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STOCHASTIC INTEGRALS AND DERIVATIVES[1]

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0. Introduction. The question we consider in this paper is whether or not the stochastic integral has a property analogous to the Fundamental Theorem of Calculus. That is, if $Y(t, \omega) = \int_0^t \phi(s, \omega)\, dM(s, \omega)$ and if $\Delta Y(t, \omega) = Y(t + \Delta t, \omega) - Y(t, \omega)$, does $\lim_{\Delta t \to 0} \Delta Y(t, \omega)/\Delta M(t, \omega) = \phi(t, \omega)$ and in what sense does the limit exist?

1. Stochastic integrals using Brownian motion as integrator. In this section we consider only stochastic integrals using Brownian motion as integrator. Throughout this paper we will let $X = (X_t, \mathfrak{F}_t, t \geqq 0)$ denote one dimensional standard Brownian motion defined on $(\Omega, \mathfrak{F}, P)$, a complete probability space. Let $\mathfrak{F}_t$ be the complete sub σ-field of $\mathfrak{F}$ generated by $\{X_s : s \leqq t\}$. (By standard Brownian motion we mean the process is normalized so that

$$\text{Var}\,[X(t, \omega) - X(s, \omega)] = t - s \quad \text{for} \quad s < t.)$$

For notational purposes we let $\text{Plim}_{\Delta t \to 0} \Delta Y(t, \omega) = H(t, \omega)$ mean that $\Delta Y(t, \omega)$ converges in probability to $H(t, \omega)$ as $\Delta t \to 0$ where we always take $\Delta t > 0$.

DEFINITION. A real valued process, $\phi(s, \omega)$, is *stochastically integrable* on R^+ with respect to $X(t, \omega)$ if:
(i) $\phi(s, \omega)$ is adapted to $\{\mathfrak{F}_s\}$.
(ii) $\phi(s, \omega)$ is measurable on $(R^+ \times \Omega, \beta(R^+) \times \mathfrak{F})$.
(iii) $\int_0^t E\,|\phi(s, \omega)|^2\, ds < \infty$ for all finite $t \geqq 0$.
Let $M_1(X)$ denote the space of all processes stochastically integrable with respect to $X(t, \omega)$. For $\phi(s, \omega) \varepsilon M_1(X)$ one can define the stochastic integral $\int_0^t \phi(s, \omega)\, dX(s, \omega)$. For a discussion of this integral see [2] or [3].

To motivate the type of answer one should expect to our question, consider the case where $\phi(s, \omega) = X(s, \omega)$. i.e., as an integrand we take Brownian motion itself. One can easily show that $X(s, \omega)$ is stochastically integrable. In fact, the integral can be evaluated.

$$\int_0^t X(s, \omega)\, dX(s, \omega) = (X^2(t, \omega)/2) - (t/2) \quad ([2] \text{ page } 444).$$

Hence, if $Y(t, \omega) = (X^2(t, \omega)/2) - (t/2)$, then

$$\Delta Y(t, \omega)/\Delta X(t, \omega) = X(t, \omega) + (\Delta X(t, \omega)/2) - (\Delta t/2\Delta X(t, \omega)).$$

We now must show the last two terms on the right-hand side go to zero. Fix $t \geqq 0$. Now as $\Delta t \to 0$, one easily sees that $\Delta X(t, \omega)/2 \to 0$ a.s. and in L_2. How-

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ever, $\Delta t/2\Delta X(t, \omega)$ does not converge to zero in either of these senses. Fortunately, it does converge to zero in probability. This is easily shown since $\Delta X(t, \omega)/(\Delta t)^{\frac{1}{2}}$ is a normal random variable with mean zero and variance one independent of t and Δt.

We now turn to the case of more general integrands. In the remainder of this section we will make repeated use of the following property of the stochastic integral.

$$E\left|\int_0^t \phi(s, \omega)\, dX(s, \omega)\right|^2 = \int_0^t E\left|\phi(s, \omega)\right|^2 ds. \tag{1.1}$$

This property is discussed in [3].

