The Annals of Mathematical Statistics
1970, Vol. 41, No. 3, 1016–1026

SEQUENTIAL CONFIDENCE INTERVALS BASED ON RANK TESTS[1]

BY J. C. GEERTSEMA

Potchefstroom University

1. Introduction. In this paper the problem of finding a bounded length confidence interval for the mean of a statistical population will be considered. In the case of a normal population, Stein [13] gave what is now a well-known two-stage procedure. Recently Chow and Robbins [4] proposed a truly sequential procedure for the mean of a population with finite variance. They considered properties of this procedure when the prescribed bound on the length of the confidence interval tends to zero and proved that the procedure can be chosen in such a way that asymptotically a prescribed coverage probability is obtained. Furthermore, they found an asymptotic expression for the expected sample size. Starr [12] investigated the behavior of this procedure and modifications of it in the case of a normal population for various values of the prescribed bound on the length of the interval. He presented numerical evidence to show that the coverage probability of such a procedure in these cases differs very little from the asymptotic coverage probability (if the latter is .95 or .99). He also compared the Stein procedure with that of Chow and Robbins and found that the latter is usually more efficient.

The methods used by Chow and Robbins to construct a sequential confidence interval procedure and to investigate its asymptotic properties, can be applied quite generally. This is discussed in the next section.

In Sections 3 and 4 these methods are used to construct and investigate two procedures which are derived from the sign test and the Wilcoxon one-sample test respectively. It may be noted that the asymptotic theory derived by Chow and Robbins is under the assumption of a finite second moment. No such assumption will be needed in the procedures discussed in this paper.

An asymptotic comparison between the three procedures is carried out in Section 5 with respect to asymptotic expected sample sizes. It is shown that when the prescribed bound on the length of the confidence interval tends to zero the asymptotic ratio of expected sample sizes of two procedures is the same as the Pitman efficiency between the corresponding tests (if the procedure of Chow and Robbins is thought of as corresponding to the t-test). The main results of some Monte Carlo studies are mentioned in Section 6.

2. A method for constructing bounded length confidence intervals. Denote by $X_1, X_2, \cdots, X_n$ a sample of fixed size n from a population with cdf F and let θ be a parameter of the population.

Received January 27, 1969; revised January 9, 1970.

[1] This research was supported by the South African Council for Scientific and Industrial Research, Shell (South Africa), the Computer Center of the University of California, Berkeley, and the U.S. Army Research Office (Durham), Grant DA-ARO-D-31-124-G816.

In this paper, sequential procedures for finding a confidence interval of length not larger than $2d$ are constructed in the following way: For each positive integer n, consider two statistics L_n and U_n (not depending on d) based on the first n observations, such that $L_n < U_n$ a.s. and $\lim_{n\to\infty} P(L_n \leqq \theta \leqq U_n) = 1-2\alpha$ (so that, for n large, (L_n, U_n) is a confidence interval for θ with coverage probability approximately $1-2\alpha$). Define a stopping variable N to be the first integer $n \geqq n_0$ such that $U_n - L_n \leqq 2d$, where n_0 is a positive integer. Take as confidence interval (L_N, U_N).

The procedure of Chow and Robbins [4] is of this type. In the two procedures to be considered in this paper, the statistics L_n and U_n are the lower and upper confidence bounds, respectively, of fixed sample size confidence intervals based on families of rank tests for the one-sample problem.

Having defined a sequential procedure in the above way, two questions immediately arise: (1) What is the coverage probability of the procedure? (2) What is the expected sample size?

These questions can, under certain assumptions, be answered asymptotically as $d \to 0$. This forms the content of the remainder of this section, for which purpose a set of assumptions will now be introduced. In Sections 3 and 4 it is shown that the procedures defined there satisfy these assumptions.

ASSUMPTION 2.1.

(i) $L_n < U_n$ a.s. (L_n and U_n independent of d).

(ii) $n^{\frac{1}{2}}(U_n - L_n) \to 2K_\alpha/A$ a.s. as $n \to \infty$ where $A > 0$ and $\Phi(K_\alpha) = 1-\alpha$ and where Φ is the standard normal cdf.

(iii) $n^{\frac{1}{2}}(L_n - \theta) = Z_n/A - K_\alpha/A + o(1)$ a.s. as $n \to \infty$ where Z_n is a standardized average of i.i.d. random variables with finite second moment.

(iv) The set $\{Nd^2\}_{d>0}$ is uniformly integrable.

THEOREM 2.1. *Under the Assumptions* 2.1

(1) *N is well defined, $EN < \infty$ for all $d > 0$, $N(=N(d))$ is a function of d which is non decreasing as d decreases,* $\lim_{d\to 0} N = \infty$ a.s. *and* $\lim_{d\to 0} EN = \infty$.

