The Annals of Applied Probability
1994, Vol. 4, No. 4, 1083–1097

SPREADING OF A PULSE TRAVELLING IN RANDOM MEDIA

BY J. F. CLOUET AND J. P. FOUQUE

Ecole Polytechnique

This paper investigates the deformation of an acoustic pulse travelling in a slab of random medium when its width is large compared to the size of the random inhomogeneities of the medium. A limit theorem is shown that explains how the shape of the transmitted pulse can be obtained as a result of a deterministic Gaussian convolution of the initial pulse. Since the random fluctuations are not supposed to be small, this gives a new rigorous formulation of the O'Doherty–Anstey result, which is well known in geophysical literature theory.

1. Introduction. In this paper we present a rigorous solution of a problem arising in wave propagation in random media: this is the now classical result from O'Doherty and Anstey which maintains that in proper conditions a pulse transmitted through a random medium emerges with a deterministic shape but at a random time. In fact their derivation was essentially heuristic. Furthermore, it involved the summing of reflection series and was implicitly for small reflection coefficients. On the other hand, many numerical simulations agree with their prediction, even in the case where the random noise is of the same order as the macroscopic fluctuations of the medium.

This analysis takes place in the general framework, based on the separation of scales introduced by Papanicolaou and his co-workers (see, e.g., [2] for the one-dimensional case and [1] for the three-dimensional case). We consider here the problem of acoustic wave propagation in a one-dimensional random medium when the incident pulse wavelength is long compared to the correlation length of the random inhomogeneities, but short compared to the size of the slab.

In this framework, it has already been proved in [1] (see also [3] for more details) that when the random fluctuations are weak, the O'Doherty–Anstey theory is valid, that is, the travelling pulse retains its shape up to a low spreading; furthermore, its shape is deterministic when observed from the point of view of an observer travelling at the same random speed as the wave, whereas it is stochastic when the observer's speed is the mean speed of the wave.

We do not assume the fluctuations to be small, but as we are mainly concerned with the shape of the transmitted pulse, we suppose that the

Received July 1993; revised February 1994.

AMS 1991 *subject classifications*. Primary 73D70; secondary 60B10, 60H10.

Key words and phrases. Waves in random media, transmitted pulse, diffusion approximation, stochastic equations.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Applied Probability*. www.jstor.org

incident pulse has a constant amplitude but its energy is small. The methods we use are based on those of [2] and [1]. We study the problem in the Fourier domain and get stochastic equations. However, we derive a complex linear stochastic differential equation which enables us to couple distinct frequencies and to get a limit for the whole process in time.

Our main result consists in a complete description of the asymptotic law of the emerging pulse. We prove a limit theorem which shows that the pulse spreads in a deterministic way and that the emerging time is the sum of the deterministic travel time corresponding to the effective medium and of a Gaussian random variable properly scaled.

2. Problem formulation. We recall the main features of our model. We shall only present the mathematical framework because it has been fully described in [2]. We consider an acoustic wave travelling in a one-dimensional random medium located in the region $0 \le x \le L$, satisfying the linear conservation laws

$$\begin{aligned} \rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial p}{\partial x}(x,t) &= 0, \\ \frac{1}{K(x)}\frac{\partial p}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) &= 0. \end{aligned} \tag{1}$$

Here $u(x,t)$ and $p(x,t)$ are, respectively, the speed and the pressure of the wave, whereas $\rho(x)$ and $1/(K(x))$ are the density and bulk modulus of the medium and admit the following representation:

$$\begin{aligned} \rho(x) &= \rho_0(x)\left(1 + \eta\left(\frac{x}{\varepsilon^2}\right)\right), \\ \frac{1}{K(x)} &= \frac{1}{K_0(x)}\left(1 + \nu\left(\frac{x}{\varepsilon^2}\right)\right). \end{aligned}$$

Here ρ_0 and K_0 represent the slow varying deterministic parameters of the medium, and $\eta(x/\varepsilon^2)$ and $\nu(x/\varepsilon^2)$ are the rapidly varying random coefficients describing the inhomogeneities. We shall use the acoustic impedance ζ and the acoustic speed c which are the macroscopic functions defined by

$$\begin{aligned} \zeta(x) &= \sqrt{\rho_0(x)K_0(x)}, \\ c(x) &= \sqrt{\frac{K_0(x)}{\rho_0(x)}}. \end{aligned}$$

It is relevant to make a change of variable and to introduce a right-going wave A and a left-going wave B in order to work with a hyperbolic system of equations and to make our boundary conditions precise. So let

$$\begin{aligned} A &= \zeta^{-1/2}p + \zeta^{1/2}u, \\ B &= -\zeta^{-1/2}p + \zeta^{1/2}u. \end{aligned} \tag{2}$$

