The Annals of Mathematical Statistics
1972, Vol. 43, No. 3, 948–967

ON MIXTURE, QUASI-MIXTURE AND NEARLY NORMAL RANDOM PROCESSES[1]

BY R. D. MARTIN AND S. C. SCHWARTZ

University of Washington and Princeton University

1. Summary. The properties of both a mixture random process, specified by the multi-dimensional simple mixtures, $\hat{F}(x_1, x_2, \cdots, x_n) = a_1F_1(x_1, x_2, \cdots, x_n) + a_2F_2(x_1, x_2, \cdots, x_n)$, $a_1 + a_2 = 1$, and a related quasi-mixture process are investigated. It is shown that if the set of random variables of the component cdf's (cumulative distribution functions) are independent, then the random variables of the resulting mixture are independent if and only if the mixture cdf $\hat{F}$ is degenerate. The quasi-mixture process, on the other hand, does have the property that factorization of the component cdf's implies factorization of the resulting mixture cdf. Specializing to the case of Gaussian cdf's, it is further shown that the GMP (Gaussian Mixture Process) never satisfies the strong mixing condition, while with reasonable assumptions on the component correlation functions the GQMP (Gaussian Quasi-Mixture Process) does satisfy the strong mixing condition. These, and other properties of the resulting mixture cdf's are of importance when mixture processes are used as models in various estimation and hypothesis testing problems. Some examples are also given for generating GMP and GQMP processes.

2. Introduction. The study of the identifiability of finite mixtures initiated by Teicher (e.g., [10]) has recently been extended to include multi-dimensional cdf's [14], [15]. Finite mixtures are appropriate models in many statistical problems and the identifiability of such mixtures is of prime significance. Of equal importance in problems of hypothesis testing and estimation are the properties enjoyed by mixture cdf's and the corresponding mixture random process. In this paper we investigate three relevant properties of the simple mixture (Section 3) and the quasi-mixture (Section 4) processes: the relationship between factorization of the component cdf's and the mixture cdf; the strong mixing property; and invariance under linear transformation. The latter two properties are studied with the component cdf's taken as Gaussian. The resulting nearly normal processes appear to be useful models for testing and estimation problems, in both continuous and discrete time, when the noise process is non-white, as can be seen from [7] and the following discussion.

The need for nonnormal random processes arises when one is interested in determining the robustness of hypothesis testing or estimation procedures. For this situation, the statistic, $\mathscr{T}_T$, upon which the test or estimate is based, is a

Received April 29, 1970; revised May 12, 1971.

[1] Research supported in part by NSF under grants NSF-GK-1439, NSF-GK-5338 and NSF-GK-13193.

functional of the observed process, $x(t)$, on $[0, T]$. When the process is non-Gaussian, there is generally little hope of finding the cdf of $\mathscr{T}_T$. An exceptional circumstance would be when $\mathscr{T}_T$ is a linear functional and the nth order cdf's of $x(t)$ are invariant under linear transformations. Hence, the importance of invariance properties. It is shown in Section 8 that the Gaussian mixture process is invariant under linear transformation.

When the cdf of the statistic $\mathscr{T}_T$ cannot be found for finite T, an alternate approach is to obtain sufficient conditions under which $\mathscr{T}_T$ is asymptotically normal. The test functionals for optimal procedures in the Gaussian case are of the form $W^{-1}(T) \int_0^T q_T(t)x(t)\,dt$ in the continuous time case and $W^{-1}(T) \sum_{t=0}^T q_T(t)x(t)$ in the discrete time case. Since one of the sufficient conditions for asymptotic normality of functionals of this form is that $x(t)$ satisfies the strong mixing condition [8], it is reasonable to attempt to find nonnormal processes which satisfy the strong mixing condition. As might be expected from the factorization property discussed above and in Sections 3, 4, the GMP never satisfies the strong mixing condition while the GQMP does. In fact, it is the lack of factorization of the simple mixture which leads to the definition of the quasi-mixture and the subsequent specialization to the GQMP.

3. The mixture process. Let $F_1(x_1, \cdots, x_n)$ and $F_2(x_1, \cdots, x_n)$ be the joint distribution functions of two stochastic processes, x_t^1 and x_t^2, $-\infty < t < +\infty$, for all time sets $\{t_1, \cdots, t_n\}$, $n \geqq 1$. In the discrete time case we take $-\infty < t < +\infty$ to mean that t takes on all integer values. Let a_1 and a_2 be given such that $a_1, a_2 \geqq 0$ and $a_1 + a_2 = 1$. Then

$$\hat{F}(x_1, \cdots, x_n) = a_1 F_1(x_1, \cdots, x_n) + a_2 F_2(x_1, \cdots, x_n) \tag{1}$$

is a distribution function which we refer to as a simple mixture. Since the cdf's F_1 and F_2 are those of a stochastic process, they satisfy the Kolmogorov symmetry and consistency conditions. It is clear that $\hat{F}$ satisfies these conditions by virtue of its definition and, hence, there exists a stochastic process $\hat{x}_t$ having $\hat{F}$ for its cdf's. We refer to the process $\hat{x}_t$ as a mixture process.

DEFINITION. The simple mixture $\hat{F}$ will be called nondegenerate if $a_1 > 0$, $a_2 > 0$ and $F_1 \not\equiv F_2$; otherwise $\hat{F}$ is called degenerate. If $\hat{F}(x_1, \cdots, x_n)$ is nondegenerate for all sets $\{t_1, \cdots, t_n\}$, $n \geqq 1$ then $\hat{x}_t$ is said to be a nondegenerate mixture process.

At first thought one might hope that independence of a set of random variables of the processes x_t^1 and x_t^2 would imply independence for the corresponding random variables of the mixture process. Unfortunately this is not the case; the pertinent result is stated as

LEMMA 1. *Suppose* $\hat{F}(x_1, \cdots, x_n) = a_1 \prod_1^n F_1(x_i) + a_2 \prod_1^n F_2(x_i)$. *Then* $\hat{F}(x_1, \cdots, x_n) = \prod_1^n \hat{F}(x_i)$ *if and only if* $\hat{F}$ *is degenerate.*

PROOF. The "if" part is obvious. For the "only if" part let

$$\hat{F}(x_1, \cdots, x_n) = \prod_1^n \hat{F}(x_i) = \prod_1^n [a_1 F_1(x_i) + a_2 F_2(x_i)] .$$

By assumption

$$\hat{F}(x_1, \cdots, x_n) = a_1 \prod_1^n F_1(x_i) + a_2 \prod_1^n F_2(x_i)$$

so that for $n = 2$

$$\prod_1^2 [a_1 F_1(x_i) + a_2 F_2(x_i)] - [a_1 \prod_1^2 F_1(x_i) + a_2 \prod_1^2 F_2(x_i)] = 0$$

or

$$a_1 a_2 [F_1(x_1) - F_2(x_1)] \cdot [F_1(x_2) - F_2(x_2)] = 0 .$$

Since the above relation must hold for all x_1, x_2, $\hat{F}$ must degenerate.

4. The quasi-mixture process. We now introduce a "quasi-mixture" process which has the property, which the mixture process lacks, that factorization of the component df's implies the factorization of the df's of the process itself.

Let $F_1^n(x_1, \cdots, x_n)$ and $F_2^n(x_1, \cdots, x_n)$ be the cdf's of two continuous parameter stochastic processes x_t^1 and x_t^2, for the time sets $\{t_1, \cdots, t_n\}$. These cdf's necessarily satisfy the Kolmogorov symmetry and consistency conditions. Now let

$$G_k^n = G_k^n(x_1, \cdots, x_n) =_\Delta \sum_{\Delta_k^n} F_1^k(x_{i_1}, \cdots, x_{i_k}) F_2^{n-k}(x_{i_{k+1}}, \cdots, x_{i_n}) \tag{2}$$

where $0 \leqq k \leqq n$; $F_1^0 = F_2^0 =_\Delta 1$; $i_1 < i_2 < \cdots < i_k$; $i_{k+1} < i_{k+2} < \cdots < i_n$ and where Δ_k^n indicates the summation over all the $\binom{n}{k}$ ways of selecting k integers from the set $\{1, 2, \cdots, n\}$. Now, with $a_1, a_2 \geqq 0$, $a_1 + a_2 = 1$, we define the quasi-mixture process by

$$\tilde{F}^n(x_1, \cdots, x_n) =_\Delta \sum_{k=0}^n a_1^k a_2^{n-k} G_k^n(x_1 \cdots, x_n) \tag{3}$$

$\tilde{F}^n$ as defined is obviously an nth order probability distribution function. The existence of a stochastic process, $\tilde{x}_t$, having $\tilde{F}^n(x_1, \cdots, x_n)$ as its cdf's for the time sets $\{t_1, \cdots, t_n\}$ is guaranteed provided the Kolmogorov symmetry and consistency conditions are satisfied. In the following theorem we show that these conditions are in fact satisfied.

THEOREM 1. *There exists a stochastic process, $\tilde{x}_t$, having $\tilde{F}^n(x_1, \cdots, x_n)$ as defined by* (3) *for its* df's *for the time sets* $\{t_1, \cdots, t_n\}$.

