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# LIMITING DISTRIBUTIONS OF SOME VARIATIONS OF THE CHI-SQUARE STATISTIC

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**1. Summary.** The interarrival times past a point at the intersection of cars that queued up during the stop phase were considered by the authors of [1]. Denoting by $\{T_i\}$, $i = 1, 2, \cdots$, the process of these consecutive interarrival times or time headways, an examination of experimental data indicated that the random variables $\{T_{n_0+i}, i = 1, 2, \cdots\}$ were independently identically distributed. For $n_0 = 2$ the $\chi^2$-test for homogeneity showed this hypothesis to be tenable. Therefore the data were pooled to estimate the parameters of a suitable density function. As a final step, each of the subpopulations was tested, using the $\chi^2$-statistic, against the population whose parameters were determined using the pooled data. Two methods of estimation, namely, the modified minimum $\chi^2$ and the method of maximum likelihood, were considered. In each case, this paper shows that these $\chi^2$-statistics do not have the usual limiting $\chi^2$-distribution, but are stochastically larger than would be expected under the $\chi^2$-theory. More generally and precisely, let $\Pi_i$, $i = 1, 2, \cdots, k$ populations and let the mathematical form of $\Pi_i$ be known except for the unknowns $\boldsymbol{\alpha}_i = (\alpha_{1i}, \alpha_{2i}, \cdots, \alpha_{si})$. Assume that the hypothesis $H_0: \Pi_1 = \Pi_2 = \cdots = \Pi_k$ is true, and also that a random sample of size $n_i$ from the $i$th population ($i = 1, 2, \cdots, k$) is available. The authors propose to treat in a subsequent communication a more complicated case in which the samples are tested separately, initially, and the null hypothesis merely fails to be rejected (so that it is not known whether or not it is true). Let $\boldsymbol{\alpha}^{(N)}$ and $\boldsymbol{\alpha}^{*(N)}$ be respectively the modified minimum $\chi^2$ and maximum likelihood estimates of $\boldsymbol{\alpha} = \boldsymbol{\alpha}_1 = \cdots = \boldsymbol{\alpha}_k$ based on the pooled sample of size $N = n_1 + n_2 + \cdots + n_k$. Let $\hat{\Pi}$ and $\Pi^*$ denote respectively $\Pi(\hat{\boldsymbol{\alpha}}^{(N)})$ and $\Pi(\boldsymbol{\alpha}^{*(N)})$. The hypothesis $H_{01}: \Pi_i = \hat{\Pi}$ and $H_{02}: \Pi_i = \Pi^*$ are considered for any specified $i$. The following two theorems for the case $k = 2$ are proven.

THEOREM 1. *$\chi^2_{n,m}$ given by* (24) *is distributed as*

$$\chi^2 = \xi_1{}^2 + \xi_2{}^2 + \cdots + \xi^2_{r-s-1} + (1-\tau)\{\eta^2 + \eta_2{}^2 \cdots + \eta_s{}^2\}$$

*where the $\xi$'s and $\eta$'s are independently identically distributed normal variables with zero mean and unit variance, and $\tau$ is defined by* (28).

THEOREM 2. *$\chi^{*2}_{n,m}$ given by* (60) *is distributed as*

$$\xi_1{}^2 + \xi_2{}^2 \cdots + \xi^2_{r-s-1} + \textstyle\sum_{i=1}^{s}(1-\tau\mu_i)\eta_i{}^2, \qquad 0 \leqq \mu_i \leqq 1,$$

*where $\xi$'s and $\eta$'s are as in Theorem* 1.

