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ON STRONG CONSISTENCY OF DENSITY ESTIMATES[1]

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1. Introduction and summary. Let $X_1, X_2, \cdots, X_n$, be a sample of n independent observations of a random variable X with distribution $F(x) = F(x_1, \cdots, x_m)$ or R^m and Lebesgue density $f(x) = f(x_1, \cdots, x_m)$. To estimate the density $f(x)$ consider estimates of the form

$$(1) \quad f_n(x) = n^{-1} \sum_{j=1}^{n} K_n(x, X_j), \qquad K_n(x, X_j) = h_n^{-m} K(h_n^{-1}(x - X_j));$$

where $K(u) = K(u_1, \cdots, u_m)$ is a real-valued Borel-measurable function on R^m such that

$$(2) \qquad K(u) \quad \text{is a density on} \quad R^m$$

$$(3) \qquad \sup_{u \varepsilon R^m} K(u) < \infty$$

$$(4) \qquad \|u\|^m K(u) \to 0 \quad \text{as} \quad \|u\|^2 = \sum_{i=1}^{m} u_i^2 \to \infty$$

and $\{h_n\}$ is a sequence of numbers such that

$$(5) \quad h_n > 0, \qquad n = 1, 2, \cdots; \qquad \lim_{n\to\infty} h_n = 0 \quad \text{and} \quad \lim_{n\to\infty} n h_n^m = \infty.$$

Such density estimates have been shown to be weakly consistent (that is, $f_n(x) \to f(x)$ in probability as $n \to \infty$) on the continuity set, $C(f)$, of the density $f(x)$ by Parzen [4] for $m = 1$ and by Cacoullos [1] for $m > 1$. In Theorem 1, we state conditions under which strong consistency (that is, $f_n(x) \to f(x)$ with probability one as $n \to \infty$) of such estimates obtains.

Theorem 2 gives conditions under which uniform (in x) strong consistency of the estimates (1) is valid. In this respect, our results are very similar in the case $m = 1$ to those of Nadaraya [4], although the method of proof and conditions imposed are different. Theorem 3 concerns the estimation of the unique mode of the density $f(x)$ when it exists.

2. A pointwise strong consistency theorem. Before stating our first theorem, we shall give a lemma which we will need and is of some interest in its own right.

LEMMA. *Let $\{Y_n\}$ and $\{Y_n'\}$ be two sequences of random variables on a probability space $(\Omega, \mathfrak{F}, P)$. Let $\{\mathfrak{F}_n\}$ be a sequence of Borel fields, $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$, where Y_n and*

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Y_n' are measurable with respect to $\mathfrak{F}_n$. If

(i) $0 \leqq Y_n$ a.e.

(ii) $EY_1 < \infty$

(iii) $E[Y_{n+1} \mid \mathfrak{F}_n] \leqq Y_n + Y_n'$ a.e.

(iv) $\sum_{n=1}^{\infty} E|Y_n'| < \infty$,

then Y_n converges a.e. *to a finite limit.*

PROOF. Let $Z_n = Y_n + S_{n-1}$, where $S_0 \equiv 0$ and $S_n = \sum_{i=1}^{n} \{Y_i - E[Y_{i+1} \mid \mathfrak{F}_i]\}$. Observe that by (i) and (iii), $Z_n \geqq S_{n-1} \geqq -\sum_{i=1}^{n-1} Y_i'$ a.e., and we have

$$\sup_n EZ_n^- \leqq \sup_n \sum_{i=1}^{n-1} E|Y_i'| \leqq \sum_{i=1}^{\infty} E|Y_i'| < \infty.$$

However, it is easy to verify that $\{Z_n, \mathfrak{F}_n\}$ is a martingale and that $\sup_n E|Z_n| = 2\sup_n EZ_n^- + EY_1 < \infty$ by the above observation and condition (ii). Hence, by a standard martingale theorem (e.g., see Theorem 9.4.1 in Chung [2]), $\{Z_n\}$ converges a.e. to a finite limit.