THEOREM 1.1. *Let $\phi(s, \omega)$ be a continuous process in $M_1(X)$. Then if $Y(t, \omega) = \int_0^t \phi(s, \omega)\, dX(s, \omega)$, $\text{Plim}_{\Delta t \to 0} \Delta Y(t, \omega)/\Delta X(t, \omega) = \phi(t, \omega)$ for every $t \geqq 0$.*

PROOF. Clearly it suffices to show

$$\text{Plim}_{\Delta t \to 0} \int_t^{t+\Delta t} [\phi(s, \omega) - \phi(t, \omega)]\, dX(s, \omega)/\Delta X(t, \omega) = 0. \tag{1.2}$$

i.e., that if $\epsilon > 0$, there exists $\delta = \delta(t, \epsilon) > 0$ such that $0 \leqq \Delta t < \delta$ implies

$$P\left[\left|\int_t^{t+\Delta t} [\phi(s, \omega) - \phi(t, \omega)]\, dX(s, \omega)/\Delta X(t, \omega)\right| > \epsilon\right] \leqq \epsilon.$$

Now

$$\begin{aligned} P\left[\left|\int_t^{t+\Delta t} [\phi(s, \omega) - \phi(t, \omega)]\, dX(s, \omega)/\Delta X(t, \omega)\right| > \epsilon\right] \leqq \\ P\left[\left|\int_t^{t+\Delta t} [\phi(s, \omega) - \phi(t, \omega)]\, dX(s, \omega)/(\Delta t)^{\frac{1}{2}}\right| > \epsilon/K\right] \\ + P\left[\left|(\Delta t)^{\frac{1}{2}}/\Delta X(t, \omega)\right| > K\right] \end{aligned} \tag{1.3}$$

for $K > 0$. Fix $K = K_\epsilon$ so that $P[|(\Delta t)^{\frac{1}{2}}/\Delta X(t, \omega)| > K] \leqq \epsilon/2$. This can be done independently of t and Δt since $\Delta X(t, \omega)/(\Delta t)^{\frac{1}{2}}$ is normal $(0, 1)$. From the continuity of $\phi(s, \omega)$ one can choose $\delta > 0$ such that

$$P\{\omega: \sup_{t \leqq s \leqq t+\delta} |\phi(s, \omega) - \phi(t, \omega)|^2 > \epsilon^3/(4K^2)\} \leqq \epsilon/4.$$

Now define a stopping time by

$$T(\omega) = \inf\{s\epsilon[t, t+\delta]: |\phi(s, \omega) - \phi(t, \omega)|^2 \geqq \epsilon^3/(4K^2)\}.$$

$$T(\omega) = \infty \quad \text{if no such } s \text{ exists.}$$

T is a stopping time since $\{T \leqq s\} = \{\omega: \sup_{t \leqq r \leqq s} |\phi(r, \omega) - \phi(t, \omega)|^2 \geqq \epsilon^3/(4K^2)\} = \{\omega: \sup_{t \leqq r_j \leqq s} |\phi(r_j, \omega) - \phi(t, \omega)|^2 \geqq \epsilon^3/(4K^2)\}$ where r_j ranges through a countable set since $\phi(s, \omega)$ is separable. The right-hand side is clearly in $\mathfrak{F}_s$. Define a new process

$$\phi^*(s, \omega) = \phi(s, \omega) \qquad \text{if} \quad s \leqq T$$

$$\phi^*(s, \omega) = \phi(T(\omega), \omega) \quad \text{if} \quad s > T.$$

It follows that $\phi^*(s, \omega) - \phi(t, \omega)$ is stochastically integrable on $[t, t + \delta]$ with respect to $X(s, \omega)$. Use the fact that $\phi^*(s, \omega) = \phi(s \wedge T(\omega), \omega)$ where $s \wedge T$ is a stopping time ([4], pages 70–73). Now

$$E\,\left|\int_t^{t+\Delta t}[\phi^*(s,\omega)-\phi(t,\omega)]\,dX(s,\omega)/(\Delta t)^{\frac{1}{2}}\right|^2 = \int_t^{t+\Delta t}E\,|\phi^*(s,\omega)-\phi(t,\omega)|^2\,ds/\Delta t \leqq \epsilon^3/(4K^2)$$

if $\Delta t \leqq \delta$. Therefore

$$P[|\int_t^{t+\Delta t}[\phi^*(s,\omega)-\phi(t,\omega)]\,dX(s,\omega)/(\Delta t)^{\frac{1}{2}}| > \epsilon/K] < \epsilon/4 \quad \text{if} \quad \Delta t \leqq \delta.$$

Using this and the facts that $P[|(\Delta t)^{\frac{1}{2}}/\Delta X(t,\omega)| > K] \leqq \epsilon/2$ and

$$P[\phi^*(s,\omega) \neq \phi(s,\omega)] \leqq \epsilon/4$$

we get (1.2) from (1.3).

