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ON AN EXTENDED COMPOUND DECISION PROBLEM

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0. Summary. In this paper we develop some theorems for an extended compound decision problem which has as the component a general statistical decision problem. The results generalize and strengthen some results previously reported by Swain (1965) and Johns (1967). From the outset it is assumed that the reader is familiar with the compound decision problem.

1. Introduction. Swain (1965) has at the suggestion of Professor M. V. Johns, Jr. investigated the use of more stringent standards in the compound decision problem and has called the resulting version the extended compound decision problem. The usual standard is $R(G_n)$, the Bayes envelope of the component problem evaluated at the empirical distribution of player I's first n pure strategies $\theta_1, \theta_2, \cdots, \theta_n$. In the extended version, the idea is to take advantage of higher order empirical dependencies in the parameter sequence as measured by $G_n^{\ k}$, the empirical distribution of the vectors $(\theta_1, \cdots, \theta_k)$, $(\theta_2, \cdots, \theta_{k+1})$, $\cdots$, $(\theta_{n-k+1}, \cdots, \theta_n)$, which is defined for $1 \leqq k \leqq n$, $n \geqq 1$.

In order to be more explicit we introduce more notation by defining R_k to be the Bayes envelope in the following game, which we call the Γ_k game. Player I picks an $\boldsymbol{\omega}^k = (\omega_1, \cdots, \omega_k) \,\varepsilon\, \Omega^k$, and Player II picks an action $a \,\varepsilon\, A$ and suffers loss $L(\omega_k, a) \geqq 0$. In the statistical version Player II gets to observe $\underset{\sim}{X}^k = (X_1, \cdots, X_k)$ distributed $P_{\omega_1} \times \cdots \times P_{\omega_k}$ before choosing an action. Here a strategy is a (randomized) decision function φ which is a mapping from $\mathfrak{X}^k$, the range space of the random vector $\underset{\sim}{X}^k$, to the set of probability measures on a suitable σ-field of subsets of A. We require that a decision function φ is such that the random loss

$$(1) \qquad L(\omega_k, \varphi(\mathbf{x}^k)) = \int L(\omega_k, \cdot)\, d(\varphi)\mathbf{x}^k))$$

is jointly measurable in ω_k and $\mathbf{x}^k$. (A dot is used to denote the dummy variable of integration within an integrand whenever it is convenient.) We define the risk,

$$(2) \qquad R_k(\boldsymbol{\omega}^k, \varphi) = \int L(\omega_k, \varphi(\cdot))\, d(P_{\omega_1} \times \cdots \times P_{\omega_k})$$

and the Bayes risk of φ versus a probability measure G,

$$(3) \qquad R_k(G, \varphi) = \int R_k(\cdot\,, \varphi)\, dG.$$

The Bayes envelope in the Γ_k game is given by

$$(4) \qquad R_k(G) = \inf_\varphi R_k(G, \varphi).$$

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The decision problem Γ_1 is the component game in the compound decision problem we are considering. The sequence $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots)$ is the sequence of player I pure strategies in the repetitions of the component game. The extended compound decision problem, as conceived by Johns, has a standard $R_k(G_n^k)$ for some fixed k. For $k = 1$ we have $R_1(G_n^1) = R(G_n)$, the usual standard.

In special cases sequence compound procedures have been demonstrated which achieve the standards $R_k(G_n^k)$ asymptotically (see Swain (1965) and Johns (1967)). Also, in these special cases it has been shown that $R_{k+1}(G_n^{k+1})$ is asymptotically a more stringent standard than $R_k(G_n^k)$; that is, for each fixed $k \geqq 1$ and $\boldsymbol{\theta} \varepsilon \Omega^\infty$,

$$\limsup_n \{R_{k+1}(G_n^{k+1}) - R_k(G_n^k)\} \leqq 0. \tag{5}$$

In fact, Swain (1965, Theorem 1) has proved that for estimation with squared error loss and $A = \Omega$ a bounded subset of the reals, $R_{k+1}(G_n^{k+1}) - R_k(G_n^k) = f(k, n, \boldsymbol{\theta}) + h(k, n, \boldsymbol{\theta})$ where $f(k, n, \boldsymbol{\theta}) \leqq 0$ and $|h(k, n, \boldsymbol{\theta})| \leqq M(n-k)^{-1}$ with M a constant depending only upon the supremum of the loss function in the component game. Johns (1967, Theorem 2) has proved the same result for the two-action problem with bounded loss function. The proofs given in both cases are peculiar to the special component problems being considered. Theorem 1 to follow yields an improved bound for the most general situation with an easy proof.

