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CAUCHY'S EQUATION AND SUFFICIENT STATISTICS ON ARCWISE CONNECTED SPACES[1]

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1. Summary. We show that a measure-theoretic extension of Cauchy's functional equation, namely, $g(x_1)+g(x_2) = h(f(x_1, x_2))$ a.e., for real-valued functions defined on measure spaces equipped with a "reasonably compatible" arcwise connected topology is equivalent to a theorem which characterizes one-parameter exponential families on such measure spaces in terms of a real-valued sufficient statistic.

2. Introduction. P. Erdös [6] raised the following question: when does the functional equation $g(x)+g(y) = g(x+y)$, for almost all pairs (x, y) in the plane (Lebesgue measure), imply that $g(x) = cx$ almost everywhere? W. B. Jurkat [7] and N. G. de Bruijn [2] independently answered this and related problems. The present author, at the time unaware of the results of Jurkat and de Bruijn, considered the same problem for Euclidean x and y in establishing conditions on a Euclidean-valued sufficient statistic, defined on an arbitrary probability space, which ensure that a family of probability distributions is a k-parameter exponential family ([5], in particular, Lemma 2). In this paper we study further the relation between an extension of Cauchy's functional equation and sufficient statistics, in Theorems 4.2 and 4.3. The results are in part "local" theorems.

From the point of view of sufficient statistics this paper may be regarded as related to the work of L. Brown [1]. There it is shown that suitable measure-theoretic conditions on a continuous real-valued sufficient statistic for n independent identically distributed real observations ensure that the class of probability distributions is a one-parameter exponential family. Here we work with a continuous real-valued sufficient statistic defined on set endowed with a topology which is arcwise connected and locally arcwise connected. Theorem 4.1 is obtained under the assumption that there is a continuous version of the densities. In Corollary 4.1, where it is assumed that "locally, lower-semicontinuous functions are measurable," no continuity assumptions on the densities are employed. While this paper contains no applications (see, however, Section 5), we note that some stochastic processes are identified with measures on Banach spaces and Lie groups, and Banach spaces and components of Lie groups are arcwise connected and locally arcwise connected.

Each topology is assumed to be Hausdorff. We recall that a topological space $(\mathfrak{X}, \mathscr{T})$ is *arcwise connected* if with each x and y in $\mathfrak{X}$, $x \neq y$, there can be associated a bicontinuous mapping (continuous, one-one, and with a continuous inverse) $\phi: [0, 1] \to \mathfrak{X}$ so that $\phi(0) = x$ and $\phi(1) = y$. Then the image $\phi([0, 1])$ of ϕ by $[0, 1]$

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is called a *simple arc*, x and y are *endpoints* of $\phi([0,1])$, and $\phi([0,1])$ *joins* x and y. A topology $\mathscr{T}$ is *locally arcwise connected* if there is a base for $\mathscr{T}$ which consists of arcwise connected sets (note: a different definition for local arcwise connectedness is sometimes used). Each locally convex linear topological space is clearly arcwise connected and locally arcwise connected, and these properties are possessed by each connected, locally connected, locally compact metrizable space (see Corollary 4.11 on page 27 of [9] which also implies the local arcwise connectedness).

Given $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, \mu)$ where $\mathfrak{X}$ is a set, $\mathscr{T}$ a topology for $\mathfrak{X}$, $\mathscr{A}$ is a sigma-algebra for $\mathfrak{X}$, and μ is a nonnegative measure on $\mathscr{A}$, the following assumptions will be in force throughout this paper: (i) for each $x \in \mathfrak{X}$, $\inf\{\mu(A): A \in \mathscr{A}, x \in A\} = 0$; (ii) if $\mu(A^c) = 0$ and $U \in \mathscr{T}$ is not empty then $A \cap U$ is not empty. The first assumption is used in the density point arguments, e.g. Lemma 3.10, and the second appears in Lemma 3.6.

3. Cauchy's functional equation. In the next three lemmas we assume that continuous $f: \mathfrak{X} \times \mathfrak{X} \to R$ and continuous $h: R \to R$ satisfy for a fixed $x_0 \in \mathfrak{X}$

$$h(f(x_1, x_0)) + h(f(x_2, x_0)) = h(f(x_1, x_2)) \tag{3.1}$$

for each $(x_1, x_2) \in \mathfrak{X} \times \mathfrak{X}$.

The proof of Lemma 3.1 is taken from Lemma 4.1 and the proof of Lemma 3.2 is taken from Lemmas 4.2 and 4.3 of [1]. We give the proof of Lemma 3.1 for completeness.

LEMMA 3.1. *Let h and f satisfy* (3.1) *and let $\mathfrak{X}$ be arcwise connected. Fix $x' \in \mathfrak{X}$ and let $A_2 \subset \mathfrak{X}$ be a simple arc which has x' as an endpoint. Assume that for each simple arc $B \subset A_2$ which has x' as an endpoint the restriction of $h(f(\cdot, x_0))$ to B is not constant. If $A_1 \subset \mathfrak{X}$ is an arbitrary simple arc and if the restriction of $h(f(\cdot, x_0))$ to A_1 is constant then the restriction of $f(\cdot, x')$ and $f(x', \cdot)$ to A_1 is constant.*

