The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 133–145

THE JOINT DISTRIBUTION OF TRACES OF WISHART MATRICES AND SOME APPLICATIONS[1]

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0. Summary. Let $\mathbf{W}_{jj}$ and $\mathbf{\Sigma}_{jj}$, $1 \leqq j \leqq q$, respectively denote the diagonal blocks of a partitioned Wishart matrix $\mathbf{W}$ and its matrix $\mathbf{\Sigma}$ of parameters. A Laguerrian expansion is given for the joint distribution of $v_j = \operatorname{tr} \mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1}$, $1 \leqq j \leqq q$, which is a generalization of known multivariate chi-square distributions. Approximations to the joint distribution function are discussed, and probability inequalities are given for this and a related multivariate F-distribution. Applications are made to some simultaneous multivariate test procedures.

1. Introduction. Let $\mathbf{W}(p \times p)$ be a central Wishart matrix having ν degrees of freedom, positive definite parameter matrix $\mathbf{\Sigma}(p \times p)$, and rank $\min(p, \nu)$. The distribution is nonsingular when $\nu \geqq p$, its probability density function (pdf) well known (cf. [2], page 154), and it is singular otherwise; in either case we shall require only its characteristic function (ch.f.), which exists for arbitrary integer ν. Block partitions of $\mathbf{W}$ and $\mathbf{\Sigma}$ are $\mathbf{W}_{jk}$ and $\mathbf{\Sigma}_{jk}$, respectively, both $(p_j \times p_k)$, where $1 \leqq j, k \leqq q$ and $p_1 + \cdots + p_q = p$. The random scalar $v = \operatorname{tr} \mathbf{W}\mathbf{\Sigma}^{-1}$, known as the Lawley-Hotelling [9], [21] statistic, occupies an important place in multivariate inference as well as in the analysis of univariate linear models, where it occurs as a quadratic form when $\nu = 1$. Equally important to the development of some simultaneous multivariate procedures are the scalars $v_j = \operatorname{tr} \mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1}$, $1 \leqq j \leqq q$. Accordingly, we shall be concerned here with the joint distribution of $\{v_1, \cdots, v_q\}$, the marginals of which clearly are chi-square (χ^2) distributions with $\nu p_1, \cdots, \nu p_q$ degrees of freedom, respectively.

When $q = p$ and thus $\mathbf{W}_{11}, \mathbf{W}_{22}, \cdots, \mathbf{W}_{qq}$ all are scalars, expressions for their joint pdf were given in the bivariate [11], [29] and multivariate [20] cases, and some properties of their joint ch.f. have been studied [19] also. These multivariate distributions all have the same number (ν) of degrees of freedom marginally. Attempts to remove this restriction through modifications of the ch.f. have failed at times (cf. [11] and [20]) because the modified functions are not necessarily ch.f.'s. Such problems are avoided here through exclusive use of Fourier transforms for known random variables.

Our principal findings follow. Inversion of the joint ch.f. of $\{v_1, \cdots, v_q\}$ yields an expression for their joint pdf as a series in Laguerre polynomials of vector argument, thereby extending results in [11] and [20]. From this expression is derived a series for the pdf of an associated multivariate F-distribution of the Snedecor-Fisher

Received November 27, 1968; revised July 24, 1969.

[1] This investigation was supported by a Public Health Service Research Career Development Award, No. 5-K03-GM37209-02, from the National Institute of General Medical Sciences.

type which extends all of those reported in [6], [15], [16], and [25]. Unfortunately, however, both series are quite intractable in form. In view of this, some consideration is given here to the problem of approximating the joint pdf of $\{v_1, \cdots, v_q\}$ using multivariate Edgeworth series (cf. [3], for example). Of much more practical consequence are probability inequalities, which are given for both the multivariate χ^2-distributions and F-distributions, based largely on work by Khatri [13], [14]. These inequalities provide conservative solutions to some problems in simultaneous multivariate inference as noted later.

We remark in passing that the multivariate χ^2-distribution given here also generalizes the multivariate Rayleigh distribution (cf. [22], for example), which has applications to signal detection [18], [23] and other engineering problems. Further details will be omitted here.

2. Preliminaries. In the section following this, the joint pdf for $\{v_1, \cdots, v_q\}$ is obtained upon expanding its ch.f. and inverting the expanded form. That development will be aided by results given here.

(i) *Notation.* Bold-faced characters will represent arrays, lower case for vectors and upper case for matrices unless otherwise indicated. The determinant, transpose, and inverse of a matrix $\mathbf{A}$ will be indicated respectively by $|\mathbf{A}|$, $\mathbf{A}'$, and $\mathbf{A}^{-1}$ when appropriate. Let $\mathbf{x}$ and $\mathbf{r}$ be $(q \times 1)$ vectors, the latter containing non-negative integers. Then we define $\int d\mathbf{x} = \int \cdots \int dx_1 \cdots dx_q$ and the operator

$$\left(\frac{d}{d\mathbf{x}}\right)^{\mathbf{r}} = \left(\frac{\partial}{\partial x_1}\right)^{r_1} \cdots \left(\frac{\partial}{\partial x_q}\right)^{r_q}. \tag{2.1}$$

By $\sum_r^\rho$ is meant the sum over all indices $r_1, \cdots, r_q$ which are partitions of the integer ρ, and this notation should be distinguished from the sum $\sum_r^m = \sum_{r_1=0}^{m_1} \cdots \sum_{r_q=0}^{m_q}$ in which the indices independently range over the values indicated. Recall the Maclaurin expansion for a function of q variables as

$$g(x_1, \cdots, x_q) = g(0, \cdots, 0) + \sum_{n \geqq 1} \sum_r^n \prod_{j=1}^q \frac{x_j^{r_j}}{r_j!} \left(\frac{d}{d\mathbf{x}}\right)^{\mathbf{r}} g(x_1, \cdots, x_q)\bigg|_{\mathbf{x}=0} \tag{2.2}$$

where derivatives of the function are to be evaluated at $\mathbf{x} = \mathbf{0}$. In particular, when $\sum_{j=1}^q r_j = n$ let

$$C_{n,r} = -\left(\prod_{j=1}^q r_j!\right)^{-1} \left(\frac{d}{d\mathbf{x}}\right)^{\mathbf{r}} g(x_1, \cdots, x_q)\bigg|_{\mathbf{x}=0} \tag{2.3}$$

whereupon (2.2) becomes

$$g(x_1, \cdots, x_q) = g(0, \cdots, 0) - \textstyle\sum_{n \geqq 1} \sum_r^n C_{n,r} x_1^{r_1} \cdots x_q^{r_q}. \tag{2.4}$$

(ii) *Matric series.* Distinguish the scalar-valued logarithmic function $\log(\cdot)$ from the matric-valued function $\mathbf{Log}(\cdot)$, and let the latent roots $\{\alpha_1, \cdots, \alpha_m\}$ of $\mathbf{A}(m \times m)$ satisfy $|\alpha_j| < 1$, $1 \leqq j \leqq m$. Then we write as convergent series (cf. Householder [10], for example)

(2.5) $$\mathbf{Log}\,(\mathbf{I}-\mathbf{A}) = -[\mathbf{A}+(\tfrac{1}{2})\mathbf{A}^2+(\tfrac{1}{3})\mathbf{A}^3+\cdots]$$

(2.6) $$\begin{aligned}\log|\mathbf{I}-\mathbf{A}| &= \operatorname{tr}\mathbf{Log}\,(\mathbf{I}-\mathbf{A})\\ &= -\operatorname{tr}[\mathbf{A}+(\tfrac{1}{2})\mathbf{A}^2+(\tfrac{1}{3})\mathbf{A}^3+\cdots].\end{aligned}$$

(iii) *Laguerre polynomials with vector argument.* Let $\psi(\mathbf{x};\boldsymbol{\theta})$ be the joint pdf of q independent gamma variates having unit scale parameters and shape parameters $\theta_1,\cdots,\theta_q$, i.e.

