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STOPPING TIME OF A RANK-ORDER SEQUENTIAL PROBABILITY RATIO TEST BASED ON LEHMANN ALTERNATIVES II[1]

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1. Summary. We are motivated by Stein's proof (Stein (1946), Wald (1947), pages 157–158) of the termination of a sequential probability ratio test in the case of independent and identically distributed random variables. Extending his ideas to take certain "dependencies" into account we examine the rank-order sequential probability ratio test based on a Lehmann alternative studied in a paper with the above title by I. R. Savage and the author (1966) (referred to as SS I in the rest of this paper). We prove that this test terminates with probability one and that the stopping time has a finite moment generating function under a very mild condition on the bivariate random variables which resembles the Stein-condition, namely that a certain random variable $V(X_1, Y_1)$, defined in (32), is not identically equal to 0. Finally the asymptotic normality of the logarithm of the likelihood ratio of the rank order is established using the well-known Chernoff–Savage Theorem.

2. Notation, test procedure and preliminaries. $(X_1, Y_1), \cdots, (X_n, Y_n), \cdots$ are independently and identically distributed bivariate random variables with a joint distribution function $H(\cdot, \cdot)$ which has marginal distribution functions $F(\cdot)$ and $G(\cdot)$. All distribution functions are taken to be right continuous in this paper. We wish to test the null hypothesis $H_0: (X_1, Y_1)$ *are independent, and* $G = F$ *continuous* against the alternative hypothesis $H_1: (X_1, Y_1)$ *are independent, and* $G = F^A$, *with* F *continuous* where $A > 0$, $A \neq 1$ is a known constant. The nth stage of experimentation yields $\mathbf{Z}(n) = ((X_1, Y_1), \cdots, (X_n, Y_n))$, though only the ordered ranks $(s_1, \cdots, s_n)$ of $(Y_1, \cdots, Y_n)$ in the pooled sample $(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ are available for testing. We use the sequential probability ratio test based on $(s_1, \cdots, s_n)$ which is described later in (2). This test has been studied in our earlier paper, SS I.

Let $F_n(\cdot)$, $F_{n,r}(\cdot)$, $G_n(\cdot)$ and $G_{n,r}(\cdot)$ be the (right-continuous) empirical distribution functions of $(X_1, \cdots, X_n)$, $(X_{n+1}, \cdots, X_{n+r})$, $(Y_1, \cdots, Y_n)$ and $(Y_{n+1}, \cdots, Y_{n+r})$, respectively. Let $I(x; z) = 1$ if $x \leqq z$, $= 0$ if $x > z$. The above empirical distribution functions may be written in terms of $I(x; z)$ as follows:

$$F_n(z) = \textstyle\sum_1^n I(X_i; z)/n, \qquad F_{n,r}(z) = \sum_{n+1}^{n+r} I(X_i; z)/r \quad \text{etc.}$$

Let $W(z) = F(z) + AG(z)$, $W_n(z) = F_n(z) + AG_n(z)$ and $W_{n,r}(z) = F_{n,r}(z) + AG_{n,r}(z)$. Again, if $W(x, y; z) = I(x; z) + AI(y; z)$ then $W_n(z) = \sum_1^n W(X_i, Y_i; z)/n$, etc.

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Let $L(A, F_n, G_n) = P_{H_1}(s_1, \cdots, s_n)/P_{H_0}(s_1, \cdots, s_n)$. From relation (2) in SS I or directly from Savage (1956, Corollary 7.a.1) we have the basic relation

$$L(A, F_n, G_n) = A^n(2n)!/[n^{2n}\textstyle\prod_1^n (W_n(X_i)W_n(Y_i))]. \tag{1}$$

Let $l(n) = \log L(A, F_n, G_n)$. The rank-order sequential probability ratio test based on a Lehmann alternative may be described as follows:

$$\begin{aligned}&\text{'Take one more observation if} \quad a < l(n) < b, \quad \text{accept} \quad H_0\\ &\text{if} \quad l(n) \leqq a, \quad \text{reject} \quad H_0 \quad \text{if} \quad l(n) \geqq b\,;\, n = 1, 2, \cdots,\\ &\text{where} \quad a < 0 < b \quad \text{are suitable constants.'}\end{aligned} \tag{2}$$

The number of states before termination, N, is defined to be

$$\begin{aligned}&r \quad \text{if} \quad a < l(n) < b, n = 1, \cdots, r-1 \quad \text{and} \quad l(n) \leqq a \quad \text{or}\\ &\qquad l(n) \geqq b \quad \text{for} \quad n = r\\ &\infty \quad \text{if} \quad a < l(n) < b \quad \text{for all} \quad n.\end{aligned} \tag{3}$$

Our final aim is to prove that $P(N < \infty) = 1$ and $E(e^{tN}) < \infty$ for some positive t under the condition that $V(X_1, Y_1)$ (defined in (32) and (33)) is not equal to zero with probability one. The proof is rather long and it runs into several sections. In Section 3 we describe our generalizations of Stein's proof (Stein (1946), Wald (1947) page 157) of the termination of a sequential test. Section 4 and Section 5 are devoted to obtain several inequalities and to estimate several probabilities. Section 6 gathers all these results and proves the main theorem. Section 7 presents a discussion and a comparison with the result in SS I and contains our best result in the form of Theorem 6. Section 8 proves the asymptotic normality of $l(n)$, the logarithm of the likelihood ratio of the ranks.