PROOF. We argue to the contrary and suppose that $f(\cdot, x')$ is not constant while $h(f(\cdot, x_0))$ is constant. Let $\phi_1: [0,1] \to A_1$ and $\phi_2: [2,3] \to A_2$ be bicontinuous with $\phi_2(2) = x'$. Define $H: [0,1] \cup [2,3] \to R$ by $H(t_1) = h(f(\phi_1(t_1), x_0))$, $t_1 \in [0,1]$, and $H(t_2) = h(f(\phi_2(t_2), x_0))$, $t_2 \in [2,3]$. Define $F: [0,1] \times [2,3] \to R$ by $F(t_1, t_2) = f(\phi_1(t_1), \phi_2(t_2))$. Then for $(t_1, t_2) \in [0,1] \times [2,3]$

$$H(t_1) + H(t_2) = h(F(t_1, t_2)). \tag{3.2}$$

Now $f(\cdot, x')$ is not constant on A_1 if and only if $F(\cdot, 2)$ is not constant on $[0,1]$. Let D be the interior of $\{F(t_1, 2): t_1 \in [0,1]\}$. Since $h(f(\cdot, x_0))$ being constant on A_1 is equivalent to $H(\cdot)$ being constant on $[0,1]$ it follows that $h(\cdot)$ is constant on D. By continuity of F there is $\delta > 0$ so that for each t_2, $t_2 - 2 \leqq \delta$, there is $t_1(t_2) \in [0,1]$ for which $F(t_1(t_2), t_2) \in D$ and therefore $H(t_1(t_2)) + H(t_2) = h(F(t_1(t_2), t_2)) = h(F(t_1, 2)) = H(t_1) + H(2)$ for all $t_1 \in [0,1]$. Thus $H(t_2) = H(2)$ and the simple arc $B \subset A_2$ may be taken to be $\phi_2([2, 2+\delta])$.

LEMMA 3.2. *Assume the first three sentences of Lemma* 3.1. *Then,* (i) *if $A_1 \subset \mathfrak{X}$ is an arbitrary simple arc the restriction of h to $\{f(y, x'): y \in A_1\}$ and to*

$\{f(x',y): y \in A_1\}$ is strictly monotone and, (ii) *the restriction of h to $\{f(x',y): y \in \mathfrak{X}\}$ is strictly monotone.*

We omit the proof of

LEMMA 3.3. *Let $\mathfrak{X}$ be arcwise connected and let continuous $g: \mathfrak{X} \to R$ be not constant. Then there exists $x' \in \mathfrak{X}$ and a simple arc $A \subset \mathfrak{X}$ which has x' as an endpoint, so that for each simple arc $B \subset A$ which has x' as an endpoint, the restriction of g to B is not constant.*

A continuous real-valued function ϕ defined on an interval I is said to be piecewise strictly monotone if there is a decomposition $I = \bigcup_{i=1}^{k} I_i$ where $1 \leqq k \leqq \infty$ and each I_i is an interval so that the restriction of ϕ to each I_i is strictly monotone.

THEOREM 3.1. *Let $\mathfrak{X}$ be arcwise connected and locally arcwise connected and let h and f satisfy* (3.1). *Let $U \subset \mathfrak{X}$ be an arcwise connected open set so that for each open $V \subset U$ the restriction of $h(f(\cdot, x_0))$ to V is not constant. Then h is strictly monotone on the interval $\{f(x,y): x \in U, y \in \mathfrak{X}\}$.*

PROOF. $\{f(x,y): x \in U, y \in \mathfrak{X}\}$ is an interval since U and $\mathfrak{X}$ are connected. By Lemma 3.3 and the local arcwise connectedness there is a set $D \subset V$ which is dense in U so that for each $x' \in D$ the restriction of h to $\{f(x',y): y \in \mathfrak{X}\}$ is strictly monotone. We claim that h is strictly monotone on $\{f(x,y): y \in \mathfrak{X}\}$ for each $x \in U$. It suffices to prove that h is strictly monotone on each compact subinterval of the interior of $\{f(x,y): y \in \mathfrak{X}\}$ and this follows from the denseness of D. Clearly h is piecewise strictly monotone, and it remains to be shown that h is strictly monotone. Now for each $x \in \mathfrak{X}$, $\{h(f(x,y)): y \in \mathfrak{X}\} = h(f(x,x_0)) + h(f(y,x_0): y \in \mathfrak{X}\}$ (the algebraic sum) and thus h maps $\{f(x,y): y \in \mathfrak{X}\}$ onto an interval of positive length whose length is independent of x. Suppose h is not strictly monotone. Then by the preceding fact it easily follows that there exist two intervals I_1 and I_2 so that (i) the right endpoint of I_1 is the left endpoint of I_2; (ii) h is strictly increasing on I_1 and strictly decreasing on I_2 (or else the opposite); (iii) $I_i = \{f(x_i, y): y \in \mathfrak{X}\}$ where $x_i \in U$, $i = 1, 2$. The contradiction is seen to follow by joining x_1 and x_2 by a simple arc which lies in U and by invoking Lemma 3.2.

Clearly, for fixed $(x_2{}^0, \cdots, x_n{}^0) \in \mathfrak{X}^{n-1}$, $n \geqq 2$, the equation

$$\textstyle\sum_{i=1}^{n} h(f(x_i, x_2{}^0, \cdots, x_n{}^0)) = h(f(x_1, \cdots, x_n))$$

may be written as $\sum_{i=1}^{n} g(x_i) = h(f(x_1, \cdots, x_n))$. Conversely, the latter equation may be written as $\sum_{i=1}^{n} \bar{h}(f(x_i, x_2{}^0, \cdots, x_n{}^0)) = \bar{h}(f(x_1, \cdots, x_n))$ where $\bar{h}(z) = h(z) - (n/n-1)\sum_{i=2}^{n} g(x_i{}^0)$ and $g(x) = h(f(x, x_2{}^0, \cdots, x_n{}^0)) - \sum_{i=2}^{n} g(x_i{}^0)$.