(2.7) $$\psi(\mathbf{x};\boldsymbol{\theta}) = \prod_{j=1}^{q} \frac{x_j^{\theta_j-1}e^{-x_j}}{\Gamma(\theta_j)}, \qquad 0 \leqq x_j < \infty;\quad 0 < \theta_j < \infty.$$

This function now is used as a weight function for defining the Laguerre polynomial of vector argument, $L_h(\mathbf{x};\boldsymbol{\theta})$, by the q-dimensional analog of Rodrigues' formula, namely,

(2.8) $$\left(\prod_{j=1}^{q} h_j!\right)\psi(\mathbf{x};\boldsymbol{\theta})L_h(\mathbf{x};\boldsymbol{\theta}) = \left(\frac{-d}{d\mathbf{x}}\right)^h [x_1^{h_1}\cdots x_q^{h_q}\psi(\mathbf{x};\boldsymbol{\theta})]$$

which reduces in the scalar case to the series

(2.9) $$L_h(x;\theta) = \textstyle\sum_{m=0}^{h}(-1)^m\binom{h+\theta-1}{h-m}x^m/m!$$

(cf. [1], where conventional notation for our $L_h(x;\theta)$ is $L_h^{(\theta-1)}(x)$). At this point we observe that $L_h(\mathbf{x};\boldsymbol{\theta})$ is simply the product of q Laguerre polynomials, each having a scalar argument, and its orthogonality properties follow immediately from this construction in the q-dimensional product space which is the domain of $L_h(\mathbf{x};\boldsymbol{\theta})$. More general Laguerrian polynomials with matrix argument have been investigated by Constantine [4] using as weight function the central Wishart pdf.

The polynomials $L_h(\mathbf{x};\boldsymbol{\theta})$ associate in a natural way with the inverse Fourier transforms needed later, for we observe that

$$x_1^{h_1}\cdots x_q^{h_q}\psi(\mathbf{x};\boldsymbol{\theta}) = \prod_{j=1}^{q}\frac{\Gamma(\theta_j+h_j)}{\Gamma(\theta_j)}\psi(\mathbf{x};\boldsymbol{\theta}+\mathbf{h})$$

and the ch.f. of $\psi(\mathbf{x};\boldsymbol{\theta}+\mathbf{h})$ is known to be $\phi_x(\mathbf{t}) = \prod_{j=1}^{q}(1-it_j)^{-\theta_j-h_j}$ from properties of the gamma distribution. Then $\psi(\mathbf{x};\boldsymbol{\theta}+\mathbf{h})$ is the integral transform

(2.10) $$(\tfrac{1}{2}\pi)^q\textstyle\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}[\prod_{j=1}^{q}(1-it_j)^{-\theta_j-h_j}]\exp(-i\mathbf{t}'\mathbf{x})\,d\mathbf{t} = \psi(\mathbf{x};\boldsymbol{\theta}+\mathbf{h})$$

and, upon differentiating both sides with respect to $\mathbf{x}$, we find

(2.11) $$\begin{aligned}&(\tfrac{1}{2}\pi)^q\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}(-1)^{\Sigma_j h_j}\prod_{j=1}^{q}\left[\frac{it_j}{(1-it_j)^{\theta_j}}\right]^{h_j}\exp(-i\mathbf{t}'\mathbf{x})\,d\mathbf{t}\\ &\qquad = \prod_{j=1}^{q}\frac{\Gamma(\theta_j)}{\Gamma(\theta_j+h_j)}\left(\frac{d}{d\mathbf{x}}\right)^h[x_1^{h_1}\cdots x_q^{h_q}\psi(\mathbf{x};\boldsymbol{\theta})].\end{aligned}$$

From the definition (2.8) we conclude that $\prod_{j=1}^{q}[it_j/(1-it_j)^{\theta_j}]^{h_j}$ is the Fourier transform of the function

$$f_h(\mathbf{x};\boldsymbol{\theta}) = \prod_{j=1}^{q} \frac{h_j!\Gamma(\theta_j)}{\Gamma(\theta_j+h_j)} \psi(\mathbf{x};\boldsymbol{\theta})L_h(\mathbf{x};\boldsymbol{\theta}). \tag{2.12}$$

(iv) *Expansion of a determinantal function.* Consider the polynomial of total order p in the scalars $\{z_1, \cdots, z_q\}$ given by the determinant

$$g(z_1,\cdots,z_q) = |\mathbf{I}_p-\mathbf{A}(\mathbf{z})| \tag{2.13}$$

where $\mathbf{A}(\mathbf{z})$ is defined

$$\mathbf{A}(\mathbf{z}) = \begin{bmatrix} \mathbf{0} & z_1\mathbf{R}_{12} & \cdots & z_1\mathbf{R}_{1q} \\ z_2\mathbf{R}_{21} & \mathbf{0} & \cdots & z_2\mathbf{R}_{2q} \\ \vdots & \vdots & & \vdots \\ z_q\mathbf{R}_{q1} & z_q\mathbf{R}_{q2} & \cdots & \mathbf{0} \end{bmatrix} \tag{2.14}$$

and $\mathbf{R}_{jk}$ is of order $(p_j \times p_k), p_1+\cdots+p_q = p$. We seek to expand the function $g(z_1, \cdots, z_q)$ in a terminating Maclaurin series of the form (2.4), where $1 \leqq n \leqq p$ and clearly $g(0, \cdots, 0) = 1$. A special case of this problem, in which $\mathbf{R}_{jk}$ is replaced by a scalar ρ_{jk} for all j and k, was considered in [20], where the Maclaurin coefficients were found directly upon differentiating a cofactor expansion of the determinant. This approach is not instructive when applied to block-partitioned determinants, and we consider another instead.

