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ABSTRACTS

(*Abstract of a paper presented at the Eastern Regional meeting, Chapel Hill, North Carolina, May* 5–7, 1970. *Additional abstracts appeared in earlier issues.*)

125-54. Some problems in regression design and approximation theory. GRACE WAHBA, University of Wisconsin. (Invited)

Let $X(t)$, $t \in [0, 1]$, be a zero mean stochastic process with $EX(s)X(t) = R(s, t)$ and let $Y(t) = \theta f(t) + X(t)$, $t \in [0, 1]$, where θ is a parameter to be estimated and f is known. Suppose $Y(t)$ possess $m-1$ quadratic mean derivatives. For each n, it is desired to choose a design $T_n{}^1 T_n = \{t_{0n} < t_{1n^1} < \cdots, t_{nn}\}$ so that the Gauss–Markov estimate for θ, based on $Y^{(\nu)}(t_{in})$, $\nu = 0, 1, \cdots, m-1$, $i = 0, 1, \cdots, n$, is as small as possible. Let $X(t)$ formally satisfy the stochastic differential equation $L_m X(t) = dW(t)/dt$, $X^{(\nu)}(0) = \xi_\nu$, $\nu = 0, 1, \cdots, m-1$, where $W(t)$ is a Wiener process and the $\{\xi_\nu\}$ are normal, zero mean random variables, independent of $W(t)$. L_m is an mth order linear differential operator whose null space is spanned by an extended, complete, Tchebychev system of continuity class C^{2m}. Let $f(t) = \int_0^1 R(t, u)\rho(u)\, du$, where ρ is strictly positive and possess a bounded first derivative. Let $\lim_{s \downarrow t} \partial^{2m-1}/\partial s^{2m-1} R(s, t) - \lim_{s \uparrow t} \partial^{2m-1}/\partial s^{2m-1} R(s, t) = (-1)^m \alpha(t)$. Then $T_n^* = \{t_{0n}^*, t_{1n}^*, \cdots, t_{nn}^*\}$, $n = 1, 2, \cdots$ with t_{in}^* given by $t_{0n}^* = 0$, $\int_0^{t_{in}^*} [\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\, du = i/n \int_0^1 [\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\, du$, $i = 1, 2, \cdots, n$, is an asymptotically optimum sequence of designs and $(1/\sigma^2 - 1/\sigma_{T_n^*}^2) = [(m!)^2/n^{2m}(2m)!(2m+1)!] \times [\int_0^1 [\rho^2(u)\alpha(u)]^{(2m+1)^{-1}}\, du] + o(1/n^{2m})$, where σ^2 and $\sigma_{T_n^*}^2$ are the variances of the Gauss–Markov estimates of θ based on $Y(t)$, $t \in [0, 1]$, and $Y^{(\nu)}(t)$, $\nu = 0, 1, 2, \cdots m-1$, $t \in T_n^*$, respectively. This extends some results of Sacks and Ylvisaker, [*Ann. Math. Statist.* **37**, 66–89; **39**, 49–69; and unpublished manuscript], whose definition of asymptotically optimum we use. This regression design problem is shown to be equivalent to some problems in approximation theory, in particular the establishment of optimal quadrature formulae of a certain type. (Received October 7, 1970.)

(*Abstracts of papers to be presented at the Eastern Regional meeting, University Park, Pennsylvania, April* 21–23, 1971. *Additional abstracts will appear in future issues.*)

129-1. Construction of maximal designs of resolution VII and VIII. BODH RAJ GULATI, Southern Connecticut State College.

A symmetrical factorial design based on N runs, k factors each operating at two levels, is said to be of resolution t if no main effect or $(t-1)$-factor $(t > 2)$ or its lower order interaction is confounded with block effects. Such a design may symbolically be denoted by (N, k, t)-design. An (N, k, t)-design is said to be *maximal* if there exists no other (N, k^*, t)-design with $k^* > k$. This paper produces the following maximal resolution VII designs: (i) (64, 7, 7), (ii) (128, 8, 7), (iii) (192, 7, 7), (iv) (256, 9, 7), (v) (320, 7, 7), (vi) (384, 8, 7), (vii) (512, 11, 7), (viii) (1024, 15, 7) and (ix) (2048, 23, 7). Maximal designs of resolution VIII follow immediately, since it is known that if (N, k, t)-design with largest k and $t = 2u+1$ exists, so does $(2N, k+1, t+1)$. (Received August 10, 1970.)