(2) $\lim_{d\to 0} Nd^2 = K_\alpha^2/A^2$ a.s. *and* $\lim_{d\to 0} ENd^2 = K_\alpha^2/A^2$.

(3) $\lim_{d\to 0} P(L_N \leqq \theta \leqq U_N) = 1-2\alpha$.

PROOF.

(1) That N is well defined, i.e. finite a.s. for all $d > 0$ follows from (ii), while $EN < \infty$ for all $d > 0$ follows from (iv). The definition of N and the Monotone Convergence Theorem imply the remaining statements.

(2) By definition of N, one gets $U_N - L_N \leqq 2d$ and $U_{N-1} - L_{N-1} > 2d$. Using the fact that $N \to \infty$ a.s. as $d \to 0$ together with (ii), we obtain:

$$\liminf_{d\to 0} 2dN^{\frac{1}{2}} \geqq \liminf_{d\to 0} N^{\frac{1}{2}}(U_N - L_N) = 2K_\alpha/A \quad \text{a.s.}$$

$$\limsup_{d\to 0} 2d(N-1)^{\frac{1}{2}} \leqq \limsup_{d\to 0} (N-1)^{\frac{1}{2}}(U_{N-1} - L_{N-1}) = 2K_\alpha/A \quad \text{a.s.}$$

These together yield the first statement. The second statement follows immediately from (iv). See Loève [10] page 163.

(3) A theorem of Anscombe [1] together with the first statement of part (2) of the present theorem implies that Z_N has asymptotically a $N(0, 1)$ distribution as $d \to 0$. From (iii) it follows that the asymptotic distribution of $N^{\frac{1}{2}}(L_N - \theta)$ is $N(-K_\alpha/A, 1/A^2)$ as $d \to 0$ so that $\lim_{d\to 0} P(0 \leqq N^{\frac{1}{2}}(\theta - L_N) \leqq 2K_\alpha/A) = 1-2\alpha$. But $N^{\frac{1}{2}}(U_N - L_N) = 2K_\alpha/A + o_P(1)$ as $d \to 0$ by (ii) and the first part of (2), so that the statement follows.

3. A procedure based on the sign test. Suppose that $X_1, X_2, \cdots, X_n$ are observations from a population with unique median γ. For testing the hypothesis $\gamma = 0$, the sign test may be used. It is based on the statistic $\sum_{i=1}^n I(X_i > 0)$ (where $I(B)$ is the indicator function of the set B).

In the case of a sample of fixed size n, a confidence interval for γ can be derived from the sign test in a standard way. This confidence interval is of the form $(X_{n,b(n)}, X_{n,a(n)})$ where $X_{n,1} \leqq X_{n,2} \leqq \cdots \leqq X_{n,n}$ are the ordered X's and where $a(n)$ and $b(n)$ are integers depending on n. The limiting coverage probability as $n \to \infty$ of such a confidence interval is $1-2\alpha$ if

$$a(n) \sim n/2 + K_\alpha n^{\frac{1}{2}}/2$$
$$b(n) \sim n/2 - K_\alpha n^{\frac{1}{2}}/2.$$

From this confidence interval one can thus obtain a sequential procedure as indicated in Section 2.

DEFINITION 3.1. Procedure based on the sign test. Let N be the first integer $n \geqq n_0$ for which $X_{n,a(n)} - X_{n,b(n)} \leqq 2d$ and choose as resulting confidence interval $(X_{N,b(N)}, X_{N,a(N)})$ where $\{a(n)\}$ and $\{b(n)\}$ are sequences of positive integers satisfying Assumptions 3.1 below and n_0 is some integer.

A similar sequential procedure was introduced by Farrell [5].

The following assumptions will be used throughout the present section.

ASSUMPTIONS 3.1. $X_1, X_2, \cdots$ is a sequence of independent random variables with common cdf $F(x-\gamma)$, where $F(x)$ is symmetric about 0. F has two derivatives in a neighborhood if 0 and the second derivative is bounded in the neighborhood, so that also γ is the unique median of the X's. The sequences $a(n)$ and $b(n)$ are defined by

$$b(n) = \max\{1, [n/2 - K_\alpha n^{\frac{1}{2}}/2]\}$$
$$a(n) = n - b(n) + 1$$

where $[x]$ is the largest integer less than or equal to x.

We now show that the above procedure satisfies the Assumptions 2.1. Assume without loss of generality that $\gamma = 0$.