3. General theorems on the termination of a sequential test. We are familiar with Stein's proof (Stein (1946), Wald (1947) page 157) of the termination of the Wald sequential test based on sums of independent random variables. We give here three theorems which apply to more general cases of sequential tests.

Let $\mathbf{Z}(n)$ be the vector of variables observed up to the nth stage of experimentation. Let $l(n)$ be a statistic based on $\mathbf{Z}(n)$, $n = 1, 2, \cdots$. The general sequential test, ST(l; a, b), we are referring to is of the form:

"Take one more observation if $a < l(n) < b$; stop and decide for or against the hypothesis if $l(n) \leqq a$ or $l(n) \geqq b$; $n = 1, 2, \cdots$," where $a < 0 < b$ are fixed constants. The number of stages before termination, N, is r if $a < l(n) < b$ for $n = 1, \cdots, r-1$ and $l(n) \leqq a$ or $l(n) \geqq b$ for $n = r$ and is ∞ if $a < l(n) < b$ for all n.

We shall say that the sequential test *terminates and has a finite moment generating function under P* if $P(N < \infty) = 1$ and $E(e^{tN}) < \infty$ for some $t > 0$. (Here E stands for expectation under P.)

THEOREM 1. *Let $K = b-a$. Suppose that there exists a $\theta > 0$ and positive integers m_0 and k such that*

$$P\{l(mk) - l((m-1)k) \leqq -K \,|\, \mathbf{Z}((m-1)k)\} \geqq \theta \tag{4}$$

with probability one for all $m \geqq m_0$ *then the sequential test* ST(l; a, b) *terminates and has a moment generating function under* P. (*Here* $P(\cdot \mid \cdot)$ *denotes conditional probability.*)

PROOF. Let m_1 and r be positive integers with $m_1 \geqq m_0$.

$$\begin{aligned}\{N > (m_1+r)k\} &= \{a < l(n) < b, n = 1, \cdots, (m_1+r)k\} \\ &\subset \{l(mk)-l((m-1)k) > -K, m = m_1+1, \cdots, m_1+r\} \\ &= E(-K; m_1, r) \quad \text{say.}\end{aligned} \tag{5}$$

$E(-K; m_1, 0)$ will stand for the whole sample space of $\mathbf{Z}(m_1k)$. Let I_E denote the indicator function of E. Now,

$$\begin{aligned}P\{E(-K; m_1, r)\} &= E\{I_{E(-K;m_1,r-1)}P\{l((m_1+r)k)-l((m_1+r-1)k) \\ &\qquad > -K \mid \mathbf{Z}((m_1+r-1)k)\}\} \\ &\leqq (1-\theta)P\{E(-K; m_1, r-1)\}\end{aligned} \tag{6}$$

from (4). Applying (6) repeatedly and using (5) we have

$$P\{N > (m_1+r)k\} \leqq (1-\theta)^r \tag{7}$$

if $m_1 \geqq m_0$. Putting $r = m_1$ we have, for $m_1 \geqq m_0$,

$$P\{N \geqq 2m_1 k\} \leqq (1-\theta)^{m_1}. \tag{8}$$

This implies that $P(N < \infty) = 1$ and $E(e^{tN}) < \infty$ for some positive $t > 0$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $K = b-a$. *Let there exist positive integers* m_0 *and* k, *a number* $\theta > 0$ *and events* $A(mk)$ *and* $B(mk)$ *defined on the random variables* $\mathbf{Z}(mk)$ *for* $m \geqq m_0-1$ *such that, for* $m \geqq m_0$,

$$P\{l(mk)-l((m-1)k) \leqq -K \mid \mathbf{Z}((m-1)k)\} \geqq \theta - P\{\bar{B}(mk) \mid \mathbf{Z}((m-1)k)\} \tag{9}$$

almost everywhere for $\mathbf{Z}((m-1)k)$ *in* $A((m-1)k)$. *Further let there be numbers* A_1, A_2, ρ_1, ρ_2 *with* $0 \leqq A_1, A_2 < \infty$ *and* $0 \leqq \rho_1, \rho_2 < 1$ *such that, for* $m \geqq m_0$,

$$P\{\bar{A}(mk)\} \leqq A_1 \rho_1^{mk}, \qquad P\{\bar{B}(mk)\} \leqq A_2 \rho_2^{mk} \tag{10}$$

where for any event A, $\bar{A}$ *denotes its complement.*

Then, the sequential test ST(l; a, b) *terminates and has a finite moment generating function under* P.