129-2. On the sign test for symmetry. JOSEPH L. Gastwirth, The Johns Hopkins University.

The sign test of the null hypothesis that n observations are from a density which is symmetric about a specified value μ is based on the number of observations which are less than μ and rejects the symmetry hypothesis if this number is too far from its expected value $(n/2)$. In practice one does not know μ and might estimate it by the sample mean $(\bar{X})$. The modified sign test counts the number of observations which are less than $\bar{X}$. The asymptotic distribution of the modified sign test statistic (S^*) is derived. An interesting fact is that the asymptotic variance of S^* can be greater than, equal to or less than the corresponding asymptotic variance of the ordinary sign test. (Received August 31, 1970.)

(Abstracts of papers contributed by title.)

71T-1. On the distribution of $D^2_{p+q} - D_q{}^2$ and $D^2_{p+q}/D_q{}^2$ statistics (preliminary report). K. SUBRAHMANIAM AND KATHLEEN SUBRAHMANIAM, University of Manitoba.

In this paper, we discuss the distribution of the statistics ($D^2_{p+q} - D_q{}^2$) and $D^2_{p+q}/D_q{}^2$ (C. R. Rao, *Sankhyā, Ser. A* **9** (1949) 343-366). Both the central and noncentral distributions of those statistics have been considered by the second author (1969) unpublished doctoral dissertation, Johns Hopkins University). Rao, *Sankhyā, Ser. A* **10** (1950) 257–268) has looked at some aspects of the approximate distribution of the first statistic. We develop, by a different argument, the approximation to the distribution of this statistic in the central and noncentral cases. It is shown that under some restrictions the density functions of the statistic can be approximated by those of the chi-square and the variance ratio statistics. Extensive tables of the 1%, 5%, and 10% points of the distribution of $D^2_{p+q} - D_q{}^2$ are obtained in the exact case. Power of the various tests proposed in this connection are presented. (Received September 15, 1970.)

71T-2. Limit laws for the record values of an i.i.d. sequence of random variables (preliminary report). SIDNEY I. RESNICK, Technion—Israel Institute of Technology.

Let $\{X_n, n \geqq 1\}$ be i.i.d. random variables with continuous df $F(x)$. X_j is a record value of this sequence if $X_j > \max\{X_1, \cdots, X_{j-1}\}$. Consider the sequence of such record values $\{X_{L_n}, n \geqq 1\}$. Set $\phi(y) = -\log(1-F(y))$. There exist $\beta_n > 0$ such that $X_{L_n}/B_n \to 1$ in probability (i.p.) iff $X_{L_n}/\phi^{-1}(n) \to 1$ i.p. iff $\{\phi(kx) - \phi(x)\}/\phi^{\frac{1}{2}}(kx) \to \infty$ as $x \to \infty$ for all $k > 1$. Similar criteria hold for the existence of constants A_n such that $X_{L_n} - A_n \to 0$ i.p. For suitably chosen constants $a_n > 0$, b_n, $n \geqq 1$, $P[X_{L_n} \leqq a_n x + b_n]$ can converge to only three types of nondegenerate df's: (i) $N(x)$, the standard normal df; (ii) $N_1(x) = 0$, $x \leqq 0$, $N(\log x^\alpha)$, $x \geqq 0$; (iii) $N_2(x) = N(\log(-x)^{-\alpha})$, $x \leqq 0$, 1 if $x \geqq 0$ where $\alpha > 0$. Domain of attraction criteria can be given for each case. For example the limit law $N_1(x)$ is obtained iff $\{\phi(tx) - \phi(x)\}/\phi^{\frac{1}{2}}(x) \to \alpha.\log t$ as $x \to \infty$ for all $t > 0$. Repeated use is made of the following Lemma. *If* $P[X_n \leqq x] = 1 - e^{-x}$, $x \geqq 0$ *then* $X_{L_m} = y_0 + \cdots + y_m$ *where the* y_j*'s are* i.i.d. *and* $P[y_j \leqq x] = 1 - e^{-x}$. (Received September 21, 1970.)