In what follows strong use is made of the following result of Bahadur [2]. Under the Assumptions 3.1

$$X_{n,k(n)} = [k(n)/n - F_n(\xi)]/f(\xi) + O(n^{-\frac{3}{4}} \log n) \quad \text{a.s.} \tag{3.1}$$

where $\{k(n)\}$ is a sequence of positive integers satisfying $k(n) = np + o(n^{\frac{1}{2}} \log n)$, $0 < p < 1$, $F(\xi) = p$, $F'(\xi) = f(\xi)$ and F_n is the empirical cdf of the X's.

LEMMA 3.1.

$$n^{\frac{1}{2}}(X_{n,a(n)} - X_{n,b(n)}) \to K_\alpha/f(0)\,\text{a.s.} \quad \textit{as} \quad n \to \infty.$$

PROOF. It follows immediately from (3.1) that

$$n^{\frac{1}{2}}(X_{n,a(n)} - X_{n,b(n)}) = \frac{a(n)-b(n)}{n^{\frac{1}{2}}}\frac{1}{f(0)} + O(n^{-\frac{1}{4}}\log n) \quad \text{a.s.}$$

$$\to K_\alpha/f(0) \quad \text{as} \quad n \to \infty \quad \text{a.s.}$$

LEMMA 3.2.

$$n^{\frac{1}{2}}X_{n,b(n)} = Z_n/A - K_\alpha/A + o(1)\,\text{a.s.} \quad \textit{as} \quad n \to \infty$$

where $A = -2f(0)\ Z_n = 2n^{\frac{1}{2}}(F_n(0) - \frac{1}{2})$.

This follows immediately from (3.1).

LEMMA 3.3. *The set* $\{Nd^2\}_{d>0}$ *is uniformly integrable.*

PROOF. According to a result of Bickel and Yahav ([3] Lemma 3.2) it is sufficient to prove that $\sum_{m=1}^{\infty} \sup_{0<d<d_0} P[N(d)d^2 > m] < \infty$ for some d_0. Now

$$\begin{aligned} P[N(d)d^2 > m] &= P[N(d) > m(d)] \quad \text{where} \quad m(d) = [m/d^2] \\ &\leqq P[X_{m(d),a(m(d))} - X_{m(d),b(m(d))} > 2d] \\ &\leqq P[S_{m(d)}(d) < a(m(d))] + P[S_{m(d)}(-d) \geqq b(m(d))] \\ &\quad \text{where} \quad S_n(t) = \textstyle\sum_{i=1}^n I(X_i \leqq t) \\ &= 2P[B(m(d), F(-d)) \geqq b(m(d))] \end{aligned}$$

where $B(n, p)$ denotes a binomial random variable with parameters n and p. Then (see Hoeffding [9] Theorem 1)

$$P[N(d)d^2 > m] \leqq 2\exp\{-2m(d)t_{m(d)}^2\} \qquad \text{where}$$

$$\begin{aligned} t_{m(d)} &= \{b(m(d)) - m(d)F(-d)\}/m(d) \\ &\geqq d\delta \end{aligned}$$

for some $\delta > 0$ for $d \leqq d_0$ and $m \geqq M_0$ where $d_0 > 0$ and M is sufficiently large. Hence for $d \leqq d_0$ and $m \geqq M_0$ and a constant A

$$P[N(d)d^2 > m] \leqq A\exp\{-2m\delta^2\}$$

which proves the uniform integrability since $\sum_{m=M_0}^{\infty} \exp\{-2m\delta^2\} < \infty$.

The following theorem is now a direct consequence of Theorem 2.1 and the above lemmas:

THEOREM 3.1. *The confidence interval procedure based on the sign test* (*as defined in Definition* 3.1) *has asymptotic coverage probability* $1-2\alpha$ *as* $d \to 0$. *The stopping variable* N *satisfies* $\lim_{d\to 0} ENd^2 = \frac{1}{4}K_\alpha{}^2/f^2(0)$.

4. A procedure based on the Wilcoxon one-sample test. The Wilcoxon one-sample test is based on the statistic

$$\sum_{i=1}^{n} \sum_{j=i}^{n} I(\tfrac{1}{2}(X_i+X_j) > 0)$$

where $X_1, \cdots, X_n$ is a sample from a symmetric distribution with center of symmetry γ. The test is used to test the hypothesis $\gamma = 0$ against shift alternatives.

A confidence interval for γ in the case of a sample of fixed size n, derived from the above test in the standard way, is of the form $(Z_{n,b(n)}, Z_{n,a(n)})$ where $Z_{n,1} \leqq Z_{n,2} \leqq \cdots \leqq Z_{n,n(n+1)/2}$ are ordered averages $\frac{1}{2}(X_i+X_j)$ for $i, j = 1, \cdots, n$ and $i \leqq j$.