This fact together with Theorem 3.1. gives

THEOREM 3.2. *Let $\mathfrak{X}$ be arcwise connected and locally arcwise connected. Let continuous $g: \mathfrak{X} \to R$, continuous $f: \mathfrak{X}^n \to R$, and continuous $h: R \to R$ satisfy*

$$\textstyle\sum_{i=1}^{n} g(x_i) = h(f(x_1, \cdots, x_n)) \tag{3.2$'$}$$

for $(x_1, \cdots, x_n) \in \mathfrak{X}^n$, $n \geqq 2$. Let $U_i \subset \mathfrak{X}$ be open arcwise connected sets so that the *restriction of g to each open $V_i \subset U_i$ is not constant, $i = 1, \cdots, n-1$. Then h is strictly monotone on the interval $\{f(x_1, \cdots, x_n): x_i \in U_i,\ i = 1, \cdots, n-1,\ x_n \in \mathfrak{X}\}$. Consequently*

$$f(x_1, \cdots, x_n) = \phi(\textstyle\sum_{i=1}^n g(x_i) + c_0) \tag{3.3}$$

for $(x_1, \cdots, x_n) \in \prod_{i=1}^{n-1} U_i \times \mathfrak{X}$ where c_0 is a constant and ϕ is continuous and strictly monotone.

COROLLARY 3.1. *If, in addition, the restriction of g is not constant on each open subset of $\mathfrak{X}$ then* (3.3) *holds on $\mathfrak{X}^n$.*

COROLLARY 3.2. *Let $\sum_{i=1}^n g_j(x_i) = h_j(f(x_1, \cdots, x_n))$, $j = 1, 2$, where g_j, h_j, f, and $\mathfrak{X}$ satisfy the hypotheses of the first two sentences of Theorem* 3.2. *Let $U \subset \mathfrak{X}$ be an open arcwise connected set so that the restriction of g_1 to each $V \subset U$ is not constant. Then there are real numbers a_1 and b_1 so that $a_1 g_1 + b_1 = g_2$ on $\mathfrak{X}$.*

PROOF. Define $\bar{g}_1(x) = g_1(x) - g_1(x_0)$ where $g_1(x_0)$ lies in the interior of $g_1(U)$. Clearly $\sum_{i=1}^n \bar{g}_1(x_i) = \bar{h}_1(f(x_1, \cdots, x_n))$ for continuous $\bar{h}$ on $U^{n-1} \times \mathfrak{X}$ and thus $f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^n \bar{g}_1(x_i))$ since we clearly may choose ϕ so that $c_0 = 0$. Thus $\sum_{i=1}^n g_2(x_i) = h_2(\phi(\sum_{i=1}^n \bar{g}_1(x_i)))$ and hence $\sum_{i=1}^n \bar{g}_2(x_i) = \bar{h}_2(\phi(\sum_{i=1}^n \bar{g}_1(x)))$ on $U^{n-1} \times \mathfrak{X}$ where $\bar{g}_2(x) = g_2(x) - g_2(y_0)$, for fixed $y_0 \in U$. Thus for $x \in \mathfrak{X}$, $\bar{g}_2(x) = \bar{h}_2(\phi(\bar{g}_1(x) + (n-1)\bar{g}_1(y_0)))$. Finally, for $x \in \mathfrak{X}$, we have $\sum_{i=1}^n \psi(\bar{g}_1(x_i) + (n-1)\bar{g}_1(y_0)) = \psi(\sum_{i=1}^n \bar{g}_1(x_i)$ where $\psi = \bar{h}_2 \circ \phi$, and since the interval $\bar{g}_1(\mathfrak{X})$ contains the origin in its interior, it follows that $\psi(t) = at + b$. This gives the assertion.

We introduce a sigma-algebra $\mathscr{A}$ on $\mathfrak{X}$ and a nonnegative measure μ on $\mathscr{A}$ and recall the assumption made in the introduction that if $\mu(A^c) = 0$ and $U \in \mathscr{T}$ is not empty then $A \cap U$ is not empty.

DEFINITION 3.1. A function $f: \mathfrak{X} \to R$ *preserves ample sets* (relative to μ) if for each pair of non-void $U_i \in \mathscr{T}$, $i = 1, 2$, such that $f(U_1) = f(U_2)$ it is true that $f(U_1 \cap A)$ has a non-empty intersection with $f(U_2 \cap A)$ whenever $\mu(A^c) = 0$.

A substantial part of the argument used in the proofs of the next three lemmas and the next theorem is borrowed from Lemmas 1 and 2 of [4]—however, inclusion of the following proofs may be helpful. A referee suggested we point out the relation between f preserving ample sets and f satisfying Lusin's condition (N) when $\mathfrak{X} \subset R$ is an interval. In general the relations are not comparable. However, for a measure μ on the Borel sets of the interval $\mathfrak{X} \subset R$ which dominates Lebesgue measure, if continuous f satisfies Lusin's condition (N) then f preserves ample sets. From Lemma 3.3 through Theorem 3.3 it is assumed that $\mathfrak{X}$ is arcwise connected and locally arcwise connected.