71T-3. On a semi-stochastic model arising in a biological system. W. J. PADGETT AND C. P. TSOKOS, Virginia Polytechnic Institute.

A stochastic version of a mathematical model for chemotherapy which was developed deterministically by Bellman, Jacquez, and Kalaba (*Bull. Math. Biophys.* **22** (1960) 181–197 and 309–322) is given. After an injection at the heart entrance, the concentration of drug in the blood plasma of a biological system with one organ and a simplified heart results in the semi-stochastic integral equation $u_L(t;\omega) = (-c/V^*)\int_0^t [u_L(y;\omega) - u_R(y;\omega)]\,dy$, $t \geqq 0$, where $u_L(t;\omega)$ is the concentration of drug in plasma leaving the heart and $u_R(t;\omega)$ is the concentration of drug in plasma entering the heart at time t. The function $u_L(t;\omega)$ is a deterministic function of time for $0 \leqq t < \tau$, where τ is the blood recirculation time lag, and a random function of t, $\tau \leqq t \leqq M$, for some $M \in (\tau, \infty)$. It is shown that the integral equation has a solution both for $t \in [0, \tau)$, using the method of successive approximations, and for $t \in [\tau, M]$, using some methods of probabilistic functional analysis. (Received September 23, 1970.)

71T-4. On the existence of a solution of a stochastic integral equation in turbulence theory. W. J. PADGETT AND C. P. TSOKOS, Virginia Polytechnic Institute.

In the theory of turbulence, the random position of a tagged point in a continuous fluid in turbulent motion, $\mathbf{r}(t;\omega)$, is a vector-valued random function of time $t \geqq 0$, $\omega \in \Omega$, where Ω is the supporting set of the underlying probability space (Ω, B, P). If $\mathbf{u}(\mathbf{r}, t;\omega)$ is the Eulerian velocity field, then $\mathbf{r}(t;\omega)$ satisfies the stochastic integral equation $\mathbf{r}(t;\omega) = \int_0^t \mathbf{u}(\mathbf{r}(\Upsilon;\omega), \Upsilon;\omega)\,d\Upsilon$, $t \geqq 0$. General conditions under which a random solution of this stochastic integral equation exists are given in the form of a theorem, and the theorem is proved using the concepts of admissibility with respect to an operator on a Banach space and fixed point methods of functional analysis. The results of the paper are generalizations of the work of Lumley, J. L. (*J. Math. and Phys.* **3** (1962) 309–312.) (Received September 23, 1970.)

71T-5. On a stochastic integral equation of the Volterra type in telephone traffic theory. W. J. PADGETT AND C. P. TSOKOS, Virginia Polytechnic Institute.

Consider a telephone exchange which has $m < \infty$ channels with calls arriving at random time instants $0 < t_1 < t_2 < \cdots < \infty$. Let $a(t)$ be the arrival time density function, and let $K(t, \Upsilon;\omega)$ be a random function which is equal to one if a conversation from a call arising at time Υ is still being held at time $t \geqq \Upsilon$ and is equal to zero otherwise. If $V[k]$ is a function with value one whenever $k = 0, 1, \cdots, m-1$ and value zero otherwise, then the number of conversations held at time $t \geqq 0$, $x(t;\omega)$, is a random function satisfying the stochastic nonlinear integral equation of the Volterra type of the form $x(t;\omega) = \int_0^t K(t, \Upsilon;\omega) V[x(\Upsilon;\omega)]a(\Upsilon)\,d\Upsilon$. A theorem of Tsokos (*Math. Systems Theory* **3** (1969) 222–231) is applied to this equation to show that a unique random solution exists, where a random solution is defined to be a second order stochastic process satisfying the equation almost surely. A similar equation was studied by Fortet (*Proc. Third Berkeley Symposium Math. Statist. Prob.* (1956) 81–88). (Received September 23, 1970.)

71T-6. On the existence of a unique solution of a stochastic integral equation in hereditary mechanics. W. J. PADGETT AND C. P. TSOKOS, Virginia Polytechnic Intitute.