Using the series (2.6), we formally expand the determinant (2.13) and write

$$g(z_1,\cdots,z_q) = \exp\{-\textstyle\sum_{m=1}^{\infty}(1/m)\operatorname{tr}\mathbf{A}^m(\mathbf{z})\} \tag{2.15}$$

which now is to be differentiated with respect to its arguments. To this end we employ Good's [8] generalization of Faá di Bruno's formula for the repeated differentiation of a function of a function, for which we require the partitions of a vector $\mathbf{r}$ of integers. Let $\{\mathbf{S}_j\} \equiv \{\mathbf{s}_{j1}, \cdots, \mathbf{s}_{jm_j}\}$ be a set of m_j *distinct* vector partitions of $\mathbf{r}$ and let $\{\mathbf{i}_j\} \equiv \{i_{j1}, \cdots, i_{jm_j}\}$ be their multiplicities $(i_{jk} > 0)$ such that $\Sigma_k i_{jk}\mathbf{s}_{jk} = \mathbf{r}$ for each j, where k ranges from 1 to m_j and j ranges over the total number $N \equiv N(\mathbf{r})$ of such sets. For example, if $\mathbf{r}' = [4, 3]$ we may take the distinct vectors in one such set to be $\{\mathbf{S}_j\} = \{[1, 1], [1, 0]\}$, in which case $[4, 3] = 3[1, 1]+[1, 0]$ and $\{\mathbf{i}_j\} = \{3, 1\}$. Then from Good [8] we obtain

$$\begin{aligned} G(\mathbf{r})\left(\frac{d}{d\mathbf{z}}\right)^{\mathbf{r}} \exp\left[-\sum_{m=1}^{\infty}(1/m)\operatorname{tr}\mathbf{A}^m(\mathbf{z})\right] &= \sum_{j=1}^{N} \frac{\prod_{k=1}^{m_j}\{G(\mathbf{s}_{jk})(d/d\mathbf{z})^{\mathbf{s}_{jk}}[-\sum_{m=1}^{\infty}(1/m)\operatorname{tr}\mathbf{A}^m(\mathbf{z})]\}^{i_{jk}}}{\prod_{k=1}^{m_j} i_{jk}!} \\ &\quad \times \exp[-\textstyle\sum_{m=1}^{\infty}(1/m)\operatorname{tr}\mathbf{A}^m(\mathbf{z})] \end{aligned} \tag{2.16}$$

where $G^{-1}(\mathbf{t}) = t_1!\cdots t_q!$.

To obtain the Maclaurin coefficients $C_{n,r}$, we perform the operations indicated in (2.16) and then evaluate the resulting expression at $\mathbf{z} = \mathbf{0}$. Further simplification can be achieved upon noting that the ultimate effect of the operation

$$\left(\frac{d}{d\mathbf{z}}\right)^{s}\left[-\sum_{m=1}^{\infty}(1/m)\operatorname{tr}\mathbf{A}^{m}(\mathbf{z})\right],$$

after products of such terms are formed and the resulting expression is evaluated at $\mathbf{z} = \mathbf{0}$, is simply to extract the coefficient $K_{\sigma,s}$ of $z_1^{s_1}\cdots z_q^{s_q}$ in $(1/\sigma)\operatorname{tr}\mathbf{A}^{\sigma}(\mathbf{z})$, where $\sigma = s_1+\cdots+s_q$. Thus from (2.3) and (2.16) we write

$$C_{n,r} = -\sum_{j=1}^{N}\frac{\prod_{k=1}^{m_j}[-G(\mathbf{s}_{jk})K_{\sigma,s_{jk}}]^{i_{jk}}}{\prod_{k=1}^{m_j} i_{jk}!}. \tag{2.17}$$

Observe that $K_{1,s} = 0$ by virtue of the fact that $\operatorname{tr}\mathbf{A}(\mathbf{z}) = 0$, and consequently the first-order terms vanish.

As a partial check on our computations, when $q = 4$ and $n = 4$ we find the Maclaurin coefficient for $z_1z_2z_3z_4$ to be $C_{4,r} = 2\operatorname{tr}(\mathbf{R}_{12}\mathbf{R}_{23}\mathbf{R}_{34}\mathbf{R}_{41}+\mathbf{R}_{12}\mathbf{R}_{24}\mathbf{R}_{43}\mathbf{R}_{31}+\mathbf{R}_{13}\mathbf{R}_{32}\mathbf{R}_{24}\mathbf{R}_{41})-(\operatorname{tr}\mathbf{R}_{12}\mathbf{R}_{21})(\operatorname{tr}\mathbf{R}_{34}\mathbf{R}_{43})-(\operatorname{tr}\mathbf{R}_{13}\mathbf{R}_{31})(\operatorname{tr}\mathbf{R}_{24}\mathbf{R}_{42})-(\operatorname{tr}\mathbf{R}_{14}\mathbf{R}_{41})(\operatorname{tr}\mathbf{R}_{23}\mathbf{R}_{32})$, which agrees with that of Krishnamoorthy and Parthasarathy [20] when our matrices $\mathbf{R}_{jk}$ are replaced by their scalars ρ_{jk}, $1 \leqq j, k \leqq q$. We further compute the coefficient of $z_1^2z_2z_3$ as $C_{4,r} = \operatorname{tr}\mathbf{R}_{12}\mathbf{R}_{21}\mathbf{R}_{13}\mathbf{R}_{31}-(\operatorname{tr}\mathbf{R}_{12}\mathbf{R}_{21})(\operatorname{tr}\mathbf{R}_{13}\mathbf{R}_{31})$, which vanishes when the matrices $\mathbf{R}_{jk}$ are replaced by scalars as noted in [20].

For later reference we summarize the developments of this section in Theorem 1.

THEOREM 1. *The Maclaurin expansion for the determinantal function* $g(z_1,\cdots,z_q)$ *given by* (2.13) *is*

$$g(z_1,\cdots,z_q) = 1-\textstyle\sum_{n=2}^{p}\sum_{r}^{n} C_{n,r}\, z_1^{r_1}\cdots z_q^{r_q} \tag{2.18}$$

where the coefficients $C_{n,r}$ *are given by* (2.17) *and the first-order terms vanish.*

3. Multivariate χ^2-distributions and F-distributions. First we consider the joint distribution of $v_j = \operatorname{tr}\mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1}$, $1 \leqq j \leqq q$. The ch.f. of the central Wishart distribution with parameters ν and $\mathbf{\Sigma}$ is (cf. [2])

$$\phi_W(\mathbf{T}) = |\mathbf{I}_p-2i\mathbf{T}\mathbf{\Sigma}|^{-\frac{1}{2}\nu} \tag{3.1}$$

where $\mathbf{T}$ is real and symmetric. By direct methods, either from (3.1) using convolutions and linear transformations of the elements of $\mathbf{W}$, or by finding the expectation $\mathscr{E}\exp[i\mathbf{t}'\mathbf{v}]$ in terms of the original Gaussian variates, we find

$$\phi_v(\mathbf{t}) = |\mathbf{I}_p-2i\mathbf{H}^*(\mathbf{t})\mathbf{\Sigma}|^{-\frac{1}{2}\nu} \tag{3.2}$$

where

$$\mathbf{H}^*(\mathbf{t}) = \operatorname{Diag}(t_1\mathbf{\Sigma}_{11}^{-1},\cdots,t_q\mathbf{\Sigma}_{qq}^{-1}) \tag{3.3}$$

is a block-diagonal matrix.