The limiting coverage probability of such an interval is $1-2\alpha$ if

$$a(n) \sim n(n+1)/4 + K_\alpha[n(n+1)(2n+1)/24]^{\frac{1}{2}} \tag{4.1}$$
$$b(n) \sim n(n+1)/4 - K_\alpha[n(n+1)(2n+1)/24]^{\frac{1}{2}}.$$

We can now define a sequential procedure from such a confidence interval.

DEFINITION 4.1. Procedure based on the Wilcoxon one-sample test: let N be the first integer $n \geqq n_0$ for which $Z_{n,a(n)} - Z_{n,b(n)} \leqq 2d$ and choose as resulting confidence interval $(Z_{N,b(N)}, Z_{N,a(N)})$. $\{a(n)\}$ and $\{b(n)\}$ are sequences of positive integers satisfying (4.1) and n_0 is some positive integer.

The asymptotic analysis of this procedure is much more complicated than the analysis in Section 3, because the present procedure is based on ordered dependent random variables, namely the ordered $\frac{1}{2}(X_i+X_j)$ for $i \leqq j$, $i, j = 1, \cdots, n$.

Fortunately the theory of U-statistics can be applied. See Hoeffding [8] and [9].

The statistic

$$\binom{n}{2}^{-1} \sum_{i=1}^{n} \sum_{j=i+1}^{n} I(\tfrac{1}{2}(X_i+X_j) > 0) \tag{4.2}$$

is a one-sample U-statistic and the test based on it is asymptotically equivalent to the Wilcoxon one-sample test. For these reasons a procedure based on (4.2) will be considered first. It is defined as follows:

DEFINITION 4.2. Procedure based on a test which is asymptotically equivalent to the Wilcoxon one-sample test: let N be the first integer $n \geqq n_0$ for which $W_{n,a(n)} - W_{n,b(n)} \leqq 2d$ and choose as resulting confidence interval $(W_{N,b(N)}, W_{N,a(N)})$. $\{a(n)\}$ and $\{b(n)\}$ are sequences of integers satisfying (4.1), n_0 is some positive integer and $W_{n,1} \leqq W_{n,2} \leqq \cdots \leqq W_{n,n(n-1)/2}$ are the ordered averages $\frac{1}{2}(X_i+X_j)$ for $i < j$ and $i, j = 1, \cdots, n$.

Except for the last remark of this section, the procedure to which reference will be made in this section is that of Definition 4.2.

The following assumptions will be used throughout the present section.

ASSUMPTIONS 4.1. $X_1, X_2, \cdots$ is a sequence of independent random variables with common cdf $F(x-\gamma)$, where F is symmetric about 0. F has a density f which satisfies $\int f^2(x)\,dx < \infty$. $G(x-\gamma)$ denotes the cdf of $\frac{1}{2}(X_1+X_2)$ and G has a second derivative in some neighborhood of 0 with G'' bounded in the neighborhood. G'

is sometimes denoted by g where it exists. $\{a(n)\}$ and $\{b(n)\}$ are sequences of positive integers defined by

$$b(n) = \max\{1, [n(n+1)/4 - K_\alpha(n(n+1)(2n+1/24)^{\frac{1}{2}}]\}$$
$$a(n) = n(n+1)/2 - b(n) + 1$$

where $[x]$ is the largest integer less than or equal to x.

REMARKS. The following facts can be established without difficulty (for details here and elsewhere in this paper, refer to [6]).

(1) The above assumptions on F guarantee the existence of a derivative for G.

(2) If f has a Radon–Nikodym derivative f' satisfying $\int |f'| < \infty$ and $\int (f')^2 < \infty$, then the assumptions on G are satisfied.

(3) Assumptions 4.1 imply that $G'(0) > 0$, since

$$G'(0) = 2\int f^2(x)\,dx. \tag{4.3}$$

In the rest of this section it will be assumed without loss of generality that $\gamma = 0$. The next lemma states two inequalities derived by Hoeffding [9] for one-sample U-statistics.

LEMMA 4.1. *Denote by U the statistic*

$$\binom{n}{r}^{-1}\sum\nolimits_C \psi(X_{i_1}, \cdots, X_{i_r})$$

where $X_1, X_2, \cdots, X_n$ are i.i.d. *random variables, ψ is a function symmetric in its r variables and C denotes summation over all $\binom{n}{r}$ combinations of rX's. Then*

$$P(U - EU \geqq t) \leqq \exp\{-2kt^2\} \tag{4.4}$$

if $0 \leqq \psi(x_1, \cdots, x_r) \leqq 1$, $t \geqq 0$, and $k = [n/r]$, the largest integer less than or equal to n/r. Also

$$P(|U - EU| \geqq t) \leqq 2e^{-h} \tag{4.5}$$

where

$$h = \frac{kt^2}{2(\sigma^2 + \frac{1}{3}t\max(z, 1-z))}$$

if $|\psi(x_1, \cdots, x_r)| \leqq 1$, $t \geqq 0$, $z = E\psi(X_1, \cdots, X_r)$, $\sigma^2 = \operatorname{Var}\psi(X_1, \cdots, X_r)$ and k is as above.