PROOF. (i) *Symmetry*—Consider the effect on $G_k^n(x_1, \cdots, x_n)$ of interchanging two of its arguments, say x_l and x_m. This exchange occurs in every term of $\sum_{\Delta_k^n} F_1^k(x_{i_1}, \cdots, x_{i_k}) F_2^{n-k}(x_{i_{k+1}}, \cdots, x_{i_n})$. If in a given term this exchange occurs in one of the sets $\{i_1, \cdots, i_k\}$ or $\{i_{k+1}, \cdots, i_n\}$ then this term is not changed since F_1^k and F_2^{n-k} satisfy the symmetry condition by hypothesis. Suppose the exchange is between an element of $\{i_1, \cdots, i_k\}$ and an element of $\{i_{k+1}, \cdots, i_n\}$ for some term. For every such term there is a matching term for which an opposite exchange takes place. Such pairs of terms merely replace one another and hence

$$G_k{}^n(x_1, \cdots, x_n) = \sum_{\Delta_k{}^n} F_1{}^k(x_{i_1}, \cdots, x_{i_k})F_2{}^{n-k}(x_{i_{k+1}}, \cdots, x_{i_n})$$

is invariant under any permutation of its arguments. The same is true of $\tilde{F}^n(x_1, \cdots, x_n)$ and the symmetry condition is satisfied.

(ii) *Consistency*—Now consider $\lim_{x_n\to\infty} \tilde{F}^n(x_1, \cdots, x_n)$. We have

$$\begin{aligned}\lim_{x_n\to\infty} \tilde{F}^n(x_1, \cdots, x_n) &= \lim_{x_n\to\infty} \sum_{k=0} a_1{}^k a_2{}^{n-k} G_k{}^n(x_1, \cdots, x_n)\\ &= \sum_{k=0}^n a_1{}^k a_2{}^{n-k} \sum_{\Delta_k{}^n} \lim_{x_n\to\infty} F_1{}^k(x_{i_1}, \cdots, x_{i_k})\\ &\qquad \times F_2{}^{n-k}(x_{i_{k+1}}, \cdots, x_{i_n})\,.\end{aligned}$$

Let

$$\begin{aligned}G_k{}^n(\infty) &=_\Delta \lim_{x_n\to\infty} G_k{}^n(x_1, \cdots, x_n)\\ &= G_k{}^n(x_1, \cdots, x_{n-1}, \infty)\,.\end{aligned}$$

Then

$$\begin{aligned}G_0{}^n(\infty) &= F_2{}^n(x_1, \cdots, x_{n-1}, \infty)\\ &= F_2{}^{n-1}(x_1, \cdots, x_{n-1})\\ &= G_0{}^{n-1}\\ G_n{}^n(\infty) &= F_1{}^n(x_1, \cdots, x_{n-1}, \infty)\\ &= F_1{}^{n-1}(x_1, \cdots, x_{n-1})\\ &= G_{n-1}^{n-1}\,.\end{aligned}$$

Now consider $G_k{}^n(\infty)$ for $1 \leqq k \leqq n-1$. $G_k{}^n(\infty)$ contains $\binom{n-1}{k-1}$ terms for which $x_{i_k} = \infty$, and for each of these terms we have $F_1{}^k(x_{i_1}, \cdots, x_{i_{k-1}}, \infty) = F_1{}^{k-1}(x_{i_1}, \cdots, x_{i_{k-1}})$. Since for each of these terms $i_k = n$, it must be that $\{i_1, \cdots, i_{k-1}\}$ is some combination of the set $\{1, 2, \cdots, n-1\}$, and $\{i_{k+1}, \cdots, i_n\}$ are the remaining integers. Thus for these terms we may write

$$F_2{}^{n-k}(x_{i_{k+1}}, \cdots, x_{i_n}) = F_2{}^{(n-1)-(k-1)}(x_{i_k}, \cdots, x_{i_{n-1}})$$

with $\{i_k, \cdots, i_{n-1}\} = \{1, 2, \cdots, n-1\} - \{i_1, \cdots, i_{k-1}\}$.

Hence there are $\binom{n-1}{k-1}$ terms of the form $F_1{}^{k-1}(x_{i_1}, \cdots, x_{i_{k-1}})F_2{}^{(n-1)-(k-1)}(x_{i_k}, \cdots, x_{i_{n-1}})$ with each $\{i_1, \cdots, i_{k-1}\}$ being a distinct combination of $k-1$ integers from the set $\{1, 2, \cdots, n-1\}$.

Thus

$$\begin{aligned}G_k{}^n(\infty) &= \sum_{\Delta_{k-1}^{n-1}} F_1{}^{k-1}(x_{i_1}, \cdots, x_{i_{k-1}})F_2{}^{(n-1)-(k-1)}(x_{i_k}, \cdots, x_{i_{n-1}})\\ &\qquad + \sum \quad (\text{terms with} \quad i_n = \infty)\\ &= G_{k-1}^{n-1}(x_1, \cdots, x_{n-1}) + \sum \quad (\text{terms with} \quad i_n = \infty)\,.\end{aligned}$$

The number of terms for which $i_n = \infty$ is $\binom{n-1}{n-k-1} = \binom{n-1}{k}$, and each of these terms is of the form

$$\begin{aligned}F_1{}^k(x_{i_1}, \cdots, x_{i_k})&F_2{}^{n-k}(x_{i_{k+1}}, \cdots, x_{i_{n-1}}, \infty)\\ &= F_1{}^k(x_{i_1}, \cdots, x_{i_k})F_2{}^{n-1-k}(x_{i_{k+1}}, \cdots, x_{i_{n-1}})\end{aligned}$$

with the $\{i_1, \cdots, i_k\}$ of each term being a distinct combination of k integers from the set $\{1, 2, \cdots, n-1\}$. This means that

$$\sum \text{ (terms with } \ i_n = \infty) = G_k^{n-1}(x_1, \cdots, x_n)$$

and so

$$G_k^n(\infty) = G_{k-1}^{n-1}(x_1, \cdots, x_{n-1}) + G_k^{n-1}(x_1, \cdots, x_{n-1}) .$$

Dropping arguments for convenience we have

$$\begin{aligned}
\lim_{x_n \to \infty} \tilde{F}^n(x_1, \cdots, x_n) &= \textstyle\sum_{k=0}^n a_1^k a_2^{n-k} G_k^n(\infty) \\
&= a_2^n G_0^{n-1} + \textstyle\sum_1^{n-1} a_1^k a_2^{n-k}[G_{k-1}^{n-1} + G_k^{n-1}] + a_1^n G_{n-1}^{n-1} \\
&= a_2^n G_0^{n-1} + \textstyle\sum_{k=0}^{n-2} a_1^{k+1} a_2^{n-k-1} G_k^{n-1} \\
&\qquad + \textstyle\sum_{k=1}^{n-1} a_1^k a_2^{n-k} G_k^{n-1} + a_1^n G_{n-1}^{n-1} \\
&= a_2^n G_0^{n-1} + a_1 a_2^{n-1} G_0^{n-1} \\
&\qquad + \textstyle\sum_{k=1}^{n-2} [a_1^{k+1} a_2^{n-k-1} + a_1^k a_2^{n-k}] G_k^{n-1} \\
&\qquad + a_1^{n-1} a_2^{n-n+1} G_{n-1}^{n-1} + a_1^n G_{n-1}^{n-1} \\
&= a_2^{n-1}(a_1 + a_2) G_0^{n-1} + \textstyle\sum_{k=1}^{n-2} a_1^k a_2^{n-1-k}(a_1 + a_2) G_k^{n-1} \\
&\qquad + a_1^{n-1}(a_1 + a_2) G_{n-1}^{n-1} \\
&= \textstyle\sum_{k=0}^{n-1} a_1^k a_2^{(n-1)-k} G_k^{n-1} \\
&= \tilde{F}^{n-1}(x_1, \cdots, x_{n-1}) .
\end{aligned}$$

Therefore the consistency condition is satisfied.

It is to be noted that for a quasi-mixture process the condition $F_1 = F_2 = F$ does not in general imply that $\tilde{F} = F$, in contrast to the situation for a mixture process.

Returning to the quasi-mixture process we note that it has

$$\tilde{F}^1(x_1) = a_1 F_1(x_1) + a_2 F_2(x_1) \tag{4}$$

$$\begin{aligned}\tilde{F}^2(x_1, x_2) = {} & a_1^2 F_1(x_1, x_2) + a_2^2 F_2(x_1, x_2) \\ & + a_1 a_2 [F_1(x_1) F_2(x_2) + F_1(x_2) F_2(x_1)]\end{aligned} \tag{5}$$

as univariate and bivariate distributions. The univariate distribution is identical to that of a mixture process having the same a_1, a_2, F_1 and F_2. The bivariate distribution differs from that of the corresponding mixture process by the inclusion of the terms $F_1(x_1)F_2(x_2)$ and $F_1(x_2)F_2(x_1)$ and by the smaller weighting of the terms $F_i(x_1, x_2)$. It is this aspect of the structure which results in the following lemma.