LEMMA 3.4. *Let f and g be continuous mappings of $\mathfrak{X}$ into R and let f preserve ample sets. Let $h: f(\mathfrak{X}) \to R$ be such that for a fixed $A \in \mathscr{A}$ where $\mu(A^c) = 0$, $g(x) = h(f(x))$ for each $x \in A$. If $x_i \in \mathfrak{X}$ and $f(x_1) = f(x_2)$ and if the restriction of f to each of two fixed neighborhoods of x_1 and x_2, respectively, is constant, then $g(x_1) = g(x_2)$.*

PROOF. This follows from the continuity of g and the assumption that $U \cap A$ is not empty if U is not empty.

LEMMA 3.5. *Under the assumptions of Lemma* 3.4, *if* $f(x_1) = f(x_2)$ *and each neighborhood* U_i *of* x_i *contains a* y_i *such that* $f(y_i) > f(x_i)$, *then* $g(x_1) = g(x_2)$.

PROOF. Choose arcwise connected neighborhoods U_i of x_i, $i = 1, 2$, such that $|g(z_i) - g(x_i)|$ is small for all $z_i \in U_i$. Choose $y_i \in U_i$ such that $f(y_i) > f(x_i)$ and let $\alpha = \min(f(y_1), f(y_2))$. Then $U_i \cap f^{-1}(f(x_1), \alpha) \equiv C_i$ is a non-empty open set, $i = 1, 2$, and moreover $f(C_1) = f(C_2)$. Since f preserves ample sets, there is $t_i \in C_i \cap A$ such that $f(t_1) = f(t_2)$. Then $g(t_1) = g(t_2)$, which gives the assertion.

LEMMA 3.6. *Under the assumptions of Lemma* 3.4, *if the restriction of* f *to a neighborhood* U *of* x_i *is constant and if* z *belongs to the frontier of* $f^{-1}(f(x_1))$, *then* $g(x_1) = g(z)$.

PROOF. This follows from the fact that for each neighborhood V of z, $V \cap U \cap A$ is not empty, the continuity of g, and Lemma 3.4.

THEOREM 3.3. *Let* f *and* g *be continuous mappings of* $\mathfrak{X}$ *into* R *and let* f *preserve ample sets. Let* $h: f(\mathfrak{X}) \to R$ *be such that for a fixed* $A \in \mathscr{A}$ *where* $\mu(A^c) = 0$, $g(x) = h(f(x))$ *for each* $x \in A$. *Then there is a function* $H: f(\mathfrak{X}) \to R$ *so that* $g(x) = H(f(x))$ *for each* $x \in \mathfrak{X}$.

PROOF. We assume $f(x_1) = f(x_2)$ and we prove $g(x_1) = g(x_2)$. By Lemmas 3.4 and 3.6 it follows that we may assume each neighborhood U_i of x_i contains a y_i such that $f(y_i) \neq f(x_i)$. By Lemma 3.5 we may assume $f(z_1) \geqq f(x_1)$ and $f(z_2) \leqq f(x_2)$ for all $z_i \in U_i$, for sufficiently small U_i. Let A be a simple arc joining x_1 and x_2 where $\phi(0) = x_1$, ϕ the homeomorphism. For the remainder of this proof when we refer to an inequality and an infimum (or supremum) over a subset of A it is taken via the homeomorphism ϕ; however, open sets are in $\mathfrak{X}$ and not relativized to A. Let $z_3 = \sup\{z : z \in A$, each neighborhood of z contains a point y such that $f(y) > f(x_1)\}$, and let $z_4 = \inf\{z : z \in A, z \geqq z_3$, each neighborhood of z contains a point y such that $f(y) < f(x_2)\}$. Clearly $x_1 \leqq z_3 \leqq z_4 \leqq x_2, f(z_3) \geqq f(x_1)$ and $f(z_4) \leqq f(x_2)$, and a brief argument can establish that the latter inequalities are equalities: $f(z_3) = f(x_1) = f(x_2) = f(z_4)$. By Lemma 3.5 and its analogue with the inequality sign reversed, it follows that $g(z_3) = g(x_1)$ and $g(z_4) = g(x_2)$. Since the proof is finished if $z_3 = z_4$, we assume $z_3 < z_4$. If $z_3 < t < z_4$, then it follows that there is a neighborhood V of t such that f is constant on V. Since $\phi(s)$ lies in V for all s sufficiently close to $\phi^{-1}(t)$, it follows that $f \circ \phi$ is constant on an open interval containing $\phi^{-1}(t)$. This implies $f \circ \phi$ is constant on the interval $(\phi^{-1}(z_3), \phi^{-1}(z_4))$. Therefore, f is constant on $\{t : z_3 < t < z_4\}$ and by continuity $f(t) = f(x_1)$, $z_3 < t < z_4$. By Lemma 3.4, for $z_3 < t_1 < t_2 < z_4$ we obtain $g(t_1) = g(t_2)$. By continuity of g it follows that $g(z_3) = g(z_4)$ and hence $g(x_1) = g(x_2)$.

If $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, \mu)$ is given, then as usual $(\mathfrak{X}^n, \mathscr{T}^n, \mathscr{A}^n, \mu^n)$ denotes the product set-up where $2 \leqq n < \infty$.