REMARK. If the integrand is not random, we need only assume the path to be right continuous and the conclusion of the above theorem still holds. The proof is a simple modification of the above proof.

We now treat the case where $\phi(s,\omega)$ is only assumed to be stochastically integrable. Let $\mathfrak{F}_\infty$ denote the σ-field generated by $\bigcup_{n=1}^\infty \mathfrak{F}_n$.

THEOREM 1.2. *Let $L_2(\Omega, \mathfrak{F}_\infty, P)$ be separable. Let $\phi(s,\omega) \in M_1(X)$. If $Y(t,\omega) = \int_0^t \phi(s,\omega)\,dX(s,\omega)$, then* $\text{Plim}_{\Delta t\to 0}\,\Delta Y(t,\omega)/\Delta X(t,\omega) = \phi(t,\omega)$ *for $t \not\in N''$ where N'' is a Lebesgue null set.*

PROOF. We must show there exists a null set, N'', such that for $\epsilon > 0$ and $t \not\in N''$ there is a $\delta = \delta_t > 0$ such that

$$(1.4)\qquad P[|\int_t^{t+st}[\phi(s,\omega)-\phi(t,\omega)]\,dX(s,\omega)/\Delta X(t,\omega)| > \epsilon] < \epsilon$$

for $\Delta t \leqq \delta$. As in the previous theorem, fix $K = K_\epsilon > 0$ such that

$$P[|(\Delta t)^{\frac{1}{2}}/\Delta X(t,\omega)| > K] \leqq \epsilon/2.$$

Now (1.4) is proved from (1.3) by showing

$$(1.5)\qquad P[|\int_t^{t+\Delta t}[\phi(s,\omega)-\phi(t,\omega)]\,dX(s,\omega)/(\Delta t)^{\frac{1}{2}}| > \epsilon/K] \leqq \epsilon/2$$

for $t \not\in N''$ and $\Delta t \leqq \delta$. By separability let $Q = \{\gamma_i(\omega)\}$ be a countable dense subset of $L_2(\Omega, \mathfrak{F}_\infty)$. Fix $\gamma_i(\omega) \in Q$. Let $S > 0$ be fixed and consider

$$\lim_{\Delta t\to 0}\int_t^{t+\Delta t}E\,|\phi(s,\omega)-\gamma_i(\omega)|^2\,ds/\Delta t \quad \text{for} \quad t \in [0, S).$$

$E\,|\phi(s,\omega)-\gamma_i(\omega)|^2$ is Lebesgue integrable on $[0, S)$ so by classical Lebesgue theory

$$\lim_{\Delta t\to 0}\int_t^{t+\Delta t}E\,|\phi(s,\omega)-\gamma_i(\omega)|^2\,ds/\Delta t = E\,|\phi(t,\omega)-\gamma_i(\omega)|^2$$

for a.e. $t \in [0, S)$. Let N_i denote the Lebesgue null set where convergence fails for the fixed random variable $\gamma_i(\omega)$. Let $N' = \{t \in [0,S) : E\,|\phi(t,\omega)|^2 = \infty\}$. N' is a Lebesgue null set since by hypothesis $\int_0^t E\,|\phi(s,\omega)|^2\,ds < \infty$. Let $N'' = N' \cup (\bigcup_{i=1}^\infty N_i)$. Now fix $t \not\in N''$, where $t \in [0, S)$.

$$\lim_{\Delta t\to 0} E\,|\int_t^{t+\Delta t} [\phi(s,\omega) - \phi(t,\omega)]\, dX(s,\omega)/(\Delta t)^{\frac{1}{2}}|^2$$
$$= \lim_{\Delta t\to 0} \int_t^{t+\Delta t} E\,|\phi(s,\omega) - \phi(t,\omega)|^2\, ds/\Delta t$$
$$\leqq 2\lim_{\Delta t\to 0} \int_t^{t+\Delta t} E\,|\phi(s,\omega) - \gamma_i(\omega)|^2\, ds/\Delta t$$
$$+ 2\lim_{\Delta t\to 0} \int_t^{t+\Delta t} E\,|\gamma_i(\omega) - \phi(t,\omega)|^2\, ds/\Delta t$$
$$= 4E\,|\phi(t,\omega) - \gamma_i(\omega)|^2.$$

Now $\phi(t,\omega)\ \varepsilon\ L_2(\Omega, \mathfrak{F}_\infty)$ since $\phi(t,\omega)$ is adapted. Therefore by the denseness of Q the left-hand side is zero so $\int_t^{t+\Delta t} [\phi(s,\omega) - \phi(t,\omega)]\, dX(s,\omega)/(\Delta t)^{\frac{1}{2}}$ converges to zero in L_2 and hence in probability. Now $S > 0$ is arbitrary so (1.5) is true and hence the theorem is proved.