For convenience we now make the change of scale $u_j = \frac{1}{2}v_j$, $1 \leqq j \leqq q$, and we exploit its symmetry and definiteness to write $\boldsymbol{\Sigma}_{jj} = \boldsymbol{\Sigma}_{jj}^{\frac{1}{2}}\boldsymbol{\Sigma}_{jj}^{\frac{1}{2}}$ in terms of its symmetric, positive definite square root. Starting with (3.2), by elementary operations we determine the ch.f. of $\{u_1, \cdots, u_q\}$ in the form

$$\phi_u(\mathbf{t}) = |\mathbf{I}_p - i\mathbf{H}(\mathbf{t})\mathbf{R}|^{-\frac{1}{2}\nu} \tag{3.4}$$

where now

$$\mathbf{H}(\mathbf{t}) = \mathrm{Diag}(t_1 \mathbf{I}_{p_1}, \cdots, t_q \mathbf{I}_{p_q}) \tag{3.5}$$

and $\mathbf{R}$ is a block-partitioned matrix with elements $\mathbf{R}_{jj} = \mathbf{I}_{p_j}$ and $\mathbf{R}_{jk} = \boldsymbol{\Sigma}_{jj}^{-\frac{1}{2}}\boldsymbol{\Sigma}_{jk}\boldsymbol{\Sigma}_{kk}^{-\frac{1}{2}}$, $1 \leqq j, k \leqq q$. Upon forming the product $\mathbf{H}(\mathbf{t})\mathbf{R}$, letting $z_j = it_j/(1-it_j)$ for $1 \leqq j \leqq q$, and factoring terms out of the determinant, we write

$$\phi_u(\mathbf{t}) = \prod_{j=1}^q (1-it_j)^{-\frac{1}{2}\nu p_j}[g(z_1, \cdots, z_q)]^{-\frac{1}{2}\nu} \tag{3.6}$$

where $g(z_1, \cdots, z_q)$ is given by (2.13). Details parallel those of [20] with no additional complications arising from our block-partitioned form.

From (2.18) we now write $[g(z_1, \cdots, z_q)]^{-\frac{1}{2}\nu} = [1-B(\mathbf{z})]^{-\frac{1}{2}\nu}$, which we formally expand in the binomial series

$$[1-B(\mathbf{z})]^{-\frac{1}{2}\nu} = \sum_{m=0}^{\infty} \frac{\Gamma[(\frac{1}{2}\nu)+m]}{\Gamma(\frac{1}{2}\nu)m!} B^m(\mathbf{z}) \tag{3.7}$$

where, by the multinomial theorem, $B^m(\mathbf{z})$ is the finite sum

$$B^m(\mathbf{z}) = \sum_{a_1=0}^m \cdots \sum_{a_q=0}^m A_a z_1^{a_1} \cdots z_q^{a_q} \tag{3.8}$$

and the coefficients A_a are functions of both the multinomial coefficients and the Maclaurin coefficients $C_{n,r}$; in particular, their generating function is

$$\begin{aligned} &\sum_{a_1=0}^m \cdots \sum_{a_q=0}^m A_a z_1^{a_1} \cdots z_q^{a_q} \\ &\quad = (\textstyle\sum_r^2 C_{2,r} z_1^{r_1} \cdots z_q^{r_q} + \cdots + \sum_r^p C_{p,r} z_1^{r_1} \cdots z_q^{r_q})^m. \end{aligned} \tag{3.9}$$

Here $\{a_1, \cdots, a_q\}$ are non-negative integers and unless at least two of them are positive, the corresponding coefficient A_a of such first-order terms vanishes. In agreement with our conventions of notation, we now write

$$\sum_a^{m\mathbf{1}} = \sum_{a_1=0}^m \cdots \sum_{a_q=0}^m$$

where $m\mathbf{1}' = [m, \cdots, m]$.

Combining (3.6)–(3.8) and replacing $z_1^{a_1} \cdots z_q^{a_q}$ by $\prod_{j=1}^q [it_j/(1-it_j)]^{a_j}$, we finally write the ch.f. $\phi_u(\mathbf{t})$ as

$$\phi_u(\mathbf{t}) = \sum_{m=0}^{\infty} \frac{\Gamma[(\frac{1}{2}\nu)+m]}{\Gamma(\frac{1}{2}\nu)m!} \sum_a^{m\mathbf{1}} A_a \prod_{j=1}^q [it_j/(1-it_j)^{\frac{1}{2}\nu p_j}]^{a_j}. \tag{3.10}$$

Upon applying (2.11) and (2.12) term-by-term to the series (3.10) to find its inverse Fourier transform, we obtain the pdf given in Theorem 2.

THEOREM 2. *The joint* pdf *of* $u_j = (\frac{1}{2})\operatorname{tr}\mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1}$, $1 \leqq j \leqq q$, *is expressible in the series*

$$(3.11) \qquad (\mathbf{u}) = \sum_{m=0}^{\infty} \frac{\Gamma[(\frac{1}{2}\nu)+m]}{\Gamma(\frac{1}{2}\nu)m!} \sum_{a}^{m1} A_a f_a(\mathbf{u}; \tfrac{1}{2}\nu\mathbf{p})$$

where the functions $f_a(\mathbf{u}; \frac{1}{2}\nu\mathbf{p})$ *are defined in* (2.12) *and the coefficients* A_a *depend on the matrix* $\mathbf{\Sigma}$ *through* $\mathbf{R}_{jk} = \mathbf{\Sigma}_{jj}^{-\frac{1}{2}}\mathbf{\Sigma}_{jk}\mathbf{\Sigma}_{kk}^{-\frac{1}{2}}$, $1 \leqq j, k \leqq q$.

From (3.11) we can obtain the joint pdf for the scalars $\{v_1, \cdots, v_q\}$ directly upon making a simple change of scale. Special cases of the expression (3.11) were given when $p = q$ by Kibble [11] ($p = 2$) and more generally by Krishnamoorthy and Parthasarathy [20]. Absolute convergence of the series (3.11) can be established along the lines of the development in [20].

Starting from the joint pdf for $\{v_1, \cdots, v_q\}$, we now derive a multivariate F-distribution of the Snedecor-Fisher type. In particular, let the random scalar v_0 be distributed independently of $\{v_1, \cdots, v_q\}$ as central χ^2 with λ degrees of freedom, and form the ratios $f_j = v_j/v_0$, $1 \leqq j \leqq q$ (equivalently $f_j = u_j/u_0$, where $u_0 = \frac{1}{2}v_0$).

From our definition (2.8) of the Laguerre polynomials $L_h(\mathbf{x}; \boldsymbol{\theta})$ of total order $h_1 + \cdots + h_q$ in several arguments, it follows that the analog of the series (2.9) is

$$(3.12) \qquad L_h(\mathbf{x}; \boldsymbol{\theta}) = \sum_{m}^{h} \sum_{j=1}^{q} \frac{(-1)^{m_j}\Gamma(\theta_j + h_j)x_j^{m_j}}{\Gamma(h_j - m_j + 1)\Gamma(\theta_j + m_j)m_j!}.$$

Using this expression and the definitions of $\psi(\mathbf{x}; \boldsymbol{\theta})$ and $f_h(\mathbf{x}; \boldsymbol{\theta})$ (cf. (2.7) and (2.12)), we reduce the latter to the form

$$(3.13) \qquad f_h(\mathbf{x}; \boldsymbol{\theta}) = \sum_{m}^{h} \prod_{j=1}^{q} \frac{(-1)^{m_j}h_j!}{(h_j - m_j)!m_j!}\psi(\mathbf{x}; \boldsymbol{\theta} + \mathbf{m}).$$

Thus from (3.11) $f(\mathbf{u})$ becomes

$$(3.14) \qquad f(\mathbf{u}) = \sum_{m=0}^{\infty} \frac{\Gamma[(\frac{1}{2}\nu)+m]}{\Gamma(\frac{1}{2}\nu)m!} \sum_{a}^{m1} A_a \sum_{r}^{a} \prod_{j=1}^{q} \frac{(-1)^{r_j}a_j!}{(a_j - r_j)!r_j!}\psi(\mathbf{u}; \tfrac{1}{2}\nu\mathbf{p} + \mathbf{r}).$$