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**2. Introduction.** Without loss of generality, the case when $k = 2$ (i.e., when there are two populations, $\Pi_1$ and $\Pi_2$) is considered. The extension to the general case is straightforward. Under the hypothesis $H_0 : \Pi_1 = \Pi_2 = \Pi(\alpha_1, \alpha_2, \cdots, \alpha_s)$ let $x_1, x_2, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}$ be a random sample of size $N = n+m$ from the common distribution function $F = F(x, \alpha_1, \cdots, \alpha_s)$. Let the sample space be divided into $r$ mutually exclusive and exhaustive subsets and let $p_i^0 = p_i(\alpha_1^0, \cdots, \alpha_s^0)$ denote the probability of obtaining an observation belonging to the $i$th group, $i = 1, 2, \cdots, r$, where $\boldsymbol{\alpha}^0 = (\alpha_1^0, \alpha_2^0, \cdots, \alpha_s^0)$, $(s < r)$, is the interior point of a nondegenerate interval $A$ in the $s$-dimensional Euclidean space of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \cdots, \alpha_s)$. The $p_i = p_i(\alpha_1, \alpha_2, \cdots, \alpha_s)$ satisfy

$$\sum_{i=1}^{r} p_i(\alpha_1, \alpha_2, \cdots, \alpha_s) = 1. \tag{1}$$

Following Cramér ([3] pages 426 to 429) let us further assume that

$$p_i(\alpha_1, \alpha_2, \cdots, \alpha_s) > c^2 > 0, \qquad \text{for all } i, \tag{2}$$

$$\partial p_i / \partial \alpha_j \text{ and } \partial^2 p_i / \partial \alpha_j \partial \alpha_k \text{ are all continuous,} \tag{3}$$

and the matrix

$$D = (\partial p_i / \partial \alpha_j), \qquad i = 1, 2, \cdots, r; \qquad j = 1, 2, \cdots, s, \tag{4}$$

is of rank $s$.

We will present the details of our results for the case when the parameters are estimated by the modified minimum chi-square method. The analogous results when the parameters are estimated by the method of maximum likelihood will be briefly summarized, the detailed calculations being similar to those of Chernoff and Lehmann [2]. The modified minimum $\chi^2$ estimates $\hat{\boldsymbol{\alpha}}^{(N)}$ based on a random sample of size $N$ of $\boldsymbol{\alpha}$ are obtained by solving for $\boldsymbol{\alpha}$ the following system of $s$ equations in the $s$ unknowns $\boldsymbol{\alpha}$ given by

$$\sum_{i=1}^{r} \frac{v_i^{(N)} - Np_i}{p_i} \frac{\partial p_i}{\partial \alpha_j} = 0, \qquad j = 1, 2, \cdots, s, \tag{5}$$

where $v_i^{(N)}$ is the number of observations in the sample of size $N$ belonging to the $i$th group, $i = 1, 2, \cdots, r$.

In view of (1), equation (5) can be written as

$$\sum_{i=1}^{r} \frac{v_i^{(N)}}{p_i} \frac{\partial p_i}{\partial \alpha_j} = 0, \qquad j = 1, 2, \cdots, s. \tag{6}$$

Let $(\partial p_i / \partial \alpha_j)_0$ denote the value assumed by $(\partial p_i / \partial \alpha_j)$ at the point $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$. Let $B$ denote the $r \times s$ matrix

$$B = \begin{vmatrix} \frac{1}{\sqrt{p_1^0}} \left( \frac{\partial p_1}{\partial \alpha_1} \right)_0 & \cdots & \frac{1}{\sqrt{p_1^0}} \left( \frac{\partial p_1}{\partial \alpha_s} \right)_0 \\ \vdots & & \vdots \\ \frac{1}{\sqrt{p_r^0}} \left( \frac{\partial p_r}{\partial \alpha_1} \right)_0 & \cdots & \frac{1}{\sqrt{p_r^0}} \left( \frac{\partial p_r}{\partial \alpha_s} \right)_0 \end{vmatrix}. \tag{7}$$