Theorem 2 is a restatement of a game theoretic result due to Hannan (1957) in a form pertinent to the extended compound decision problem. It has been proved several times in special contexts since the original result was published. Theorems 3 and 4 relate the extended sequence compound and empirical Bayes problems. Theorem 3 was stated by Swain (1965, Theorem 4) for the squared error loss estimation problem.

2. Results. We begin with a simple remark.

Remark 1. Let G be any probability measure on Ω^k and let G_* be the marginal of G on any ordered subset of coordinates $(i_1, \cdots, i_j)$ where $i_j = k$. (a) It follows that $R_k(G) \leqq R_j(G_*)$. (b) If in addition G is the product of G_* and the marginal H_* on the other coordinates, then $R_k(G) = R_j(G_*)$.

Proof. Without loss of generality we let $(i_1, \cdots, i_j) = (k - j + 1, \cdots, k)$. Also we let φ^* be generic for decision functions in the game Γ_k which are $(x_{k-j+1}, \cdots, x_k)$ measurable. Then $R_k(G) = \inf_\varphi R_k(G, \varphi) \leqq \inf_{\varphi^*} R_k(G, \varphi^*) = R_j(G_*)$ since $R_k(G, \varphi^*) = R_j(G_*, \varphi^*)$. We now suppose that $G = H_* \times G_*$ and write

$$R_k(G, \varphi) = \int R_k(\cdot, \varphi)\, dG = \iint R_k(\cdot, \varphi)\, dG_*\, dH_*.$$

An interchange in the order of integration with respect to the two measures G_* and $P_{\omega_1} \times \cdots \times P_{\omega_{k-j}}$ in the inner integral, $\int R_k(\cdot, \varphi)\, dG_*$, shows it to be greater than or equal to $R_j(G_*)$ for every $\boldsymbol{\omega}^{k-j}$. Thus, $R_k(G, \varphi) \geqq R_j(G_*)$ for every φ which together with (a) implies (b).

Theorem 1. *For all* $1 \leqq k < n, n \geqq 1$ *and* $\boldsymbol{\theta}$,

$$(n-k)R_{k+1}(G_n^{k+1}) \leqq (n-k+1)R_k(G_n^k). \tag{6}$$

PROOF. Consider H_n^{k+1} the empirical distribution of $(\theta_n, \theta_1, \cdots, \theta_k)$, $(\theta_1, \theta_2, \cdots, \theta_{k+1}), \cdots, (\theta_{n-k}, \theta_{n-k+1}, \cdots, \theta_n)$. The marginal it induces on the last k coordinates is G_n^k so by Remark 1 (a),

$$R_{k+1}(H_n^{k+1}) \leqq R_k(G_n^k). \tag{7}$$

For any decision function φ in the Γ_{k+1} game,

$$\begin{aligned}
&R_{k+1}(G_n^{k+1}, \varphi) - R_{k+1}(H_n^{k+1}, \varphi) \\
&\quad = (n-k)^{-1} \textstyle\sum_{i=k+1}^n R_{k+1}((\theta_{i-k}, \cdots, \theta_i), \varphi) \\
&\qquad - (n-k+1)^{-1}\{\textstyle\sum_{i=k+1}^n R_{k+1}((\theta_{i-k}, \cdots, \theta_i), \varphi) + R_{k+1}((\theta_n, \theta_1, \cdots, \theta_k), \varphi)\} \\
&\quad = (n-k+1)^{-1}(n-k)^{-1} \textstyle\sum_{i=k+1}^n \{R_{k+1}((\theta_{i-k}, \cdots, \theta_i), \varphi) \\
&\qquad - R_{k+1}((\theta_n, \theta_1, \cdots, \theta_k), \varphi)\} \\
&\quad \leqq (n-k+1)^{-1}(n-k)^{-1} \textstyle\sum_{i=k+1}^n R_{k+1}((\theta_{i-k}, \cdots, \theta_i), \varphi) \\
&\quad = (n-k+1)^{-1} R_{k+1}(G_n^{k+1}, \varphi), \text{ so that}
\end{aligned}$$

$$(n-k)R_{k+1}(G_n^{k+1}, \varphi) \leqq (n-k+1)R_{k+1}(H_n^{k+1}, \varphi), \tag{8}$$

From (8) it follows that $(n-k)R_{k+1}(G_n^{k+1}) \leqq (n-k+1)R_{k+1}(H_n^{k+1})$ which together with (7) implies (6).