THEOREM 3.4. *Let* $(\mathfrak{X}, \mathcal{T}, \mathcal{A}, \mu)$ *be given where* $\mathcal{T}$ *is an arcwise connected and locally arcwise connected topology for* $\mathfrak{X}$. *Let continuous* $f: \mathfrak{X}^n \to R$ *preserve ample sets relative to* μ^n *and let continuous* $g: \mathfrak{X} \to R$. *Let* $A \in \mathcal{A}^n$ *be such that* $\mu(A^c) = 0$ *and let* $h: f(\mathfrak{X}^n) \to R$ *be such that*

$$\sum_{i=1}^n g(x_i) = h(f(x_1, \cdots, x_n)) \tag{3.4}$$

holds for each $(x_1, \cdots, x_n) \in A$. *Then there is a continuous* $h: f(\mathfrak{X}^n) \to R$ *so that* (3.4) *holds for each* $(x_1, \cdots, x_n) \in \mathfrak{X}^n$. *Moreover, if for a non-void arcwise connected* $U \in \mathcal{T}$ *the restriction of* g *to each open* $V \subset U$ *is not constant then* (i) *if a function* h_1, *a continuous function* g_1, *and a* $B \in \mathcal{A}$ *for which* $\mu(B^c) = 0$ *satisfy* $\sum_{i=1}^n g_1(x_1) = h_1(f(x_1, \cdots, x_n))$ *for* $(x_1, \cdots, x_n) \in B$ *then* $g_1(x) = a_1 g(x) + b_1$ *for fixed real numbers* a_1 *and* b_1 *and every* $x \in \mathfrak{X}$; (ii) $f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^n g(x_i) + c_0)$ *for* $(x_1, \cdots, x_n) \in U^{n-1} \times \mathfrak{X}$.

PROOF. Theorem 3.3 implies (3.4) for some function h, for each $(x_1, \cdots, x_n) \in \mathfrak{X}^n$. Choose distinct points x and y in $\mathfrak{X}^n$ and let A be a simple arc joining these points. Then A is compact, (3.4) evidently holds on A, and by Theorem 9 on page 95 of [8] h is continuous on $f(A)$. Since the points are arbitrary it easily follows that h is continuous on $f(\mathfrak{X}^n)$. The second assertion now follows from Theorem 3.2 and Corollary 3.2.

We provide a definition of the *essential limit supremum* of a real-valued measurable function. For non-empty $U \in \mathcal{T}$ and $A \in \mathcal{A}$, $P(A) = 1$, let $g(U, A) = \sup\{g(y): y \in U \cap A\}$. Let $g(U) = \inf\{g(U, A): P(A) = 1\}$. If $x \in \mathfrak{X}$ and $\mathcal{U}(x)$ is a base for the neighborhood system of x then the essential limit supremum of g at x, $(\text{ess lim sup}\, g)(x)$, is defined to be $\lim\{g(U): U \in \mathcal{U}(x)\}$ where $\mathcal{U}(x)$ is regarded as the directed set. It is routine to verify that $(\text{ess lim sup}\, g)(x)$ is well-defined in $[-\infty, \infty]$. We define $(\text{ess lim inf}\, g)$ analogously. Now, $(\text{ess lim sup}\, g)$ is upper-semicontinuous and $(\text{ess lim inf}\, g)$ and also I_U, the indicator function of an open set U, are lower-semicontinuous and this is used in the next assertions which now require a "local inclusion" of $\mathcal{T}$ in $\mathcal{A}$.

DEFINITION 3.4. If for each $x \in \mathfrak{X}$ there is a base for the neighborhood system $\mathcal{U}(x)$ of x such that $\mathcal{U}(x) \subset \mathcal{A}$ then we say that enough local base sets are measurable and that $\mathcal{U}(x)$ is a measurable local base.

We require a condition on P which ensures that measurable g is equal to a continuous function almost everywhere if and only if the essential limits coincide.

DEFINITION 3.5. Given $(\mathfrak{X}, \mathcal{T}, \mathcal{A}, P)$ where enough local base sets are measurable we say that P has some local density if for each $A \in \mathcal{A}$ such that $P(A) > 0$ there is $A' \subset A$ with $P(A') = P(A)$ such that for each $x \in A'$ there is a local base $\mathcal{U}(x)$ for which $P(A' \cap U) > 0$ for each $U \in \mathcal{U}(x)$. The elements $x \in A'$ are said to be points with some density.

LEMMA 3.7. *Let* $(\mathfrak{X}, \mathcal{T}, \mathcal{A}, P)$ *be given and let* $U \in \mathcal{T} \cap \mathcal{A}$ *be such that each real-valued lower-semicontinuous function defined on* U *is measurable with respect to the relativized sigma-algebra. Let measurable* $g: \mathfrak{X} \to R$ *and let* P *have some local* *density. Then there is a continuous $G: U \to R$ such that $G(x) = g(x)$ for almost every $x \in U$ if and only if for every $x \in U$, $-\infty <$ (ess lim inf $g)(x) =$ (ess lim sup $g)(x) < \infty$.*

PROOF. Let continuous $G = g$ almost everywhere on U. Since $A \cap V$ is not empty whenever open V is not empty and $P(A) = 1$ it follows that the essential limits coincide on U. Conversely, if the (finite) two essential limits coincide on U then they are continuous. A brief argument shows it suffices to prove that $g(x) \leqq$ (ess lim sup)$g(x)$ for almost every $x \in U$. Suppose not. Then it follows that there is a set A such that $P(A) > 0$ and each point of A has some local density and that there are real constants M_1 and M_2 and a positive real δ such that $M_1 <$(ess lim sup $g)(x) + 2\delta < g(x) < M_2$ for every $x \in A$. Then for each $x \in A$ there is $A(x)$ with $P(A(x)) = 1$ and $U(x) \in \mathscr{U}(x)$ so that $g(x) > \delta + \sup\{g(y): y \in U(x) \cap A(x)\}$. Since each point of A has some local density it follows by a simple argument that g is unbounded from below on A. This is the contradiction.