LEMMA 2. *If the component random variables, $x_1, \cdots, x_n$, are independent, i.e.,*

$$F_j^n(x_1, \cdots, x_n) = \textstyle\prod_{i=1}^n F_j^1(x_i) , \qquad j = 1, 2$$

then the corresponding quasi-mixture random variables are also independent, i.e.,

$$\tilde{F}^n(x_1, \cdots, x_n) = \textstyle\prod_{i=1}^n \tilde{F}^1(x_i) .$$

PROOF. It follows from the hypothesis that

$$F_j^k(x_1, \cdots, x_k) = \prod_{i=1}^k F_j^1(x_i)\,, \qquad j = 1, 2 \quad k \leqq n$$

so that

$$\begin{aligned}\tilde{F}^n(x_1, \cdots, x_n) &= \sum_{k=0}^n a_1{}^k a_2{}^{n-k} G_k{}^n(x_1, \cdots, x_n) \\ &= \sum_{k=0}^n a_1{}^k a_2{}^{n-k} \sum_{\Delta_k{}^n} F_1^1(x_{i_1}) \cdots F_1^1(x_{i_k}) F_2^1(x_{i_{k+1}}) \cdots F_2^1(x_{i_n}) \\ &= \prod_{i=1}^n [a_1 F_1^1(x_i) + a_2 F_2^1(x_i)] \\ &= \prod_{i=1}^n \tilde{F}^1(x_i)\,.\end{aligned}$$

5. The Gaussian mixture process (GMP). Let $x_t{}^1$ and $x_t{}^2$ be stationary, zero mean, Gaussian random processes having correlation functions $R_1(\cdot)$ and $R_2(\cdot)$. Then the df's $F_1(x_1, \cdots, x_n) = \Phi_1(x_1, \cdots, x_n)$ and $F_2(x_1, \cdots, x_n) = \Phi_2(x_1, \cdots, x_n)$ for the various time sets $\{t_1, \cdots, t_n\}$ are Gaussian df's and the set of df's

$$\tilde{F}_G(x_1, \cdots x_n) = a_1 \Phi_1(x_1, \cdots, x_n) + a_2 \Phi_2(x_1, \cdots, x_n) \tag{6}$$

define a Gaussian mixture process (GMP), $\hat{x}_t$. We assume throughout that $\hat{x}_t$ is nondegenerate.

Since zero correlation implies independence for a Gaussian process, it is natural to ask what is the implication of zero correlation for a Gaussian mixture process. Let $x_1 = \hat{x}_t$ and $x_2 = \hat{x}_{t+\tau}$. With $\hat{R}(\tau) =_\Delta E_{\hat{F}_G} x_1 x_2$ it is clear that

$$\hat{R}(\tau) = a_1 R_1(\tau) + a_2 R_2(\tau) \qquad \text{for all} \quad \tau\,. \tag{7}$$

For zero correlation between $\hat{x}_t$ and $\hat{x}_{t+\tau}$ we have

$$a_1 R_1(\tau) + a_2 R_2(\tau) = 0\,. \tag{8}$$

The possible solutions of (8) are

$$R_1(\tau) = R_2(\tau) = 0\,; \tag{9}$$

$$R_1(\tau)/R_2(\tau) = -a_2/a_1\,. \tag{10}$$

DEFINITION. If $R_1(\mu)/R_2(\mu) = -a_2/a_1$ at $\mu = \tau$ then $\hat{x}_t$ is said to be singular at τ; otherwise $\hat{x}_t$ is said to be nonsingular at τ. If $R_1(\mu)/R_2(\mu) \neq -a_2/a_1$ for every μ, then $\hat{x}_t$ is said to be completely nonsingular.

Thus $R_1(\tau) = R_2(\tau) = 0$ is the only solution of (8) for a completely nonsingular Gaussian mixture process, $\hat{x}_t$; conversely, if $\hat{x}_t$ is not completely nonsingular then for some τ there exists a solution of the form (10).

Suppose $\hat{x}_t$ is nonsingular at τ and $\hat{R}(\tau) = 0$. Then $\Phi_i(x_1, x_2) = \Phi_i(x_1)\Phi_i(x_2)$, $i = 1, 2$, i.e., the component random variables are independent. However, according to Lemma 1, the random variables $\hat{x}_t$ and $\hat{x}_{t+\tau}$ are not independent. This feature may be undesirable in some situations. For instance in some cases we may demand that $\hat{R}(\tau) \to 0$ as $\tau \to \infty$ and for such cases asymptotic independence may be expected on an intuitive basis but is not obtained in this case. On the other hand, it may sometimes be desirable to have model which exhibits long term dependence effects (e.g., Van Ness [12]).

6. The Gaussian quasi-mixture process (GQMP). If x_t^1 and x_t^2 are stationary, zero mean, Gaussian random processes having correlation functions $R_1(\cdot)$ and $R_2(\cdot)$, then the cdf's $F_1(x_1, \cdots, x_n) = \Phi_1(x_1, \cdots, x_n)$ and $F_2(x_1, \cdots, x_n) = \Phi_2(x_1, \cdots, x_n)$ for the time sets $\{t_1, \cdots, t_n\}$ are Gaussian cdf's and the cdf's

$$\tilde{F}_G(x_1, \cdots, x_n) = \textstyle\sum_{k=0}^n a_1^k a_2^{n-k} \sum_{\Delta_k^n} \Phi_1(x_{i_1}, \cdots, x_{i_k})\Phi_2(x_{i_{k+1}}, \cdots, x_{i_n}) \tag{11}$$

define a stationary, zero mean, Gaussian quasi-mixture process (GQMP), $\tilde{x}_t$. We assume throughout that $\tilde{x}_t$ is nondegenerate, the definition of degeneracy being the same as for the mixture process.

Let $\hat{R}(\tau) =_\Delta E_{\tilde{F}_G} x_1 x_2$ with $x_1 = x_t$ and $x_2 = x_{t+\tau}$. Then, noting the zero mean assumption on the component processes, we have

$$\tilde{R}(\tau) = a_1^2 R_1(\tau) + a_2^2 R_2(\tau)\,, \qquad |\tau| > 0 \tag{12}$$

$$\operatorname{Var} \tilde{x}_t = \tilde{R}(o) = a_1 \sigma_1^2 + a_2 \sigma_2^2\,. \tag{13}$$

The definition of singularity and nonsingularity for a GQMP is the same as for a GQMP when (10) is replaced by

$$R_1(\tau)/R_2(\tau) = -(a_1/a_2)^2\,. \tag{14}$$

The following theorem is an immediate consequence of Lemma 2.

THEOREM 2. *Let $\tilde{x}_t$ be a stationary* GQMP *with component correlation functions $R_1(\cdot)$ and $R_2(\cdot)$ and with component* cdf's *having zero expectations. If $R_1(\tau) = R_2(\tau) = 0$, or if $\tilde{x}_t$ is singular at τ and $\tilde{R}(\tau) = 0$, then $\tilde{F}_G(x_t x_{t+\tau}) = \tilde{F}_G(x_t)\tilde{F}_G(x_{t+\tau})$.*

EXAMPLE. Suppose

$$R_1(\tau) = \sigma_1^2 e^{-\alpha_1|\tau|}$$
$$R_2(\tau) = \sigma_2^2 e^{-\alpha_2|\tau|}$$

so that

$$\tilde{R}(\tau) = a_1^2\sigma_1^2 e^{-\alpha_1|\tau|} + a_2^2\sigma_2^2 e^{-\alpha_2|\tau|} \qquad |\tau| > 0\,.$$

Then the corresponding GQMP, $\tilde{x}_t$, is completely nonsingular; thus $\tilde{x}_t$ and $\tilde{x}_{t+\tau}$ are asymptotically independent as $\tau \to \infty$.

One way of measuring the deviation from normality for both the GMP and the GQMP is to define a distance from normality, using, for example, the Kolmogorov distance. Let $\hat{\delta}_n = \sup_{\mathbf{x}} |\hat{F}_G(x_1, \cdots, x_n) - \Phi_1(x_1, \cdots, x_n)|$ and $\tilde{\delta}_n = \sup_{\mathbf{x}} |\tilde{F}_G(x_1, \cdots, x_n) - \Phi_1(x_1, \cdots, x_n)|$, i.e., $\hat{\delta}_n$ and $\tilde{\delta}_n$ are the Kolmogorov distances of the nth order GMP and GQMP cdf's from the nominal Gaussian cdf.

Since $\hat{\delta}_n \leqq a_2$, it is clear that a normal process can be approximated by a GMP in the sense that all of its nth order cdf's will be within a distance a_2 of the Gaussian cdf's. It appears that closeness of approximation by a GQMP can be easily specified only in terms of lower order df's; certainly good approximations are easily obtained for the univariate and bivariate df's ($\tilde{\delta}_1 \leqq a_2$ and $\tilde{\delta}_2 \leqq \frac{1}{2}a_2$).

It is because we can approximate a Gaussian process by a GMP or a GQMP in the sense given above that we refer to these as nearly normal processes. Closeness of approximation in terms of $\hat{\delta}_n$ and $\tilde{\delta}_n$ does not, however, imply closeness of moments, and this is precisely why such nearly normal processes are useful in robustness studies [11], [7].

7. Strong-mixing. The strong mixing condition may be stated as follows [9]. Let $x_t = x(t) = x(t, \omega)$, $-\infty < t < \infty$, be a random process jointly measurable in t and ω. As before we take $-\infty < t < \infty$ is to mean t is any integer in the discrete time case. Let $U_{-\infty}^{t_1}$ be the σ-algebra generated by x_t, $t \leqq t_1$, and $U_{t_2}^{+\infty}$ the σ-algebra generated by x_t, $\tau \geqq t_2$. The process x_t is said to satisfy a strong mixing condition if there is some positive function $\alpha(\mu)$ defined for $0 \leqq \mu < \infty$ with $\alpha(\mu) \to 0$ as $\mu \to \infty$ such that for any $A_1 \in U_{-\infty}^{t_1}$, $A_2 \in U_{t_2}^{+\infty}$, $t_1 < t_2$

$$|P(A_1 \cap A_2) - P(A_1)P(A_2)| \leqq \alpha(t_2 - t_1) . \tag{15}$$

For a stationary process we let $t_1 = t$, $t_2 = t + \tau$, and then $t_2 - t_1 = \tau$.

