lemma.

The term $s^{(i)}(\theta) = a_i c_i^{t_i}(\theta)$ is called the asymptotic slope of the sequence of test statistics. A standard argument shows that under our regularity conditions the likelihood ratio test has maximum slope among all tests of H vs. K, that it satisfies assumption 4.1 of [3] with $t_2 = 1$, $a_2 = .1$, $b^{(2)}(N) = N$, and that its slope is

$$(5.20)\qquad s^{(2)}(\theta) = \lambda(1 - \lambda) \inf (f)\theta^2 + o(\theta^2).$$

(See Chernoff and Savage [2] and Bahadur [1].) Hence it is reasonable to use the likelihood ratio test as standard for comparison of non-parametric competitors, and we obtain

THEOREM 5.1. *Under assumptions* (1.1), (1.2), (1.4) *and* (5.1) *the* (*approximate*) *local* ARE *of the* LMP *invariant test is given by*

$$(5.21)\qquad \text{eff } (T_N^{(1)},\ \text{best test } \mid f) = (\textstyle\sum_{k=-\infty}^{\infty} |c_k|^4)(\max_l |c_l|^2 \sum_{k=-\infty}^{\infty} |c_k|^2)^{-1},$$

where $\{c_k\}$ *are the Fourier coefficients of* $\psi(\cdot)$.

PROOF. Using the results of the previous lemma we obtain

$$(5.22)\qquad s^{(1)}(\theta) = a_1 c_1^{t_1}(\theta) = \lambda(1 - \lambda) \textstyle\sum |c_k|^4\theta^2(\max_l |c_l|^2)^{-1} + o(\theta^2).$$

By (5.20)

$$(5.23)\qquad s^{(2)}(\theta) = \lambda(1 - \lambda) \inf (f)\, \theta^2 + o(\theta^2) = \lambda(1 - \lambda) \textstyle\sum |c_k|^2\theta^2 + o(\theta^2).$$

Assumption 4.3 of [3] is obviously satisfied, and hence, as $\theta \to 0$

$$(5.24)\qquad \lim s^{(1)}(\theta)/s^{(2)}(\theta) = \{\textstyle\sum |c_k|^4\}/\max_l |c_l|^2 \sum |c_k|^2\}.$$

6. Applications. There seems to exist no "nice" parametric class of circular distributions for which the h function and the numerical value of the efficiency term in (5.21) are readily obtainable. In particular, for the class of von Mises-distributions given by

$$(6.1)\qquad f_{\kappa,\theta}(x) = C(\kappa)e^{\kappa\cos(x-\theta)}, \qquad \kappa \geqq 0, -\pi < \theta \leqq \pi,$$

the evaluation of the $\psi(\cdot)$ function (which depends now on κ) and of its Fourier series requires numerical integrations which we have not carried out.

For values of κ close to zero the test statistic

$$(6.2)\qquad T_{WW} = N^{-1} \textstyle\sum_{i=1}^{N} \sum_{j=1}^{N} \cos 2\pi((i - j)/N)Z_iZ_j$$

has asymptotic efficiency close to 1 in the case of a von Mises-distribution. For more details see Schach [9].

The test statistic T_{WW} is of the form (4.1) if h_N is the step function corresponding to the cosine function, i.e., $h_N(i/N) = \cos 2\pi i/N$. However, it is easy to see that this $h(\cdot)$ does not correspond to a density satisfying our regularity conditions.

The test with critical region $T_{WW} \geqq C_{N,\alpha}$ was proposed by Wheeler and Watson [13] on intuitive grounds as a non-parametric test for the two-sample circular distribution problem.

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