For every fixed $i$, the random variable $v_i^{(N)}$ has mean value $Np_i^0$ and standard deviation $[Np_i^0(1-p_i^0)]^{\frac{1}{2}}$; and therefore, by Tchebycheff's inequality, for every $\lambda_N > 0$

$$P(|v_i^{(N)} - Np_i^0| \geqq \lambda_N N^{\frac{1}{2}}) \leqq \frac{p_i^0(1-p_i^0)}{\lambda_N^2} < \frac{p_i^0}{\lambda_N^2}\,.$$

Hence with a probability greater than $1-\lambda_N^{-2}$ we have

$$|v_i^{(N)} - Np_i^0| < \lambda_N N^{\frac{1}{2}} \qquad \text{for } all \quad 1, 2, \cdots, r. \tag{8}$$

Let

$$\mathbf{x}'^{(N)} = (x_1^{(N)}, \quad x_2^{(N)}, \cdots, x_r^{(N)}), \qquad \text{where} \tag{9}$$

$$x_i^{(N)} = (Np_i^0)^{-\frac{1}{2}}(v_i^{(N)} - Np_i^0) \qquad i = 1, 2, \cdots, r. \tag{10}$$

In view of (2) and (8), with probability greater than $1-\lambda_N^{-2}$, we have

$$|x_i^{(N)}| < c^{-1}\lambda_N, \qquad \text{for all} \quad i = 1, 2, \cdots, r. \tag{11}$$

After a straightforward calculation (for details see [3], pages 428–431), one can write the solution $\alpha^{(N)}$ of the system of simultaneous equations for $\boldsymbol{\alpha}$ given by (5) as

$$\hat{\boldsymbol{\alpha}}^{(N)} - \boldsymbol{\alpha}_0 = N^{-\frac{1}{2}}(B'B)^{-1}B'\mathbf{x}^{(N)} + N^{-1}K\lambda_N^2\boldsymbol{\theta} \qquad \text{where} \tag{12}$$

$$\boldsymbol{\theta}' = (\theta_1, \theta_2, \cdots, \theta_s), \qquad |\theta_j| \leqq 1, \qquad j = 1, 2, \cdots, s; \tag{13}$$

$K$ is independent of $N$ and $\lambda N \to \infty$ as $N \to \infty$ in such a way that $N^{-\frac{1}{2}}\lambda_N^2 \to 0$ as $N \to \infty$. In fact we shall assume, without loss of generality, that $\lambda_N = N^q$ where $0 < q < \frac{1}{4}$. We shall also assume that $n \leqq N$ is a function of $N$ such that $\lim_{N\to\infty} N^{-1}n$ exists.

**3. A modified $\chi^2$-statistic for testing $H_{01}$.** The usual $\chi^2$-statistic based on the given sample of size $N = m+n$ is given by

$$\chi_N^2 = \sum_{i=1}^r (y_i^{(N)})^2 \tag{14}$$

where

$$y_i^{(N)} = [v_i^{(N)} - Np_i(\hat{\boldsymbol{\alpha}}^{(N)})]/[Np_i(\hat{\boldsymbol{\alpha}}^{(N)})]^{\frac{1}{2}}. \tag{15}$$

It was shown in [3] that

$$\chi_N^2 \to_D \chi_{r-s-1}^2 \qquad N \to \infty. \tag{16}$$

Consider now a modified $\chi^2$-statistic for testing $H_{01}$ defined by

$$\chi_{n,m}^2 = \sum_{i=1}^r (y_i^{(n,m)})^2, \qquad \text{where} \tag{17}$$

$$y_i^{(n,m)} = [v_i^{(n)} - np_i(\hat{\boldsymbol{\alpha}}^{(N)})]/[np_i(\hat{\alpha}^{(N)})]^{-\frac{1}{2}}, \qquad i = 1, 2, \cdots, r, \tag{18}$$

where $v_1^{(n)}$ is the number of observations in the sample $x_1, x_2, \cdots, x_n$ belonging to the $i$th group.