REMARK. In the above case where $\mathfrak{F}_t$ is chosen to be minimal one has $L_2(\Omega, \mathfrak{F}_\infty)$ is separable so it need not be assumed.

It is possible to define a stochastic integral for integrands that do not have all the properties of the above theorems. That is, instead of property (iii) one need only assume $P[\int_0^t |\phi(s,\omega)|^2\, ds < \infty] = 1$ for each $t > 0$. The above two theorems can be proved in this case also. To do this define $\phi_N(t,\omega) = \phi(t,\omega)$ if $\int_0^t |\phi(s,\omega)|^2\, ds < N$ and $\phi_N(t,\omega) = 0$ otherwise. $\phi_N(t,\omega)$ has the properties of the previous theorems and $\lim_{N\to\infty} \phi_N(t,\omega) = \phi(t,\omega)$ a.s.

2. Stochastic integrals using general integrators. The stochastic integral has been defined using integrators that are much more general than the Brownian motion used in Section One. In particular, Meyer has defined the integral using right continuous square integrable martingales as integrators [5]. Hence, if $N(t,\omega) = \int_0^t \phi(s,\omega)\, dM(s,\omega)$, it is reasonable to ask whether

$$\operatorname{Plim}_{\Delta t\to 0} \Delta N(t,\omega)/\Delta M(t,\omega) = \phi(t,\omega).$$

Unfortunately, this is not true in general. In this section we will discuss what is known about this question but many proofs will be omitted. We will only consider the question for continuous integrands although similar results hold in general.

For the reader's benefit we will now state a few standard definitions and known results.

DEFINITION. A process $A(t,\omega)$ is called a time change for a Brownian motion $(X_t, \mathfrak{F}_t, t \geqq 0)$ if

(i) $A(t,\omega)$ is right continuous and nondecreasing as a function of t for a.e. ω.

(ii) $A(t,\omega)$ is a stopping time for each fixed t.

(iii) $E\,|A(t,\omega)| < \infty$ for each finite t.

In the remainder of this paper we will restrict our attention to the special class of martingales that can be written as a Brownian motion with a time change. According to Dambis, we know this includes all continuous martingales with $M_0 = 0$ [1]. Now if we define $M(t,\omega] = X(A(t,\omega),\omega)$ the process $M(t,\omega)$ is a right continuous square integrable martingale. (i.e., $E[M_t^2] < \infty$ for all finite t.) ([2] page 365). In fact, $E[M^2(t,\omega) - M^2(s,\omega)|\,\mathfrak{F}_s] = E[A(t,\omega) - A(s,\omega)|\,\mathfrak{F}_s]$ so

$\langle M, M\rangle_t = A(t, \omega)$ where $\langle M, M\rangle_t$ is the natural increasing process associated with $M(t, \omega)$ when $A(t, \omega)$ is continuous [5]. The martingale $M(t, \omega)$ is adapted to the family of σ-fields $\{\mathcal{F}_{(A\,t,\omega)}\}$ which we will denote by $\{\mathcal{G}_t\}$.

DEFINITION. A stochastic process $Y(s, \omega)$ defined on $(R^+ \times \Omega)$ is called *very well measurable* if the function $(t, \omega) \rightarrow Y_t(\omega)$ is measurable with respect to the σ-field on $R^+ \times \Omega$ generated by the processes having left continuous paths.

DEFINITION. A real valued process, $\phi(s, \omega)$, is stochastically integrable with respect to $(M_s, \mathcal{G}_s)$ if

(i) $\phi(s, \omega)$ is adapted to $\{\mathcal{G}_s\}$.

(ii) $\phi(s, \omega)$ is very well measurable on $(R^+ \times \Omega)$.

(iii) $E \int_0^t |\phi(s, \omega)|^2 \, d\langle M, M\rangle_s < \infty$ for all finite $t > 0$.

Let $\hat{L}^2(M)$ denote the space of processes stochastically integrable with respect to M. For $\phi(s, \omega) \in \hat{L}^2(M)$ one can define the stochastic integral $\int_0^t \phi(s, \omega)\, dM(s, \omega)$. For a discussion of this integral see [5].

If we take a nonrandom time change, we get the following theorem.