From the identity arctan(ess lim sup g) = (ess lim sup arctan g) we obtain

LEMMA 3.8. *Let $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, P)$ be given and let $U \in \mathscr{T} \cap \mathscr{A}$ be such that each extended real-valued lower-semicontinuous function defined on U is measurable with respect to the relativized sigma-algebra. Let measurable $g: \mathfrak{X} \to [-\infty, \infty]$ and let P have some local density. Then there is a continuous $G: U \to [-\infty, \infty]$ such that $G(x) = g(x)$ for almost every $x \in U$ if and only if for every $x \in U$,* (ess lim inf $g)(x) =$ (ess lim sup $g)(x)$.

DEFINITION 3.6. Let $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, P)$ satisfy the condition that enough local sets are measurable. The subsets $A_i \in \mathscr{A}$, $i = 1, 2$, are said to have the point $x \in \mathfrak{X}$ as a common point of some density if for each $U \in \mathscr{U}(x)$, the measurable local base, $P(A_i \cap U) > 0$ for $i = 1, 2$.

LEMMA 3.9. *Let $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, P)$ satisfy the condition that enough local base sets are measurable. Let $\mathscr{Y}$ be a set and let $h: \mathscr{Y} \to R$, $f: \mathfrak{X}^n \to \mathscr{Y}$, and measurable $g: \mathfrak{X} \to R$ satisfy $\sum_{i=1}^n g(x_i) = h(f(x_1, \cdots, x_n))$, $(x_1, \cdots, x_n) \in A$ with $P(A) = 1$. If there exists $x_0 \in \mathfrak{X}$ such that* (ess lim sup $g)(x_0) >$ (ess lim inf $g)(x_0)$ *then there are $B_i \in \mathscr{A}$, $i = 1, 2$, which have x_0 as a common point of density and $C \in \mathscr{A}$ with $P(C) > 0$ such that the image by f of $A \cap (B_1 \times \cdots \times B_1 \times C)$ is disjoint from the image of $A \cap (B_2 \times \cdots \times B_2 \times C)$.*

PROOF. It is easy to verify that there are real constants c_1 and c_2 so that the inequality (ess lim inf $g)(x_0) < c_1 < c_2 <$ (ess lim sup $g)(x_0)$ is satisfied together with x_0 being a common point of some density of $B_1 = \{y: g(y) < c_1\}$ and $B_2 = \{y: g(y) > c_2\}$. The assertion follows upon choosing a $C \in \mathscr{A}$ such that $\sup\{g(y): y \in C\} - \inf\{g(y): y \in C\}$ is sufficiently small.

DEFINITION 3.7. Let continuous $f: \mathfrak{X}^n \to R$. We say that f does not isolate sets with a common density point if for each $C \in \mathscr{A}$ with $P(C) > 0$, each $A \in \mathscr{A}$ with $P(A) = 1$, and each pair B_i with a common point of some density $f(A \cap (B_1 \times \cdots \times B_1 \times C))$ has a non-void intersection with $f(A \cap (B_2 \times \cdots \times B_2 \times C))$.

The following theorem extends Cauchy's functional equation.

THEOREM 3.5. *Let* $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, P)$ *be given where* (i) $\mathscr{T}$ *is an arcwise connected and locally arcwise connected topology*; (ii) P *is a probability on the sigma-algebra* $\mathscr{A}$; (iii) *for each* $x \in \mathfrak{X}$ *there is a local base* $\mathscr{U}(x)$ for x *such that* $\mathscr{U}(x) \subset \mathscr{A}$ *and for each* $U \in \mathscr{U}(x)$ *it is true that each extended real-valued lower-semicontinuous function defined on* U *is measurable with respect to the relativized sigma-algebra*; (iv) P *has some local density. Let* $(\mathfrak{X}^n, \mathscr{T}^n, \mathscr{A}^n, P^n)$ *denote the product set-up. Let measurable* $g: \mathfrak{X} \to [-\infty, \infty]$, *let the function* $h: R \to [-\infty, \infty]$, *and let continuous* $f: \mathfrak{X}^n \to R$ *satisfy*

$$\sum_{i=1}^{n} g(x_i) = h(f(x_1, \cdots, x_n)) \tag{3.5}$$

for each $(x_1, \cdots, x_n) \in A$ *with* $P(A) = 1$, *where* $g(x_i)$ *is finite on each coordinate of each point of* A. *If* f *does not isolate sets with a common point of some density and if* f *preserves ample sets relative to* P^n *then there is continuous* $G: \mathfrak{X} \to [-\infty, \infty]$ *such that* $G = g$ *almost everywhere* P. *Moreover, for each arcwise connected open measurable set* $U \subset \{x: -\infty < G(x) < \infty\}$

$$\sum_{i=1}^{n} G(x_i) = h(f(x_1, \cdots, x_n)) \tag{3.6}$$

for each $(x_1, \cdots, x_n) \in U^n$ *and continuous* $h: f(\mathfrak{X}^n) \to R$. *Also, if the restriction of* G *to each open subset of* U *is not constant then* $f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^{n} G(x_i) + c_0)$ *for each* $(x_1, \cdots, x_n) \in U^n$, *a real constant* c_0, *and continuous strictly monotone* ϕ, *and consequently* f *is measurable in the relativized sigma-algebra when restricted to* U^n. G *is unique in the sense that if continuous* G_1 *satisfies* (3.5) *then on* U, $G_1 = a_1 G + b_1$ *for real constants* a_1 *and* b_1.