Consider now the identity

$$(19)\qquad y_i^{(n,m)} \equiv \frac{v_i^{(n)}-np_i^0}{\sqrt{np_i^0}}-\sqrt{n}\,\frac{p_i(\hat{\alpha}^{(N)})-p_i^0}{\sqrt{p_i^0}}+\frac{v_i^{(n)}-np_i(\hat{\alpha}^{(N)})}{\sqrt{n}}\left(\frac{1}{\sqrt{p_i(\hat{\alpha}^{(N)})}}-\frac{1}{\sqrt{p_i^0}}\right).$$

Using Taylor expansion for $p_i(\hat{\alpha}^{(N)})$ in the neighborhood of $\boldsymbol{\alpha}^0$ and (11) and (12), we can write (19) as

$$(20)\qquad y_i^{(n,m)} = x_i^{(n)}-(n/p_i^0)^{\frac{1}{2}}\textstyle\sum_{j=1}^s(\partial p_i/\partial\alpha_j)_0(\hat{\alpha}_j^{(N)}-\alpha_j^0)+O(\lambda_n^2/n^{\frac{1}{2}})$$

where

$$(21)\qquad x_i^{(n)} = (np_i^0)^{-\frac{1}{2}}(v_i^{(n)}-np_i^0), \qquad i = 1, 2, \cdots, r$$

and $\lambda_n \to \infty$ as $n \to \infty$ such that $n^{-\frac{1}{2}}\lambda_n^2 \to 0$ as $n \to \infty$. Expressing (20) in matrix notation we obtain

$$(22)\qquad \mathbf{y}^{(n,m)} = \mathbf{x}^{(n)}-n^{\frac{1}{2}}B(\hat{\alpha}^{(N)}-\alpha^0)+n^{-\frac{1}{2}}K'\lambda_n^2\boldsymbol{\theta}'$$

where $K'$ is independent of $n$ and each component $\theta_j', j = 1,2,\cdots,r$ of $\boldsymbol{\theta}'$ is absolutely less than one. Substituting for $(\hat{\boldsymbol{\alpha}}^{(N)}-\boldsymbol{\alpha}^0)$ from (12) we obtain

$$(23)\qquad \mathbf{y}^{(n,m)} = \mathbf{x}^{(n)}-n^{\frac{1}{2}}N^{-\frac{1}{2}}B(B'B)^{-1}B'\mathbf{x}^{(N)}+n^{-\frac{1}{2}}K\lambda_n^2\boldsymbol{\theta},$$

where $K$ is a constant independent of $n$, $\lambda_n \to \infty$ as $n \to \infty$ such that $\lambda_n^2/n^{\frac{1}{2}} \to 0$ as $n \to \infty$ and each component of $\theta$ is less than one in absolute value.

**4. Limiting distribution of $\chi^2_{n,m}$.**
We will now attempt to find the limiting distribution of

$$(24)\qquad \chi^2_{n,m} = \mathbf{y}'^{(n,m)}\mathbf{y}^{(n,m)} = \textstyle\sum_{i=1}^r\{y_i^{(n,m)}\}^2,$$

where $\mathbf{y}'^{(n,m)}$ denotes the transpose of the $(r\times 1)$ column vector $\mathbf{y}^{(n,m)}$ given by (23).
Now equation (23) can be written as

$$(25)\qquad \mathbf{y}^{(n,m)} = \xi_{n,N}+\eta_{n,N}+Kn^{-\frac{1}{2}}\lambda_n^2\boldsymbol{\theta}, \qquad \text{where}$$

$$(26)\qquad \xi_{n,N} = (I-B(B'B)^{-1}B')\mathbf{x}^{(N)}n^{\frac{1}{2}}N^{-\frac{1}{2}},$$

$$(27)\qquad \eta_{n,N} = (\mathbf{x}^{(n)}-\mathbf{x}^{(N)}n^{\frac{1}{2}}N^{-\frac{1}{2}}) \quad\text{and}\quad n = n(N)$$

such that $\lim_{N\to\infty}n(N) = \infty$ and

$$(28)\qquad \lim_{N\to\infty}\alpha_{n,N} = \lim_{N\to\infty}n/N = \tau,\quad\text{say,}\qquad 0 \leqq \tau \leqq 1.$$