Now letting

$$(3.15) \qquad B_{a,r} = \prod_{j=1}^{q} \frac{(-1)^{r_j}a_j!}{(a_j - r_j)!r_j!} A_a$$

and using (2.7) together with the independence of u_0 and $\{u_1, \cdots, u_q\}$, we write their joint pdf as

$$(3.16) \quad f(u_0, \mathbf{u}) = \sum_{m=0}^{\infty} \frac{\Gamma[(\frac{1}{2}\nu)+m]}{\Gamma(\frac{1}{2}\nu)m!} \sum_{a}^{m1} \sum_{r}^{a} B_{a,r} \frac{u_0^{\frac{1}{2}\lambda - 1}}{\Gamma(\frac{1}{2}\lambda)} \prod_{j=1}^{q} \frac{u_j^{\frac{1}{2}\nu p_j + r_j - 1}\exp(-\sum_{j=0}^{q} u_j)}{\Gamma[(\frac{1}{2}\nu p_j) + r_j]}.$$

At this point we make the transformation $f_j = u_j/u_0$, $1 \leqq j \leqq q$, and then integrate term-by-term using a result of Ghosh [6] (see also [25]) to obtain the joint pdf for $\{f_1, \cdots, f_q\}$. We finally encounter the form given in Theorem 3.

THEOREM 3. *Let the joint distribution of* $u_j = (\frac{1}{2})\operatorname{tr}\mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1}$, $1 \leqq j \leqq q$, *be given by Theorem* 2, *and suppose that* $v_0 = 2u_0$ *is distributed independently of* $\{u_1, \cdots, u_q\}$ *as central* χ^2 *with* λ *degrees of freedom. Then the joint* pdf *of* $\{f_1, \cdots, f_q\}$, *where* $f_j = u_j/u_0$, $1 \leqq j \leqq q$, *can be written as the series*

$$g(f_1, \cdots, f_q) = \sum_{m=0}^{\infty} \frac{\Gamma[(\frac{1}{2}v)+m]}{\Gamma(\frac{1}{2}v)m!} \sum_{a}^{m1} \sum_{r}^{a} G_{a,r} \frac{\prod_{j=1}^{q} f_j^{\frac{1}{2}vp_j + r_j - 1}}{(1+\sum_{j=1}^{q} f_j)^{\Sigma_j[(\frac{1}{2}vp_j)+r_j]+\frac{1}{2}\lambda}} \tag{3.17}$$

where

$$G_{a,r} = \frac{\Gamma[\Sigma_j(\frac{1}{2}vp_j + r_j) + \frac{1}{2}\lambda]}{\Gamma(\frac{1}{2}\lambda)} \prod_{j=1}^{q} \frac{(-1)^{r_j} a_j!}{(a_j - r_j)! r_j! \Gamma[(\frac{1}{2}vp_j) + r_j]} A_a$$

and the coefficients depend on $\mathbf{\Sigma}$ *only through* $\mathbf{R}_{jk} = \mathbf{\Sigma}_{jj}^{-\frac{1}{2}}\mathbf{\Sigma}_{jk}\mathbf{\Sigma}_{kk}^{-\frac{1}{2}}$, $1 \leqq j, k \leqq q$.

We remark that special cases of Theorem 3 have been reported elsewhere in somewhat different notation. When $\mathbf{\Sigma}$ is block-diagonal and thus $\mathbf{R}_{jk} = 0$, $j \neq k$, then $\{u_1, \cdots, u_q\}$ are mutually independent and all terms vanish in the series (3.17) except the first ($m = 0$), which then agrees with Ghosh [6] and Ramachandran [25]. Some earlier references to this Dirichlet distribution and its applications are given in Wilks [30]. A more general distribution of this type was given by Olkin and Rubin [24], who considered the joint distribution of $\mathbf{S}_0^{-\frac{1}{2}}\mathbf{S}_j\mathbf{S}_0^{-\frac{1}{2}}$, $1 \leqq j \leqq q$, where $\mathbf{S}_0, \mathbf{S}_1, \cdots, \mathbf{S}_q$ are independent Wishart matrices. When $p_1 = p_2 = \cdots = p_q = 1$ ($p = q$), a special case of (3.17) was given by Krishnaiah [15].

4. Approximations and inequalities. Series for the multivariate χ^2-distributions and F-distributions are obviously intractable, sufficiently so to limit seriously their applications. Several problems are immediate, including the required partitions of a vector of integers. In some of the more important applications the joint distribution is beset by numerous nuisance parameters. In view of such difficulties we now examine some approximations, first to the pdf of $\{v_1, \cdots, v_q\}$ itself, and then to the joint probabilities for both distributions as required in the construction of simultaneous multivariate test procedures. In what follows let $\mathbf{y}_1, \cdots, \mathbf{y}_v$ be independent, identically distributed Gaussian p-vectors with zero means and covariance matrix $\mathbf{\Sigma}$ (denoted $\mathbf{y}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma})$), and partition $\mathbf{y}_i' = [\mathbf{y}_{i1}', \cdots, \mathbf{y}_{iq}']$ and $\mathbf{\Sigma} = [\mathbf{\Sigma}_{jk}]$ as before, where $\mathbf{y}_{ij}(p_j \times 1)$, $1 \leqq j \leqq q$, and $p_1 + \cdots + p_q = p$. Then we write $\mathbf{W} = \mathbf{y}_1\mathbf{y}_1' + \cdots + \mathbf{y}_v\mathbf{y}_v'$ and $\mathbf{W}_{jk} = \mathbf{y}_{1j}\mathbf{y}_{1k}' + \cdots + \mathbf{y}_{vj}\mathbf{y}_{vk}'$, $1 \leqq j, k \leqq q$.

Chambers [3] has given a detailed account of approximations to multivariate pdf's by means of multidimensional Edgeworth series. Under regularity conditions on orders of magnitude of the cumulants, he provided the asymptotic bounds $o(n^{\varepsilon - \frac{1}{2}(r+1)})$ on the error of approximation afforded by r terms of the series, for $\varepsilon > 0$ and some parameter n. Algorithms are available [3] for expressing the series in terms of cumulants, and for evaluating its coefficients numerically. The regularity conditions are that the cumulants of order s for properly standardized variables be $O(n^{-\frac{1}{2}})$ for $s = 1$, $O(1)$ for $s = 2$, and $O(n^{-\frac{1}{2}s+1})$ for $s > 2$. Chambers pointed out that these conditions are satisfied by the (standardized) central Wishart distribution for $v = n$. Upon recalling that $\{v_1, \cdots, v_q\}$ result from a linear transformation and convolutions involving the elements of $\mathbf{W}$, we conclude immediately that cumulants of the (standardized) multivariate χ^2-distribution are of proper order to justify Chambers' development. Consequently its pdf can be developed in a multivariate Edgeworth series as in [3].