(4.5) is an extension of the so-called Bernstein inequality (stated by Hoeffding [9] for sums of independent random variables) using the methods indicated by Hoeffding.

In the following lemma it is proved that Bahadur's result [2] can be extended to apply also to the ordered averages of pairs of observations that are being considered. The proof follows closely the method of proof of Bahadur.

LEMMA 4.2. *If* $\{k(n)\}$ *is any sequence of integers satisfying* $k(n)/\binom{n}{2} = \frac{1}{2}+o(n^{-\frac{1}{2}}\log n)$ *as* $n \to \infty$ *then, with probability one,*

$$W_{n,k(n)} = \frac{k(n)/\binom{n}{2}-G_n(0)}{g(0)} + O(n^{-\frac{3}{4}}\log n) \tag{4.6}$$

where

$$G_n(x) = \binom{n}{2}^{-1}\textstyle\sum_{i<j} I(\frac{1}{2}(X_i+X_j) \leqq x).$$

PROOF. Let $B_n(x) = [G_n(x)-G_n(0)]-[G(x)-G(0)]$. $\{c_n\}$ is a sequence of constants with $c_n \sim \log n/n^{\frac{1}{4}}$ as $n \to \infty$. $I_n = (-c_n, c_n)$, $H_n = \sup\{|B_n(x)| : x \in I_n\}$. First prove that

$$H_n = O(n^{-\frac{3}{4}}\log n) \quad \text{a.s.} \tag{4.7}$$

Let $\{d_n\}$ be a sequence of positive integers with $d_n \sim n^{\frac{1}{2}}$. Put $\eta_{r,n} = rc_n/d_n$ where r is an integer

$$J_{r,n} = [\eta_{r,n}, \eta_{r+1,n}] \qquad \alpha_{r,n} = G(\eta_{r+1,n})-G(\eta_{r,n}).$$

Then, for $x \in J_{r,n}$, since G_n and G are nondecreasing,

$$\begin{aligned} B_n(x) &\leqq G_n(\eta_{r+1,n})-G_n(0)-G(\eta_{r,n})+G(0) \\ &= B_n(\eta_{r+1,n})+\alpha_{r,n}. \end{aligned}$$

Similarly $B_n(x) \geqq B_n(\eta_{r,n})-\alpha_{r,n}$ for x in $J_{r,n}$. Therefore

$$\begin{aligned} H_n &\leqq \max\{|B_n(\eta_{r,n})| : -d_n \leqq r \leqq d_n\}+\max\{\alpha_{r,n} : -d_n \leqq r \leqq d_n-1\} \\ &= K_n+\beta_n \quad \text{say.} \end{aligned}$$

Now

$$\begin{aligned} \beta_n &= \max_r[G(\eta_{r+1,n})-G(\eta_{r,n})] \\ &= \max_r(\eta_{r+1,n}-\eta_{r,n})G'(\zeta_{r,n}) \end{aligned}$$

where

$$\begin{aligned} \eta_{r,n} \leqq \zeta_{r,n} \leqq \eta_{r+1,n} &= (c_n/d_n)\max_r G'(\zeta_{r,n}) \\ &= O(n^{-\frac{3}{4}}\log n) \end{aligned}$$

because G' is continuous, hence bounded in a neighborhood of 0.

By the Borel–Cantelli Lemma it is now sufficient, in order to complete the proof of (4.7) to prove that for some constant p_1 $\sum P(K_n \geqq \xi_n) < \infty$ where $\xi_n = p_1 n^{-\frac{3}{4}}\log n$. First consider $P(|B_n(\eta_{r,n})| \geqq \xi_n)$. $|B_n(\eta_{r,n})|$ can be written in the form $|U_n-EU_n|$ where

$$\begin{aligned} U_n &= |G_n(\eta_{r,n})-G_n(0)| \\ &= \binom{n}{2}^{-1}\textstyle\sum_{i<j}|I(\frac{1}{2}(X_i+X_j) \leqq \eta_{r,n})-I(\frac{1}{2}(X_i+X_j) \leqq 0)|, \end{aligned}$$

a one-sample U-statistic with $EU_n = |G(\eta_{r,n})-G(0)| = z_{r,n}$, say. Applying inequality (4.5) and noting that $h \geqq \frac{1}{2}kt^2/(z_{r,n}+t)$, one obtains $P(|B_n(\eta_{r,n})| \geqq \xi_n) \leqq 2\exp\{-\delta_n\}$ where $\delta_n = \frac{1}{2}[n/2]\xi_n{}^2/(z_{r,n}+\xi_n)$. Fix p_2 such that $G'(0) < p_2$. Then since