PROOF. As in the previous theorem let m_1, r be positive integers with $m_1 \geqq m_0$. With the same notation as in Theorem 1

$$\begin{aligned}P\{E(-K; m_1, r)\} &\\ &= E\{I_{E(-K;m_1,r-1)}(I_{A((m_1+r-1)k)}+I_{\bar{A}((m_1+r-1)k)}) \cdot \\ &\qquad \cdot P\{l((m_1+r)k)-l((m_1+r-1)k) > -K \mid \mathbf{Z}((m_1+r-1)k)\}\} \\ &\leqq E\{I_{E(-K;m_1,r-1)}[1-\theta+P\{\bar{B}((m_1+r)k) \mid \mathbf{Z}((m_1+r-1)k)\}]\} \\ &\qquad + P(\bar{A}((m_1+r-1)k)) \\ &\leqq (1-\theta)P\{E(-K; m_1, r-1)\}+P\{\bar{B}((m_1+r)k)\}+P\{\bar{A}((m_1+r-1)k)\} \\ &\leqq (1-\theta)P\{E(-K; m_1, r-1)\}+A_2\rho_2^{(m_1+r)k}+A_1\rho_1^{(m_1+r-1)k}.\end{aligned} \tag{11}$$

Applying (11) repeatedly,

$$P\{E(-K;m_1,r)\} \leqq (1-\theta)^r + \frac{A_2\rho_2^{m_1k+k}}{1-\rho_2^k} + \frac{A_1\rho_1^{m_1k}}{1-\rho_1^k}.$$

Putting $r = m_1$, we have for all $m_1 \geqq m_0$,

$$P\{N > 2km_1\} \leqq (1-\theta)^{m_1} + \frac{A_2\rho_2^{m_1k+k}}{1-\rho_2^k} + \frac{A_1\rho_1^{m_1k}}{1-\rho_1^k}. \tag{12}$$

Inequality (12) establishes the theorem.

We state the following theorem without proof since it is now immediate. The reason for not stating the previous theorems in this form is that our application (see Section 6) does not utilize this form of the result.

THEOREM 3. *Theorem* 1 *and Theorem* 2 *remain valid if in* (4) *and* (9), *respectively, the event* $\{l(mk)-l((m-1)k) \leqq -K\}$ *is replaced by the event* $\{|l(mk)-l((m-1)k)| \geqq K\}$.

4. Preliminary inequalities. Going back to the definition of $l(n)$ and (1) and applying Stirling's formula we have

$$l(n) = \textstyle\sum_1^n [-\log(W_n(X_i)W_n(Y_i)) + \log 4A - 2] + (\log n)/2 + (\log 4\pi)/2 + O(1/n). \tag{13}$$

Let k be a positive integer which we shall choose later, see (43). For any integer $m \geqq 2$,

$$\begin{aligned} &l(mk)-l((m-1)k) \\ &\quad= \textstyle\sum_{(m-1)k+1}^{mk}[-\log(W_{mk}(X_i)W_{mk}(Y_i)) + \log 4A - 2] \\ &\qquad+ \textstyle\sum_1^{(m-1)k}[-\log(W_{mk}(X_i)W_{mk}(Y_i)/W_{(m-1)k}(X_i)W_{(m-1)k}(Y_i))] \\ &\qquad+ [\log(m/(m-1))/2 + O(1/(m-1)k)] \\ &\quad= A_{mk} + B_{mk} + C_{mk} \quad \text{say.} \end{aligned} \tag{14}$$

Let $\delta > 0$. Let r be a positive integer. Let $h(\cdot,\cdot)$ be any function with $E(h(X_1, Y_1)) < \infty$. We now define three events:

$$A(\delta; n) = \{\sup_x |W_n(x) - W(x)| < \delta\} \tag{15}$$

$$B(\delta; n, r) = \{W(X_i) > \delta,\ W(Y_i) > \delta,\ i = n+1, \cdots, n+r\}, \quad \text{and} \tag{16}$$

$$C(\delta; h(\cdot,\cdot), n) = \{\textstyle\sum_1^n h(X_i, Y_i)/n - E(h(X_1, Y_1)) \leqq \delta\}. \tag{17}$$