Since $Y_n = Z_n - S_{n-1}$ a.e., the proof will now be completed if we show S_n converges a.e. to a finite limit. Observe that $S_n = -\sum_{i=1}^{n} Y_i' + \sum_{i=1}^{n} W_i$, where $W_i = Y_i + Y_i' - E[Y_{i+1} \mid \mathfrak{F}_i] \geqq 0$ a.e. by (iii) and that under (iv) $\sum_{i=1}^{\infty} Y_i'$ converges a.e. (see Chung [2, Ex. 7, p. 116]). Therefore, S_n converges a.e. to a finite limit if and only if $\sum_{i=1}^{\infty} W_i < \infty$ a.e. Note that from the inequality $S_n \leqq Z_{n+1}$ a.e. (recall (i)) and the fact that $\{Z_n\}$ converges a.e. to a finite limit, we have $\limsup_n S_n \leqq \lim_n Z_n < \infty$ a.e. Finally, since the W_i are non-negative, we have $0 \leqq \sum_{i=1}^{\infty} W_i = \lim_n \sum_{i=1}^{n} W_i \leqq \limsup_n S_n + \sum_{i=1}^{\infty} Y_i' < \infty$ a.e. This completes the proof.

DEFINITION. A real-valued function on the real line $g(c)$ is said to be locally Lipschitz of order α, $\alpha > 0$, at 1 if there exists an $\epsilon > 0$ and $0 < M < \infty$ such that $|g(c) - g(1)| \leqq M|c - 1|^{\alpha}$ for all $c \in (1 - \epsilon, 1 + \epsilon)$.

THEOREM 1. *Let $K(u)$ satisfy (2), (3) and (4) and let $\{h_n\}$ satisfy (5) and*

$$h_n/h_{n+1} \to 1 \quad \text{as} \quad n \to \infty. \tag{6}$$

Assume that $K(u)$ and $\{h_n\}$ are such that

$$g_1(c) = \sup_{\|u\| \geqq a} \|u\|^m \{K(cu) - K(u)\}^2 \quad \textit{is locally Lipschitz of order } \alpha \textit{ at } c = 1 \textit{ for some } a > 0, \tag{7}$$

$$g_2(c) = \int \{K(cu) - K(u)\}^2 \, du \quad \textit{is locally Lipschitz of order } \alpha \textit{ at } c = 1, \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2 h_n^m} < \infty, \quad \textit{and} \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{1}{n h_n^{m-\beta}} \left| \frac{1}{h_{n+1}} - \frac{1}{h_n} \right|^{\beta} < \infty, \qquad \beta = \min\{\tfrac{1}{2}\alpha, 1\}. \tag{10}$$

Then, $f_n(x) \to f(x)$ with probability one if $x \in C(f)$.

PROOF. Since $Ef_n(x) \to f(x)$ for $x \in C(f)$ by Theorem 3.1 of Cacoullos [1], it suffices to show that $f_n(x) - Ef_n(x) \to 0$ with probability one as $n \to \infty$ on $C(f)$. The proof of this relies on the lemma with $Y_n = \{f_n(x) - Ef_n(x)\}^2$, $\mathfrak{F}_n =$ Borel field generated by $X_1, \cdots, X_n$, and the fact that for each $x \varepsilon C(f)$, $\lim_n EY_n(x) = 0$ (Cacoullos [1, Lemma 2.1]). Observe that

$$f_{n+1}(x) - Ef_{n+1}(x) = f_n(x) - Ef_n(x) + \sum_{j=1}^{n} W_n(x, X_j) + (n+1)^{-1}\{K_{n+1}(x, X_{n+1}) - EK_{n+1}(x, X)\}$$

where

$$W_n(x, X_j) = (n+1)^{-1}\{K_{n+1}(x, X_j) - EK_{n+1}(x, X)\} - n^{-1}\{K_n(x, X_j) - EK_n(x, X)\}.$$

Hence,

$$E[Y_{n+1} \mid \mathfrak{F}_n] = Y_n + U_n(x) + V_n(x) + \alpha_n(x), \tag{11}$$

with $U_n(x) = \{\sum_{j=1}^{n} W_n(x, X_j)\}^2$, $V_n(x) = 2\{f_n(x) - Ef_n(x)\}\{\sum_{j=1}^{n} W_n(x, X_j)\}$ and $\alpha_n(x) = (n+1)^{-2} \operatorname{Var}\{K_{n+1}(x, X)\}$.