$\lim_{n\to\infty} \{G(c_n)-G(0)\}/c_n = G'(0)$, there is an integer n_1 such that for $n > n_1$ $G(c_n)-G(0) < p_2 c_n$ and also $G(0)-G(-c_n) < p_2 c_n$. From $\eta_{r,n} = r(c_n/d_n) \leqq c_n$ and $\eta_{r,n} \geqq -c_n$ then follows that $z_{r,n} < p_2 c_n$ because G is monotone nondecreasing. One now finds $\delta_n \geqq \frac{1}{8}n\xi_n^2/(p_2 c_n + \xi_n) = \delta_n'$ say, for n large. Then $P(K_n \geqq \xi_n) \leqq \sum_{r=-d_n}^{d_n} P(|B_n(\eta_{r,n})| \geqq \xi_n) \leqq 2(2d_n+1)\exp\{-\delta_n'\}$ and

$$\begin{aligned}\frac{\log P(K_n \geqq \xi_n)}{\log n} &\leqq \frac{1}{4} - \frac{p_1^2}{16p_2} \quad \text{for } n \text{ large}\\ &< -1-\delta \quad \text{for some } \delta > 0 \text{ and for } p_1 \text{ large enough.}\end{aligned}$$

This proves $\sum P(K_n \geqq \xi_n) < \infty$ and so (4.7) has been established.

The second part of the proof consists in proving that

(4.8) $\qquad W_{n,k(n)}$ is in I_n with probability one for n large enough.

Now

$$\text{(4.9)} \qquad \begin{aligned} P(W_{n,k(n)} \leqq -c_n) &= P[G_n(-c_n) \geqq k(n)/m] \quad \text{with} \quad m = \tbinom{n}{2}\\ &\leqq \exp\{-nt_n^2/2\} \quad \text{by} \quad (4.4),\end{aligned}$$

provided $t_n = k(n)/m - G(-c_n) \geqq 0$. But

$$\begin{aligned} G(-c_n) &= G(0) - c_n G'(0) + c_n^2 G''(\zeta_n)/2 \quad \text{where} \quad -c_n \leqq \zeta_n \leqq 0\\ &= G(0) - c_n G'(0) + o(c_n)\end{aligned}$$

because G'' is bounded in a neighborhood of 0. Therefore $t_n \sim n^{-\frac{1}{2}} \log n G'(0)$ as $n \to \infty$ which is positive, since $G'(0) > 0$. The right-hand side of (4.9) becomes $\exp\{-\frac{1}{2}(G'(0)\log n)^2\}$ so that $\sum P(W_{n,k(n)} < -c_n) < \infty$. Similarly $\sum P(W_{n,k(n)} > c_n) < \infty$. This proves (4.8).

The proof of (4.6) can now be completed. By (4.8), with probability one, there is an n_2 such that $W_{n,k(n)}$ is in I_n for $n > n_2$. Also, n_2 can be chosen such that for $n > n_2$ $G''(0)$ is defined and bounded on I_n. Let p_3 be such that

$$\tfrac{1}{2}|G''(x)| \leqq p_3 \quad \text{for} \quad x \quad \text{in} \quad I_n \quad \text{and} \quad n > n_2.$$

Now $G_n(W_{n,k(n)}) = k_n/\binom{n}{2}$ so that, by definition of B_n,

$$\text{(4.10)} \qquad k(n)/\tbinom{n}{2} = G_n(0) + G(W_{n,k(n)}) - G(0) + \theta_n H_n \quad \text{with} \quad |\theta_n| \leqq 1.$$

But

$$\begin{aligned} G(W_{n,k(n)}) &= G(0) + W_{n,k(n)} G'(0) + W_{n,k(n)}^2 G''(\zeta_n)/2\\ &\qquad \text{with} \quad \zeta_n \quad \text{between} \quad 0 \quad \text{and} \quad W_{n,k(n)}\\ &= G(0) + W_{n,k(n)} G'(0) + p_3 \phi_n c_n^2 \quad \text{with} \quad |\phi_n| \leqq 1.\end{aligned}$$

Substitute in (4.10). Then one obtains

$$k_n/\tbinom{n}{2} = G_n(0) + W_{n,k(n)} g(0) + O(n^{-\frac{3}{4}} \log n) \text{ a.s.}$$

This completes the proof.