PROOF. If U is a measurable open set such that each real-valued lower-semicontinuous function defined on U is measurable, then each open subset $V \subset U$ is measurable and each real-valued lower-semicontinuous function defined on V is measurable. Choose the arcwise connected open measurable U. The proof of the theorem follows from the hypotheses and conclusions of Lemma 3.8, Lemma 3.9, Theorem 3.4, and Theorem 3.2, employed in that order.

4. Sufficient statistics. In this section we obtain consequences of Theorems 3.4 and 3.5 in terms of sufficient statistics. We continue the notation and definitions of Section 3.

DEFINITION 4.1. Let $(\mathfrak{X}, \mathscr{A}, \{Q_t: t \in T\})$ be an experiment, that is, $\mathscr{A}$ is a sigma-algebra of subsets of $\mathfrak{X}$ and the Q_t are probabilities on $\mathscr{A}$. Let Q_{t_0}, fixed $t_0 \in T$, dominate $\{Q_t\}$. Let F, G, and H be functions so that for each $t \in T$ the mapping $x \to G(F(x), t)$ is measurable, the mapping $x \to H(x)$ is measurable, and for each $A \in \mathscr{A}$, $t \in T$, $Q_t(A) = \int_A G(F(\cdot), t) H(\cdot) Q_{t_0}(dx)$. In this case we say that F *is essentially sufficient, but not necessarily measurable for* $\{Q_t\}$.

THEOREM 4.1. *Let* $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, \{P_t: t \in T\})$ *be given where* $\mathscr{T}$ *is an arcwise connected and locally arcwise connected topology for* $\mathfrak{X}$, *and* $(\mathfrak{X}, \mathscr{A}, \{P_t: t \in T\})$ *is an experiment where* $P_t(N) = 0$ *if and only if* $P_{t'}(N) = 0$, $(t, t') \in T \times T$. *For fixed* $n \geqq 2$, *let continuous* $f: \mathfrak{X}^n \to R$ *satisfy* (i) f *preserves ample sets relative to* $P_{t_0}^n$ *for a fixed* t_0, *and* (ii) *f is essentially sufficient but not necessarily measurable for $\{P_t^n\}$. Assume that there is a version of dP_t/dP_{t_0} which is continuous and positive for each $t \in T$. Then there is a function $H: R \times T \to R$ which is continuous on R for each fixed $t \in T$ so that*

$$\prod_{i=1}^{n} dP_t/dP_{t_0}(x_i) = H(f(x_1, \cdots, x_n), t) \quad \text{for each} \quad (x_1, \cdots, x_n) \in \mathfrak{X}^n$$

for the continuous versions of dP_t/dP_{t_0}. Moreover, if there is $t' \in T$ and non-void arcwise connected $U \in \mathscr{T}$ so that for each open $V \subset U$ the restriction of the continuous version of $dP_{t'}/dP_{t_0}$ to V is not constant, then there is a continuous measurable $g: \mathfrak{X} \to R$ so that (i) *$dP_t/dP_{t_0}(\cdot) = c_1(t) \exp c_2(t) g(\cdot)$ almost everywhere P_{t_0} for each version of dP_t/dP_{t_0}, and* (ii) *$f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^{n} g(x_i) + c_0)$ for $(x_1, \cdots, x_n) \in U^{n-1} \times \mathfrak{X}$ for continuous strictly monotone ϕ and real c_0, and thus for each such $U \subset \mathfrak{X}$, f is measurable when restricted to $U^{n-1} \times \mathfrak{X}$ with the relativized sigma-algebra on $U^{n-1} \times \mathfrak{X}$.*

To prove Theorem 4.1 it suffices to prove

THEOREM 4.2. *The following assertions are equivalent:*

(i) *Theorem* 3.4 *with the additional assumptions that μ is finite, g is measurable, and $\int \exp cg(\cdot)\mu(dx) < \infty$ for nonzero real c;*

(ii) *Theorem* 4.1.

PROOF. We first show that (ii) implies (i). Let $\{P_t\}$ be the family of probabilities defined by $dP_t/dP_{t_0}(\cdot) = c_1(t) \exp c_2(t) g(\cdot)$ where $P_{t_0} = c_1(t_0)\mu$ and the c_i are the normalizing constants. Since g is assumed to be related to f by (3.4) almost everywhere, it follows that f is essentially sufficient, but not necessarily measurable, for $\{P_t^n\}$. It follows by Theorem 4.1 that $\log c_1(t) + c_2(t) \sum_{i=1}^{n} g(x_i) = \bar{H}(f(x_1, \cdots, x_n), t)$, $(x_1, \cdots, x_n) \in \mathfrak{X}^n$, $t \in T$, and this implies (3.4). Let U be the set where g is not constant. Then dP_t/dP_{t_0} is also nonconstant. If $\sum_{i=1}^{n} g_1(x_i) = h_1(f(x_1, \cdots, x_n))$ almost everywhere $P_{t_0}^n$ then f is essentially sufficient for $\{Q_t^n\}$ where $dQ_t/dQ_{t_0} = d_1(t) \exp d_2(t) g_1$. Then, $\sum_{i=1}^{n} g_1(x_i) = h_1(\phi(\sum_{i=1}^{n} g(x_i) + c_0))$ on $\mathfrak{X}^n$ and this implies that $g_1 = a_1 g + b_1$. That $f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^{n} g(x_i) + c_0)$ is true. Conversely, suppose (i) holds. Let $\{g_s : s \in S\}$ be a basis for the smallest vector space containing the continuous versions of $\log dP_t/dP_{t_0}$. If f satisfies the hypotheses of Theorem 4.1 then each g_s and f satisfy (3.4) except for a set of probability zero depending on g_s. Since (i) holds each g_s and f satisfy (3.4) on $\mathfrak{X}^n$. This implies the existence of the function $H: R \times T \to R$ of Theorem 4.1. Let $dP_{t'}/dP_{t_0}$ be nonconstant on U. Since we may assume with no loss of generality that $\log dP_{t'}/dP_{t_0}$ is a member of $\{g_s : s \in S\}$, this implies, by Theorem 3.4, that each g_s is of the form $a_1(\log dP_{t'}/dP_{t_0}) + b_1$ everywhere on $\mathfrak{X}$ for real constants a_1 and b_1 and this leads directly to a completion of the assertion.