With the notations

$$\Lambda = \frac{d \ln \zeta^{1/2}}{dx},$$

$$m\left(\frac{x}{\varepsilon^2}\right) = \frac{\eta(x/\varepsilon^2) + \nu(x/\varepsilon^2)}{2},$$

$$n\left(\frac{x}{\varepsilon^2}\right) = \frac{\eta(x/\varepsilon^2) - \nu(x/\varepsilon^2)}{2},$$

from (1) and (2), we obtain the system of equations for A and B:

$$\frac{d}{dx}\begin{bmatrix} A \\ B \end{bmatrix} = \Lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} \tag{3}$$
$$- \left(\frac{1}{c(x)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{c(x)} \begin{bmatrix} m\left(\frac{x}{\varepsilon^2}\right) & n\left(\frac{x}{\varepsilon^2}\right) \\ -n\left(\frac{x}{\varepsilon^2}\right) & -m\left(\frac{x}{\varepsilon^2}\right) \end{bmatrix} \right) \frac{\partial}{\partial t}\begin{bmatrix} A \\ B \end{bmatrix}.$$

The slab of medium we are considering is located in the region $0 \le x \le L$ and at $t = 0$ an incident pulse is generated at the interface $x = 0$ between the random medium and the homogenous medium on the outside. According to previous works [1, 2], we choose a pulse which is broad compared to the size of the random inhomogeneities but short compared to the macroscopic variations of the medium. There is no wave entering the medium at $x = L$:

$$A(0,t) = f\left(\frac{t}{\varepsilon}\right), \tag{4}$$
$$B(L,t) = 0,$$

where f is a function with compact support and C^∞ regularity. Note that the energy entering the medium is $\varepsilon \int f(t)^2\, dt$ and so is small when ε is small. We need to perform another change of variable adapted to our problem so that (3) becomes centered. We are interested in the transmitted wave, that is, $A(L, \tau(L))$, where $\tau(L)$ is the travel time in the macroscopic medium defined by

$$\tau(x) = \int_0^x \frac{ds}{c(s)}.$$

Moreover, in order to have a complete description of the transmitted pulse, we study the process $A(L, \tau(L) + \varepsilon\sigma)_{\sigma \in (-\infty,\infty)}$; that is, we open a window of size ε in the neighborhood of the mean travel time. So let

$$a^\varepsilon(x,\sigma) = A(x, \tau(x) + \varepsilon\sigma), \tag{5}$$
$$b^\varepsilon(x,\sigma) = B(x, -\tau(x) + \varepsilon\sigma).$$

The solution of (3) and (4) takes place in an infinite-dimensional space because of the variable t. So we perform the Fourier transform:

$$\hat{a}^{\varepsilon}(x,\omega)=\int e^{\iota\omega\sigma}a^{\varepsilon}(x,\sigma)\,d\sigma,$$

$$\hat{b}^{\varepsilon}(x,\omega)=\int e^{\iota\omega\sigma}b^{\varepsilon}(x,\sigma)\,d\sigma.$$

In the frequency domain, with the change of variable (5), (3) and (4) become

$$\frac{d}{dx}\begin{bmatrix}\hat{a}^{\varepsilon}\\ \hat{b}^{\varepsilon}\end{bmatrix}=\frac{1}{\varepsilon}P^{\varepsilon}(x,\omega)\begin{bmatrix}\hat{a}^{\varepsilon}\\ \hat{b}^{\varepsilon}\end{bmatrix}+Q^{\varepsilon}(x,\omega)\begin{bmatrix}\hat{a}^{\varepsilon}\\ \hat{b}^{\varepsilon}\end{bmatrix},\tag{6}$$

$$\begin{aligned}\hat{a}^{\varepsilon}(0,\omega)&=\hat{f}(\omega),\\ \hat{b}^{\varepsilon}(L,\omega)&=0,\end{aligned}\tag{7}$$

where

$$\begin{aligned}P^{\varepsilon}(x,\omega)&=\frac{\iota\omega}{c(x)}\begin{bmatrix}m\left(\dfrac{x}{\varepsilon^2}\right) & n\left(\dfrac{x}{\varepsilon^2}\right)\exp\left(-2\iota\omega\dfrac{\tau(x)}{\varepsilon}\right)\\ -n\left(\dfrac{x}{\varepsilon^2}\right)\exp\left(2\iota\omega\dfrac{\tau(x)}{\varepsilon}\right) & -m\left(\dfrac{x}{\varepsilon^2}\right)\end{bmatrix},\\ Q^{\varepsilon}(x,\omega)&=\Lambda\omega\begin{bmatrix}0 & \exp\left(-2\iota\omega\dfrac{\tau(x)}{\varepsilon}\right)\\ \exp\left(2\iota\omega\dfrac{\tau(x)}{\varepsilon}\right) & 0\end{bmatrix}.\end{aligned}\tag{8}$$