Notice that since $N \geqq n$, we have $0 \leqq \alpha_{n,N} \leqq 1$. Now from (25) we have

$$(29)\qquad [\mathbf{y}^{(n,m)}-(\mathbf{x}^{(n)}-\alpha_{n,N}^{\frac{1}{2}}B(B'B)^{-1}B'\mathbf{x}^{(N)})] \to_P \mathbf{0} \qquad N\to\infty.$$

Now

$$(30)\qquad \mathbf{x}^{(n)}-\alpha_{n,N}^{\frac{1}{2}}B(B'B)\mathbf{x}^{(N)} = (\mathbf{x}^{(n)}-\tau^{\frac{1}{2}}B(B'B)^{-1}B'\mathbf{x}^{(N)})+\tau^{\frac{1}{2}}-\alpha_{n,N}^{\frac{1}{2}})B(B'B)^{-1}B'\mathbf{x}^{(N)}.$$

Clearly

$$(31)\qquad (\tau^{\frac{1}{2}}-\alpha_{n,N}^{\frac{1}{2}})B(B'B)^{-1}B'\mathbf{x}^{(N)} \to_P \mathbf{0} \qquad N\to\infty.$$

Hence

$$[\mathbf{y}^{(n,m)}-(\mathbf{x}^{(n)}-\tau^{\frac{1}{2}}B(B'B)^{-1}B'\mathbf{x}^{(N)})] \to_P \mathbf{0} \qquad N \to \infty. \tag{32}$$

We will now investigate the distribution of

$$\mathbf{y}'\mathbf{y} \qquad \text{where} \tag{33}$$

$$\mathbf{y} = \mathbf{x}^{(n)}-\tau^{\frac{1}{2}}B(B'B)^{-1}B'\mathbf{x}^{(N)}. \tag{34}$$

Since $E(\mathbf{y}) = E(\mathbf{x}^{(n)})-\tau^{\frac{1}{2}}B(B'B)^{-1}B'E(\mathbf{x}^{(N)}) \equiv 0$, we have for the variance-covariance matrix of $\mathbf{y}$ the following:

$$\Sigma_{n,N} = E(\mathbf{y}\mathbf{y}') \qquad \text{where} \tag{35}$$

$$\begin{aligned}\mathbf{y}\mathbf{y}' = \mathbf{x}^{(n)}\mathbf{x}'^{(n)}-\tau^{\frac{1}{2}}B(B'B)^{-1}B'\mathbf{x}^{(N)}\mathbf{x}'^{(n)}-\tau^{\frac{1}{2}}x^{(n)}x'^{(N)}B(B'B)^{-1}B' \\ +\tau B(B'B)^{-1}B'\mathbf{x}^{(N)}\mathbf{x}'^{(N)}B(B'B)^{-1}B'.\end{aligned} \tag{36}$$

Now

$$\lim_{n\to\infty} E(\mathbf{x}^{(n)}\mathbf{x}'^{(n)}) = I-pp' \tag{37}$$

where $p' = [(p_1{}^0)^{\frac{1}{2}}, (p_2{}^0)^{\frac{1}{2}}, \cdots, (p_t{}^0)^{\frac{1}{2}}]$ (see [3] page 432) and similarly

$$\lim_{N\to\infty} E(\mathbf{x}^{(N)}\mathbf{x}'^{(N)}) = I-pp'. \qquad \text{Also} \tag{38}$$

