The Annals of Mathematical Statistics
1972, Vol. 43, No. 1, 369-370

A REMARK ON NONATOMIC MEASURES

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In this paper the following result is proved:

THEOREM. *If λ is a measure on a product σ-algebra $\mathfrak{A} \times \mathscr{B}$ and λ_1 and λ_2 are the corresponding marginals on $\mathfrak{A}$ and $\mathscr{B}$ respectively, then λ is nonatomic* iff *at least one of λ_1 and λ_2 is nonatomic.*

By a measure we always mean a nonzero finite measure defined on a σ-algebra of subsets of a space. A σ-algebra $\mathfrak{A}$ of subsets of X is said to be atomic if X is the union of all atoms of $\mathfrak{A}$. A measure μ on an atomic σ-algebra $\mathfrak{A}$ is said to be continuous if $\mu(A) = 0$ for every atom A of $\mathfrak{A}$. A σ-algebra is said to be separable if it has a countable generator. An element $A \in \mathfrak{A}$ is said to be a μ-atom if $\mu(A) > 0$ and $B \in \mathfrak{A}$, $B \subset A$ implies $\mu(B) = 0$ or $\mu(B) = \mu(A)$. μ on $\mathfrak{A}$ is said to be nonatomic if there are no μ-atoms in $\mathfrak{A}$. μ on $\mathfrak{A}$ is said to be 0–1 valued if it takes only two values 0 and 1. If $\mathscr{D}$ is any collection of subsets of X, $\sigma(\mathscr{D})$ denotes the σ-algebra generated by $\mathscr{D}$ on X.

The following well-known results will be used without mention in the sequel. Every separable σ-algebra is atomic. There is no 0–1 valued continuous measure on a separable σ-algebra. From this it follows that a measure μ on a separable σ-algebra is nonatomic iff it is continuous. For any nonatomic measure μ on $\mathfrak{A}$, $A \in \mathfrak{A}$, $\mu(A) > 0$, $0 < \alpha < \mu(A)$ implies there is a $B \in \mathfrak{A}$, $B \subset A$ such that $\mu(B) = \alpha$ (see [1] page 174).

LEMMA. *Let μ be a nonatomic measure on a σ-algebra $\mathfrak{A}$ and $A \in \mathfrak{A}$. Then there exists a separable sub σ-algebra $\mathscr{B}$ of $\mathfrak{A}$ containing A such that μ is nonatomic on $\mathscr{B}$.*

PROOF. *Case* (i). $\mu(A) > 0$ and $\mu(A^c) > 0$.

Let $A_0 = A$ and $A_1 = A^c$. We can define for every finite sequence $i_1, i_2, \cdots, i_k$ of 0's and 1's sets $A_{i_1, i_2, \cdots, i_k}$ from $\mathfrak{A}$ by induction on k satisfying

(a) $$A_{i_1, i_2, \cdots, i_k} \subset A_{i_1, i_2, \cdots, i_{k-1}},$$

(b) $$A_{i_1, i_2, \cdots, i_{k-1}, 0} \cup A_{i_1, i_2, \cdots, i_{k-1}, 1} = A_{i_1, i_2, \cdots, k-1},$$

(c) $$A_{i_1, i_2, \cdots, i_{k-1}, 0} \cap A_{i_1, i_2, \cdots, i_{k-1}, 1} = \emptyset,$$

(d) $$\mu(A_{i_1, i_2, \cdots, i_k}) = \tfrac{1}{2}\mu(A_{i_1, i_2, \cdots, i_{k-1}}).$$

Let $\mathscr{B} = \sigma\{A_{i_1, \cdots, i_k}\}$. Then $A \in \mathscr{B}$ and μ is continuous on $\mathscr{B}$. Hence μ is nonatomic on $\mathscr{B}$.

Received November 6, 1970.

Case (ii). $\mu(A) = 1$. Let $B \in \mathfrak{A}$ be such that $B \subset A$, $\mu(B) > 0$ and $\mu(A - B) > 0$. Take $A_0 = B$, $A_1 = A - B$ in the construction of Case (i).

THEOREM. *Let $\mathfrak{A}$ and $\mathscr{B}$ be σ-algebras on X and Y respectively, $\mathfrak{A} \times \mathscr{B}$ the product σ-algebra of $\mathfrak{A}$ and $\mathscr{B}$ on $X \times Y$, λ any measure on $\mathfrak{A} \times \mathscr{B}$, and λ_1 and λ_2 the marginals of λ on $\mathfrak{A}$ and $\mathscr{B}$ respectively. Then λ is nonatomic* iff *at least one of λ_1 and λ_2 is nonatomic.*

PROOF. 'if' part: Assume that λ_1 is nonatomic. Let $C \in \mathfrak{A} \times \mathscr{B}$ be such that $\lambda(C) > 0$. Then $C \in \sigma\{A_i \times B_i; i \geqq 1\}$ for some A_i's in $\mathfrak{A}$ and B_i's in $\mathscr{B}$. Using the Lemma we can find a separable sub σ-algebla $\mathscr{D}$ of $\mathfrak{A}$ such that $A_i \in \mathscr{D}$ for all $i \geqq 1$ and λ_1 is nonatomic on $\mathscr{D}$. Let $\mathscr{C}$ be the product σ-algebra $\mathscr{D} \times \sigma\{B_i\}$. Observe that '$C \in \mathscr{C}$' and '$\mathscr{C} \subset \mathfrak{A} \times \mathscr{B}$. Since $\mathscr{C}$ is separable and λ_1 is continuous on $\mathscr{D}$, λ is continuous on $\mathscr{C}$' and hence λ is nonatomic on $\mathscr{C}$.

'only if' part: It is sufficient to prove that if λ_1 and λ_2 are 0–1 valued so is λ. Look at $\mathscr{D} = \{C \in \mathfrak{A} \times \mathscr{B} : \lambda(C) = 0 \text{ or } 1\}$. $\mathscr{D}$ contains $\mathfrak{A} \times Y$ and $X \times \mathscr{B}$. Hence $\mathscr{D} = \mathfrak{A} \times \mathscr{B}$.

COROLLARY. *Let λ_1 and λ_2 be measures on $\mathfrak{A}$ and $\mathscr{B}$ respectively. Let $\lambda_1 \times \lambda_2$ be the product measure on $\mathfrak{A} \times \mathscr{B}$. Then $\lambda_1 \times \lambda_2$ is nonatomic* iff *at least one of λ_1 and λ_2 is nonatomic.*

REMARKS. (1) The previous corollary can be easily extended to countable product spaces.

(2) We shall give a proof of the following result using the Lemma which can be proved with the help of the Radon-Nikodym theorem also.

If μ and λ are two measures on a σ-algebra $\mathfrak{A}$, μ nonatomic on $\mathfrak{A}$ and λ is absolutely continuous w.r.t. μ, then λ is nonatomic on $\mathfrak{A}$.

PROOF. For $A \in \mathfrak{A}$ such that $\lambda(A) > 0$, by the Lemma we can find a separable sub σ-algebra $\mathscr{B} \subset \mathfrak{A}$ such that $A \in \mathscr{B}$ and μ is nonatomic on $\mathscr{B}$. Then μ is continuous on $\mathscr{B}$ and hence λ is continuous on $\mathscr{B}$, from which it follows that λ is nonatomic on $\mathscr{B}$.

Acknowledgments. The authors are grateful to Professors J. K. Ghosh and A. Maitra for their helpful comments.

REFERENCE

[1] HALMOS, P. R. (1950). *Measure Theory*. Van Nostrand, New York.