The transmitted pulse admits the integral representation

$$A(L,\tau(L)+\varepsilon\sigma)=a^{\varepsilon}(L,\sigma)=\frac{1}{2\pi}\int e^{-\iota\omega\sigma}\hat{a}^{\varepsilon}(L,\omega)\,d\omega$$

As the problem (6) is linear, we can replace (7) by

$$\begin{aligned}\hat{a}^{\varepsilon}(0,\omega)&=1,\\ \hat{b}^{\varepsilon}(L,\omega)&=0\end{aligned}\tag{9}$$

and in that case the following representation holds for the transmitted pulse:

$$A(L,\tau(L)+\varepsilon\sigma)=a^{\varepsilon}(L,\sigma)=\frac{1}{2\pi}\int e^{-\iota\omega\sigma}\hat{f}(\omega)\hat{a}^{\varepsilon}(L,\omega)\,d\omega\tag{10}$$

[$\hat{a}^{\varepsilon}$ is now the solution of (6) and (9)].

We want to prove an asymptotic theorem for this quantity. Instead of working with $(\hat{a}^{\varepsilon}(x,\omega),\hat{b}^{\varepsilon}(x,\omega))$, we will use its propagator, that is, the matrix $Y^{\varepsilon}(x,\omega)$ defined by

$$\begin{bmatrix}\hat{a}^{\varepsilon}(x,\omega)\\ \hat{b}^{\varepsilon}(x,\omega)\end{bmatrix}=Y^{\varepsilon}(x,\omega)\begin{bmatrix}\hat{a}^{\varepsilon}(0,\omega)\\ \hat{b}^{\varepsilon}(0,\omega)\end{bmatrix},$$

so $Y^{\varepsilon}(x, \omega)$ is solution of the linear differential equation

$$\frac{d}{dx} Y^{\varepsilon}(x, \omega) = \left(\frac{1}{\varepsilon} P^{\varepsilon}(x, \omega) + Q^{\varepsilon}(x, \omega) \right) Y^{\varepsilon}(x, \omega), \tag{11}$$
$$Y^{\varepsilon}(0, \omega) = Id_{\mathbb{C}^2}.$$

If (α, β) is a solution of (6) and (9), then $(\overline{\beta}, \overline{\alpha})$ is another solution linearly independent of the previous one, so we can write $Y^{\varepsilon}(x, \omega)$ as

$$Y^{\varepsilon}(x, \omega) = \begin{bmatrix} \alpha(x, \omega) & \overline{\beta}(x, \omega) \\ \beta(x, \omega) & \overline{\alpha}(x, \omega) \end{bmatrix}. \tag{12}$$

Now, since

$$\det Y^{\varepsilon}(x, \omega) = \exp\left(\operatorname{tr} \int_0^x \left(\frac{1}{\varepsilon} P^{\varepsilon}(y, \omega) + Q^{\varepsilon}(y, \omega) \right) dy \right) = 1,$$

we have

$$|\alpha(x, \omega)|^2 - |\beta(x, \omega)|^2 = 1 \qquad \forall\, x.$$

From

$$\begin{bmatrix} \hat{a}^{\varepsilon}(L, \omega) \\ 0 \end{bmatrix} = Y(L, \omega) \begin{bmatrix} 1 \\ \hat{b}^{\varepsilon}(0, \omega) \end{bmatrix}$$

we deduce that

$$\hat{a}^{\varepsilon}(L, \omega) = \frac{1}{\overline{\alpha}(L, \omega)}, \tag{13}$$
$$\hat{b}^{\varepsilon}(0, \omega) = -\frac{\beta(L, \omega)}{\overline{\alpha}(L, \omega)},$$

so we get the conservation of energy relation

$$|\hat{a}^{\varepsilon}(L, \omega)|^2 + |\hat{b}^{\varepsilon}(0, \omega)|^2 = 1, \tag{14}$$

which shows that $\hat{a}^{\varepsilon}(L, \omega)$ is uniformly bounded.

3. Shape of the transmitted pulse. Let us define the correlation coefficients of the noise by

$$\alpha_m = \int_0^{\infty} \mathbb{E} m(0) m(x)\, dx, \tag{15}$$
$$\alpha_n = \int_0^{\infty} \mathbb{E} n(0) n(x)\, dx.$$

Let $\theta_L = \int_0^L dy/(c^2(y))$ and let Z_L be a Gaussian variable such that

$$\mathbb{E} Z_L = 0, \qquad \mathbb{E} Z_L^2 = 2\alpha_m \theta_L. \tag{16}$$

Let $G_L(t)$ be the Gaussian kernel given by

$$G_L(t) = \frac{1}{\sqrt{4\pi\alpha_n\theta_L}} \exp\left(-\frac{t^2}{4\alpha_n\theta_L} \right).$$

Our main result is the following theorem.