In this section we show that a (nondegenerate) GMP cannot satisfy the strong mixing condition, and that with mild assumptions the GQMP does satisfy the strong mixing condition. Since the strong mixing condition is a form of asymptotic independence, these results are not surprising in view of the comments of the preceding sections concerning factorization and independence.

THEOREM 3. *A nondegenerate stationary Gaussian mixture process never satisfies the strong mixing condition.*

PROOF. Let x_t be a nondegenerate stationary Gaussian mixture process with component variance σ_1^2, σ_2^2 and component correlation functions $\rho_1(\tau)$, $\rho_2(\tau)$. Set $X_1 = x_t$, $X_2 = x_{t+\tau}$ and define the events $A_1 \in U_{-\infty}^{t}$, $A_2 \in U_{t+\tau}^{+\infty}$ by $A_1 = [X_1 \leqq x_1]$, $A_2 = [X_2 \leqq x_2]$. Then $P(A_1 \cap A_2) = \hat{F}_G(x_1, x_2)$, $P(A_j) = \hat{F}_G(x_j)$, $j = 1, 2$, and

$$\begin{aligned} P(A_1 \cap A_2) - P(A_1)P(A_2) &= a_1\Phi_1(x_1, x_2) + a_2\Phi_2(x_1, x_2) \\ &\quad - [a_1\Phi_1(x_1) + a_2\Phi_2(x_1)]\cdot[a_1\Phi_1(x_2) + a_2\Phi_2(x_2)] . \end{aligned}$$

Consider the following cases, where we assume that $\rho_i =_\Delta \lim_{\tau\to\infty} \rho_i(\tau)$ exists, $i = 1, 2$:

(i) $\rho_1 = \rho_2 = 0$. In this case

$$\begin{aligned} \lim_{\tau\to\infty} &|P(A_1 \cap A_2) - P(A_1)P(A_2)| \\ &= \lim_{\tau\to\infty} |[a_1\Phi_1(x_1, x_2) + a_2\Phi_2(x_1, x_2)] \\ &\qquad - [a_1\Phi_1(x_1) + a_2\Phi_2(x_1)][a_1\Phi_1(x_2) + a_2\Phi_2(x_2)]| \\ &= |[a_1\Phi_1(x_1)\Phi_2(x_2) + a_2\Phi_2(x_1)\Phi_2(x_2)] \\ &\qquad - [a_1\Phi_1(x_1) + a_2\Phi_2(x_1)][a_1\Phi_1(x_2) + a_2\Phi_2(x_2)]| \\ &= a_1a_2|[\Phi_1(x_1) - \Phi_2(x_2)]\cdot[\Phi_1(x_2) - \Phi_2(x_2)]| \\ &> 0 \qquad\qquad \text{for} \quad x_1 = x_2\,, \quad |x_1| > 0 \end{aligned}$$

since by hypothesis, $\sigma_1 \neq \sigma_2$.

(ii) $\rho_i \neq 0$ *for at least one i*, $i = 1, 2$. For this case take $A_2 = [x_2 \leqq 0]$. Then

$$\lim_{\tau\to\infty} \{P(A_1 \cap A_2) - P(A_1)P(A_2)\} = \lim_{\tau\to\infty} \{a_1\Phi_1(x_1, 0) + a_2\Phi_2(x_1, 0) - \tfrac{1}{2}[a_1\Phi_1(x_1) + a_2\Phi_2(x_1)]\} .$$

Let

$$H(x_1; \rho_1, \rho_2) =_\Delta \lim_{\tau\to\infty} \{a_1\Phi_1(x_1, 0) + a_2\Phi_2(x_1, 0) - \tfrac{1}{2}a_1\Phi_1(x_1) + a_2\Phi_2(x_1)]\} .$$

We now show that there exists an x_1 such that $|H(x_1; \rho_1, \rho_2)| > 0$. First note that

$$\frac{dH(x_1; \rho_1, \rho_2)}{dx_1} = a_1\frac{d}{dx_1}\Phi_1(x_1, 0) + a_2\frac{d}{dx_1}\Phi_2(x_1, 0) - \tfrac{1}{2}[a_1\phi_{\sigma_1}(x_1) + a_2\phi_{\sigma_2}(x_1)]$$

where $\phi_{\sigma_i}(\cdot)$ is the Gaussian density function with mean zero and variance σ_i^2. Now

$$\begin{aligned}\frac{d}{dx_1}\Phi_i(x_1, 0) &= \int_{-\infty}^{0} \frac{1}{2\pi\sigma_i^2(1-\rho_i^2)^{\frac{1}{2}}} \exp\left[\frac{-x_1^2 - 2\rho_i x_1 y + y^2}{2\sigma_i^2(1-\rho_i^2)}\right] dy \\ &= \int_{-\infty}^{\rho_i x_1} \frac{1}{2\pi\sigma_i^2(1-\rho_i^2)^{\frac{1}{2}}} \exp\left[\frac{-x_1^2(1-\rho_i^2) - v^2}{2\sigma_i^2(1-\rho_i^2)}\right] dv \\ &= \phi_{\sigma_i}(x_1)[\tfrac{1}{2} + \int_0^{-\rho_i x_1} \phi_{\sigma_i(1-\rho^2)^{\frac{1}{2}}}(v)\, dv] , \qquad i = 1, 2\end{aligned}$$

so

$$\frac{dH(x_1; \rho_1, \rho_2)}{dx_1} = a_1\phi_{\sigma_1}(x_1) \int_0^{-\rho_1 x_1} \phi_{\sigma_1(1-\rho_1^2)^{\frac{1}{2}}}(v)\, dv + a_2\phi_{\sigma_2}(x_1) \int_0^{-\rho_2 x_1} \phi_{\sigma_2(1-\rho_2^2)^{\frac{1}{2}}}(v)\, dv .$$

If $\rho_i = 0$, $\rho_j \neq 0$, $i \neq j$, then $|dH(x_1; \rho_1, \rho_2)/dx_1| > 0$ for $|x_1| > 0$. If $\rho_i \neq 0$, $i = 1, 2$, then there exists an x_1 such that $|dH(x_1; \rho_1, \rho_2)/dx_1| > 0$. For assuming otherwise we have

$$\frac{a_1\phi_{\sigma_1}(x_1)}{a_2\phi_{\sigma_2}(x_1)} = -\frac{\int_0^{-\rho_2 x_1} \phi_{\sigma_2(1-\rho_2^2)^{\frac{1}{2}}}(v)\, dv}{\int_0^{-\rho_1 x_1} \phi_{\sigma_2(1-\rho_2^2)^{\frac{1}{2}}}(v)\, dv}$$

for all x_1; but this is a contradiction since by the nondegeneracy assumption the left-hand side of the above equation goes to 0 or ∞ as $x_1 \to \infty$ while the right-hand side goes to ± 1. Since $dH(x_1; \rho_1, \rho_2)/dx_1$ is continuous in x_1 it follows that there exists an x_1 such that $|H(x_1; \rho_1, \rho_2)| > 0$ whenever $\rho_i \neq 0$ for at least one i, $i = 1, 2$. Hence $\lim_{\tau\to\infty} |P(A_1 \cap A_2) - P(A_1)P(A_2)| > 0$ for some x_1 where $A_1 = [x_1 \leqq x_1]$, $A_2 = [x_2 \leqq 0]$.

If both $\lim_{\tau\to\infty} \rho_1(\tau)$ and $\lim_{\tau\to\infty} \rho_2(\tau)$ exist then either case (i) or case (ii) applies with the conclusion that $|P(A_1 \cap A_2) - P(A_1)P(A_2)|$ cannot be founded for all $A_1 \in U_{-\infty}^t$, $A_2 \in U_{t+\tau}^\infty$ by a positive function $\alpha(\cdot)$ for which $\lim_{\tau\to\infty} \alpha(\tau) = 0$. When $\lim_{\tau\to\infty} \rho_i(\tau)$ does not exist for at least one i then the argument for case (ii) may be rephrased so that the result holds for some τ in every neighborhood of ∞.

Hence a nondegenerate stationary Gaussian mixture process never satisfies the strong mixing condition.

It will now be shown that the GQMP satisfies the strong mixing condition provided its component correlation functions satisfy certain sufficient conditions. The proof follows closely the approach used by Kolmogorov and Rozanov to show that under mild restrictions on the spectrum a stationary Gaussian process satisfies the strong mixing condition [6], [9].

We now follow Rozanov's terminology [9]. Given two collections of real random variables $\{x'\} = M'$ and $\{x''\} = M''$ having finite second moments, let

$$\rho(M', M'') = \sup_{x' \in M''; x'' \in M''} \frac{|E(x' - Ex')(x'' - Ex'')|}{[E(x' - Ex')^2 \cdot E(x'' - Ex'')^2]^{\frac{1}{2}}} . \tag{16}$$

Given two σ-algebras, U' and U'', let M' and M'' be families of all real random variables on U' and U'', respectively, which have finite second moments. Then the *maximal correlation coefficient*, $\rho(U', U'')$, is defined by (Rozanov):

$$\rho(U', U'') =_{\Delta} \rho(M', M'') . \tag{17}$$

Let

$$\alpha(U', U'') =_{\Delta} \sup_{A' \in U'; A'' \in U''} |P(A'A'') - P(A')P(A'')| . \tag{18}$$

We have in mind, of course, that with $U' = U_{-\infty}^{t_1}$ and $U'' = U_{t_2}^{+\infty}$, $\alpha(U', U'')$ becomes the function $\alpha(t_2 - t_1)$ of (15).