Now, on $A(\delta; mk) \cap B(\delta; (m-1)k, k)$

$$\begin{aligned} A_{mk} &= \textstyle\sum_{(m-1)k+1}^{mk}[-\log(W_{mk}(X_i)W_{mk}(Y_i)) + \log 4A - 2] \\ &= \textstyle\sum_{(m-1)k+1}^{mk}[-\log(W(X_i)W(Y_i)) + \log 4A - 2 \\ &\qquad -\log(W_{mk}(X_i)W_{mk}(Y_i)/W(X_i)W(Y_i))] \\ &= \textstyle\sum_{(m-1)k+1}^{mk}(-\log(W(X_i)W(Y_i)) + \log 4A - 2 \\ &\qquad -\log(1 + (W_{mk}(X_i) - W(X_i))/W(X_i)) \\ &\qquad\quad \cdot(1 + (W_{mk}(Y_i) - W(Y_i))/W(Y_i))] \\ &\leqq \textstyle\sum_{(m-1)k+1}^{mk} V_1(\delta; X_i, Y_i) \end{aligned} \tag{18}$$

where

$$V_1(\delta; X_i, Y_i) = -\log(W(X_i)W(Y_i)) + \log 4A - 2 \quad (19)$$
$$-\log((1-(\delta/W(X_i)))(1-(\delta/W(Y_i))))$$
$$\text{if} \quad W(X_i) > \delta, \quad W(Y_i) > \delta$$
$$= -\log(W(X_i)W(Y_i)) + \log 4A - 2 \quad \text{otherwise.}$$

We now proceed to obtain an inequality for B_{mk}. Note that

$$\log(1+x) \geqq x/(1+x) \qquad \text{if} \quad x > -1. \quad (20)$$

Let $x^+ = x$ if $x \geqq 0$ and $= 0$ if $x < 0$; $x^- = 0$ if $x \leqq 0$ and $= x$ if $x < 0$. Let $-1 < \beta < 0 < \alpha$. If $\beta \leqq x \leqq \alpha$, then

$$\begin{aligned} \log(1+x) &\geqq x^+/(1-\alpha) + x^-/(1+\beta) \\ &= x/(1+\beta) + x^+[1/(1+\alpha) - 1/(1+\beta)] \\ &= x/(1+\beta) - x^+(\alpha-\beta)/[(1+\alpha)(1+\beta)] \\ &\geqq x/(1+\beta) - \alpha(\alpha-\beta)/[(1+\alpha)(1+\beta)]. \end{aligned} \quad (21)$$

Let $\phi > 0$. On $A(\phi; (m-1)k)$

$$\begin{aligned} W_{mk}(x)/W_{(m-1)k}(x) &= ((m-1)W_{(m-1)k}(x) + W_{(m-1)k,k}(x))/mW_{(m-1)k}(x) \\ &= 1 - (1/m) + (W_{(m-1)k,k}(x)/mW_{(m-1)k}(x)) \\ &\geqq 1 - (1/m) + (W_{(m-1)k,k}(x)/m(W(x)+\phi)). \end{aligned} \quad (22)$$

Also $-1/m \leqq -1/m + (W_{(m-1)k,k}(x)/m(W(x)+\phi)) \leqq -1/m + (1+A)/m\phi$. Using inequality (21) in (22) we have

$$\begin{aligned} \log(W_{mk}(x)/W_{(m-1)k}(x)) &\geqq -1/(m-1) + (W_{(m-1)k,k}(x)/(m-1)(W(x)+\phi)) \\ &\quad - [(1+A-\phi)(1+A)/(m-1)((m-1)\phi+1+A)\phi] \\ &= (1/(m-1))[((-W(x) - \phi + W_{(m-1k,k}(x))/(W(x)+\phi)) \\ &\quad - ((1+A-\phi)(1+A)/((m-1)\phi+1+A)\phi)]. \end{aligned} \quad (23)$$

Thus on $A(\phi; (m-1)k)$

$$\begin{aligned} B_{mk} &= -\textstyle\sum_1^{(m-1)k} \log[W_{mk}(X_i)W_{mk}(Y_i)/W_{(m-1)k}(X_i)W_{(m-1)k}(Y_i)] \\ &\leqq (1/(m-1))\textstyle\sum_1^{(m-1)k}[(W(X_i)+\phi - W_{(m-1)k,k}(X_i))/(W(X_i)+\phi) \\ &\quad + (W(Y_i)+\phi - W_{(m-1)k,k}(Y_i))/(W(Y_i)+\phi) \\ &\quad + 2((1+A-\phi)(1+A)/((m-1)\phi+1+A)\phi)] \\ &= (1/k(m-1))\textstyle\sum_{i=1}^{(m-1)k}\sum_{j=(m-1)k+1}^{mk} \\ &\quad \cdot[(W(X_i)+\phi - W(X_j, Y_j; X_i))/(W(X_i)+\phi) \\ &\quad + (W(Y_i)+\phi - W(X_j, Y_j; Y_i))/(W(Y_i)+\phi) \\ &\quad + 2((1+A-\phi)(1+A)/((m-1)\phi+1+A)\phi)] \\ &= (1/k(m-1))\textstyle\sum_{i=1}^{(m-1)k}\sum_{j=(m-1)k+1}^{mk}[V_2(\phi, X_j, Y_j; X_i, Y_i) \\ &\quad + 2((1+A-\phi)(1+A)/((m-1)\phi+1+A)\phi)] \end{aligned} \quad (24)$$

where

$$(25)\quad V_2(\phi, x, y; z, w)$$
$$= (W(z)+\phi-W(x, y; z))/(W(z)+\phi)+(W(w)+\phi-W(x, y; w))/(W(w)+\phi).$$

($W(x, y; z)$ is defined in Section 2). We also put

$$(26)\qquad V_3(\phi, x, y; z) = (W(z)+\phi-W(x, y; z))/(W(z)+\phi)$$

so that $V_2(\phi, x, y; z, w) = V_3(\phi, x, y; z)+V_3(\phi, x, y; w)$.