COROLLARY 4.1.

$$n^{\frac{1}{2}}(W_{n,a(n)} - W_{n,b(n)}) \to \frac{K_\alpha}{3^{\frac{1}{2}} \int f^2(x)\,dx} \text{ a.s.} \quad as \quad n \to \infty.$$

PROOF. The result follows at once from Lemma 4.2. in conjunction with (4.1) and (4.3).

LEMMA 4.3.

$$n^{\frac{1}{2}} W_{n,b(n)} = Z_n/A - K_\alpha/A + o(1) \text{ a.s.} \quad as \quad n \to \infty$$

where $A = 3^{\frac{1}{2}} g(0)$, $Z_n = 12^{\frac{1}{2}} n^{\frac{1}{2}}[n^{-1} \sum_{i=1}^n F(X_i) - \frac{1}{2}]$.

This result follows from Lemma 4.2 together with the following lemma and its corollary (which were kindly supplied by the referee) and on noting that $G_n(0)$ is a U-statistic.

LEMMA 4.4.

Let $V_m, V_{m+1}, \cdots$ *be a reverse martingale with* $EV_m = 0$. *Suppose for some* $r \geqq 1$ *and some* $s > 0$ *that* $E|V_n^r| = o(n^{-s})$. *Then*

$$V_n = o(n^{-s/r}(\log n)^\varepsilon) \text{ a.s.} \quad for\ any \quad \varepsilon > 1/r.$$

PROOF.

For $l \geqq m$ we have

$$P(|V_n| > \lambda \quad \text{for some} \quad n \geqq l) \leqq \lambda^{-r} E|V_l|^r \qquad (\lambda > 0).$$

(See [10], page 391), and hence, by assumption,

$$P(|V_n| > \lambda \quad \text{for some} \quad n \geqq l) \leqq \lambda^{-r} O(l^{-s}).$$

Letting $\eta > 0$ be arbitrary and $\lambda = \eta(2l)^{-s/r}(\log l)^\varepsilon$ we see there is an $M > 0$ such that for $l > m$, $P(|V_n| > \eta n^{-s/r}(\log n)^\varepsilon$ for some $n = l, l+1, \cdots, 2l-1) \leqq \eta^{-r} M(\log l)^{-\varepsilon r}$.

Since $\sum_{l=2,4,8,\ldots} (\log l)^{-\varepsilon r} < \infty$ for $\varepsilon r > 1$ and η arbitrary, the Borel–Cantelli lemma implies $V_n = o(n^{-s/r}(\log n)^\varepsilon)$ a.s. for any $\varepsilon > 1/r$.

COROLLARY 4.2. *Let* $U_m, U_{m+1}, \cdots (m \geqq 1)$ *be a sequence of U-statistics generated by the* i.i.d. *random variables* $X_1, X_2, \cdots$. *If* $EU_m^2 < \infty$, *then there exists a function* $h(x)$ *with* $E(h(X_1))^2 < \infty$ *such that*

$$U_n = n^{-1} \sum_{j=1}^n h(X_j) + o(n^{-1} \log n) \quad \text{a.s.}$$

PROOF. Let $h(x) = mE(U_m \mid X_1 = x) - (m-1)EU_m$; then $Eh(X_1) = EU_m$ and $E(h(X_1))^2 \leqq m^2 EU_m^2 < \infty$. Now $V_n = U_n - n^{-1} \sum_{j=1}^n h(X_j)$ $n = m, m+1, \cdots$ is a reverse martingale, and from Hoeffding [8], $EV_n^2 = O(n^{-2})$. Letting $r = s = 2$ in Lemma 4.4, we obtain a slightly stronger conclusion than the required.

LEMMA 4.5. $\{Nd^2\}_{d>0}$ *is uniformly integrable.*

The proof is similar to that given in Lemma 3.3.

REMARK. From the fact that

$$\tilde{G}_n(x) = G_n(x) + O(1/n),$$

where

$$\tilde{G}_n(x) = \frac{2}{n(n+1)} \sum_{i \leq j} I\left(\frac{X_i + X_j}{2} \leq x\right)$$

one can prove that all the results of this section have exact analogues for the procedure of Definition 4.1.

The following theorem now follows at once from Theorem 2.1 and the above results and remarks.

THEOREM 4.1. *Both confidence interval procedures based on the Wilcoxon one-sample test (as defined in Definitions* 4.1 *and* 4.2) *have asymptotic coverage probability* $1-2\alpha$ *as* $d \to 0$. *The stopping variables* N *satisfy*:

$$\lim_{d \to 0} ENd^2 = \frac{K_\alpha^{\,2}}{12(\int f^2(x)\,dx)^2}\,.$$

EXTENSION. The theory of this section can be extended immediately to a procedure based on the Wilcoxon two-sample test where observations are then taken in pairs.