LEMMA 3.

$$\alpha(U', U'') \leqq \rho(U', U'') . \tag{19}$$

PROOF. Let $\psi_{A'}$ be the characteristic function of $A' \in U'$.

Then

$$E(\psi_{A'} - E\psi_{A'})^2 = P(A')[1 - P(A')] < 1 .$$

Similarly

$$E(\psi_{A''} - E\psi_{A''})^2 < 1,\ A'' \in U'' .$$

Thus

$$|P(A'A'') - P(A')P(A'')| \leqq \frac{|P(A'A'') - P(A')P(A'')|}{[E(\psi_{A'} - E\psi_{A'})^2 \cdot E(\psi_{A''} - E\psi_{A''})^2]^{\frac{1}{2}}}$$

$$|P(A'A'') - P(A')P(A'')| \leqq \frac{|E(\psi_{A'} - E\psi_{A'})(\psi_{A''} - E\psi_{A''})|}{[E(\psi_{A'} - E\psi_{A'})^2 \cdot E(\psi_{A''} - E\psi_{A''})^2]^{\frac{1}{2}}}$$

$$|P(A'A'') - P(A')P(A'')| \leqq \sup_{x' \in M'; x'' \in M''} \frac{|E(x' - Ex')(x'' - Ex'')|}{[E(x' - Ex')^2 E(x'' - Ex'')^2]^{\frac{1}{2}}}$$

$$|P(A'A'') - P(A')P(A'')| \leqq \rho(U', U'')$$

$$\begin{aligned} \alpha(U', U'') &= \sup_{A' \in U'; A'' \in U''} |P(A'A'') - P(A')P(A'')| \\ &\leqq \rho(U' \cdot U'') . \end{aligned}$$

If $\{x'\}$ and $\{x''\}$ are families of random variables having finite second moments we let $H_{x'}$ and $H_{x''}$ be the closed (in mean square) linear manifolds generated by $\{x'\}$ and $\{x''\}$, respectively. Let $U_{x'}$ and $U_{x''}$ be the σ-algebras generated by $\{x'\}$ and $\{x''\}$.

Furthermore let every finite set of random variables comprised of members of $\{x'\}$ and $\{x''\}$ collectively have joint GQM distributions such that the component correlation coefficients are of the same sign. Then we have the following.

THEOREM 4. *There exists a constant C, $1 \leqq C < \infty$, such that*

$$\rho(U_{x'}, U_{x''}) \leqq C \cdot \rho(H_{x'}, H_{x''}) . \tag{20}$$

In order to prove this theorem the following result is needed.

LEMMA 4. *If the random variables x_1 and x_2 have a bivariate Gaussian quasi-mixture distribution with density $\tilde{f}_G$, parameters $a_1, a_2, \sigma_1, \sigma_2 > 0$, correlation coefficient $\tilde{\rho}$, and component correlation coefficients ρ_1 and ρ_2 of the same sign, then there exists a constant C, $0 < C < \infty$, such that*

$$\sup_{g,h} |E_{\tilde{f}_G} g(x_1)h(x_2)| \leqq C \cdot |\tilde{\rho}| \tag{21}$$

where the supremum is taken over all functions g and h for which

$$E_{\tilde{f}_G} g(x_1) = E_{\tilde{f}_G} h(x_2) = 0 ; \tag{22}$$

$$E_{\tilde{F}_G} g^2(x_1) = E_{\tilde{f}_G} h^2(x_2) = 1 . \tag{23}$$

PROOF. We assume without loss of generality that $Ex_1 = Ex_2 = 0$. Then $\tilde{f}_G$ is given by

$$\tilde{f}_G(x_1, x_2) = a_1^2 f_{G_1}(x_1, x_2) + a_2^2 f_{G_2}(x_1, x_2) + a_1 a_2[f_{G_1}(x_1) f_{G_2}(x_2) + f_{G_1}(x_2) f_{G_2}(x_1)]$$

where

$$f_{G_i}(x_1, x_2) = \frac{1}{2\pi\sigma_i^2(1-\rho_i^2)^{\frac{1}{2}}} \exp\left\{-\frac{x_1^2 + x_2^2 - 2\rho_i x_1 x_2}{2\sigma_i^2(1-\rho_i^2)}\right\}, \qquad i = 1, 2$$

$$f_{G_i}(x_j) = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_i} \exp\left\{-\frac{x_j^2}{2\sigma_i^2}\right\}, \qquad i, j = 1, 2 .$$

The $f_{G_i}(x_1, x_2)$ have the following expansions in terms of Hermite polynominals (see, for example, Rozanov, page 182):

$$f_{G_i}(x_1, x_2) = \frac{1}{2\pi\sigma_i^2} \exp\left\{-\frac{x_1^2 + x_2^2}{2\sigma_i^2}\right\} \cdot \sum_{\nu=0}^{\infty} \frac{\rho_i^\nu}{\nu!} H_\nu\left(\frac{x_1}{\sigma_i}\right) H_\nu\left(\frac{x_2}{\sigma_i}\right), \qquad i = 1, 2$$

$$H_\nu(x) =_\Delta (-1)^n e^{\frac{1}{2}x^2} \frac{d^n e^{-\frac{1}{2}x^2}}{dx^n} .$$

Since

$$E_{\tilde{f}_G} g^2(x) = a_1 E_{f_G} g^2(x) + a_2 E_{f_{G_2}} g^2(x) = 1$$
$$E_{\tilde{f}_G} h^2(x) = a_1 E_{f_{G_1}} h^2(x) + a_2 E_{f_{G_2}} h^2(x) = 1$$

it follows that $g(\cdot)$ and $h(\cdot)$, being square integrable with respect to $e^{-\frac{1}{2}x^2}$, have the following alternate expansions (in the mean square sense):

$$g(x) = \sum_{\nu=0}^{\infty} \frac{\alpha_\nu{}^1 H_\nu\left(\frac{x}{\sigma_1}\right)}{\nu!}\,; \qquad g(x) = \sum_{\nu=0}^{\infty} \frac{\alpha_\nu{}^2 H_\nu\left(\frac{x}{\sigma_2}\right)}{\nu!}$$

$$h(x) = \sum_{\nu=0}^{\infty} \frac{\beta_\nu{}^1 H_\nu\left(\frac{x}{\sigma_1}\right)}{\nu!}\,; \qquad h(x) = \sum_{\nu=0}^{\infty} \frac{\beta_\nu{}^2 H_\nu\left(\frac{x}{\sigma_2}\right)}{\nu!}$$

where

$$\alpha_\nu{}^i = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_i} \int_{-\infty}^{\infty} g(x) H_\nu\left(\frac{x}{\sigma_i}\right) \exp\left[-x^2/2\sigma_i{}^2\right] dx$$

$$\beta_\nu{}^i = \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_i} \int_{-\infty}^{\infty} h(x) H_\nu\left(\frac{x}{\sigma_i}\right) \exp\left[-x^2/2\sigma_i{}^2\right] dx \qquad i = 1, 2\,.$$

Now

$$E_{\tilde{f}_G} g(x_1)h(x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{a_1{}^2 g(x_1)h(x_2) f_{G_1}(x_1, x_2) + a_2{}^2 g(x_1)h(x_2) f_{G_2}(x_1, x_2) + a_1 a_2 g(x_1)h(x_2)[f_{G_1}(x_1)f_{G_2}(x_2) + f_{G_1}(x_2)f_{G_2}(x_1)]\}\, dx_1 dx_2\,.$$

Let

$$E_{\tilde{f}_G} g(x_1)h(x_2) = I_1 + I_2 + I_3 + I_4 \qquad \text{where} \quad I_1, I_2, I_3$$

and I_4 represent the four integrals comprising $E_{\tilde{f}_G} g(x_1)h(x_2)$.