In this connection we give explicitly the cumulant generating function for the joint distribution of $\{v_1, \cdots, v_q\}$. Starting from (3.4) and applying (2.5) and (2.6), we find

$$\log \phi_v(\mathbf{t}) = v \textstyle\sum_{m=1}^{\infty} 2^{m-1} i^m \operatorname{tr} [\mathbf{H}(\mathbf{t})\mathbf{R}]^m / m \tag{4.1}$$

and the joint cumulant $\kappa_{s_1 s_2 \cdots s_q}$ of total order $s = s_1 + \cdots + s_q$, can be obtained from the sth term of (4.1) upon multiplying the coefficient of $i^s t_1^{s_1} \cdots t_q^{s_q}$ by $s_1! \cdots s_q!$ using a standard procedure. In particular, we easily establish the means, variances, and covariances to be $\varepsilon v_j = v p_j$, $\operatorname{Var}(v_j) = 2vp_j$, and $\operatorname{Cov}(v_j, v_k) = 2v \operatorname{tr} \mathbf{R}_{jk}\mathbf{R}_{kj}$, $1 \leqq j, k \leqq q$. From the definitions of $\mathbf{R}_{jk}$ and the canonical correlations [2] between the Gaussian vectors $\mathbf{y}_{ij}$ and $\mathbf{y}_{ik}$, it follows that the correlation of v_j with v_k is equal to the sum of squares of canonical correlations between $\mathbf{y}_{ij}$ and $\mathbf{y}_{ik}$. For the special case $p_1 = \cdots = p_q = 1$, the correlation between v_j and v_k is ρ_{jk}^2, the square of the simple correlation parameter, as noted by Cramér [5], page 317, when $p = 2$ and by Krishnaiah and Rao [19]. Higher order cumulants follow similarly.

We shall give useful probability inequalities for both the multivariate χ^2-distributions and F-distributions in terms of their marginal distributions. To this end we require the following definition and corollary from Khatri [13], [14].

DEFINITION. (Khatri [13]). A region $D(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ is separately symmetric in $\mathbf{x}_1, \cdots, \mathbf{x}_n$ about the origin if $(\mathbf{x}_1, \cdots, \mathbf{x}_n) \in D$ implies $(\varepsilon_1 \mathbf{x}_1, \cdots, \varepsilon_n \mathbf{x}_n) \in D$ for all $\varepsilon_1, \cdots, \varepsilon_n$ such that $\varepsilon_j = +1$ or $\varepsilon_j = -1$, $1 \leqq j \leqq n$.

COROLLARY 1. (Khatri [14]). *Let* $\mathbf{z}_i \sim N_p(\mathbf{0}, \mathbf{\Sigma}_i)$, $1 \leqq i \leqq n$, *and let them be independent. Partition* $\mathbf{z}_i' = [\mathbf{z}_{i1}', \cdots, \mathbf{z}_{iq}']$, $1 \leqq i \leqq n$, *let* $D_k = D_k(\mathbf{z}_{1k}, \cdots, \mathbf{z}_{nk})$ *be convex and separately symmetric in* $\mathbf{z}_{1k}, \cdots, \mathbf{z}_{nk}$ *about the origin for* $1 \leqq k \leqq q$, *and let* $\bar{D}_k$ *be the complement of* D_k. *Then*

$$P\{\textstyle\bigcap_{k=1}^q D_k\} \geqq \prod_{k=1}^q P\{D_k\} \quad \text{and} \quad P\{\bigcap_{k=1}^q \bar{D}_k\} \geqq \prod_{k=1}^q P\{\bar{D}_k\}.$$

Using the foregoing results, we now state and prove the following.

THEOREM 4. *Let* $\mathbf{W} = [\mathbf{W}_{jk}]$ *be a central Wishart matrix with* v *degrees of freedom and parameter matrix* $\mathbf{\Sigma} = [\mathbf{\Sigma}_{jk}]$. *Further let* $v_j = \operatorname{tr} \mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1}$, $1 \leqq j \leqq q$. *Then*

$$P\{v_1 \leqq a_1, \cdots, v_q \leqq a_q\} \geqq \textstyle\prod_{j=1}^q P\{v_j \leqq a_j\} \tag{4.2}$$

for arbitrary scalars $a_1, \cdots, a_q$.

PROOF. As before write $\mathbf{W}_{jk} = \sum_{i=1}^{v} \mathbf{y}_{ij}\mathbf{y}_{ik}'$ in terms of the partitioned Gaussian vectors $\mathbf{y}_1, \cdots, \mathbf{y}_v$, each $(p \times 1)$. In view of earlier developments we can take $\mathbf{\Sigma} = \mathbf{R}$ without loss of generality, in which case $v_j = \operatorname{tr}\mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1} = \operatorname{tr}\mathbf{W}_{jj} = \operatorname{tr}\sum_{i=1}^{v} \mathbf{y}_{ij}\mathbf{y}_{ij}'$ Now let D_j be the region D_j: $\{(\mathbf{y}_{1j}, \cdots, \mathbf{y}_{vj}) \mid \operatorname{tr}\sum_{i=1}^{v} \mathbf{y}_{ij}\mathbf{y}_{ij}' \leqq a_j\}$,

$1 \leqq j \leqq q$, and observe that they are separately symmetric regions, for $\mathbf{y}_{ij}$ can be replaced by $-\mathbf{y}_{ij}$ without consequence. Corollary 1 of Khatri [14] thus applies immediately to establish the theorem.

Our final theorem deals with the multivariate F-distribution treated in Theorem 3. The symbol $P_0\{\cdot\}$ indicates the probability measure applicable when the numerators of the ratios $f_j = v_j/v_0$, $1 \leqq j \leqq q$, are mutually independent (cf. [6], [12], and [25]).

THEOREM 5. *Let* $\{v_1, \cdots, v_q\}$ *be as in Theorem* 4, *and let* v_0 *be distributed independently of* $\{v_1, \cdots, v_q\}$ *as central* χ^2 *with* λ *degrees of freedom. Then*

$$P\{f_1 \leqq b_1, \cdots, f_q \leqq b_q\} \geqq P_0\{f_1 \leqq b_1, \cdots, f_q \leqq b_q\} \geqq \textstyle\prod_{j=1}^{q} P\{f_j \leqq b_j\} \tag{4.3}$$

for arbitrary $b_1, \cdots, b_q$, *where* $f_j = v_j/v_0$, $1 \leqq j \leqq q$.

PROOF. The first inequality in (4.3) easily follows an application of Theorem 4 conditionally to obtain the inequality

$$P\{v_1 \leqq b_1 v_0, \cdots, v_q \leqq b_q v_0 \mid v_0\} \geqq P_0\{v_1 \leqq b_1 v_0, \cdots, v_q \leqq b_q v_0 \mid v_0\}$$

which is shown to hold unconditionally as well upon taking expectations on both sides using a standard argument (cf. [28], for example). The second inequality was given by Kimball [12].

5. Some applications. Multivariate distributions from earlier sections occur naturally as the joint distributions of statistics useful for simultaneous hypothesis tests in a variety of cases. The probability inequalities provided in Theorems 4 and 5 are particularly important, as they facilitate the construction of conservative simultaneous procedures free of nuisance parameters. In the context of multivariate linear models (cf. [2], Chapter 8), we now discuss simultaneous tests involving subsets of the responses, as well as subsets of the factors, using Lawley-Hotelling (cf. [7], [9], and [21]) statistics in large samples.