Let $\varepsilon > 0$ and

$$(27)\qquad C(\varepsilon; \phi, (m-1)k) = \textstyle\bigcap_{x,y} C(\varepsilon; V_2(\phi, x, y; \cdot, \cdot), (m-1)k).$$

(See (17) for the definition of the elements of the r.h.s.). Thus on $A(\phi; (m-1)k)\bigcap C(\varepsilon; \phi, (m-1)k)$ we have, from (24), that

$$(28)\quad B_{mk} \leqq \textstyle\sum_{j=(m-1)k+1}^{mk} [\int V_3(\phi, X_j, Y_j; z)(dF(z)+dG(z))$$
$$+\varepsilon+2((1+A-\phi)(1+A)/((m-1)\phi+1+A)\phi)].$$

We can now give the final inequality of this section. (14), (18) and (28) yield the following: on $A(\phi; (m-1)k)\bigcap C(\varepsilon; \phi, (m-1)k)\bigcap A(\delta; mk)\bigcap B(\delta; (m-1)k, k)$

$$(29)\quad l(mk)-l((m-1)k) \leqq \textstyle\sum_{j=(m-1)k+1}^{mk} [V(\delta, \phi; X_j, Y_j)$$
$$+\varepsilon+2(1+A-\phi)(1+A)/((m-1)\phi+1+A)\phi)]+C_{mk}$$

where

$$(30)\qquad V(\delta, \phi; X_j, Y_j) = V_1(\delta; X_j, Y_j)+\textstyle\int V_3(\phi, X_j, Y_j; z)(dF(z)+dG(z)).$$

Inequality (29) cannot be used directly in our later calculations since we have not demonstrated that $C(\varepsilon; \phi, (m-1)k)$ is an event (a measurable set), though it seems that this would not be difficult to prove this. However, as will be seen in Section 5 (40), we can choose a measurable subset $D(\varepsilon; \phi, (m-1)k)$ of $C(\varepsilon; \phi, (m-1)k)$ which contains a large probability and this result will suffice for our purposes.

Finally we note that as δ and ϕ tend to 0

$$(31)\qquad V(\delta, \phi; X_j, Y_j) \to V(X_j, Y_j)$$

almost everywhere, where

$$(32)\quad V(X_j, Y_j)$$
$$= -\log W(X_j)W(Y_j)+\log 4A-2+\int \frac{W(z)-W(X_j,Y_j; z)}{W(z)}(dF(z)+dG(z)).$$

We can simplify the r.h.s. further:

$$\int \frac{W(z)-W(X_j, Y_j; z)}{W(z)}(dF(z)+dG(z))$$
$$= \int \frac{(F(z)-I(X_j; z))+A(G(z)-I(Y_j; z))}{W(z)}(dF(z)+dG(z))$$
$$= \int \frac{F(z)-I(X_j; z)+G(z)-I(Y_j; z)}{W(z)}dW(z)$$
$$+(A-1)\left[\int \frac{G(z)-I(Y_j; z)}{W(z)}dF(z)-\int \frac{F(z)-I(X_j; z)}{W(z)}dG(z)\right]$$
$$= (F(z)+G(z))\log W(z)\,|_{-\infty}^{\infty} - \int \log W(z)(dF(z)+dG(z))$$
$$-\log W(z)\,|_{X_j}^{\infty} - \log W(z)\,|_{Y_j}^{\infty}$$
$$+(A-1)\left[\int \frac{G(z)-I(Y_j; z)}{W(z)}dF(z)-\int \frac{F(z)-I(X_j; z)}{W(z)}dG(z)\right]$$
$$= \log W(X_j)W(Y_j) - \int \log W(z)(dF(z)+dG(z))$$
$$+(A-I)\left[\int \frac{G(z)-I(Y_j; z)}{W(z)}dF(z)-\int \frac{F(z)-I(X_j; z)}{W(z)}dG(z)\right].$$

Thus

$$V(X_j, Y_j) = (A-1)\left[\int \frac{G(z)-I(Y_j; z)}{W(z)}dF(z) - \int \frac{F(z)-I(X_j; z)}{W(z)}dG(z)\right] + S(A, F, G) \tag{33}$$

where

$$S(A, F, G) = -\int \log W(z)(dF(z)+dG(z)) + \log 4A - 2 = E(-\log(W(X_1)W(Y_1)) + \log 4A - 2). \tag{34}$$

5. Exponentially converging estimates of certain probabilities. In this section P stands for the probability measure induced by the distribution function $H(\cdot, \cdot)$ Relation (7) of SS I states that for each $\delta > 0$, there exists a $\rho_3(\delta) < 1$ such that

$$P\{\bar{A}(\delta; n)\} \leqq \rho_3{}^n \tag{35}$$

for large n. This can also be directly deduced from Sethuraman (1964, Theorem 1) and can be seen to hold for general distributions F and G, not necessarily continuous.