Using the expansions of $g(x_1)$ and $h(x_2)$ with respect to $f_{G_1}(x_i)$, $i = 1, 2$ respectively, we have

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1{}^2 g(x_1)h(x_2) f_{G_1}(x_1, x_2)\, dx_1 dx_2$$

$$I_1 = a_1{}^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha_j{}^1 H_j\left(\frac{x_1}{\sigma_1}\right)}{j!} \cdot \sum_{k=0}^{\infty} \frac{\beta_k{}^1 H_k\left(\frac{x_2}{\sigma_1}\right)}{k!} \times \frac{1}{2\pi\sigma_1{}^2} \exp\left\{-\frac{x_1{}^2 + x_2{}^2}{2\sigma_1{}^2}\right\} \cdot \sum_{l=0}^{\infty} \frac{\rho_1{}^2}{l!} H_l\left(\frac{x_1}{\sigma_1}\right) H_l\left(\frac{x_2}{\sigma_2}\right) dx_1 dx_2$$

$$I_1 = a_1{}^2 \sum_{l=0}^{\infty} \frac{\rho_1{}^l}{l!} \alpha_l{}^1 \beta_l{}^1\,.$$

Similarly, using the expansions of $g(x_1)$ and $h(x_2)$ with respect to $f_{G_2}(x_i)$, we obtain

$$I_2 = a_2{}^2 \sum_{l=0}^{\infty} \frac{\rho_2{}^l}{l!} \alpha_l{}^2 \beta_l{}^2\,.$$

Using mixed expansions for $g(x_1)$ and $h(x_2)$ we have

$$I_3 = a_1 a_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)h(x_2) f_{G_1}(x_1) f_{G_2}(x_2)\, dx_1 dx_2$$

$$I_3 = a_1 a_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha_j^1 H_j\left(\frac{x_1}{\sigma_1}\right)}{j!} \cdot \sum_{k=0}^{\infty} \frac{\beta_k^2 H_k\left(\frac{x_2}{\sigma_2}\right)}{k!}$$
$$\times \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)\right\} dx_1 dx_2$$
$$I_3 = a_1 a_2 \alpha_0^1 \beta_0^2$$

and similarly $I_4 = a_1 a_2 \alpha_0^2 \beta_0^1$ so that

$$E_{\tilde{f}_G} g(x_1)h(x_2) = a_1^2 \sum_{l=0}^{\infty} \frac{\rho_1^l}{l!} \alpha_l^1 \beta_l^1 + a_2^2 \sum_{l=0}^{\infty} \frac{\rho_2^l}{l!} \alpha_l^2 \beta_l^2 + a_1 a_2 [\alpha_0^l \beta_0^2 + \alpha_0^2 \beta_0^1]$$
$$E_{\tilde{f}_G} g(x_1)h(x_2) = \sum_{l=1}^{\infty} \left[a_1^2 \frac{\rho_1^l}{l!} \alpha_l^1 \beta_l^1 + a_2^2 \frac{\rho_2^l}{l!} \alpha_l^2 \beta_l^2 \right]$$
$$+ [a_1 \alpha_0^1 + a_2 \alpha_0^2] \cdot [a_1 \beta_0^1 + a_2 \beta_0^2] .$$

But

$$[a_1 \alpha_0^1 + a_2 \alpha_0^2] \cdot [a_1 \beta_0^1 + a_2 \beta_0^2] = E_{\tilde{f}_G} g(x) \cdot E_{\tilde{f}_G} h(x)$$
$$= 0$$

so

$$E_{\tilde{f}_G} g(x_1)h(x_2) = a_1^2 \sum_{\nu=1}^{\infty} \frac{\rho_1^\nu}{\nu!} \alpha_\nu^1 \beta_\nu^1 + a_2^2 \sum_{\nu=1}^{\infty} \frac{\rho_2^2}{\nu!} \alpha_\nu^2 \beta_\nu^2$$
$$|E_{\tilde{f}_G} g(x_1)h(x_2)| \leqq a_1^2 |\rho_1| \sum_1^{\infty} \frac{|\alpha_\nu^1 \beta_\nu^1|}{\nu!} + a_2^2 |\rho_2| \sum_1^{\infty} \frac{|\alpha_\nu^2 \beta_\nu^2|}{\nu!}$$
$$\leqq a_1^2 |\rho_1| \left[\sum_1^{\infty} \frac{|\alpha_\nu^1|^2}{\nu!} \right]^{\frac{1}{2}} \cdot \left[\sum_1^{\infty} \frac{|\beta_\nu^1|^2}{\nu!} \right]^{\frac{1}{2}}$$
$$+ a_2^2 |\rho_2| \left[\sum_1^{\infty} \frac{|\alpha_\nu^2|^2}{\nu!} \right]^{\frac{1}{2}} \cdot \left[\sum_1^{\infty} \frac{|\beta_\nu^2|^2}{\nu!} \right]^{\frac{1}{2}} .$$

The normalization to unity variance leads to

$$E_{\tilde{f}_G} g^2(x) = a_1 E_{f_{G_1}} g^2(x) + a_2 E_{f_{G_2}} g^2(x)$$
$$= a_1 \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha_k^1 H_k\left(\frac{x}{\sigma_1}\right)}{j!} \cdot \sum_{k=0}^{\infty} \frac{\alpha_k^1 H_k\left(\frac{x}{\sigma_1}\right)}{k!} \cdot \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_1} \exp\left[-\frac{x^2}{2\sigma_1^2}\right] dx$$
$$+ a^2 \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{\alpha_j^2 H_j\left(\frac{x}{\sigma_2}\right)}{j!} \cdot \sum_{k=0}^{\infty} \frac{\alpha_k^2 H_k\left(\frac{x}{\sigma_2}\right)}{k!} \cdot \frac{1}{(2\pi)^{\frac{1}{2}}\sigma_2} \exp\left[-\frac{x^2}{2\sigma_2^2}\right] dx$$
$$= a_1 \sum_{k=0}^{\infty} \frac{|\alpha_k^1|^2}{k!} + a_2 \sum_{k=0}^{\infty} \frac{|\alpha_k^2|^2}{k!}$$
$$= 1$$

and in the same manner

$$a_1 \sum_{k=0}^{\infty} \frac{|\beta_k^1|^2}{k!} + a_2 \sum_{k=0}^{\infty} \frac{|\beta_k^2|^2}{k!} = 1 .$$

Thus

$$\sum_{k=0}^{\infty} \frac{|\alpha_k^i|^2}{k!} \leqq 1/a_i$$

$$\sum_{k=0}^{\infty} \frac{|\beta_k^i|^2}{k!} \leqq 1/a_i \qquad i = 1, 2 .$$

With $m =_\Delta \max \{1/a_1, 1/a_2\}$

$$|E_{\tilde{f}_G} g(x_1)h(x_2)| < m \cdot \{a_1^2|\rho_1| + a_2^2|\rho_2|\} .$$

Now

$$\begin{aligned}\tilde{\rho} &= E_{\tilde{f}_G} \frac{x_1 x_2}{a_1\sigma_1^2 + a_2\sigma_2^2} \\ &= \frac{a_1^2 E_{f_{G_1}} x_1 x_2 + a_2^2 E_{f_{G_2}} x_1 x_2}{a_1\sigma_1^2 + a_2\sigma_2^2} \\ &= \frac{a_1^2\rho_1\sigma_1^2 + a_2^2\rho_2\sigma_2^2}{a_1\sigma_1^2 + a_2\sigma_2^2} .\end{aligned}$$

Rearranging and noting that ρ_1 and ρ_2 have the same sign we have

$$\begin{aligned}\left[\frac{a_1}{\sigma_2^2} + \frac{a_2}{\sigma_1^2}\right]\cdot[\sigma_1^2 + \sigma_2^2]\cdot|\tilde{\rho}| &= a_1^2|\rho_1|\frac{\sigma_1^2 + \sigma_2^2}{\sigma_2^2} + a_2^2|\rho_2|\frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2} \\ &\geqq a_1^2|\rho_1| + a_2^2|\rho_2| .\end{aligned}$$

With

$$C =_\Delta m \cdot \left[\frac{a_1}{\sigma_2^2} + \frac{a_2}{\sigma_1^2}\right]\cdot[\sigma_1^2 + \sigma_2^2]$$

we get

$$|E_{\tilde{f}_G} g(x_1)h(x_2)| \leqq C \cdot |\tilde{\rho}|$$

and the result follows.

PROOF OF THEOREM 4. The proof follows very closely the proof of Theorem 10.1 of Rozanov [9]. His proof utilizes Lemmas 10.1, 10.2 and 10.3 of [9]. Our Lemma 4 replaces his Lemma 10.2 with obvious modifications in the resulting proof.

Now let $H_{-\infty}^t$ and $H_{t+\tau}^{+\infty}$ be the closed linear manifolds generated by $x(u)$ for $u \leqq t$ and $\mu \geqq t + \tau$, respectively.

THEOREM 5. *If $\tilde{x}_t$ is a stationary nondegenerate Gaussian quasi-mixture process having component correlation functions of the same sign in some neighborhood of ∞, and if*

$$\lim_{\tau\to\infty} \rho(H_{-\infty}^t, H_{t+\tau}^{+\infty}) = 0 \tag{24}$$

then $\tilde{x}_t$ satisfies the strong mixing condition.

PROOF. The proof is an immediate consequence of Lemma 3 and Theorem 4 with

$$U_{x'} = U' = U_{-\infty}^{t}, \qquad H_{x'} = H_{-\infty}^{t}$$
$$U_{x''} = U'' = U_{t+\tau}^{+\infty}, \qquad H_{x''} = H_{t+\tau}^{+\infty}$$
$$\alpha(t_2 - t_1) = \alpha(\tau) = \rho(U_{-\infty}^{t}, U_{t+\tau}^{+\infty}) .$$

Let $\tilde{\phi}(\lambda)$ be the spectrum of the GQMP, $\tilde{x}_t$. From (12) and (13) we have

$$\tilde{\phi}(\lambda) = a_1^2\phi_1(\lambda) + a_2^2\phi_2(\lambda) \tag{25}$$

where $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are the spectrum corresponding to $R_1(\cdot)$ and $R_2(\cdot)$, respectively. Sufficient conditions for (24) to hold have been given by Rozanov. In terms of the spectrum, $\tilde{\phi}(\lambda)$, they are as follows [9]:
If $\phi(\lambda)$ is uniformly continuous on the entire real line, does not vanish, and satisfies the inequality

$$\frac{m}{\lambda^k} \leqq \phi(\lambda) \leqq \frac{M}{\lambda^{k-1}} \tag{26}$$

for sufficiently large λ, for some positive m and M and integer k, then

$$\lim_{\tau\to\infty} \rho(H_{-\infty}^{t}, H_{t+\tau}^{+\infty}) = 0 .$$

Conditions (26) are satisfied, for instance, when the spectrum is rational; the corresponding correlation function, $\tilde{R}(\cdot)$, then consists of a sum of exponentials. Conversely, if the component spectra, $\phi_1(\lambda)$ and $\phi_2(\lambda)$, are rational, then $\tilde{\phi}(\lambda)$ satisfies conditions (26) and the process is strong mixing.