Similarly, the next theorem is equivalent to Theorem 3.5.

THEOREM 4.3. *Let $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, \{P_t : t \in T\})$ be given where $(\mathfrak{X}, \mathscr{A}, \{P_t : t \in T\})$ satisfies the hypotheses of Theorem* 4.1. *Assume that for some $t_0 \in T$, $(\mathfrak{X}, \mathscr{T}, \mathscr{A}, P_{t_0})$ satisfies the hypotheses of Theorem* 3.5. *Let continuous $f: \mathfrak{X}^n \to R$ satisfy the hypo-*

theses of Theorem 3.5 and in addition be essentially sufficient but not necessarily measurable for $\{P_t^n\}$. Then there is a version of extended real-valued dP_t/dP_{t_0} which is everywhere continuous. Moreover, for each arcwise connected open measurable U on which the continuous versions of dP_t/dP_{t_0} are positive and finite and for which there is a $t' \in T$ such that the restriction of the continuous version of $dP_{t'}/dP_{t_0}$ to each open subset $V \subset U$ is not constant there is a continuous measurable $g: U \to R$ so that $dP_t/dP_{t_0}(\cdot) = c_1(t) \exp c_2(t) g(\cdot)$ almost everywhere P_{t_0} on U, for each $t \in T$. Then $f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^n g(x_i) + c_0)$ for each $(x_1, \cdots, x_n) \in U^n$, continuous strictly monotone ϕ and real c_0, and consequently f is measurable when restricted to U^n.

PROOF. Theorems 3.5 and 4.1 produce the assertion except for the continuity everywhere. Define $\phi_t(x) = (\text{ess lim sup} \log dP_t/dP_{t_0})(x)$. Then $dP_t/dP_{t_0}: \mathfrak{X} \to [0, \infty]$ has a continuous version if and only if $\phi_t: \mathfrak{X} \to [-\infty, \infty]$ is continuous and the continuity is therefore obtained by Lemma 3.8.

COROLLARY 4.1. *In the hypotheses of Theorem 4.3 assume only that for a given arcwise connected open measurable U there is (possibly only one) $t' \in T$ for which the continuous version of $dP_{t'}/dP_{t_0}$ is positive and finite on U and for which the restriction of $dP_{t'}/dP_{t_0}$ to each open subset $V \subset U$ is not constant. Then the conclusions of Theorem 4.3 hold on this U and in particular $dP_t/dP_{t_0}(\cdot) = c_1(t) \exp c_2(t) g(\cdot)$ almost everywhere P_{t_0} on U, for each $t \in T$.*

PROOF. This follows from the fact that $f(x_1, \cdots, x_n) = \phi(\sum_{i=1}^n g(x_i) + c_0)$ where $g(x) = \log dP_{t'}/dP_{t_0}(x)$, $x \in U$.

5. Remarks. Theorem 4.3 is related to Theorem 2.1 of [1] where $\mathfrak{X} \subset R$ is an interval and where in addition the family of probabilities are assumed to be dominated by Lebesgue measure λ_1. In this case, it is shown in [1], at least implicitly, that the hypotheses on f in Theorem 4.3 of this paper will be satisfied if the following condition is satisfied: there is a Borel set $A \subset \mathfrak{X}$ with $\lambda_1(A) > 0$ such that if $B \subset A$ and $\lambda_1(B) > 0$ then for each $(x_1, \cdots, x_{n-1}) \in \mathfrak{X}^{n-1}$ the linear Lebesgue measure of $\{f(x_1, \cdots, x_{n-1}, y): y \in B\}$ is positive.

Such a condition easily generalizes if $\mathfrak{X}$ is a rectangle, say, in R^m and the family of probabilities are dominated by Lebesgue measure. In another paper we will discuss differentiability conditions on mappings $f: \mathfrak{X}^n \to R$, where $\mathfrak{X}$ is a Banach space and the family of probabilities is regular, which ensure that the conditions of Theorem 4.3 are satisfied—for example, consider a family $\{P_t\}$ of probabilities equivalent to Weiner measure, which we may regard as defined on the Borel sets of $L_2([0, 1])$, and the L_2-continuous mapping $f(x_1, \cdots, x_n) = \sum_{i=1}^n \int_0^1 x_i^2(t)\, dt$.

Finally, for examples of continuous functions which are sufficient statistics, which fail the hypotheses of the theorems of this paper, and for which the conclusions of the theorems also fail, we mention page 1460 of [1], [3], and [5].

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