$$\begin{aligned} x_i{}^{(n)}x_j{}^{(N)} &= \frac{(v_i{}^{(n)}-np_i{}^0)(v_j{}^{(N)}-Np_j{}^0)}{(nNp_i{}^0p_j{}^0)^{\frac{1}{2}}} \\ &= \frac{(v_i{}^{(n)}-np_i{}^0)(v_j{}^{(n)}+v_j{}^{(m)}-np_j{}^0-mp_j{}^0)}{(nNp_i{}^0p_j{}^0)^{\frac{1}{2}}} \\ &= \frac{(v_i{}^{(n)}-np_i{}^0)(v_j{}^{(n)}-np_j{}^0)+(v_i{}^{(n)}-np_i{}^0)(v_j{}^{(m)}-mp_j{}^0)}{(nNp_i{}^0p_j{}^0)^{\frac{1}{2}}}. \end{aligned} \tag{39}$$

Taking expected values and limits on both sides of (39) we readily obtain

$$\lim_{N\to\infty} E(x_i{}^{(n)}x_i{}^{(N)}) = \tau^{\frac{1}{2}}(1-p_i{}^0), \tag{40}$$

$$\lim_{N\to\infty} E(x_i{}^{(n)}x_j{}^{(N)}) = -\tau^{\frac{1}{2}}(p_i{}^0p_j{}^0)^{\frac{1}{2}} \qquad (j \neq i). \tag{41}$$

Combining (40) and (41) we have

$$\lim_{N\to\infty} E(\mathbf{x}^{(N)}\mathbf{x}'^{(n)}) = \tau^{\frac{1}{2}}(I-PP'). \tag{42}$$

Finally, combining (37), (38), (42) and (35) we discover that

$$\begin{aligned}\lim_{N\to\infty} \Sigma_{n,N} &= \lim_{N\to\infty} E(\mathbf{y}\mathbf{y}') \\ &= (I-PP')-\tau B(B'B)^{-1}B'(I-PP')-\tau(I-PP')B(B'B)^{-1}B' \\ &\quad +\tau B(B'B)^{-1}B'(I-PP')B(B'B)^{-1}B'.\end{aligned} \tag{43}$$

Since $\sum_{i=1}^r p_i(\alpha_1, \alpha_2, \cdots, \alpha_s) = 1$, the $j$th element of the vector $B'P$ is

$$\sum_{i=1}^r (\partial p_i/\partial\alpha_j)_0 = 0, \qquad \text{so that}$$

$$B'P \equiv 0. \tag{44}$$

Substituting (44) in (43), we discover that the limiting variance-covariance matrix of $\mathbf{y}$ reduces to

$$\lim_{N\to\infty}\Sigma_{n,N} = I - PP' - \tau B(B'B)^{-1}B'. \tag{45}$$

It was shown by Cramér ([3] pages 433, 434) that the symmetric matrix of order $r \times r$ given by

$$I - PP' - B(B'B)^{-1}B' \tag{46}$$

has exactly $r-s-1$ characteristic roots equal to one, while the rest are all zero. From (45) and (46) it follows that the $r \times r$ symmetric matrix

$$I - PP' - \tau B(B'B)^{-1}B' \tag{47}$$

has exactly one characteristic root equal to zero, $r-s-1$ characteristic roots equal to one, and $s$ characteristic roots equal to $1-\tau$. Hence the limiting distribution of the statistic $\mathbf{y}'^{(n,m)}\mathbf{y}^{(n,m)}$ is the distribution of

$$X^2 = \xi_1{}^2+\xi_2{}^2+\cdots+\xi^2_{r-s-1}+(1-\tau)\{\eta_1{}^2+\eta_2{}^2+\cdots+\eta_s{}^2\} \tag{48}$$

where the $\xi$'s and $\eta$'s are independently identically distributed normal variables with zero mean and unit variance.