Distefano (*J. Math. Anal. Appl.* **23** (1968) 365–383) studied a pair of random integral equations of the Volterra type occurring in a simple hereditary system. He showed that there exist solutions to the equations by a method of successive approximation and also by a technique of "truncated hierarchy." This paper is a generalization of the results of Distefano in that the random functions in the pair of stochastic integral equations may be more general functions, enabling one to consider more general hereditary systems. A theorem of Tsokos (*Math. Systems Theory* **3** (1969) 222–231) is applied to the stochastic integral equations to obtain general conditions under which a unique random solution exists. (Received September 23, 1970.)

71T-7. A stochastic model for chemotherapy: computer simulation. W. J. PADGETT AND C. P. TSOKOS, Virginia Polytechnic Institute.

A stochastic version of a deterministic mathematical model for chemotherapy of Bellman, Jacquez, and Kalaba (*Bull. Math. Biophys.* **22** (1960) 181–197 and 309–322) was formulated by the authors (*Math. Biosci.* same issue) for a one-organ biological system and, due to the random nature of diffusion of a drug from the blood plasma into the extracellular space in a capillary bed, resulted in a *semi-stochastic* integral equation with the concentration of drug in plasma leaving the heart at time t as the unknown function. In this paper the diffusion of drug into the extracellular space of the organ is simulated using a two-dimensional diffusion process. Values of the concentration of drug entering the heart at time t are obtained, and using numerical integration a realization of the *semi-random* solution of the semi-stochastic integral equation is found. The solution compares favorably with experimental results of Bassingthwaighte (*Science* **167** (1970) 1347–1353) and Bellman, *et al.* (RAND Corp., RM-3463). (Received September 23, 1970.)

71T-8. Progressively censored bivariate samples. G. BAIKUNTH NATH, University of Queensland.

In numerous biological and physical situations for the two-component systems, progressively censored samples arise when at various stages of an experiment some though not all of the sample specimens are eliminated from further observation. The sample specimens remaining after each stage of censoring are continued on test until a subsequent stage of screening or censoring. Using the above information, the maximum likelihood estimating equations of the distribution parameters are derived for the two-dimensional correlated surface in which each of the variables has a univariate normal marginal distribution. It is shown that both the method of moments and the method of maximum likelihood lead to identical results. For solving the above estimating equations a procedure is suggested to get rapid first approximations. The information matrix is given from which the asymptotic variances and covariances of the estimates of the parameters might be obtained. (Received September 28, 1970.)

71T-9. A method for studying the integral functionals of stochastic processes with applications: I. Markov chain case. PREM. S. PURI, Purdue University.

A method for studying the distribution of integrals of the type $Y(t) = \int_0^t f(X(\tau), \tau)\,d\tau$ is introduced where $\{X(t); t \geqq 0\}$ is some appropriately defined continuous time parameter stochastic process, and f is a suitable nonnegative function of its arguments. The method is based on an auxiliary process $Z(t)$ called "Quantal Response Process," defined for a hypothetical animal as: $Z(t)$ equals one if the animal is alive at time t and is equal to zero otherwise. In particular it is assumed that $P(Z(t+\tau) = 0 \mid Z(t) = 1, X(t) = x) = \delta f(x, t)\tau + o(\tau)$, with $Z(0) = 1$ and $\delta \geqq 0$. Here "zero" is an absorption state for the process $Z(t)$. From this we have $P(Z(t) = 1) = E\{\exp[-\delta \int_0^t f(X(\tau), \tau)\,d\tau]\}$, which in turn gives the Laplace transform (L.T.) of the integral $Y(t)$. Thus the study of the distribution of $Y(t)$ is shown to be equivalent to that of the process $Z(t)$. The method is then applied to the case of time homogeneous denumerable Markov chains with f depending only on $X(t)$ and not explicitly on t where, for the joint distribution of $X(t)$ and $Y(t)$, it is sufficient to study the probabilities $\tilde{P}_{ij}(t) = P(X(t) = j, Z(t) = 1 \mid X(0) = i)$. We give here the conditions for existence and uniqueness of solutions of certain system of equations (backward and forward type) involving L.T. of $\tilde{P}_{ij}$. The solution of these in turn determine the joint distribution of $X(t)$ and $Y(t)$. For the case of finite Markov chains an explicit solution is presented. (Received October 1, 1970.)