THEOREM 2.1. *Let* $M(t, \omega) = X(A(t), \omega)$ *where* $A(t)$ *is a right continuous strictly increasing function. If* $\phi(s, \omega) \in \hat{L}^2(M)$ *is continuous and* $N(t, \omega) = \int_0^t \phi(s, \omega)\, dM(s, \omega)$, *then* $\operatorname{Plim}_{\Delta t \to 0} \Delta N(t, \omega)/\Delta M(t, \omega) = \phi(t, \omega)$ *for every* $t \geqq 0$.

PROOF. The proof is very similar to that of Theorem 1.1, so it will be omitted.

Now consider a time change of the form $A(t, \omega) = \sum_{i=1}^{\infty} 1_{B_i}(\omega) A_i(t)$ where $\bigcup_{i=1}^{\infty} B_i = \Omega$, $B_i \cap B_j = \phi$ for $i \neq j$, and $A_i(t)$ is a right continuous, strictly increasing function of t. Let $M(t, \omega) = X(A(t, \omega), \omega)$. The analogue of Theorem 1.1 is still true but the proof requires the following lemmas.

LEMMA 2.1. *Let* $B \in \mathcal{F}$ *and let* $A(s, \omega)$ *be a time change of the form* $A(s, \omega) = 1_B(\omega) A_1(s) + 1_{B^c}(\omega) A_2(s)$. *Let* $\phi(s, \omega) \in \hat{L}^2(M)$ *be continuous. Then*

$$1_B \int_t^{t+\Delta t} [\phi(s, \omega) - \phi(t, \omega)]\, dX(A(s, \omega), \omega)$$

$$= 1_B \int_t^{t+\Delta t} [\phi(s, \omega) - \phi(t, \omega)]\, dX(A_1(s), \omega) \text{ a.s.}$$

HINT OF PROOF. Use step processes to approximate $\phi(s, \omega)$ and easily show the result for integrands that are step processes.

LEMMA 2.2. *Let* $Z(t, \omega)$ *be a stochastic process for which*

$$\operatorname{Plim}_{t \to 0} Z(A_i(t), \omega) = 0$$

for each of denumerably many real valued functions A_i. *Let* $\bigcup_{i=1}^{\infty} B_i = \Omega$ *where* $B_i \cap B_j = \phi$ *for* $i \neq j$. *Let* $A(t, \omega) = \sum_{i=1}^{\infty} 1_{B_i}(\omega) A_i(t)$. *If* $1_{B_i} Z(A(t, \omega), \omega) = 1_{B_i} Z(A_i(t), \omega)$ a.s. *then* $\operatorname{Plim}_{t \to 0} Z(A(t, \omega), \omega) = 0$.

PROOF. The proof is straightforward and will be omitted.

Using these two lemmas, one can prove:

THEOREM 2.2. *Let* $A(s, \omega) = \sum_{i=1}^{\infty} 1_{B_i} A_i(s)$ *as described above,* $M(s, \omega) = X(A(s, \omega), \omega)$ *and* $\phi(s, \omega) \in \hat{L}^2(M)$ *be continuous. Then if* $Y(t, \omega) = \int_0^t \phi(s, \omega)\, dM(s, \omega)$, $\operatorname{Plim}_{\Delta t \to 0} \Delta Y(t, \omega)/\Delta M(t, \omega) = \phi(t, \omega)$ *for every* $t \geqq 0$.

Now one would hope that by approximation, the desired convergence could be proved for all integrators of the form $M(s, \omega) = X(A(s, \omega), \omega)$. However, we have an example where $A(s, \omega)$ is right continuous and convergence fails. We will sketch the ideas of the example here. Choose a sequence of points converging to zero as follows. Let $b_1 = 1$, $b_n = \sup\{s < b_{n-1}/2 : P[X(t, \omega) = 0 \text{ for some } t \in [s, b_{n-1}/2]] \geqq \frac{15}{16}\}$. Define

$$A(b_n, \omega) = \inf\{s : b_n < s \leqq b_{n-1}/2 \quad \text{and} \quad X(s, \omega) = 0\}$$

$$A(b_n, \omega) = b_{n-1}/2 \quad \text{if no such} \quad s \quad \text{exists.}$$