Letting $Y_n' = Y_n'(x) = U_n(x) + V_n(x) + \alpha_n(x)$ for each $x \varepsilon C(f)$ and applying the above lemma the proof is then completed by merely verifying that $\sum_{n=1}^{\infty} E|Y_n'| < \infty$ for each $x \varepsilon C(f)$. We show that for each $x \varepsilon C(f)$: (i) $\sum_{n=1}^{\infty} EU_n(x) < \infty$, (ii) $\sum_{n=1}^{\infty} E|V_n(x)| < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n(x) < \infty$.

(i) By double use of the inequality $(a+b)^2 \leqq 2a^2 + 2b^2$ we obtain

$$\begin{aligned} EU_n(x) &= nEW_n^2(x, X) \\ &\leqq nE\left\{\frac{K_{n+1}(x, X)}{n+1} - \frac{K_n(x, X)}{n}\right\}^2 \\ &= \frac{1}{nh_n^m}\int\left\{\frac{nh_n^m}{(n+1)h_{n+1}^m} K\left(\frac{h_n}{h_{n+1}}u\right) - K(u)\right\}^2 f(x - h_n u)\,du \\ &\leqq \frac{2}{nh_n^m}\int\left\{K\left(\frac{h_n}{h_{n+1}}u\right) - K(u)\right\}^2 f(x - h_n u)\,du \\ &\quad + \frac{4}{n(n+1)^2 h_{n+1}^m}\int K^2(u) f(x - h_{n+1}u)\,du \\ &\quad + \frac{4}{nh_{n+1}^m}\left\{1 - \left(\frac{h_{n+1}}{h_n}\right)^m\right\}^2 \int K^2(u) f(x - h_{n+1}u)\,du. \end{aligned} \tag{12}$$

Note that since

$$(1 - t^m) = \left(\sum_{j=1}^{m} t^{j-1}\right)(1 - t), \tag{13}$$

we have under (6) that

$$\left\{1 - \left(\frac{h_{n+1}}{h_n}\right)^m\right\}^2 \sim (mh_{n+1})^2\left(\frac{1}{h_{n+1}} - \frac{1}{h_n}\right)^2 \quad \text{as } n \to \infty. \tag{14}$$

Also, (2), (3), (4) and (5), Theorem 2.1 of Cacoullos [1] yields

$$\lim_n \int K^2(u) f(x - h_{n+1}u)\,du = f(x)K^*, \qquad K^* = \int K^2(u)\,du, \tag{15}$$

for all $x \varepsilon C(f)$. This result (6) and (14) combine to show that the second and third terms in the upper bound of (12) are respectively asymptotically equivalent for $x \varepsilon C(f)$ to

$$(16) \qquad \frac{4K^*f(x)}{n^3 h_n^m} \quad \text{and} \quad \frac{4m^2 K^* f(x)}{n h_n^{m-2}} \left(\frac{1}{h_{n+1}} - \frac{1}{h_n} \right)^2 .$$

Furthermore, for each $x \varepsilon C(f)$ we have the following inequality for every $\epsilon > 0$ and some $\delta = \delta(\epsilon, x) > 0$,

$$\begin{aligned}
\int \left\{ K\left(\frac{h_n}{h_{n+1}} u \right) - K(u) \right\}^2 & f(x - h_n u)\, du \\
&\leqq \int_{\|h_n u\| \geqq \delta} \left\{ K\left(\frac{h_n}{h_{n+1}} u \right) - K(u) \right\}^2 f(x - h_n u)\, du \\
&\quad + \int_{\|h_n u\| < \delta} \left\{ K\left(\frac{h_n}{h_{n+1}} u \right) - K(u) \right\}^2 f(x - h_n u)\, du \\
&\leqq \delta^{-m} \sup_{\|u\| \geqq \delta / h_n} \left[\|u\|^m \left\{ K\left(\frac{h_n}{h_{n+1}} u \right) - K(u) \right\}^2 \right] \\
&\quad + \{\epsilon + f(x)\} \int \left\{ K\left(\frac{h_n}{h_{n+1}} u \right) - K(u) \right\}^2 du.
\end{aligned}$$

Hence, for n sufficiently large such that $\delta / h_n \geqq a$ (see condition (7)), we have

$$\begin{aligned}
\int \left\{ K\left(\frac{h_n}{h_{n+1}} u \right) - K(u) \right\}^2 & f(x - h_n u)\, du \\
&\leqq \delta^{-m} g_1\left(\frac{h_n}{h_{n+1}} \right) + \{\epsilon + f(x)\} g^2 \left(\frac{h_n}{h_{n+1}} \right).
\end{aligned}$$