COROLLARY. *If* $B = \sup_{\omega, a} L(\omega, a)$, *then for all* $1 \leqq k < n$, $n \geqq 1$ *and* $\boldsymbol{\theta}$,

$$R_{k+1}(G_n^{k+1}) - R_k(G_n^k) \leqq B(n-k+1)^{-1}. \tag{9}$$

PROOF. Inequality (6) is equivalent to $(n-k+1)\{R_{k+1}(G_n^{k+1}) - R_k(G_n^k)\} \leqq R_{k+1}(G_n^{k+1})$ from which (9) is immediate.

The bound $B(n-k+1)^{-1}$ is an improvement over those established in the special cases cited earlier. For squared error loss estimation Swain obtained $7B[4(n-k+1)]^{-1}$, and for the two-action problem Johns obtained $4B(n-k)^{-1}$.

In compound problems with non-trivial components there exist parameter sequences $\boldsymbol{\theta}$ for which $\lim\sup_n \{R_{k+1}(G_n^{k+1}) - R_k(G_n^k)\} < 0$ so that asymptotically $R_{k+1}(G_n^{k+1})$ is truly more stringent than $R_k(G_n^k)$.

REMARK 2. In the set compound problem, which involves abeyance of all of the first n actions until the first n random variables have been observed, it seems that a more natural set of standards is provided by $R_k(I_n^k)$, $k \geqq 1$, where I_n^k is the empirical distribution of $(\theta_{n-k+2}, \cdots, \theta_n, \theta_1)$, $(\theta_{n-k+3}, \cdots, \theta_1, \theta_2), \cdots, (\theta_{n-k+1}, \cdots, \theta_{n-1}, \theta_n)$ and the coordinate arithmetic is mod n. We note that I_n^k depends only upon $\boldsymbol{\theta}^n$ and that $I_n^1 = G_n^1 = G_n$. Asymptotically $R_k(I_n^k)$ and $R_k(G_n^k)$ are equivalent when R_k is bounded. However, with these standards we get an improvement over (6). Since I_n^{k+1} induces the marginal I_n^k on the last k coordinates, Remark 1 (a) implies that for all $1 \leqq k < n$, $n \geqq 1$ and $\boldsymbol{\theta}$,

$$R_{k+1}(I_n^{k+1}) \leqq R_k(I_n^k). \tag{6*}$$

Our next theorem is a restatement of results (8.8) and (8.11) of Hannan (1957) in the notation of the extended sequence compound decision problem. These results have often been given and proved in special cases since the original paper [see Samuel (1963, Lemma 2), (1965a, Lemma 1), Swain (1965, Lemma 1) and Van Ryzin (1966, Lemma 3.2)]. Relation (8.8) shows that across N repetitions of any game Γ, the average risk resulting from playing Bayes versus the empirical distribution of player I's up-to-date history of pure strategies at each stage is no greater than the Bayes envelope of Γ evaluated at the empirical distribution of all N strategies. The decomposition (8.11) shows that the average risk resulting from playing Bayes versus the empirical distribution of player I's past history of pure strategies at each stage is no less than the Bayes envelope of Γ evaluated at the empirical distribution of all N strategies. We now write these results for $N = n - k + 1$ repetitions of the Γ_k game where player I's pure strategies are $(\theta_1, \cdots, \theta_k), \cdots, (\theta_{n-k+1}, \cdots, \theta_n)$.

THEOREM 2. *For $i \geqq k$, let ψ_i^k denote a Bayes response versus G_i^k in the Γ_k game and let ψ_{k-1}^k be arbitrary. Then for all $1 \leqq k \leqq n$, $n \geqq 1$ and $\boldsymbol{\theta}$,*

$$(10)\qquad \textstyle\sum_{i=k}^n R_k((\theta_{i-k+1}, \cdots, \theta_i), \psi_i^k) \leqq (n - k + 1)R_k(G_n^k) \leqq \sum_{i=k}^n R_k((\theta_{i-k+1}, \cdots, \theta_i), \psi_{i-1}^k).$$

We now state some theorems which relate the extended versions of the sequence compound and empirical Bayes problems.

REMARK 3. Let $\boldsymbol{\theta}$ be a strictly stationary stochastic process with G denoting the measure on infinite sequences $\boldsymbol{\theta}$ and G_*^k denoting the marginal on $\boldsymbol{\theta}^k$. Then for all $1 \leqq k \leqq n$ and $n \geqq 1$,

$$(11)\qquad R_k(G_*^k) \geqq \int^- R_k(G_n^k)\, dG(\boldsymbol{\theta})$$

where the integral on the right is an upper integral.