8. Linear transformations of a GMP. Suppose that the random variables $\hat{x}_1, \cdots, \hat{x}_n$ have a nondegenerate Gaussian mixture distribution, $\hat{F}_G(x_1, \cdots, x_n) = a_1\Phi_1(x_1, \cdots, x_n) + a_2\Phi_2(x_1, \cdots, x_n)$ with R_1 and R_2 the correlation matrices of Φ_1 and Φ_2, respectively. Now consider the linear transformation $\hat{\mathbf{y}} = A\hat{\mathbf{x}}$. It is clear that the random variables $\hat{y}_1, \cdots, \hat{y}_n$ also have a nondegenerate Gaussian mixture distribution $\hat{F}_G{}^*(y_1, \cdots, y_n) = a_1\Phi_1{}^*(y_1, \cdots, y_n) + a_2\Phi_2{}^*(y_1, \cdots, y_n)$, where $R_i{}^* = AR_iA'$, $i = 1, 2$. However, the property of nonsingularity is not in general preserved under linear transformations.

Integrals of a GMP. If $\hat{x}_t$ is a stationary Gaussian mixture process with a finite variance and if the weighting function $q(t)$ is absolutely integrable (in the Lebesgue sense) on the finite interval $[a, b]$, then $\int_a^b q(t)\hat{x}(t)\,dt$ exists for almost all sample functions and $E[\int_a^b q(t)\hat{x}(t)\,dt] = \int_a^b q(t)E[x(t)]\,dt$. (Apply Theorem 2.7 of Doob [4], noting that $E[x(t)]$ is absolutely integrable on $[a, b]$).

THEOREM 6. *If $\hat{x}_t$ is a stationary zero mean, finite variance, Gaussian mixture process having continuous component correlation functions $R_i(\cdot)$ such that $\int_a^b \int_a^b R_i(t - \tau)q(t)a(\tau)\,dt\,d\tau$ exists, and if $q(t)$ is absolutely integrable on the finite interval $[a, b]$, then*

$$\hat{y} = \int_a^b q(t)\hat{x}(t)\,dt \tag{27}$$

has a Gaussian mixture distribution with component variances

$$\sigma_i^2 = \int_a^b \int_a^b R_i(t-\tau)q(t)q(\tau)\,dt\,d\tau\,, \qquad i = 1, 2\,. \tag{28}$$

PROOF. The proof follows that of Davenport and Root [3] for a Gaussian process, noting that $\hat{R}(\tau) = a_1 R_1(\tau) + a_2 R_2(\tau)$ and that the characteristic function of a Gaussian mixture process is a mixture of the component Gaussian characteristic functions.

Karhunen-Loève Expansion for a GMP. Theorem 6 is useful in connection with the Karhunen-Loève (K-L) expansion of GMP, $\hat{x}(t)$. Suppose

$$\hat{x}(t) = \sum_1^\infty x_i \psi_i(t) \tag{29}$$

is the K-L expansion of a zero mean GMP having a continuous correlation function $\hat{R}(\tau) = a_1 R_1(\tau) + a_2 R_2(\tau)$. It follows from Theorem 6 and its extension to vector valued $q(t)$ and $\hat{y}$ that the coefficients x_i, $i = 1, 2, \cdots, N$, have a multivariate Gaussian mixture distribution whose density, assuming it exists, is

$$\hat{f}(x_1, \cdots, x_N) = a_1 \phi_{\mathbf{R}_1}(x_1, \cdots, x_n) + a_2 \phi_{\mathbf{R}_2}(x_1, \cdots, x_N)\,. \tag{30}$$

$\mathbf{R}_1$ and $\mathbf{R}_2$ are $N \times N$ correlation matrices with elements

$$R_i^{jk} = \int_a^b \int_a^b R_i(t-\tau)\psi_j(t)\psi_k(\tau)\,dt\,d\tau\,, \qquad j, k = 1, \cdots, N\,; \quad i = 1, 2\,. \tag{31}$$

For various purposes (e.g. forming the likelihood ratio for the hypothesis testing problem $H(\theta = 0)$ against $K(\theta > 0)$ with $\hat{x}(t) = \theta s(t) + \hat{n}(t)$, $s(t)$ known and $\hat{n}(t)$ a GQMP), it is desirable to have a diagonal form for $\mathbf{R}_1$ and $\mathbf{R}_2$ for all N. This will be the case if the discrete parameter process $\{x_1, x_2, \cdots\}$ consisting of the K-L coefficients is completely nonsingular. There do not appear to be general conditions on $\hat{x}(t)$ such that $\{x_1, x_2, \cdots\}$ is completely nonsingular. For example, complete nonsingularity of $\hat{x}(t)$ does not imply complete nonsingularity of $\{x_1, x_2, \cdots\}$.

The following simple condition does imply complete singularity of $\{x_1, x_2, \cdots\}$:

$$R_2(\tau) = K \cdot R_1(\tau)\,, \qquad k > 0\,. \tag{32}$$

For then

$$\begin{aligned} Ex_i x_j &= \int_a^b \int_a^b \hat{R}(t-\tau)\psi_i(t)\psi_j(\tau)\,dt\,d\tau \\ &= (\mathrm{a}_1 + Ka_2) \int_a^b \int_a^b R_1(t-\tau)\phi_i(t)\phi_j(\tau)\,dt\,d\tau \\ &= (a_1 + Ka_2)R_1^{ij} \\ &= 0\,, \qquad\qquad i \neq j \end{aligned}$$

which implies that $R_1^{ij} = R_2^{ij} = 0$, $i \neq j$, since the K-L coefficients are uncorrelated. In this case the density $\hat{f}(x_1, \cdots, x_n)$ is of the form:

$$\hat{f}(x_1, \cdots, x_N) = a_1 c_1 \exp\left[-\sum_1^N \frac{x_i^2}{2\sigma_1^2}\right] + a_2 c_2 \exp\left[-\sum_1^N \frac{x_i^2}{2\sigma_1^2} \cdot \frac{1}{k}\right]. \tag{33}$$

9. Generation of Gaussian mixture and quasi-mixture processes.

GMP. Let (Ω, β, P) be a probability space, let $\{W_t\}_{-\infty}^{\infty}$ be i.i.d. random variables on (Ω, β, P) having a unit normal density function, and let A be a random variable on (Ω, β, P) such that: (i) A is independent of $\{W_t\}_{-\infty}^{\infty}$ and (ii) $P(A=\alpha_1)=a_1$, $P(A = \alpha_2) = a$, $0 < |\alpha_1|, |\alpha_2| < 1$, $a_1 + a_2 = 1$. Now consider the "conditional" moving average

$$\hat{X}_t = \sum_{\mu=0}^{\infty} A^{\mu} W_{t-\mu} \qquad -\infty < t < \infty . \tag{34}$$

We use the term conditional here to indicate that, conditioned on A, X_t is a moving average in the usual sense, i.e.

$$\begin{aligned} \hat{X}_t &= \textstyle\sum_{\mu=0}^{\infty} \alpha_1^{\mu} W_{t-\mu} && \text{with probability} \quad a_1 ; \\ &= \textstyle\sum_{\mu=0}^{\infty} \alpha_2^{\mu} W_{t-\mu} && \text{with probability} \quad a_2 . \end{aligned}$$

It is easy to see that conditioned on $A = \alpha_1 (A = \alpha_2)$, $\hat{X}_t$ is a stationary Gaussian process with correlation function $R_1(t) = \alpha_1^t/(1 - \alpha_1^2)$ $(R_2(t) = \alpha_2^t/(1 - \alpha_2^2))$; hence $\hat{X}_t$ is a zero mean stationary GMP with $\hat{R}(t) = a_1 \alpha_1^t/(1 - \alpha_1^2) + a_2 \alpha_2^t/(1 - \alpha_2^2)$. A first order auto-regressive form is also possible, i.e.

$$\hat{X}_{t+1} = A\hat{X}_t + W_{t+1} \qquad -\infty < t < \infty \tag{35}$$

with A and $\{W_t\}_{-\infty}^{\infty}$ as given above generates the same GMP as does (34).

GQMP. Let (Ω, β, P) be a probability space with $\{W_t\}_{-\infty}^{\infty}$ and $\{V_t\}_{-\infty}^{\infty}$ two mutually independent sequences of i.i.d. unit normal random variables. We define the discrete time process $\tilde{X}_t$ by:

$$\begin{aligned} X_t^1 &= \textstyle\sum_{\mu=0}^{\infty} \alpha_1^{\mu} V_{t-\mu} \\ X_t^2 &= \textstyle\sum_{\mu=0}^{\infty} \alpha_2^{\mu} W_{t-\mu} , \quad 0 < |\alpha_1| , \quad |\alpha_2| < 1 , \quad \textstyle\sum_{\mu=0}^{\infty} \alpha_j^{2\mu} < \infty , \quad j = 1, 2 . \end{aligned} \tag{36}$$

$$P(\tilde{X}_t = X_t^i) = a_i , \qquad i = 1, 2 , \quad a_1 + a_2 = 1 \quad -\infty < t < \infty \tag{36'}$$

$$P(\tilde{X}_{t_1} = X_{t_1}^{i_1}, \cdots, \tilde{X}_{t_n} = X_{t_n}^{i_n}) = \textstyle\prod_{j=1}^{n} P(\tilde{X}_{t_j} = X_{t_j}^{i_j}) \tag{36''}$$

for all time sets $\{t_1, \cdots, t_n\}$ for all n. Using the above definition it is easy to verify that $\tilde{X}_t$ is a zero mean GQMP with $R_1(t) = \alpha_1^t/(1 - \alpha_1^2)$ and $R_2(t) = \alpha_2^t/(1 - \alpha_2^2)$. We can of course use first order autoregressive schemes for X_t^1 and X_t^2.