Clearly, if $\tau = 0$, $X^2$ is a $\chi^2_{r-1}$ and if $\tau = 1$, $X^2$ is a $\chi^2_{r-s-1}$. Let $F(x) = P(X^2 \leqq x)$ and $F_{n,N}(x) = P(\mathbf{y}'^{(n,m)}\mathbf{y}^{(n,m)} \leqq x)$. In view of (48),

$$\lim_{N\to\infty} F_{n,N}(x) = F(x), \tag{49}$$

at every continuity point of $F(x)$. Since

$$X^2 \leqq \xi_1{}^2+\xi_2{}^2+\cdots+\xi^2_{r-s-1}+\eta_1{}^2+\eta_2{}^2+\cdots+\eta_s{}^2, \tag{50}$$

the event

$$\chi^2_{r-1} = \xi_1{}^2+\cdots+\xi^2_{r-s-1}+\eta_1{}^2+\cdots+\eta_s{}^2 \leqq x \tag{51}$$

implies the event

$$X^2 \leqq x. \tag{52}$$

Therefore

$$P(X^2 \leqq x) \geqq P(\chi^2_{r-1} \leqq x). \tag{53}$$

Similarly, one easily obtains, after a straightforward calculation, that

$$P\{\chi^2_{r-s-1} \leqq x\} \geqq P\{X^2 \leqq x\}. \tag{54}$$

Combining (53) and (54), we discover that

$$P\{\chi^2_{r-1} \leqq x\} \leqq P\{X^2 \leqq x\} \leqq P\{\chi^2_{r-s-1} \leqq x\}. \tag{55}$$

Thus we finally discover that

$$P\{\chi^2_{r-1} \leqq x\} \leqq \lim_{N\to\infty} P\{\mathbf{y}'^{(n,m)}\mathbf{y}^{(n,m)} \leqq x\} \leqq P\{\chi^2_{r-s-1} \leqq x\}. \tag{56}$$

**5. Limiting distribution of $\chi_{n,m}^{*2}$.** Using maximum likelihood estimates $\boldsymbol{\alpha}^{*(N)}$ of $\boldsymbol{\alpha}$ based on the original observations $X_1, X_2, \cdots, X_N$, Chernoff and Lehmann [2] considered the statistic

(57) $$\chi^{*2} = \sum_{i=1}^{r} (Np_i^*)^{-1}(v_i^{(N)} - Np_i^*)^2$$ where

(58) $$p_i^* = p_i(\alpha_1^{*(N)}, \cdots, \alpha_s^{*(N)}), \qquad i = 1, 2, \cdots, r.$$

It was shown in [2] that

(59) $$\chi^{*2} \to_D \xi_1^2 + \xi_2^2 + \cdots + \xi_{r-s-1}^2 + \sum_{i=1}^{s} (1-\mu_i)\eta_i^2 \qquad N \to \infty$$

where the $\xi$'s and $\eta$'s are normally, independently and identically distributed random variables with zero mean and unit variance and the $\mu_i$ are between 0 and 1 and may depend on the $s$ parameters $\alpha_1, \alpha_2, \cdots, \alpha_s$.

In our situation for testing the hypothesis $H_{02}$ let us consider the statistic

(60) $$\chi_{n,m}^{*2} = \sum_{i=1}^{r} (y_i^{*(n,m)})^2,$$ where

(61) $$y_i^{*(n,m)} = [v_i^{(n)} - np_i(\boldsymbol{\alpha}^{*(N)})]/[np_i(\boldsymbol{\alpha}^{*(N)})].$$

Extending the line of argument of the previous section to the proof advanced by Chernoff and Lehmann [2] for the maximum-likelihood case, one discovers that

(62) $$\chi_{n,m}^{*2} \to_D \xi_1^2 + \xi_2^2 + \cdots + \xi_{r-s-1}^2 + \sum_{i=1}^{s} (1-\tau\mu_i)\eta_i^2 \qquad N \to \infty.$$

From (62) it follows that

(63) $$P\{\chi_{r-1}^2 \leqq x\} \leqq \lim_{N\to\infty} P\{\chi_{n,m}^{*2} \leqq x\} \leqq P\{\chi_{r-s-1}^2 \leqq x\},$$

a result similar to the one given by (56).

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