THEOREM 3.1. *The process* $(A(L, \tau(L) + \varepsilon\sigma))_{-\infty<\sigma<\infty}$ *converges in distribution as* ε *goes to zero to the process* $(f * G_L(\sigma + Z_L))_{-\infty<\sigma<\infty}$.

This tells us that the pulse retains its shape during travel in the random medium, but that there is a deterministic spreading due to the Gaussian kernel and a random centering at Z_L which do not affect the shape. As usual, we first show the tightness, which is very easy because of the uniform boundness of $\hat{a}^\varepsilon$ and the spectral representation (10), and then we identify the limiting distribution through all its moments by the use of a diffusion approximation theorem.

LEMMA 3.2. *The family of processes* $((a^\varepsilon(L,\sigma))_{-\infty<\sigma<\infty})_{\varepsilon>0}$ *is tight in* $C[(-\infty,\infty);\mathbb{R}]$ *with the* sup norm.

PROOF. The quantity $|a^\varepsilon(L,\sigma)|$ is bounded by $(2\pi)^{-1}\int|\hat{f}(\omega)|\,d\omega$ and the modulus of continuity,

$$M^\varepsilon(\delta) = \max_{\substack{|\sigma_1-\sigma_2|\le\delta\\ -\infty<\sigma_1,\sigma_2<\infty}} |a^\varepsilon(L,\sigma_1) - a^\varepsilon(L,\sigma_2)|,$$

is bounded by

$$M^\varepsilon(\delta) \le \frac{1}{2\pi}\int \sup_{|\sigma_1-\sigma_2|\le\delta} |1-\exp(\iota\omega(\sigma_1-\sigma_2))|\cdot|\hat{f}(\omega)|\,d\omega.$$

By Lebesgue's theorem,

$$\lim_{\delta\to 0}\sup_{\varepsilon}\mathbb{P}\{M(\delta)>\alpha\} = 0, \qquad \forall\,\alpha>0,$$

which is enough for the tightness. □

We shall now characterize the limits of the finite-dimensional distributions of the process $(a^\varepsilon(L,\sigma))_{-\infty<\sigma<\infty}$ by computing the moments $\mathbb{E}[a^\varepsilon(L,\sigma_1)^{p_1}\cdots a^\varepsilon(L,\sigma_k)^{p_k}]$ for all real numbers $\sigma_1<\cdots<\sigma_k$ and all integers $p_1,\dots,p_k$. We have

$$\begin{aligned}
&\mathbb{E}\left[a^\varepsilon(L,\sigma_1)^{p_1}\cdots a^\varepsilon(L,\sigma_k)^{p_k}\right]\\
&\quad= (2\pi)^{(-\sum_{j=1}^{k}p_j)}\int \exp\left(-\iota\sum_{\substack{1\le j\le k\\1\le l\le p_j}}\sigma_j\omega_j^l\right)\left(\prod_{\substack{1\le j\le k\\1\le l\le p_j}}\hat{f}(\omega_j^l)\right)\\
&\qquad\times\mathbb{E}\left[\prod_{\substack{1\le j\le k\\1\le l\le p_j}}\hat{a}^\varepsilon(L,\omega_j^l)\right]d\omega_1^1\cdots d\omega_k^{p_k}.
\end{aligned}\tag{17}$$

Hence the quantity of interest is $\mathbb{E}[\hat{a}^\varepsilon(L,\omega_1)\cdots\hat{a}^\varepsilon(L,\omega_n)]$, where $\omega_1,\dots,\omega_n$ are n distinct frequencies. So we define an n-dimensional propagator

$$Y_x^\varepsilon = Y^\varepsilon(x,\omega_1,\omega_2,\dots,\omega_n) = \begin{bmatrix} Y^\varepsilon(x,\omega_1) & & \\ & \ddots & \\ & & Y^\varepsilon(x,\omega_n)\end{bmatrix},$$

which satisfies

$$\frac{d}{dx}Y_x^\varepsilon = \frac{1}{\varepsilon}P\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)Y_x^\varepsilon + Q\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)Y_x^\varepsilon,$$
$$Y_0^\varepsilon = Id_{\mathbb{C}^{2n}},$$

with

$$P\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) = \begin{bmatrix} P^\varepsilon(x, \omega_1) & & \\ & \ddots & \\ & & P^\varepsilon(x, \omega_n) \end{bmatrix},$$

$$Q\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) = \begin{bmatrix} Q^\varepsilon(x, \omega_1) & & \\ & \ddots & \\ & & Q^\varepsilon(x, \omega_n) \end{bmatrix},$$

where $P(x, \omega)$ and $Q(x, \omega)$ have been defined in (8). We shall use a diffusion-approximation theorem for $(Y_x^\varepsilon)_{x \geq 0}$. We recall the general result from [5], Theorem 2.8. In our case, the variable x will play the role of a time in the process $(Y_x^\varepsilon)_{x \geq 0}$, but, as usual, we shall denote it by $(X_t)_{t \geq 0}$.