71T-10. A method for studying the integral functionals of stochastic processes with applications: II. Sojourn time distributions for Markov chains. PREM S. PURI, Purdue University.

In the abstract above a method was introduced for studying the distribution problems concerning integral functionals of stochastic processes. In particular, this method was applied to time homogeneous Markov chains. The present paper is a continuation of this application. Let $X(t)$ be the Markov chain with $X(0) = i$, and let α_{iH} be the random moment in time when the chain hits a fixed nonempty taboo set of states H which is accessible from state i with $i \notin H$. Also let $Y(t) = \int_0^t f(X(\tau))\,d\tau$ where the function f and the integral are appropriately defined. Among other results, we obtain the transform of the joint distribution of the integral $Y(\alpha_{iH})$ and $X(\alpha_{iH})$. Also we obtain the joint distribution of the times spent by the process in different states of a finite set

$J = \{l_1, \cdots, l_M\}$, (with $l_r \notin H$, but H being accessible from every state $l_r, r = 1, 2, \cdots, M$), prior to the moment it hits the taboo set H. The special case with $M = 1$ has been studied earlier by Chung. Our method, however, extends to the general case of any finite M. Finally, for the case with $M = 2$, the distribution has been explicitly worked out. (Received October 1, 1970.)

71T-11. A method for studying the integral functionals of Stochastic processes with applications: III. PREM. S. PURI, Purdue University.

This paper is a continuation of the two abstracts above, where we introduced a method for obtaining the joint distribution of $X(t)$ and integral of the form $Y(t) = \int_0^t f(X(\tau), \tau)\, d\tau$. Here $X(t)$, $t \geqq 0$, is a continuous parameter stochastic process appropriately defined on a probability space and f is a nonnegative measurable function of its arguments. In this paper, this method is applied to general birth processes, both time homogeneous and time nonhomogeneous. The results so obtained are then specialised to the case of Simple Epidemic and to certain well-known processes such as Poisson process, Polya process etc. The method is also applied to certain well-known birth and death processes such as (i) Linear Birth and Death process with Immigration and (ii) $M/M/I$ Queue. The paper ends with an application to Illness and Death processes. In each of these cases, distributions of certain useful integrals are explicitly derived by using the technique introduced in (I). (Received October 1, 1970.)

70T-12. Some weighted sums of multinomial random variables. M. F. MILLER, University of Minnesota, Duluth.

This paper is concerned with a weighted sum of the relative frequencies observed in $n+1$ cells from a random sample of size n. Each cell is chosen so that it has equal probability under a simple null hypothesis. This statistic, properly standardized, is shown to have a limiting normal distribution for a general class of weight functions. It is also shown that a weight function can be selected so that the Bahadur exact slope of the resulting weighted sum is maximized for particular alternative distributions. (Received October 2, 1970.)

71T-13. Characterization of probability means by linear forms. B. L. S. PRAKASA RAO, Indian Institute of Technology.

Let X_1, X_2 and X_3 be three independent real random variables and let $Z_1 = X_1 - X_2$ and $Z_2 = X_2 - X_3$. If the characteristic function of (Z_1, Z_2) does not vanish, then the distribution of (Z_1, Z_2) determines the distributions of X_1, X_2, X_3 up to a change of location. This result was proved by Kotlarski. This result has been extended to arbitrary linear forms. We have shown that if $Z_1 = a_1 X_1 + a_2 X_2$ and $Z_2 = b_2 X_2 + b_2 X_3$ where $a_1 a_2 \neq 0$ and $b_2 b_3 \neq 0$ and if the characteristic function of (Z_1, Z_2) does not vanish, then the distribution of (Z_1, Z_2) determines the distributions of X_1, X_2, X_3 up to changes of location possibly different for different random variables. This result has been generalized to random elements in Hilbert spaces, locally compact Abelian groups and locally convex topological vector spaces. Finally, we obtained a characterization theorem for stochastic process by stochastic integrals. (Received October 8, 1970.)

71T-14. Some characterization theorems for Wiener Process in a Hilbert space. B. L. S. PRAKASA RAO, Indian Institute of Technology.