Next, for fixed r,

$$P\{\bar{B}(\delta; n, r)\} = 1 - P^r\{W(X_1) > \delta, W(Y_1) > \delta\} \to 0 \qquad \text{as} \quad \delta \to 0. \tag{36}$$

We now proceed to obtain an estimate of the outer probability measure of $\bar{C}(\varepsilon; \phi, n)$, much on the same lines as Sethuraman (1964). Let

$$-\infty = x_0 < x_1 < \cdots < x_r = \infty, \qquad -\infty = y_0 < y_1 < \cdots < y_s = \infty$$

be partitions of the real line. For each (x, y) there exist integers p, q such that $x_{p-1} < x \leqq x_p$, $y_{q-1} < y \leqq y_q$. Note that

$$W(x_p, y_q; z) \leqq W(x, y; z) \leqq W(x_{p-1}+, y+_{q-1}; z).$$

From the definition of V_3 in (26)

$$V_3(\phi, x_{p-1}+, y_{q-1}+; z) \leqq V_3(\phi, x, y; z) \leqq V_3(\phi, x_p, y_q; z) \tag{37}$$

and hence $V_2(\phi, x, y; X_i, Y_i) \leqq V_2(\phi, x_p, y_q; X_i, Y_i)$. Again,

$$\begin{aligned}&\int [V_3(\phi, x_p, y_q; z) - V_3(\phi, x, y; z)]\, dF(z)\\ &\quad = \int (W(x, y; z) - W(x_p, y_q; z))/(W(z) + \phi)\, dF(z)\\ &\quad \leqq (1/\phi)[(F(x_p-) - F(x_{p-1})) + A(G(y_q-) - G(y_{q-1}))].\end{aligned}$$

This last expression and the similar one with F replaced by G can be made smaller than $\varepsilon/4$ for all p, q by choosing the partitions suitably. Thus

$$E(V_2(\phi, x, y; X_i, Y_i)) \geqq E(V_2(\phi, x_p, y_q; X_i, Y_i)) - \varepsilon/2. \tag{38}$$

Using this and (37)

$$\bar{C}(\varepsilon; V_2(\phi, x, y; \cdot, \cdot), n) \subset \bar{C}(\varepsilon/2, V_2(\phi, x_p, y_q; \cdot, \cdot), n). \tag{39}$$

Let $D(\varepsilon, \phi, n) = \bigcap_{p,q} C(\varepsilon/2, V_2(\phi, x_p, y_q; \cdot, \cdot), n)$. $D(\varepsilon, \phi, n)$ is measurable and from (39) it follows that

$$\bar{C}(\varepsilon, \phi, n) \subset \bar{D}(\varepsilon, \phi, n). \tag{40}$$

Further, $V_2(\phi, x, y; z, w)$ is bounded and hence possesses a moment generating function. From Chernoff ((1952), Theorem 1 and Lemma 6)

$$P(\bar{C}(\varepsilon/2, V_3(\phi, x_p, y_q; \cdot, \cdot), n)) \leqq \rho_4{}^n \tag{41}$$

for all p, q for some $\rho_4 < 1$ for all large n. $\bar{D}(\varepsilon; \phi, n)$ is the union of a finite number of sets as in the left-hand side of (41) and hence

$$P(\bar{D}(\varepsilon, \phi, n)) < rs\rho_4{}^n \tag{42}$$

for large n.

6. Main theorem. Assume now, that $P(V(X_j, Y_j) < 0)$ where $V(X_j, Y_j)$ is as defined in (32). Given $K > 0$, there exist an integer k and numbers $\varepsilon' > 0$ and $\theta_1' > 0$ such that

$$P(\textstyle\sum_1^k V(X_j, Y_j) + 3k\varepsilon' \leqq -K) \geqq \theta_1\, . \tag{43}$$

This follows from the fact $V(X_1, Y_1), V(X_2, Y_2), \cdots$ are independently and identically distributed, from that $\varepsilon' > 0$ can be chosen such that $P(V(X_1, Y_1) + 3\varepsilon' < 0) > 0$ and from the argument in Stein's proof (Stein (1946), Wald (1947) page 157) already referred to. Further since $V(\delta, \phi; X_j, Y_j)$ tends to $V(X_j, Y_j)$ almost everywhere, we can find δ_0, ϕ_0, ε, θ_1 with $0 < \varepsilon < \varepsilon'$, $0 < \theta_1 < \theta_1'$ such that for all $0 < \delta \leqq \delta_0$, $0 < \phi \leqq \phi_0$

$$P(\textstyle\sum_1^k V(\delta, \phi; X_j, Y_j) + 3k\varepsilon \leqq -K) \geqq \theta_1. \tag{44}$$