THEOREM 3.3. *Assume that*:

(i) $(q_t)_{t \geq 0}$ *is an ergodic Markov process with state space* S (*S being a compact metric space, for example*).

(ii) $P(t, h, q, x)$ *and* $Q(t, h, q, x)$ *are two smooth bounded functions from* $\mathbb{R} \times \mathbb{R} \times S \times \mathbb{R}^d$ *to* $\mathbb{R}^d$, *periodic in* h *with period* T_0 *independent of* t, q *and* x.

(iii) $\mathbb{E}P(t, h, q_0, x) = 0$, *where the expectation is taken with respect to the unique invariant probability measure of* $(q_t)_{t \geq 0}$.

Then the solution of the ordinary differential equation with stochastic coefficients

$$\frac{dX_t^\varepsilon}{dt} = \frac{1}{\varepsilon}P\left(t, \frac{t}{\varepsilon}, q_{t/\varepsilon^2}, X_t^\varepsilon\right) + Q\left(t, \frac{t}{\varepsilon}, q_{t/\varepsilon^2}, X_t^\varepsilon\right), \quad X_0^\varepsilon = x_0 \tag{18}$$

converges in distribution as ε *goes to* 0 *to a diffusion process whose infinitesimal generator is given by*

$$\begin{aligned} \mathscr{L}_t F(x) = \int_0^\infty \frac{1}{T_0}\int_0^{T_0} &\mathbb{E}P(t, h, q_0, x) \\ &\cdot \nabla_x [P(t, h, q_u, x) \cdot \nabla_x F(x)]\, dh\, du \\ &+ \frac{1}{T_0}\int_0^{T_0} \mathbb{E}Q(t, h, q_0, x) \cdot \nabla_x F(x)\, dh \end{aligned} \tag{19}$$

(*assuming that all the integrals are well-defined and finite*).

The theorem is also true under general mixing conditions on the coefficients, but we shall not enter into the details (see, e.g., [3] for the computation of the infinitesimal generator with the averaging introduced by the periodic variable in that case). We only assume that the processes $m(x)$ and $n(x)$ are either ergodic Markov processes or bounded processes with enough decorrelation so that the previous theorem applies and $(Y_x^\varepsilon)_{x\in[0,L]}$ converges in distribution as ε goes to 0 to a diffusion process $(Y_x)_{x\in[0,L]}$.

We would like to characterize $(Y_x)_{x\in[0,L]}$ as the solution of a stochastic differential equation. This is not so easy, because Y_x^ε is a complex-valued matrix, so we have to consider separately the real and imaginary parts of each coefficient. However, taking advantage of the linearity of the problem, it is possible to simplify the computation.

The first step is to find a stochastic differential equation in the real case. So let us adopt the following notation for matrices $M(t,h,q)$ and $N(t,h,q)$ such that the product MN makes sense:

$$\langle M|N\rangle_t = \int_0^\infty \frac{1}{T_0}\int_0^{T_0} \mathbb{E} M(t,h,q_0)N(t,h,q_u)\,dh\,du,$$

$$\langle M\rangle_t = \frac{1}{T_0}\int_0^{T_0} \mathbb{E} M(t,h,q_0)\,dh.$$

We also denote transposition of matrices by the superscript T. Then we have the following lemma.

LEMMA 3.4. *Let $(X_t^\varepsilon)_{t\in[0,T]}$ be the $\mathbb{R}^d$-valued random process solution of*

$$\begin{aligned} \frac{dX_t^\varepsilon}{dt} &= \frac{1}{\varepsilon}P\left(t,\frac{t}{\varepsilon},q_{t/\varepsilon^2}\right)X_t^\varepsilon + Q\left(t,\frac{t}{\varepsilon},q_{t/\varepsilon^2}\right)X_t^\varepsilon, \\ X_0^\varepsilon &= x_0, \end{aligned} \tag{20}$$

where P and Q are two $d\times d$ real matrices satisfying the hypotheses of Theorem 3.3 (i), (ii) *and* (iii). *Then there exist matrices P_t^k, $1\le k\le n$, $n\le d^2$, such that, for every vector $x\in\mathbb{R}^d$,*

$$\langle Px|(Px)^T\rangle_t = \tfrac{1}{2}\sum_{k=1}^n P_t^k x(P_t^k x)^T \tag{21}$$

and (X_t^ε) converges in distribution to the diffusion process solution of the stochastic differential equation

$$\begin{aligned} dX_t &= \sum_k P_t^k X_t\,dB_t^k + (\langle P|P\rangle_t + \langle Q\rangle_t)X_t\,dt, \\ X_0 &= x_0, \end{aligned} \tag{22}$$

where $B^1,\dots,B^n$ are n standard independent Brownian motions.