Therefore, by conditions (6), (7) and (8), the first term in the upper bound of (12) has an upper bound asymptotically equivalent as $n \to \infty$ to

$$(17) \qquad \frac{M(x)}{n h_n^{m-\alpha}} \left| \frac{1}{h_{n+1}} - \frac{1}{h_n} \right|^\alpha$$

for some $0 < M(x) < \infty$ for each $x \varepsilon C(f)$. Combining the asymptotic results (16) and (17) with inequality (12), we have now shown that for each $x \varepsilon C(f)$ there exists $M^*(x)$, $0 < M^*(x) < \infty$ such that for n sufficiently large,

$$(18) \qquad EU_n(x) \leqq \frac{M^*(x)}{n h_n^{m-2\beta}} \left| \frac{1}{h_{n+1}} - \frac{1}{h_n} \right|^{2\beta} + \frac{4K^*f(x)}{n^3 h_n^m}$$

This inequality under conditions (5), (6) and (10) completes the verification of (i).

To verify (ii) note that

$$\begin{aligned}
(19) \qquad \{E|V_n(x)|\}^2 &\leqq 4E\{f_n(x) - Ef_n(x)\}^2 EU_n(x) \\
&= 4n^{-1} \operatorname{Var}\{K_n(x, X)\} EU_n(x).
\end{aligned}$$

By Lemma 2.1 of Cacoullos [1], $\operatorname{Var}\{K_n(x, X)\} \sim h_n^{-m} K^* f(x)$ for all $x \varepsilon C(f)$. Hence, inequalities (18) and (19) yield for n sufficiently large and each $x \varepsilon C(f)$

a $M^{**}(x)$, $0 < M^{**}(x) < \infty$, and a $M'(x)$, $0 < M'(x) < \infty$, such that

$$E\,|V_n(x)| \leqq \frac{M^{**}(x)}{nh_n^{m-\beta}}\left|\frac{1}{h_{n+1}} - \frac{1}{h_n}\right|^\beta + \frac{M'(x)}{n^4h_n^{2m}}$$

This inequality and conditions (5) and (10) completes the verification of (ii).

Verification of (iii) is an immediate consequence of Lemma 2.1 of Cacoullos [1] and condition (9) of the theorem. As noted earlier, this completes the proof.

REMARKS. (1) It can be shown that *all* the kernel functions $K(u)$ given in Table 1 of Parzen [4] in the case $m = 1$ and their natural product kernel generalizations (see Cacoullos [1, page 186]) in the case $m > 1$ satisfy conditions (7) and (8) with $\alpha = 1$ in Theorem 1 as well as conditions (2), (3) and (4). It can be shown that all $K(u)$ in Table 1 of Parzen [4] except the first entry satisfy conditions (7) and (8) with $\alpha = 2$ in Theorem 1.

(2) If $h_n = \alpha_n^{-1}n^{-p}$, where $0 < p < \beta m^{-1}$, $\alpha_n > 0$, and $\alpha_{n+1} = \alpha_n + O(n^{-1})$, $n\alpha_n \to \infty$ as $n \to \infty$ and $\limsup \alpha_n < \infty$, then it is straightforward to show that conditions (5), (6), (9) and (10) hold for the sequence $\{h_n\}$. In particular $h_n = cm^{-p}$, $0 < p < \beta/m$, $c > 0$, is such a sequence.

3. Uniform strong consistency and estimation of the mode. Let

$$k(t) = k(t_1, \cdots, t_m)$$

be the characteristic function of $K(u)$, that is,

$$k(t) = \int e^{it'u}K(u)\,du$$

where $t'u = \sum_{j=1}^m t_ju_j$. Also define

$$\varphi_n(t) = n^{-1}\textstyle\sum_{j=1}^n e^{it'X_j} \tag{20}$$

and following Parzen [4, equation (3.3)] as in the case $m = 1$, we see that if

$$\int |k(t)|\,dt < \infty \tag{21}$$

then

$$\begin{aligned} f_n(x) &= n^{-1}\textstyle\sum_{j=1}^n K_n(x, X_j) \\ &= (2\pi)^{-m}\int e^{-it'x}k(h_nt)\varphi_n(t)\,dt \end{aligned} \tag{22}$$

We now state and prove

THEOREM 2. *Let $K(u)$ be such that (2), (3), (4) and (21) hold and assume*

(23) $g(c) = \int |k(ct) - k(t)|\,dt$ *is locally Lipschitz of order* 1 *at* $c = 1$.