For our purposes it is enough if we fix δ and ϕ. Let us put $\phi = \phi_0$. We shall choose $\delta = \delta_1$ with $\delta_1 \leqq \delta_0$ and

$$\theta_1 - P(\bar{B}(\delta_1; (m-1)k, k)) = \theta > 0 \tag{45}$$

which is possible in view of (36). With this choice of k, ε, δ_1 and ϕ_0, we choose m_0, such that, for $m \geqq m_0$

$$2(1+A-\phi_0)(1+A)/((m-1)\phi_0+1+A)\phi_0 < \varepsilon \quad \text{and} \quad C_{mk} < k\varepsilon. \tag{46}$$

(For the definition of C_{mk} see (14).)

For $m \geqq m_0 - 1$, put

$$A(mk) = A(\phi_0; mk) \bigcap D(\varepsilon; \phi_0, mk) \quad \text{and} \quad B(mk) = A(\delta_1; mk). \tag{47}$$

We redefine m_0 to be larger if necessary, so that

$$P(\bar{A}(mk)) \leqq \rho_3^{mk}(\phi_0) + 4rs\rho_4^{mk}, \qquad P(\bar{B}(mk)) \leqq \rho_3^{mk}(\delta_1) \tag{48}$$

for $m \geqq m_0$, which is possible as can be seen from (35) and (42). We can now state the following lemma.

LEMMA 1. *Let* $P(V(X_1, Y_1) < 0) > 0$. *With the above choice of the number* $\theta > 0$ *and integers* k, m_0 *and with the events* $A(mk)$, $B(mk)$ *defined on the random variables* $\mathbf{Z}(mk)$ *for* $m \geqq m_0 - 1$ *by* (47) *we have, for* $m \geqq m_0$,

$$P(l(mk) - l(m-1)k) \leqq -K \mid \mathbf{Z}((m-1)k)) \geqq \theta - P(\bar{B}(mk) \mid \mathbf{Z}((m-1)k))$$

almost everywhere for $\mathbf{Z}((m-1)k)$ *in* $A((m-1)k)$. *Further the probabilities of* $\bar{A}(mk)$ *and* $\bar{B}(mk)$ *satisfy* (48).

PROOF. From (29), $l(mk) - l((m-1)k) \leqq \sum_{(m-1)k+1}^{mk} (V(\delta_1, \phi_0; X_j, Y_j) + 3\varepsilon)$ on $A((m-1)k) \cap B(mk) \cap B(\delta_1; (m-1)k, k)$, in view of (46). Also, $B(\delta_1; (m-1)k, k)$ is independent of $\mathbf{Z}((m-1)k)$. Thus, almost everywhere for $\mathbf{Z}((m-1)k)$ in $A((m-1)k)$,

$$\begin{aligned}
&P(l(mk) - l((m-1)k) \leqq -K \mid \mathbf{Z}((m-1)k)) \\
&\quad \geqq P\{\{l(mk) - l((m-1)k) \leqq -K\} \textstyle\bigcap B(mk) \bigcap B(\delta_1; (m-1)k, k) \mid \mathbf{Z}((m-1)k) \\
&\quad \geqq P(\textstyle\sum_{(m-1)k+1}^{mk} (V(\delta_1, \phi_0; X_j, Y_j) + 3\varepsilon) \leqq K \mid \mathbf{Z}((m-1)k)) \\
&\qquad\quad - P(\bar{B}(mk) \mid \mathbf{Z}((m-1)k)) - P(\bar{B}(\delta_1; (m-1)k, k) \mid \mathbf{Z}((m-1)k)) \\
&\quad \geqq \theta - P(\bar{B}(mk) \mid \mathbf{Z}((m-1)k))
\end{aligned}$$

in which we note the first and third probabilities of the penultimate expression are independent of $Z((m-1)k)$ and use (45). This completes the proof.

The following is an immediate consequence of Lemma 1 and Theorem 2.