PROOF. For $f(x) \in C^2(\mathbb{R}^d)$ and $F(x) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, we write

$$
\begin{aligned}
\nabla^2 f &= \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i,j \le d}, \\
\nabla f &= \left(\frac{\partial f}{\partial x_i} \right)_{1 \le i \le d}, \\
\nabla\{F\} &= \left(\frac{\partial F_j}{\partial x_i} \right)_{1 \le i,j \le dj}, \\
\nabla\{Px\}_{i,j} &= \frac{\partial \sum_k P_{j,k} x_k}{\partial x_i} = P_{j,i}
\end{aligned}
$$

so $\nabla\{Px\} = P^T$ and

$$
\begin{aligned}
Px \cdot \nabla[Px \cdot \nabla f] &= Px \cdot \left(\nabla\{Px\} \nabla f + \left[\nabla^2 f \right] Px \right) \\
&= Px \cdot P^T \nabla f + Px \cdot \left[\nabla^2 f \right] Px \\
&= (P^2 x)^T \nabla f + \operatorname{tr}\left(Px (Px)^T \nabla^2 f \right).
\end{aligned}
$$

So in the linear case the generator (19) can be written

$$
\mathscr{L}_t F(x) = \operatorname{tr}\left(\langle Px | (Px)^T \rangle_t \nabla^2 F \right) + \left[(\langle P | P \rangle_t x)^T + \langle Q_x \rangle^T \right] \nabla F. \tag{23}
$$

There always exist matrices P_t^k such that (21) holds for all vectors x in $\mathbb{R}^d$ (see [4]) and the solution of (22) which exists and is unique has (23) for its infinitesimal generator. □

We turn now to the complex case.

LEMMA 3.5. *Suppose that $(Z_t^\varepsilon)_{t \in [0,T]}$ is a $\mathbb{C}^d$-valued random process such that*

$$
\begin{aligned}
\frac{dZ_t^\varepsilon}{dt} &= \frac{1}{\varepsilon} P\left(t, \frac{t}{\varepsilon}, q_{t/\varepsilon^2} \right) Z_t^\varepsilon + Q\left(t, \frac{t}{\varepsilon}, q_{t/\varepsilon^2} \right) Z_t^\varepsilon, \\
Z_0^\varepsilon &= z_0,
\end{aligned} \tag{24}
$$

where P and Q are two $d \times d$ complex matrices satisfying the hypotheses of Theorem 3.3 (i), (ii) *and* (iii). *We suppose that there exist complex matrices P_t^k, $1 \le k \le n$, $n \le 4d^2$, such that for all $\mathbb{C}^d$ vectors z, one has*

$$
\begin{aligned}
\langle Pz | (Pz)^T \rangle_t &= \tfrac{1}{2} \sum_{k=1}^n P_t^k z \left(P_t^k z \right)^T, \\
\langle Pz | (\overline{Pz})^T \rangle_t &= \tfrac{1}{2} \sum_{k=1}^n P_t^k z \left(\overline{P_t^k z} \right)^T.
\end{aligned} \tag{25}
$$

Then the law of (Z_t^ε) *converges to the law of the diffusion process solution of the stochastic differential equation*

$$
\begin{aligned}
dZ_t &= \sum_{k=1}^{n} P_t^k Z_t \, dB_t^k + (\langle P|P\rangle_t + \langle Q\rangle_t) Z_t \, dt, \\
Z_0 &= z_0,
\end{aligned}
\tag{26}
$$

where $B^1, \dots, B^n$ *are* n *independent standard real Brownian motions.*

PROOF. Let $C \in \mathscr{M}_{d,2d}(\mathbb{C})$ and $D \in \mathscr{M}_{2d,d}(\mathbb{C})$ be defined by

$$
C = \begin{bmatrix} 1 & \iota & 0 & \cdot \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ \cdot & 0 & 1 & \iota \end{bmatrix}, \qquad D = \frac{1}{2} \begin{bmatrix} 1 & 0 & \cdots & \cdot \\ -\iota & \ddots & \cdots & \cdot \\ \cdot & \cdot & \ddots & 1 \\ \cdot & \cdot & 0 & -\iota \end{bmatrix}.
$$