Since the above methods utilize moving average forms we may generate both GMP and GQMP processes for any given $R_1(\cdot)$ and $R_2(\cdot)$ such that the corresponding spectral distribution functions are absolutely continuous.

The GMP appears to be rather uninteresting from a structural viewpoint since the realization scheme (34) indicates that its sample functions look like those of a Gaussian process with a correlation function of either $R_1(\cdot)$ or $R_2(\cdot)$. On the other hand such a process may be useful for describing or approximating processes arising in certain physical systems. For example, the model (35) occurs in certain control problems where A is usually nonrandom, say $P(A = \alpha_1) = 1$. It may happen, however, that the "transition parameter" A is more appropriately described as originally given above. This would be the case for instance when a system overloads or the transition parameter α_1 is incorrectly set at α_2 with small probability a_2, the overload or missed setting being constant for the duration of the process which is of interest.

The realization scheme for the GQMP reveals that it is obtained by a probabilistic selection of one of two independent Gaussian processes at each instant of time. In the discrete time case this poses no real difficulty and such processes may arise in practice. In the continuous time case the GQMP is not a mean square continuous process. This follows from (12) and (13) which show that $R(\tau)$ is discontinuous at $\tau = 0$ in the nondegenerate case $a_2 > 0$. That this should be the case is also suggested by the realization scheme (36), (36′) and (36″) for a discrete time process. In continuous time what is essentially required is a selection rule, i.e., a rule analogous to (36′), for X_t^1 and X_t^2 based on the values of a continuous time white noise process. Thus it does not appear to be possible to realize a continuous time GQMP *exactly*. However, there is no difficulty in obtaining quite reasonable approximations to a GQMP by basing the selection rule on a "nearly white" continuous time process.

To be specific consider the following means of generating an approximating process:
Let

$$\begin{aligned} X^1(t) &= \textstyle\int_{-\infty}^t \alpha_1(t-\tau)\,dV(\tau) \\ X^2(t) &= \textstyle\int_{-\infty}^t \alpha_2(t-\tau)\,dW(\tau) \end{aligned} \tag{37}$$

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are square integrable, and $V(t)$ and $W(t)$ are independent stationary Gaussian white noise processes such that

$$\operatorname{Var} X^i(t) = \textstyle\int_0^\infty \alpha_i^2(t)\,dt\,, \qquad i = 1, 2\,.$$

Now define $X(t)$ by

$$\begin{aligned} X^*(t) &= X^1(t) \quad \text{if} \quad A(t) = 1 \\ &= X^2(t) \quad \text{if} \quad A(t) = 0 \end{aligned} \tag{38}$$

where $A(t)$, $-\infty < t < \infty$, is a binary process derived from a Poisson process with parameter λ as follows: at each increment time t_i of the Poisson process the process $A(t)$ changes state according to the rule $P(A(t_i) = 1) = a_1$ and $P(A(t_i) = 0) = a_2$, $a_1 + a_2 = 1$, independent of $A(t)$, $t < t_i$. The resulting first and second order distribution functions are of the form:

$$F^*(x) = a_1\Phi_1(x) + a_2\Phi_2(x) \tag{39}$$

$$\begin{aligned} F^*(x_t, x_{t+\mu}) = {} & a_1[e^{-\lambda|\mu|} + (1 - e^{-\lambda|\mu|})a_1]\Phi_1(x_t, x_{t+\mu}) \\ & + a_2[e^{-\lambda|\mu|} + (1 - e^{-\lambda|\mu|})a_2]\Phi_2(x_t, x_{t+\mu}) \\ & + a_1a_2(1 - e^{-\lambda|\mu|})[\Phi_1(x_t)\Phi_2(x_{t+\mu}) + \Phi_1(x_{t+\mu})\Phi_2(x_t)] \end{aligned} \tag{40}$$

so that

$$\operatorname{Var} X_t^* = a_1R_1(0) + a_2R_2(0) \tag{41}$$

$$R^*(\mu) = a_1[e^{-\lambda|\mu|} + (1 - e^{-\lambda|\mu|})a_1]R_1(\mu) + a_2[e^{-\lambda|\mu|} + (1 - e^{-\lambda|\mu|})a_2]R_2(\mu) . \tag{42}$$

Thus the process $X^*(t)$ is mean square continuous. Furthermore, $F^*(x_t, x_{t+\mu}) \cong \tilde{F}(x_t, x_{t+\mu})$ for large $\lambda|\mu|$, so that $X^*(t)$ looks very much like a GQMP $\tilde{X}(t)$ as far as the first and second order distributions are concerned. In fact for any set $t_1, \cdots, t_n$ for which $\min_{i\neq j} |t_i - t_j|$ is sufficiently large the corresponding nth order distribution function for $X^*(t)$ will be very close to that of a GQMP. In practice one would probably choose λ small compared with the correlation times for $R_1(\cdot)$ and $R_2(\cdot)$. It is conjectured that strong mixing for $\tilde{X}(t)$ implies strong mixing for $X^*(t)$.

10. Concluding comments. The motivation of this investigation has been the desire to extend robustness studies, in the spirit of Huber [5] and Tukey [11], to the case of a process with dependence. We have defined two "nearly" normal processes which can be used in studies of robustness [7]. The first process, the Gaussian mixture process, is the continuous analog of a simple mixture for two random variables. We have shown that the class of Gaussian mixture processes is invariant under linear transformations, but the process never satisfies a strong mixing condition. This deficiency led to the definition of the Gaussian quasi-mixture process which, under certain conditions, does satisfy the strong mixing condition. The class of Gaussian quasi-mixture processes is not, however, invariant under linear transformations. From the discussion of Section 9 it is clear that it is rather straightforward to simulate a discrete time GMP or a GQMP on the computer for Monte Carlo studies.

Although a continuous time GQMP is not mean square continuous, mean square continuous approximations to a GQMP are easily obtained and are expected to be of utility in robustness studies. As in the discrete time case, simulation for the purposes of Monte Carlo studies is possible.

In robustness studies one might well consider, in addition to the GMP and GQMP, the spherically invariant processes as models of nonnormal processes [1], [13]. Since the spherically invariant process is ergodic only in the Gaussian case it is natural to compare its attributes with those of the GMP. Both GMP's and spherically invariant processes are invariant under linear transformations. In addition, both processes may be used to model heavy tailed deviations from Gaussian-ness. However, the GMP appears to be much more attractive in terms of the computational advantages resulting from the simplicity of the mixture form.

Whether or not one is willing to tolerate the non-ergodicity mentioned above will depend to a great extent upon the physical situation for which the robustness study is intended. Statistical dependency which exists over arbitrarily long time or space intervals is known to occur in some physical situations [12]. If however one insists on using a non-Gaussian model which is ergodic, then the GQMP may be an appropriate process having a simple structure for low order c.d.f.'s.

REFERENCES

[1] Blake, I. F. (1967). *On the characterization of random processes.* Ph. D. Dissertation, Princeton Univ.

[2] Cramér, H. and Leadbetter, M. R. (1967). *Stationary and Related Stochastic Processes,* Wiley, New York.

[3] Davenport, W. B. and Root, W. L. (1958). *Random Signals and Noise.* McGraw-Hill, New York, 155.

[4] Doob, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[5] Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.* **35** 73-101.

[6] Kolmogorov, A. N. and Rozanov, Y. A. (1960). On strong mixing conditions for stationary gaussian random processes. *Theory. Probability Appl.* **5** 204-208.

[7] Martin, R. D. (1969). *Robust detection and estimation.* Ph. D. Dissertation, Princeton Univ.

[8] Rosenblatt, M. (1962). *Random Processes.* Oxford Univ. Press.

[9] Rozanov, Y. A. (1967). *Stationary Random Processes.* Holden-Day, San Francisco.

[10] Teicher, H. (1963). Identifiability of finite mixtures. *Ann. Math. Statist.* **34** 1265-1269.

[11] Tukey, J. W. (1960). A Survey of Sampling from Contaminated Distributions, *Contributions to Probability and Statistics* (Harold Hotelling Volume). Stanford Univ. Press, 448.

[12] Van Ness, J. W. and Mandelbrot, B. B. (1968). Fractional brownian motion, fractional noises and applications. *SIAM Rev.* **10** 422-437.

[13] Vershik, A. M. (1964). Some characteristic properties of Gaussian stochastic processes. *Theor. Probability Appl.* **9** 353-356.

[14] Yakowitz, S. J. (1969). A Consistent Estimator for the Identification of Finite Mixtures, *Ann. Math. Statist.* **40** 1728-1735.

[15] Yakowitz, S. J. and Spragins, J. D. (1968). On the identifiability of finite mixtures. *Ann. Math. Statist.* **39** 209-214.