PROOF. For any decision function φ in Γ_k we have

$$(12)\qquad \begin{aligned} R_k(G_*^k, \varphi) &= (n - k + 1)^{-1} \textstyle\sum_{i=k}^n R_k(G_*^k, \varphi) \\ &= (n - k + 1)^{-1} \textstyle\sum_{i=k}^n \int R_k((\theta_{i-k+1}, \cdots, \theta_i), \varphi)\, dG(\boldsymbol{\theta}) \\ &= \textstyle\int R_k(G_n^k, \varphi)\, dG(\boldsymbol{\theta}) \end{aligned}$$

by the strict stationarity of G. Since $R_k(G_n^k, \varphi)$ is measurable in $\boldsymbol{\theta}$ and satisfies $R_k(G_n^k, \varphi) \geqq R_k(G_n^k)$ for all $\boldsymbol{\theta}$, the definition of upper integral implies that $\int R_k(G_n^k, \varphi)\, dG(\boldsymbol{\theta}) \geqq \int^- R_k(G_n^k)\, dG(\boldsymbol{\theta})$. Therefore, (12) yields $R_k(G_*^k, \varphi) \geqq \int^- R_k(G_n^k)\, dG(\boldsymbol{\theta})$ for every φ so that (11) is proved.

THEOREM 3. (Swain, 1965). *Let $\boldsymbol{\theta}$ be a strictly stationary stochastic process with G denoting the measure on infinite sequences $\boldsymbol{\theta}$ and G_*^k denoting the marginal on $\boldsymbol{\theta}^k$. Suppose that $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots)$ is a solution of the extended sequence compound decision problem; that is, each φ_i is $\mathbf{x}^i$ measurable and for each $\boldsymbol{\theta}$,*

$$(13)\qquad \limsup_n \{n^{-1} \textstyle\sum_{i=1}^n R_i(\boldsymbol{\theta}^i, \varphi_i) - R_k(G_n^k)\} \leqq 0.$$

It follows that under general conditions,

$$(14)\qquad \limsup_n \{n^{-1} \textstyle\sum_{i=1}^n \int R_i(\boldsymbol{\theta}^i, \varphi_i)\, dG(\boldsymbol{\theta})\} \leqq R_k(G_*^k).$$

PROOF. By subtracting and adding $\int^{-} R_k(G_n^{\ k})\, dG(\boldsymbol{\theta})$ to the term in curly brackets in (14), invoking (11), and using the fact that $\int f - \int^{-} g = \int_{-} (f - g)$, we obtain

$$(15)\quad n^{-1} \textstyle\sum_{i=1}^{n} \int R_i(\boldsymbol{\theta}^i, \varphi_i)\, dG(\boldsymbol{\theta}) - R_k(G_*^{\ k}) \le \int_{-} \{n^{-1} \sum_{i=1}^{n} R_i(\boldsymbol{\theta}^i, \varphi_i) - R_k(G_n^{\ k})\}\, dG(\boldsymbol{\theta}).$$

If $n^{-1} \sum_{i=1}^{n} R_i(\boldsymbol{\theta}^i, \varphi_i) - R_k(G_n^{\ k})$ is dominated by a G-integrable function of $\boldsymbol{\theta}$, then (15), (13) and Fatou's lemma yield (14).

In practice, (13) is usually established by demonstrating an upper bound which is independent of $\boldsymbol{\theta}$ and converges to 0, so that the conditions of the theorem are satisfied. In reasonable problems $R_k(G_n^{\ k})$ is measurable in $\boldsymbol{\theta}$ so that upper and lower integrals are the ordinary integral.

Swain (1965, Theorem 4) has stated a specialization of Theorem 3 to the case of squared error loss estimation. The basic idea of Swain's proof does yield a proof of the general case, but we have given an alternative proof.

Theorem 3 shows that solutions of the extended sequence compound decision problem are also average risk solutions of the extended empirical Bayes problem. This result has been noted in special cases for the unextended, $k = 1$, sequence compound and empirical Bayes problems; for the finite component by Van Ryzin (1966, Theorem 6.1) and for the general component by Samuel (1965b) and Gilliland (1968).

REMARK 4. Under the hypothesis of Remark 3 and with $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots)$ any sequence compound procedure, it follows that for $1 \le i \le k$,

$$(16)\qquad \int R_i(\boldsymbol{\theta}^i, \varphi_i)\, dG(\boldsymbol{\theta}) \ge R_k(G_*^{\ k}).$$

If, in addition, $\boldsymbol{\theta}^{i-k}$ and $(\theta_{i-k+1}, \cdots, \theta_i)$ are independent for $i > k$, then (16) holds for $i > k$.