If $z = (z_1, \dots, z_d)^T$ and $x = x(z) = (\Re z_1, \Im z_1, \dots, \Re z_d, \Im z_d)^T$, then

$$
\begin{aligned}
x &= Dz + \overline{Dz}, \\
z &= Cx, \\
CD &= Id_{\mathbb{C}^d}, \\
C\overline{D} &= 0_{\mathbb{C}^d}.
\end{aligned}
$$

We can apply the diffusion-approximation theorem to the real process $X_t^\varepsilon = X(Z_t^\varepsilon)$:

$$
\frac{dX_t^\varepsilon}{dt} = \frac{1}{\varepsilon}(DPC + \overline{DPC}) X_t^\varepsilon + (DQC + \overline{DQC}) X_t^\varepsilon.
$$

By (23) and (25), the diffusion matrix is

$$
\begin{aligned}
\mathscr{D} &= \left\langle (DPC + \overline{DPC}) x \,\middle|\, ((DPC + \overline{DPC}) x)^T \right\rangle_t \\
&= D\langle Pz \mid (Pz)^T \rangle_t D^T + D\langle Pz \mid (\overline{Pz})^T \rangle_t \overline{D}^T \\
&\quad + \overline{D}\left\langle \overline{Pz} \,\middle|\, (Pz)^T \right\rangle_t D^T + \overline{D\left\langle Pz \,\middle|\, (Pz)^T \right\rangle_t D^T} \\
&= \tfrac{1}{2} \sum_{k=1}^{n} \left[\left(DP_t^k C + \overline{DP_t^k C} \right) x \right] \left[\left(DP_t^k C + \overline{DP_t^k C} \right) x \right]^T
\end{aligned}
$$

and the drift vector is

$$
\begin{aligned}
\mathscr{V} &= \langle DPC + \overline{DPC} \mid DPC + \overline{DPC} \rangle_t + \langle DQC + \overline{DQC} \rangle_t \\
&= D\langle P|P\rangle_t C + \overline{D\langle P|P\rangle_t C} + D\langle Q\rangle_t C + \overline{D\langle Q\rangle_t C}.
\end{aligned}
$$

Thus (X_t^ε) converges in law to X_t, the solution of

$$
\begin{aligned}
dX_t &= \sum_k \left(DP_t^k C + \overline{DP_t^k C} \right) X_t \, dB_t^k \\
&\quad + \left(D\langle P|P\rangle_t C + \overline{D\langle P|P\rangle_t C} + D\langle Q\rangle_t C + \overline{D\langle Q\rangle_t C} \right) X_t \, dt.
\end{aligned}
$$

Multiplying on the left by C gives (26). □

Thanks to this lemma, it is now easy to see that the limit in distribution of the process $(Y_x^\varepsilon)_{x\in[0,L]}$ is the solution of the stochastic differential equation

$$(27)\qquad \begin{aligned} dY_x &= P_x Y_x^\varepsilon \, dB_x + \sum_{k=1}^{n} \left(Q_x^k Y_x \, dB_x^k + \tilde{Q}_x^k Y_x d\tilde{B}_x^k \right) + R_x Y_x \, dx, \\ Y_0 &= Id, \end{aligned}$$

where $B, B^1, \dots, B^n, \tilde{B}^1, \dots, \tilde{B}^n$ are $2n+1$ standard real Brownian motions independent of one another and the matrices $P, Q^1, \dots, Q^n, \tilde{Q}^1, \dots, \tilde{Q}^n, R$ are defined as follows:

1. α_m and α_n are the correlation coefficients (15). They measure the randomness of the medium.
2. The drift matrix is zero when only one of the density and the bulk modulus is random:

$$R = \frac{\alpha_n - \alpha_m}{c(x)^2} \begin{bmatrix} \omega_1^2 & & & & \\ & \omega_1^2 & & & \\ & & \ddots & & \\ & & & \omega_n^2 & \\ & & & & \omega_n^2 \end{bmatrix}.$$

3. Each matrix Q^k or $\tilde{Q}^k$ acts only on $Y(x, \omega_k)$:

$$Q_x^k = \frac{\sqrt{\alpha_n}}{c(x)} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & \omega_k & 0 & 0 \\ 0 & 0 & \omega_k & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix},$$

$$\tilde{Q}^k = \iota \frac{\sqrt{\alpha_n}}{c(x)} \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & \omega_k & 0 & 0 \\ 0 & 0 & -\omega_k & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