Recently Vakhaniya and Kandelskii (*Theor. Probability Appl.* **12** (1967)) have defined stochastic integrals for operator-valued functions with respect to stochastic processes which take values in a Hilbert space. Using this definition of stochastic integral, we have obtained characterization theorems for Wiener process taking values in a Hilbert space H. Suppose Φ is a homogeneous process with independent increment with mean 0 and finite associated S-operator. Let $A(\cdot)$ and

$B(\cdot)$ be functions in the space L_2 as defined by Vakhaniya and Kandelskii. Then under some conditions, $\int_0^1 A(\lambda)\Phi(d\lambda)$ and $\int_0^1 B(\lambda)\Phi(d\lambda)$ are identically distributed if and only if Φ is a Wiener process and $A(\cdot)$ and $B(\cdot)$ satisfy the relation $\int_0^1 A(\lambda)SA'(\lambda)d\lambda = \int_0^1 B(\lambda)SB'(\lambda)d\lambda$. Two other characterization theorems have been obtained. In the course of the discussion, Kolmogorov type representation for infinitely divisible distribution with finite associated S-operator is obtained. (Received October 8, 1970.)

71T-15. Limit theorems for random number of random elements on complete separable metric spaces. B. L. S. PRAKASA RAO, Indian Institute of Technology.

Let X be a complete separable metric space with metric d and $\mathscr{B}_X$ denote the σ-field of Borel sets in X. Let $\{\mu_n\}$ be a sequence of probability measures induced by random elements $\{Y_n\}$ taking values in X. Suppose that μ_n converges weakly to μ. We have proved the following theorem. Suppose N_n is a sequence of positive integer valued random variables such that $n^{-1}N_n \to 1$ in probability as $n \to \infty$ and for every $\eta > 0$, $\varepsilon > 0$, there exists c in (0, 1) and an integer $N_0 > 0$ such that $P[\sup\{d(Y_n, Y_k): r_n \leqq k \leqq s_n\} > \varepsilon] < \eta$ for all $n \geqq N_0$ where $r_n = [n(1-c)]$ and $s_n = [n(1+c)]$. We shall call this later condition as Auscombe's condition. Then μ_{N_n} converges weakly to μ. We have also proved that if $\{Y_n\}$ is Rény i mixing with limit μ and $n^{-1}N_n \to M$ in probability where M is positive and discretely distributed, then μ_{N_n} converges weakly to μ provided $\{Y_n\}$ satisfies Auscombe's condition where μ_{N_n} is the measure induced by the element Y_{N_n}. (Received October 8, 1970.)

71T-16. On an iterated logarithm law for sequences of maxima. LAURENS DE HAAN AND ARIE HORDIJK, Mathematisch Centrum.

Let X_1 X_2 $\cdots$ be independent random variables with common distribution function $F(x)$ and suppose $F(x) < 1$ for all x. Define $Y_n = \max(X_1, \cdots, X_n)$ and write $f(x) = (1-F(x))/F'(x)$ and $b_n = \inf\{x \mid 1-F(x) \leqq n^{-1}\}$. R. von Mises (*Selected Papers* **2**) proved that for all x $\Pr[Y_n - b_n \leqq f(b_n)x] \to \Lambda(x) = \exp(-\exp - x)$ as $n \to \infty$ if $df(x)/dx \to 0$ as $x \to \infty$. Under the stronger condition $d/dx\{f(x)(\log \int_a^x ds/f(s))^2\} \to c_1 < \infty$ as $x \to \infty$ it can be proved that $\{1 - F^n(b_n + f(b_n)\alpha \log\log n)\}/\{1 - \Lambda(\alpha \log\log n)\} \to \exp(\frac{1}{2}c_1\alpha^2)$ as $n \to \infty$. Finally with prob. one the expression $(Y_n - b_n)/f(b_n)\log\log n$ has lim inf equal to 0 and lim sup equal to $(\exp c_2 - 1)/c_2$ if $d/dx\{f(x)\log \int_a^x ds/f(s)\} \to c_2 \leqq \infty$ as $x \to \infty$. This last result is an extension of a theorem stated by J. Pickands (*Ann. Math. Statist.* **38** (1967) 1570–1574) and provides an answer to one of his questions. (Received October 9, 1970.)