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A THEOREM ON SATURATED PLANS AND THEIR COMPLEMENTS[1]

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Consider a saturated main effect plan for the parametric vector β_1, then this paper shows that when this plan is optimum (in the sense of maximum determinant of the design matrix) then the complementary plan is also optimum for the complementary vector β_2. Some additional corollaries are derived for special cases.

1. Summary and introduction. Consider a full replicate of any factorial with the factors having quantitative levels, then we know that the expectation of the $N \times 1$ observation vector Y is:

$$E[Y] = X\beta \tag{1.1}$$

where X is an $N \times N$ orthogonal matrix in the sense that $X'X = \text{Diagonal}$ $(d_1, d_2, \cdots, d_N)$ and β is the $N \times 1$ vector of single degree of freedom parameters.

Suppose now that our interest lies in estimating k effect parameters with k observations. Such a plan is termed in the literature as a *saturated* plan. Denote the k parameters of interest by the $k \times 1$ vector β_1 and the remainder (or *complement of* β_1) of the parameters by the $(N-k) \times 1$ vector β_2. Also denote the corresponding partitioning of the observation vector by Y_1 and Y_2 respectively, then equation (1.1) can be written as:

$$E[Y] = E\begin{bmatrix} Y_1 \\ \cdots \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_{11} & \vdots & X_{12} \\ \cdots & & \cdots \\ X_{21} & \vdots & X_{22} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \cdots \\ \beta_2 \end{bmatrix} \tag{1.2}$$

where X_{11} is a $k \times k$, X_{12} is a $k \times (N-k)$, X_{21} is an $(N-k) \times k$ and X_{22} is an $(N-k) \times (N-k)$ matrix.

To make the ensuing discussion simpler we follow Banerjee and Federer (1966) and transform the design matrix and the parametric vector β as $X\beta = (XD^{-\frac{1}{2}}) \cdot (D^{\frac{1}{2}}\beta) = W\gamma$, where $W = XD^{-\frac{1}{2}}$ and $\gamma = D^{\frac{1}{2}}\beta$, $D^{\frac{1}{2}}$ is an $N \times N$ diagonal matrix with elements $d_i^{\frac{1}{2}}$, $(i = 1, 2, \cdots, N)$, where d_i is the ith diagonal element of D. This transformation changes equations (1.1) and (1.2) into (1.3) and (1.4) respectively:

$$E[Y] = W\gamma, \tag{1.3}$$

where now $W'W = I_N = WW'$ and

$$E[Y] = E\begin{bmatrix} Y_1 \\ \cdots \\ Y_2 \end{bmatrix} = \begin{bmatrix} W_{11} & \vdots & W_{12} \\ \cdots & & \cdots \\ W_{21} & \vdots & W_{22} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \cdots \\ \gamma_2 \end{bmatrix} \tag{1.4}$$

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where as before W_{11} is $k \times k$, W_{12} is $k \times (N-k)$, W_{21} is $(N-k) \times k$ and W_{22} is $(N-k) \times (N-k)$.

Now assume that the vector γ_1 is estimable from the Y_1 observations, i.e. rank of X_{11} is equal to k, then a commonly used criterion to call the saturated plan Y_1 optimum is when the determinant of $X'_{11} X_{11}$ is maximum. But since X_{11} is a square matrix of full rank we see that Y_1 is optimum when the absolute value of the determinant of X_{11} is maximum. Henceforth we will utilize the notation $|\det A|$ to mean the absolute value of the determinant of the matrix A.

Suppose that Y_1 is an optimum plan for γ_1, then does this imply that Y_2 is an optimum plan for γ_2? This question and some related ones are settled in the next section.

2. Optimality theorem of saturated plans. Before going into the question raised at the end of the previous section, we quote without proof the following result, which can be found in Muir (1933, page 567).

LEMMA. *Every minor M of an orthogonal matrix A is equal to its algebraic complement multiplied by the determinant of A.*

This lemma implies that in our (1.4)-setting: $|\det W_{11}| = |\det W_{22}|\,|\det W| = |\det W_{22}|$ since $|\det W| = 1$. Hence when $|\det W_{11}|$ is maximum then $|\det W_{22}|$ is maximum. This means that when the saturated plan Y_1 is optimum for γ_1 then automatically Y_2 is optimum for γ_2. This result is stated formally in the following.

THEOREM. *If Y_1 is an optimum saturated plan for the parametric vector γ_1 then Y_2 is the complementary optimum saturated plan for the complementary parametric vector γ_2.*

The following corollaries relate to the original setting (1.1) and (1.2) or to special cases. The proofs follow immediately from this theorem or from other known results.

COROLLARY 1. *If Y_1 is optimum for γ_1 (and hence Y_2 is optimum for γ_2) then Y_1 is optimum for β_1 (and hence Y_2 is optimum for β_2).*

COROLLARY 2. *If Y_1 is a singular plan for β_1 then Y_2 is also a singular plan for β_2.*

COROLLARY 3. *In the case of the 2^n-factorial* $|\det X_{22}| = 2^{n(2^{n-1}-k)}\,|\det X_{11}|$. (This follows directly from Muir (1933, page 589).)

COROLLARY 4. *The value of $|\det X_{11}|$ for any k-run saturated plan Y_1 for β_1 in the case of the 2^n-factorial is at most $k^{k/2}$.* (The proof follows from Hadamard's (1893) theorem mentioned in Muir (1933, page 761).)

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REFERENCES

[1] BANERJEE, K. S. and FEDERER, W. T. (1966). On estimation and construction in fractional replication. *Ann. Math. Statist.* **37** 1021–1032.

[2] HADAMARD, J. (1893). Resolution d'une question relative aux determinants. *Bull. Sci. Math.* **17** 240–306.

[3] MUIR, T. (1933). *Theory of Determinants.* Longmans, Green and Co., New York. 1–766.