Let $\{h_n\}$ be a sequence such that (5) and (6) holds and assume

$$\sum_{n=1}^\infty \frac{1}{(nh_n^m)^2} < \infty. \tag{24}$$

$$\sum_{n=1}^\infty \frac{1}{nh_n^{2m-1}}\left|\frac{1}{h_{n+1}} - \frac{1}{h_n}\right| < \infty. \tag{25}$$

$$\lim_{n\to\infty} nh_n^{2m} = \infty. \tag{26}$$

Then, if $f(x)$ is uniformly continuous on R^m, $\sup_x |f_n(x) - f(x)| \to 0$ *with probability one as* $n \to \infty$.

PROOF. We first show

$$\sup_x |f_n(x) - Ef_n(x)| \to 0 \text{ a.e.} \quad \text{as} \quad n \to \infty. \tag{27}$$

Let $\varphi(t) = Ee^{it'X}$ and note that from (22), we see that

$$\begin{aligned}\{\sup_x |f_n(x) - Ef_n(x)|\}^2 &= \{(2\pi)^{-m} \sup_x |\textstyle\int e^{-it'x}k(h_nt)\{\varphi_n(t) - \varphi(t)\}\, dt\}^2 \\ &\leqq \{(2\pi)^{-m} \textstyle\int |k(h_nt)|\, |\varphi_n(t) - \varphi(t)|\, dt\}^2 \\ &\leqq \{(2\pi)^{-2m} \textstyle\int |k(h_nt)|\, dt\}\{\textstyle\int |k(h_nt)|\, |\varphi_n(t) - \varphi(t)|^2\, dt\} \\ &= \{(2\pi)^{-2m} \textstyle\int |k(t)|\, dt\} Y_n\end{aligned}$$

where $Y_n = h_n^{-m} \int |k(h_nt)|\, |\varphi_n(t) - \varphi(t)|^2\, dt$.

Let $\mathfrak{F}_n$ be as in Theorem 1. Since

$$\varphi_{n+1}(t) = \varphi_n(t) - (n+1)^{-1}\{\varphi_n(t) - e^{it'X_{n+1}}\},$$

we have a.e.

$$\begin{aligned}E[Y_{n+1} \mid \mathfrak{F}_n] &= n^2\{(n+1)^2h_{n+1}^m\}^{-1} \textstyle\int |k(h_{n+1}t)|\, |\varphi_n(t) - \varphi(t)|^2\, dt \\ &\quad + \{(n+1)^2h_{n+1}^m\}^{-1} \textstyle\int (1 - |\varphi(t)|^2)|k(h_{n+1}t)|\, dt \\ &\leqq Y_n + Z_n + \delta_n\end{aligned} \tag{28}$$

where

$$\begin{aligned}Z_n &= \textstyle\int |h_{n+1}^{-m}k(h_{n+1}t) - h_n^{-m}k(h_nt)|\, |\varphi_n(t) - \varphi(t)|^2\, dt \\ \delta_n &= \{(n+1)h_{n+1}^m\}^{-2} \textstyle\int |k(t)|\, dt.\end{aligned}$$

With $Y_n' = Z_n + \delta_n$ in (28) the proof of (27) follows immediately from the lemma in Section 2 and the fact that $\sum_{n=1}^\infty \delta_n < \infty$ under (21) and (24) provided we show that

$$\textstyle\sum_{n=1}^\infty EZ_n < \infty \quad \text{and} \quad \lim_n EY_n = 0. \tag{29}$$