Let $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n$ be independent $(p \times 1)$ vector observations from $N_p(\boldsymbol{\mu}_j, \boldsymbol{\Omega})$, $1 \leqq j \leqq n$. We write $\mathbf{y}' = [\mathbf{y}_1', \cdots, \mathbf{y}_n'](1 \times np)$ and $\boldsymbol{\mu}' = [\boldsymbol{\mu}_1', \cdots, \boldsymbol{\mu}_n']$. Clearly $\mathbf{y} \sim N_{np}(\boldsymbol{\mu}, \mathbf{I}_n \times \boldsymbol{\Omega})$, $\mathbf{A} \times \mathbf{B}$ indicating the Kronecker product, and we adopt the convention that $\mathbf{Y} \sim N_{np}(\mathbf{M}, \mathbf{I}_n \times \boldsymbol{\Omega})$, where $\mathbf{Y}' = [\mathbf{y}_1, \cdots, \mathbf{y}_n](p \times n)$ and $\mathscr{E}\mathbf{Y}' = \mathbf{M}' = [\boldsymbol{\mu}_1, \cdots, \boldsymbol{\mu}_n](p \times n)$. Further assume the multivariate linear model $\mathbf{M} = \mathbf{X}\boldsymbol{\Theta}$, where $\mathbf{X}(n \times r)$ of rank $r(r < n)$ is a matrix of concomitant variables and $\boldsymbol{\Theta} = [\boldsymbol{\theta}_{jk}](r \times p)$ contains unknown parameters. It is well known that the maximum likelihood estimators $\hat{\boldsymbol{\Theta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ have the properties $\hat{\boldsymbol{\Theta}} \sim N_{rp}(\boldsymbol{\Theta}, \mathbf{T} \times \boldsymbol{\Omega})$, where $\mathbf{T} = (\mathbf{X}'\mathbf{X})^{-1}$, and that $\hat{\boldsymbol{\Theta}}'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\Theta}}$ is a possibly noncentral Wishart matrix with r degrees of freedom and parameter matrix $\boldsymbol{\Omega}$. We shall arrange the elements of $\boldsymbol{\Theta}$ (and similarly for $\hat{\boldsymbol{\Theta}}$) in matrix or vector form as convenient. First let $\boldsymbol{\Theta}' = [\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_r]$ and $\boldsymbol{\theta}' = [\boldsymbol{\theta}_1', \cdots, \boldsymbol{\theta}_r'](1 \times rp)$. Alternatively let $\boldsymbol{\Theta} = [\boldsymbol{\beta}_1, \cdots, \boldsymbol{\beta}_p]$ and $\boldsymbol{\beta}' = [\boldsymbol{\beta}_1', \cdots, \boldsymbol{\beta}_p'](1 \times rp)$; clearly $\hat{\boldsymbol{\theta}} \sim N_{rp}(\boldsymbol{\theta}, \mathbf{T} \times \boldsymbol{\Omega})$ and $\hat{\boldsymbol{\beta}} \sim N_{rp}(\boldsymbol{\beta}, \boldsymbol{\Omega} \times \mathbf{T})$. In what follows we exploit the fact that $\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}'(rp \times rp)$ and $\hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}'(rp \times rp)$ are possibly noncentral Wishart matrices having one degree of freedom and parameter matrices $\mathbf{T} \times \boldsymbol{\Omega}$ and $\boldsymbol{\Omega} \times \mathbf{T}$, respectively.

Now consider simultaneous tests for hypotheses regarding the parameters associated with subsets of the responses. In particular, partition $\mathbf{\Theta} = [\mathbf{\Theta}_1, \cdots, \mathbf{\Theta}_q]$ and write the hypotheses to be considered as $H_j: \mathbf{\Theta}_j = \mathbf{0}(r \times p_j)$, for $1 \leqq j \leqq q$ and $p_1 + \cdots + p_q = p$. The large-sample ($\mathbf{\Omega}$ known) form of the Lawley-Hotelling statistic for testing H_j is $\operatorname{tr} \hat{\mathbf{\Theta}}_j'\mathbf{X}'\mathbf{X}\hat{\mathbf{\Theta}}_j\mathbf{\Omega}_{jj}^{-1}$, $1 \leqq j \leqq q$. Upon writing $\mathbf{W} = \hat{\mathbf{\Theta}}'\mathbf{X}'\mathbf{X}\hat{\mathbf{\Theta}}$ in partitioned form as

$$\mathbf{W} = \begin{bmatrix} \hat{\mathbf{\Theta}}_1' \\ \vdots \\ \hat{\mathbf{\Theta}}_q' \end{bmatrix} \mathbf{X}'\mathbf{X}[\hat{\mathbf{\Theta}}_1, \cdots, \hat{\mathbf{\Theta}}_q]$$

and recalling its Wishart distribution with parameter matrix $\mathbf{\Omega}$, we conclude that the Lawley-Hotelling statistics are of the form $v_j = \operatorname{tr} \mathbf{W}_{jj}\mathbf{\Omega}_{jj}^{-1}$ as studied in earlier sections. An important special case occurs when $p_1 = \cdots = p_q = 1$, i.e. $q = p$, for then the hypotheses $H_j: \boldsymbol{\beta}_j = \mathbf{0}(r \times 1)$, regarding parameters associated with each of the p responses, are to be tested simultaneously, $1 \leqq j \leqq p$. In this case the likelihood ratio (equivalently, Lawley-Hotelling) statistics are of the form $\hat{\boldsymbol{\beta}}_j'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}_j/\omega_{jj}$, $1 \leqq j \leqq p$. Recalling the Wishart character of $\mathbf{W} = \hat{\boldsymbol{\beta}}\hat{\boldsymbol{\beta}}'$ and making the identification $\mathbf{\Sigma} = \mathbf{\Omega} \times \mathbf{T}$, we conclude that the large sample statistics are of the form, $\hat{\boldsymbol{\beta}}_j'\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}_j/\omega_{jj} = \operatorname{tr} \hat{\boldsymbol{\beta}}_j\hat{\boldsymbol{\beta}}_j'\mathbf{T}^{-1}/\omega_{jj} = \operatorname{tr} \mathbf{W}_{jj}\mathbf{\Sigma}_{jj}^{-1} = v_j$, treated in earlier sections.

We turn now to simultaneous tests for hypotheses regarding the parameters associated with subsets of the factors. To this end partition $\mathbf{\Theta}' = [\mathbf{\Phi}_1', \cdots, \mathbf{\Phi}_q']$ (similarly for $\hat{\mathbf{\Theta}}$) and $(\mathbf{X}'\mathbf{X})^{-1} = \mathbf{T} = [\mathbf{T}_{jk}](r \times r)$, where dimensions are $\mathbf{\Phi}_j(r_j \times p)$, $\mathbf{T}_{jk}(r_j \times r_k)$ for $1 \leqq j, k \leqq q$ and $r_1 + \cdots + r_q = r$, and consider simultaneously the hypotheses $H_j: \mathbf{\Phi}_j = \mathbf{0}(r_j \times p)$, $1 \leqq j \leqq q$. The appropriate Lawley-Hotelling statistics are $\operatorname{tr} \hat{\mathbf{\Phi}}_j'\mathbf{T}_{jj}^{-1}\hat{\mathbf{\Phi}}_j\mathbf{\Omega}^{-1}$, respectively, for $1 \leqq j \leqq q$. In order to deduce their joint distribution, we write $\boldsymbol{\phi}_1' = [\boldsymbol{\theta}_1', \cdots, \boldsymbol{\theta}_{r_1}'](1 \times r_1 p)$, $\boldsymbol{\phi}_2' = [\boldsymbol{\theta}_{r_1+1}', \cdots, \boldsymbol{\theta}_{r_1+r_2}'](1 \times r_2 p)$, $\cdots, \boldsymbol{\phi}_q' = [\boldsymbol{\theta}_{r_1+\cdots+r_{q-1}+1}', \cdots, \boldsymbol{\theta}_{r_1+\cdots+r_q}'](1 \times r_q p)$ and $\boldsymbol{\phi}' = [\boldsymbol{\phi}_1', \cdots, \boldsymbol{\phi}_q']$ $(1 \times rp)$. We note as before the Wishart character of $\hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}'(rp \times rp)$ with one degree of freedom and with parameter matrix $\mathbf{T} \times \mathbf{\Omega} = [\mathbf{T}_{jk} \times \mathbf{\Omega}]$, $1 \leqq j, k \leqq q$.