5. Asymptotic efficiencies of the procedures. Consider two bounded length confidence interval procedures, T and S, for estimating the mean of symmetric population by means of an interval of prescribed length $2d$. Denote by N_T and N_S the stopping variables of the two procedures T and S respectively, and by P_T and P_S the respective coverage probabilities.

DEFINITION 5.1. The asymptotic efficiency as $d \to 0$ of procedure T relative to procedure S is $e(T, S) = \lim_{d\to 0} EN_S/EN_T$ provided $\lim_{d \to 0} P_T = \lim_{d \to 0} P_S$ and that all the limits exist.

Denote by M the procedure of Chow and Robbins mentioned in Section 1 and by S and W the procedures of Definitions 3.1 and 4.1.

Then it follows from [4] and Theorems 3.1 and 4.1 that the asymptotic efficiencies of the above procedures relative to each other in the sense of Definition 5.1 are (under the Assumptions 3.1, 4.1 and $\sigma^2 < \infty$)

$$e(S, M) = 4\sigma^2 f^2(0)$$

$$e(W, M) = 12\sigma^2(\int f^2(x)\,dx)^2$$

$$e(S, W) = \frac{f^2(0)}{3(\int f^2(x)\,dx)^2}\,.$$

If one regards the procedures M, S and W as based on the t-test, sign test and the Wilcoxon one-sample test respectively, one sees that the above efficiencies are the same as the Pitman-efficiencies of the respective tests relative to each other.

6. Monte Carlo studies. A series of Monte Carlo studies were performed for different values of d and for a few symmetric populations (normal, "contaminated normal," uniform and double exponential) to compare the behavior of the procedures with the asymptotic results. These results are fully discussed in [7].

The studies seem to indicate that the actual coverage probability is quite close to the asymptotic coverage probability and that the actual coverage probability for the two procedures which are based on rank tests is higher than that of the procedure based on the t-test.

The results also suggest that

$$EN \leqq \frac{K_\alpha^{\ 2}}{3[G(d)-\frac{1}{2}]^2} + C$$

in the case of the procedure based on the Wilcoxon test, where C is a constant. The results illustrate the upper bound.

$$EN \leqq K_\alpha^{\ 2}\sigma^2/d^2 + n_0$$

for the procedure based on the t-test (see Simons [11]).

Acknowledgment. The content of this paper forms part of the author's Ph.D. dissertation, submitted to the University of California, Berkeley. The author wishes to express his gratitude to his adviser Professor Peter J. Bickel, for suggesting the problem and for his guidance. The author also thanks the referee for his comments and suggestions.

REFERENCES

[1] ANSCOMBE, F. J. (1952). Large sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

[2] BAHADUR, R. R. (1966). A note on quantiles in large samples. *Ann. Math. Statist.* **37** 577–580.

[3] BICKEL, P. J. and YAHAV, J. A. (1968). Asymptotically optimal Bayes and minimax procedures in sequential estimation. *Ann. Math. Statist.* **39** 442–456.

[4] CHOW, Y. S. and ROBBINS, H. (1965). On the asymptotic theory of fixed-width sequential confidence intervals for the mean. *Ann. Math. Statist.* **36** 457–462.

[5] FARRELL, R. H. (1966). Bounded length confidence intervals for the p-point of a distribution function III. *Ann. Math. Statist.* **37** 586–592.

[6] GEERTSEMA, J. C. (1968). Sequential confidence intervals based on rank tests. Unpublished Ph.D. dissertation, University of California, Berkeley.

[7] GEERTSEMA, J. C. (1970). A Monte Carlo study of sequential confidence intervals based on rank tests. *S. Afr. Statist. J.* **4**.

[8] HOEFFDING, W. (1948). A class of statistics with asymptotically normal distributions. *Ann. Math. Statist.* **19** 293–325.

[9] HOEFFDING, W. (1963). Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** 13–30.

[10] LOÈVE, M. (1962). *Probability Theory* (3rd ed.) Van Nostrand, Princeton.

[11] SIMONS, G. (1968). On the cost of not knowing the variance when making a fixed width confidence interval for the mean. *Ann. Math. Statist.* **39** 1946–1952.

[12] STARR, N. (1966). The performance of a sequential procedure for the fixed-width estimation of the mean. *Ann. Math. Statist.* **37** 36–50.

[13] STEIN, C. (1945). A two-sample test for a linear hypothesis whose power is independent of the variance. *Ann. Math. Statist.* **16** 243–258.