THEOREM 4. *Let* $P(V(X_1, Y_1) < 0) > 0$. *Then the rank-order sequential probability ratio test based on a Lehmann alternative as described in* (2) *terminates and has a moment generating function.*

7. Comparison with the result of SS I and discussion. In SS I (Theorem 3) the conclusion of our Theorem 4 was established under the condition of F and G continuous and $E(-\log W(X_1)\ W(Y_1)+\log 4A-2) \neq 0$. The condition that F and G are continuous can be dropped because it is not necessary to establish (8) and (17) of SS I which are the crucial steps in the proof. Hence we may re-state the main result of SS I as follows:

THEOREM 5. *If*

$$E(-\log W(X_1)W(Y_1)+\log 4A-2) \neq 0 \tag{49}$$

then the rank-order SPRT *based on a Lehmann alternative terminates and has a moment generating function.*

Now,

$$E\left(\int \frac{W(z)-W(X_1, Y_1; z)}{W(z)}(dF(z)+dG(z))\right) = 0.$$

Referring to (33) for the expression for $V(X_1, Y_1)$, we see that condition (49) is equivalent to $E(V(X_1, Y_1)) \neq 0$. Thus the case not covered by Theorem 5 is when $E(V(X_1, Y_1)) = 0$. Comparing this uncovered case with the condition of Theorem 4 we have our final theorem.

THEOREM 6. *Let*

$$P(V(X_1, Y_1) = 0) \neq 1. \tag{50}$$

Then the rank-order SPRT *based on a Lehmann alternative terminates and has a moment generating function.*

Theorem 6 represents the best result we have been able to establish. The condition it imposes namely $P(V(X_1, Y_1) = 0) \neq 1$ is analogous to the condition in Wald ((1947) page 157). We feel that for all joint distributions $H(x, y)$, $P(V(X_1, Y_1) = 0) < 1$ so that the condition of Theorem 6 is automatically satisfied and unnecessary. However, we have not been able to establish this conjecture. The following particular case is immediate.

LEMMA 2. *Let* X_1, Y_1 *be independently distributed. Then* $P(V(X_1, Y_1) = 0) < 1$.

We can therefore assert that if we start with independent bivariate random variables then our sequential test terminates and has a moment generating function.

We now wish to add a few remarks regarding our allowing general distributions $H(x, y)$ when using a sequential test based on ranks. A standard assumption under which the ranks are well defined is that the distributions are continuous. However, the expression $L(A, F_n, G_n)$ in (1) is one representation of the probability ratio of the ranks. It turns out that this representation is always well defined since it is given through the empirical distribution functions $F_n(x)$ and $G_n(x)$. Thus the sequential test described in this paper can be applied to samples from any distribution. A few more remarks on this may be found in the next section.

8. Asymptotic normality of $l(n)$.

THEOREM 7. *For any joint distribution $H(\cdot,\cdot)$, $(l(n)-S(A,F,G))/n^{\frac{1}{2}}$ converges in distribution to the normal distribution with mean* 0 *and variance σ^2, given by*

$$\begin{aligned}\sigma^2 = 2(A-1)^2[&\textstyle\iint_{x<y}[G(x)(1-G(y))/W(x)W(y)]\,dF(x)\,dF(y)\\ &+\textstyle\iint_{x<y}[F(x)(1-F(y))/W(x)W(y)]\,dG(x)\,dG(y)\\ &-\textstyle\iint_{x,y}[(H(x,y)-F(x)G(y))/W(x)W(y)]\,dF(y)\,dG(x)].\end{aligned} \tag{51}$$

PROOF. This is an elementary consequence of the well-known limit theorem of Chernoff and Savage (1958), Theorem 1). We now list the necessary modifications in their theorem *using their own notation* instead of evolving a new extension to their theorem.

(i) H_N which is a linear combination of F_m and G_n could be any linear combination, say $(F_m+AG_n)/(1+A)$.

(ii) The assumption that X and Y are independent is used only in a minor way in disposing of the remainder term C_{2N}. This can be done in exactly their way even if X and Y are dependent.

(iii) The assumption about the continuity of F and G was imposed to make the remarks $(s_1,\cdots,s_n)$ well defined. It is not used elsewhere. Therefore it can be dispensed with. See also the remarks in the previous section. After this preamble, we expand $n^{-1}l(n)$ as usual and obtain

$$\begin{aligned}n^{-1}l(n) &= -\textstyle\int\log W_n(x)(dF_n(x)+dG_n(x))+\log 4A-2+O(\log n/n)\\ &= -\textstyle\int\log W(x)(dF(x)+dG(x))+\log 4A-2\\ &\quad -\textstyle\int\log W(x)(d(F_n(x)-F(x))+d(G_n(x)-G(x)))\\ &\quad -\textstyle\int[(W_n(x)-W(x))/W(x)](dF(x)+dG(x))+o_p(n^{-\frac{1}{2}})\\ &= S(A,F,G)-(A-1)[\textstyle\int[(G_n(x)-G(x))/W(x)]\,dF(x)\\ &\quad -\textstyle\int[(F_n(x)-F(x))/W(x)]\,dG(x)]+o_p(n^{-\frac{1}{2}}).\end{aligned} \tag{52}$$

The standard computation shows that $(l(n)-S(A,F,G))/n^{\frac{1}{2}}$ has a limiting normal distribution with mean 0 and variance σ^2 as given in (51).

Recently, Govindarajulu (1968) has demonstrated Theorem 7 assuming X_1, Y_1 to be independent.

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