Now the Lawley-Hotelling statistic for testing H_1 can be written

$$\begin{aligned} \text{(5.1)} \qquad \operatorname{tr} \hat{\mathbf{\Phi}}_1'\mathbf{G}\hat{\mathbf{\Phi}}_1\mathbf{\Omega}^{-1} &= \operatorname{tr} [\hat{\boldsymbol{\theta}}_1, \cdots, \hat{\boldsymbol{\theta}}_{r_1}][g_{ik}][\hat{\boldsymbol{\theta}}_1, \cdots, \hat{\boldsymbol{\theta}}_{r_1}]'\mathbf{\Omega}^{-1} \\ &= \operatorname{tr} \textstyle\sum_{i=1}^{r_1} \sum_{k=1}^{r_1} \hat{\boldsymbol{\theta}}_i\hat{\boldsymbol{\theta}}_k' g_{ik}\mathbf{\Omega}^{-1} \end{aligned}$$

where we have replaced $\mathbf{T}_{11}^{-1}$ by $\mathbf{G} = [g_{ik}](r_1 \times r_1)$. Moreover, upon making the identification $\mathbf{W} = \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}'(rp \times rp)$ and $\mathbf{\Sigma} = \mathbf{T} \times \mathbf{\Omega} = [\mathbf{T}_{jk} \times \mathbf{\Omega}]$, whereupon $\mathbf{\Sigma}_{jj}^{-1} = \mathbf{T}_{jj}^{-1} \times \mathbf{\Omega}^{-1}$, we find for $j = 1$ that $v_1 = \operatorname{tr} \mathbf{W}_{11}\mathbf{\Sigma}_{11}^{-1}$ becomes

$$\begin{aligned} v_1 &= \operatorname{tr} \hat{\boldsymbol{\phi}}_1\hat{\boldsymbol{\phi}}_1'\mathbf{T}_{11}^{-1} \times \mathbf{\Omega}^{-1} \\ \text{(5.2)} \qquad &= \operatorname{tr} [\textstyle\sum_{k=1}^{r_1} \hat{\boldsymbol{\theta}}_i\hat{\boldsymbol{\theta}}_k' g_{jk}\mathbf{\Omega}^{-1}] \\ &= \operatorname{tr} \textstyle\sum_{i=1}^{r_1} \sum_{k=1}^{r_1} \hat{\boldsymbol{\theta}}_i\hat{\boldsymbol{\theta}}_k' g_{ik}\mathbf{\Omega}^{-1} \end{aligned}$$

where, as before, $\mathbf{G} = \mathbf{T}_{11}^{-1}$ and the second equality contains in square brackets the typical i, j block of the indicated block-partitioned matrix. Upon comparing

(5.1) and (5.2), we conclude that the Lawley-Hotelling statistic for testing H_1 has the form $v_1 = \operatorname{tr} \mathbf{W}_{11}\boldsymbol{\Sigma}_{11}^{-1}$. A similar development applies for all of the statistics $\operatorname{tr} \hat{\boldsymbol{\Phi}}_j' \mathbf{T}_{jj}^{-1} \hat{\boldsymbol{\Phi}}_j \boldsymbol{\Omega}^{-1}$, $1 \leqq j \leqq q$.

Two special cases will be noted. Hypotheses regarding the parameters associated with each of the r factors can be tested simultaneously upon letting $r_1 = \cdots = r_q = 1$ $(q = r)$, in which case we write $H_j : \boldsymbol{\theta}_j = \mathbf{0}(p \times 1)$, $1 \leqq j \leqq r$. Upon identifying $\mathbf{W} = \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}'$ and $\boldsymbol{\Sigma} = \mathbf{T} \times \boldsymbol{\Omega}$, we observe that $\operatorname{tr} \mathbf{W}_{jj}\boldsymbol{\Sigma}_{jj}^{-1} = \operatorname{tr} \hat{\boldsymbol{\theta}}_j\hat{\boldsymbol{\theta}}_j' t_{jj}^{-1}\boldsymbol{\Omega}^{-1} = \hat{\boldsymbol{\theta}}_j'\boldsymbol{\Omega}^{-1}\hat{\boldsymbol{\theta}}_j/t_{jj}$, where t_{jj} is the jth diagonal element of $\mathbf{T}$, $1 \leqq j \leqq r$. It follows that the joint distribution of the r quadratic forms involved is thus the limiting form of a joint distribution of Hotelling's T^2-statistics. It is apparent that this joint distribution reduces to the special case of our Theorem 2 which was treated by Krishnamoorthy and Parthasarathy [20], as noted also by Siotani [29] when $p = 2$ and by Krishnaiah [17] more generally.

Another important special case occurs when $p = 1$, for then the hypotheses $H_1, \cdots, H_q$ specify values for partitions of the vector $\boldsymbol{\beta}(r \times 1)$ of parameters in the univariate linear model, i.e. $H_j : \boldsymbol{\beta}_j = \mathbf{0}(r_j \times 1)$, $1 \leqq j \leqq q$, where $\boldsymbol{\beta}' = [\boldsymbol{\beta}_1', \cdots, \boldsymbol{\beta}_q']$. Upon noting that $\boldsymbol{\Sigma}$ now is a scalar, say σ^2, and specializing expressions given previously, we find that the (likelihood ratio) statistics for testing $H_1, \cdots, H_q$, respectively, are $\hat{\boldsymbol{\beta}}_j'\mathbf{T}_{jj}^{-1}\hat{\boldsymbol{\beta}}_j/\sigma^2$, $1 \leqq j \leqq q$. Their joint distribution thus is the multivariate χ^2-distribution considered in the preceding sections. Moreover, if σ^2 is unknown and the residual mean square $\hat{\sigma}^2$ is used instead, then the joint distribution of $\hat{\boldsymbol{\beta}}_j'\mathbf{T}_{jj}^{-1}\hat{\boldsymbol{\beta}}_j/\hat{\sigma}^2$, $1 \leqq j \leqq q$, is given by Theorem 3, and a useful probability inequality for this distribution is given in Theorem 5.

In conclusion, note that simultaneous tests regarding subsets of responses using the Lawley-Hotelling statistics are alternative to other procedures which have been proposed (cf. Krishnaiah [15] and Roy [26]); however, the latter both require a sequence of conditional statements for the null distributions to be as claimed. Some other alternatives were considered in [27].

Acknowledgments. The author acknowledges the benefit of discussions with Professors H. T. David and I. J. Good. Thanks are due Dr. C. G. Khatri and Dr. P. R. Krishnaiah for pre-publication copies of references [14] and [17]. Finally, the author acknowledges with gratitude a number of suggestions by an Associate Editor and Referee which resulted in improvements both to style and content.

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