Let

$$A(s, \omega) = s \quad \text{if} \quad s \geqq 1$$

$$A(s, \omega) = A(b_n, \omega) + [b_{n-1}/2(b_{n-1} - b_n)][s - b_n] \quad \text{if} \quad b_n \leqq s < b_{n-1}$$

$$A(0, \omega) = 0.$$

It can be shown that $A(s, \omega)$ is a time change for X. Now choose for each $n \geqq 1$ a point t_n such that $b_n \leqq t_n < b_{n-1}$ and $P[X(A(t_n, \omega), \omega) = 0] \leqq \frac{1}{16}$. This is possible since $X(A(b_n, \omega) + [b_{n-1}/2(b_{n-1} - b_n)][t - b_n], \omega)$ behaves like Brownian motion for $b_n \leqq t < b_{n-1}$. Define

$$\phi(0, \omega) = 0 \qquad \text{for all} \quad \omega \in \Omega$$

$$\phi(s, \omega) = 1/n \qquad \text{if} \quad b_n \leqq s < t_n, \qquad \omega \in \Omega$$

$$\phi(s, \omega) = -1/n \quad \text{if} \quad t_n \leqq s < b_{n-1}, \qquad \omega \in \Omega$$

$\phi(s, \omega)$ is clearly in $\hat{L}^2(M)$. In fact, it is right continuous and nonrandom so one would hope for convergence for all $t \geqq 0$ by the remark following Theorem 1.1. However,

$$P[|\textstyle\int_0^{b_n} \phi(s, \omega)\, dX(A(s, \omega), \omega)| > \epsilon\, |X(A(b_n, \omega), \omega)|] \geqq \frac{1}{4}$$

for infinitely many n, with $b_n \to 0$.

There remains a very interesting class of martingales of the form $M(t, \omega) = X(A(t, \omega), \omega)$ for which no conclusions have been drawn. That is the case where $A(t, \omega)$ is assumed to be continuous. It is still unknown to the author whether or not one gets convergence in this case.

There is one additional type of martingale for which we get convergence using the above methods.

THEOREM 2.3. *Let $\phi(s, \omega) \in M_1(X)$ be continuous. Let*

$$M(t, \omega) = \textstyle\int_0^t \phi(s, \omega)\, dX(s, \omega).$$

Let $H(t, \omega) \in \hat{L}^2(M)$ be continuous. Assume $P[\phi(t^*, \omega) = 0] = 0$. Then if $N(r, \omega) = \int_0^r H(t, \omega)\, dM(t, \omega)$, $\text{Plim}_{\Delta t \to 0} \Delta N(t^*, \omega)/\Delta M(t^*, \omega) = H(t^*, \omega)$.

PROOF. $N(r, \omega) = \int_0^r H(t, \omega)\phi(t, \omega)\, dX(t, \omega)$ so by previous theorems

$$\text{Plim}_{\Delta t \to 0} \Delta N(t^*, \omega)/\Delta M(t^*, \omega) = \text{Plim}_{\Delta t \to 0} \Delta N(t^*, \omega)/\Delta X(t^*, \omega) \cdot \text{Plim}_{\Delta t \to 0} \Delta X(t^*, \omega)/\Delta M(t^*, \omega) = H(t^*, \omega).$$

We conclude this paper with a second method for recovering the integrand of a stochastic integral. This method will not yield convergence at every t for continuous integrands but the method is applicable to all stochastic integrals in which the integrator is a right continuous square integrable martingale. The essentials of this theorem are due to Meyer ([5], pages 74 and 79).

THEOREM 2.4. *Let* $M(s, \omega)$ *be a right continuous square integrable martingale,* $\phi(s, \omega) \varepsilon \hat{L}^2(M)$, *and* $N(t, \omega) = \int_0^t \phi(s, \omega)\, dM(s, \omega)$. *Then for* $t \varepsilon [0, L]$

$$(2.1) \qquad \lim_{n\to\infty} \sum_{k=0}^{L2^n} \frac{\langle N, M\rangle_{k/2^n} - \langle N, M\rangle_{k-1/2^n}}{\langle M, M\rangle_{k/2^n} - \langle M, M\rangle_{k-1/2^n}} 1_{[k-1/2^n, k/2^n]} = \phi(t, \omega)$$

a.e. $d\langle M, M\rangle$ a.s.

PROOF. From Meyer's construction of the stochastic integral it is known that $N_t = \int_0^t \phi_s\, dM_s$ implies $\langle N, M\rangle_t = \int_0^t \phi_s\, d\langle M, M\rangle_s$ [5]. Now $\langle M, M\rangle_t$ generates a finite measure and $\langle N, M\rangle_t$ generates a signed measure of finite variation. We clearly have $\langle N, M\rangle \ll \langle M, M\rangle$. Hence, as an application of martingale theory one has (2.1). ([2] page 344.)

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