Note that $E\, |\varphi_n(t) - \varphi(t)|^2 = n^{-1}(1 - |\varphi(t)|^2) \leqq n^{-1}$ implies

$$EY_n \leqq (nh_n^m)^{-1} \textstyle\int |k(h_nt)|\, dt = (nh_n^{2m})^{-1} \textstyle\int |k(t)|\, dt$$

and hence $\lim_n EY_n = 0$ under (26). Furthermore,

$$\begin{aligned}EZ_n &\leqq n^{-1} \int |h_{n+1}^{-m}k(h_{n+1}t) - h_n^{-m}k(h_nt)|\, dt \\ &\leqq \frac{1}{nh_{n+1}^m}\left|\frac{1}{h_{n+1}^m} - \frac{1}{h_n^m}\right| \int |k(t)|\, dt + \frac{1}{nh_n^{2m}}\, g\left(\frac{h_{n+1}}{h_n}\right).\end{aligned} \tag{30}$$

From (13) and condition (6), we have

$$\left|\frac{1}{h_{n+1}^m} - \frac{1}{h_n^m}\right| = \frac{1}{h_n^m}\left|1 - \left(\frac{h_n}{h_{n+1}}\right)^m\right| \sim \frac{m}{h_n^{m-1}}\left|\frac{1}{h_{n+1}} - \frac{1}{h_n}\right|. \tag{31}$$

Therefore, the first term in the upper bound of (30) is asymptotically equivalent under (6) to

$$\frac{m}{nh_n^{2m-1}}\left|\frac{1}{h_{n+1}} - \frac{1}{h_n}\right| \int |k(t)|\, dt.$$

Observe also that under conditions (6) and (23), the second term in the upper bound of (30) is bounded by a term asymptotically equivalent to

$$M(nh_n^{2m-1})^{-1}|h_{n+1}^{-1} - h_n^{-1}|$$

for some M, $0 < M < \infty$. Using condition (25) this result and the asymptotic result (31) combine with inequality (30) to complete (29).

To complete the proof we need only show that $\lim_n \{\sup_x |Ef_n(x) - f(x)|\} = 0$. Let $\delta > 0$. Then, with $K_n(u) = h_n^{-m}K(h_n^{-1}u)$, we have

$$\begin{aligned}\sup_x |Ef_n(x) - f(x)| &\leqq \sup_x \int_{\|u\| \leqq \delta} |f(x-u) - f(x)|\, K_n(u)\, du \\ &\quad + \sup_x \int_{\|u\|>\delta} |f(x-u) - f(x)|\, K_n(u)\, du \\ &\leqq \sup_x \sup_{\|u\|\leqq\delta} |f(x-u) - f(u)| + 2 \sup_x f(x) \int_{\|u\|>\delta/h_n} K(u)\, du\end{aligned}$$

By uniform continuity of $f(x)$ the first term can be made arbitrarily small by choosing δ sufficiently small, while the second term for such a fixed δ approaches 0 as $n \to \infty$. This completes the proof of the theorem.

We point out that Theorem 2 in the case $m = 1$ is closely related to Theorem 1 of Nadaraya [3], although it can be shown that neither theorem implies the other.

Under condition (21), $K(u)$ is continuous on R^m and under (4), $K(u) \to 0$ as $\|u\| \to \infty$. Therefore, there exists an m-dimensional random variable θ_n such that

$$f_n(\theta_n) = \max_x f_n(x). \tag{32}$$

THEOREM 3. *Under the conditions of Theorem 2, if the mode of $f(x)$, θ, defined by*

$$f(\theta) = \max_x f(x)$$

is unique, then $\theta_n \to \theta$ with probability one as $n \to \infty$, where θ_n is defined in (32).

PROOF. The proof follows from the fact that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(\theta) - f(x)| \geqq \delta$ if $\|x - \theta\| \geqq \epsilon$. (See Nadaraya [3, Theorem 2] in the case $m = 1$.)

REMARKS. (1) It can be shown that except for the first entry (the uniform kernel) in Table 1 of Parzen [4] for $m = 1$ all the kernels given therein satisfy the conditions of Theorems 2 and 3. This remark clearly extends to the case $m > 1$ for product kernels of the same type (see Cacoullos [1, p. 186]).

(2) If $h_n = \alpha_n^{-1}n^{-p}$, where $0 < p < (2m)^{-1}$, $\alpha_n > 0$, $\alpha_{n+1} = \alpha_n + O(n^{-1})$, $n\alpha_n \to \infty$ as $n \to \infty$ and $\lim\sup \alpha_n < \infty$, then it is easily verified that conditions (5), (6), (24), (25) and (26) of Theorems 2 and 3 hold.

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