PROOF. In order to prove (16) for $1 \le i \le k$ we note that in this case, $\boldsymbol{\theta}^i$ and $(\theta_{k-i+1}, \cdots, \theta_k)$ each have the same distribution $G_*^{\ i}$. Since this is the marginal induced by $G_*^{\ k}$ on the last i coordinates of $\boldsymbol{\theta}^k$, Remark 1 (a) implies that $R_i(G_*^{\ i}) \ge R_k(G_*^{\ k})$ which implies (16). If for the case $i > k$, $\boldsymbol{\theta}^{i-k}$ is independent of $(\theta_{i-k+1}, \cdots, \theta_i)$, then Remark 1 (b) implies $R_i(G_*^{\ i}) = R_k(G_*^{\ k})$ which implies (16).

THEOREM 4. *Under the hypothesis of Theorem* 3 *and with* $\boldsymbol{\theta}^{i-k}$ *independent of* $(\theta_{i-k+1}, \cdots, \theta_i)$ *for all* $i > k$,

$$(17)\qquad \lim_n \{n^{-1} \textstyle\sum_{i=1}^{n} \int R_i(\boldsymbol{\theta}^i, \varphi_i)\, dG(\boldsymbol{\theta})\} = R_k(G_*^{\ k}).$$

PROOF. The proof follows directly from Theorem 3 and Remark 4.

We now give a concise summary of the conclusions of Remarks 3 and 4:

$$(18)\qquad \begin{aligned} 0 &\le n^{-1} \textstyle\sum_{i=1}^{n} \int R_i(\boldsymbol{\theta}^i, \varphi_i)\, dG(\boldsymbol{\theta}) - R_k(G_*^{\ k}) \\ &= \int_{-} [n^{-1} \textstyle\sum_{i=1}^{n} R_i(\boldsymbol{\theta}^i, \varphi_i) - R_k(G_n^{\ k})]\, dG(\boldsymbol{\theta}) \\ &\quad + [\int^{-} R_k(G_n^{\ k})\, dG(\boldsymbol{\theta}) - R_k(G_*^{\ k})]. \end{aligned}$$

Since the last term is non-positive, we see that rates of convergence in the extended sequence compound decision problem yield the same rate for average risk convergence in the extended empirical Bayes problem under rather general conditions. The rate of convergence of $\int^{-} R_k(G_n^{\ k})\, dG(\boldsymbol{\theta})$ to $R_k(G_*^{\ k})$ follows as a corollary.

In closing we note that a stronger version of Theorem 4 has been stated by Swain (1965, page 98) with the hypothesis as in Theorem 4 except that the assumption that $\boldsymbol{\theta}^{i-k}$ and $(\theta_{i-k+1}, \cdots, \theta_i)$ are independent is replaced by the less restrictive assumption that $\boldsymbol{\theta}^{i-k}$ and θ_i are independent. The attempted proof depends upon a proof of the corresponding stronger version of Remark 4. The following example shows that the less restrictive assumption is not sufficient for (16) to obtain for $i > k$.

Example. Let $\boldsymbol{\theta}^3 \sim G_*^{\ 3}$ be normal with zero mean and covariance Σ and suppose the component game Γ_1 has $X \sim N(\theta, 1)$ so that $(X_1, X_2, X_3, \theta_1, \theta_2, \theta_3)$ is normal. It follows from a double application of (8 a. 2.11) of Rao (1965, page 441) that the conditional distribution of $\boldsymbol{\theta}^3$ given $\underline{X}^3$ is normal with mean $\underline{X}^3\Sigma(I + \Sigma)^{-}$ and covariance $\Sigma - \Sigma(I + \Sigma)^{-}\Sigma$. If we specialize to

$$\Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}, \qquad \rho \neq 0,$$

a computation shows that the conditional expectation of θ_3 given $\underline{X}^3$ does depend on X_1 even though θ_3 is independent of θ_1 and hence of X_1. Therefore, quite generally, we have $R_2(G_*^{\ 2}) > R_3(G_*^{\ 3})$ so that the stronger version of Remark 4 fails in this case with $i = 3$, $k = 2$. To be specific we compute for squared error loss estimation $R_2(G_*^{\ 2}) = \frac{7}{15} > \frac{91}{196} = R_3(G_*^{\ 3})$.

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