4. The matrix P is the only one that creates a coupling between distinct frequencies:

$$P_x = \frac{\iota\sqrt{2\alpha_m}}{c(x)} \begin{bmatrix} \omega_1 & & & & \\ & -\omega_1 & & & \\ & & \ddots & & \\ & & & \omega_n & \\ & & & & -\omega_n \end{bmatrix}.$$

From (13), we need to compute

$$\hat{a}(L, \omega_k) = \frac{1}{\overline{\alpha}_k} = \frac{\Re\,\alpha_k + \iota\Im\,\alpha_k}{(\Re\,\alpha_k)^2 + (\Im\,\alpha_k)^2}.$$

From the stochastic differential equation (27) we get

$$\begin{aligned}
d(\Re\,\alpha_k)_x &= -\frac{\sqrt{2\alpha_m}}{c(x)}\omega_k \Im\,\alpha_k\, dW_x + \frac{\sqrt{\alpha_n}}{c(x)}\omega_k\big(\Re\,\beta_k\, dB_x^k - \Im\,\beta_k\, d\tilde{B}_x^k\big)\\
&\quad + \frac{\alpha_n - \alpha_m}{c(x)^2}\omega_k^2\, dx,\\
(\Re\,\alpha_k)_0 &= 1,\\
d(\Im\,\alpha_k)_x &= \frac{\sqrt{2\alpha_m}}{c(x)}\omega_k \Re\,\alpha_k\, dW_x + \frac{\sqrt{\alpha_n}}{c(x)}\omega_k\big(\Im\,\beta_k\, dB_x^k - \Re\,\beta_k\, d\tilde{B}_x^k\big)\\
&\quad + \frac{\alpha_n - \alpha_m}{c(x)^2}\omega_k^2\, dx,\\
(\Im\,\alpha_k)_0 &= 0.
\end{aligned}$$

Now a long but straightforward use of Itô's formula leads to the following equation for $\hat{a}$:

$$\begin{aligned}
d\hat{a}(x, \omega_k) &= \frac{\iota\sqrt{2\alpha_m\omega_k}}{c(x)}\hat{a}(x, \omega_k)\, dB_x - \frac{\omega_k^2(\alpha_n + \alpha_m)}{c(x)^2}\hat{a}(x, \omega_k)\, dx\\
&\quad + F(Y(x, \omega_k))\big(dB_x^k - \iota d\tilde{B}_x^k\big),
\end{aligned} \tag{28}$$

$$\hat{a}(0, \omega_k) = 1,$$

where $F(Y(x, \omega_k))$ is only a function of α_k and β_k. Now we compute $\mathbb{E}[\hat{a}(L, \omega_1) \cdots \hat{a}(L, \omega_n)]$ for distinct frequencies. Itô's formula gives

$$\begin{aligned}
d\prod_i \hat{a}(x, \omega_i) &= \sum_i \prod_{j\neq i} \hat{a}(x, \omega_j)\, d\hat{a}(x, \omega_i)\\
&\quad + \sum_{i\neq j} \prod_{k\neq i, j} \hat{a}(x, \omega_k) d\langle \hat{a}(x, \omega_i)\hat{a}(x, \omega_j)\rangle\\
&= \Big(\prod_i \hat{a}(x, \omega_i)\Big)\sum_i \Bigg(\frac{\iota\sqrt{2\alpha_m\omega_i}}{c(x)}\hat{a}(x, \omega_k)\, dB_x\\
&\qquad - \frac{\omega_k^2(\alpha_n + \alpha_m)}{c(x)^2}\hat{a}(x, \omega_k)\, dx\Bigg)\\
&\quad + \Big(\prod_i \hat{a}(x, \omega_i)\Big)\sum_i \Big(F(Y(x, \omega_i))\big(dB_x^i - \iota\, d\tilde{B}_x^i\big)\\
&\quad + \Big(\prod_i \hat{a}(x, \omega_i)\Big)\sum_{i\neq j} \frac{\iota\sqrt{2\alpha_m\omega_i}}{c(x)}\,\frac{\iota\sqrt{2\alpha_m\omega_j}}{c(x)}\, dx.
\end{aligned}$$

Therefore,

$$d\mathbb{E}\left[\prod_i \hat{a}(x,\omega_i)\right] = -\frac{(\alpha_n+\alpha_m)\Sigma_i\omega_i^2 + 2\alpha_m\Sigma_{i\neq j}\omega_i\omega_j}{c(x)^2}\mathbb{E}\left[\prod_i \hat{a}(x,\omega_i)\right]dx.$$

This equation has a unique solution, but instead of solving it we can easily see it is also satisfied by $\mathbb{E}[\prod_i \tilde{a}(x,\omega_i)]$, where $\tilde{a}(x,\omega)$ is a solution of

$$d\tilde{a}(x,\omega_k) = \frac{\iota\sqrt{2\alpha_m\omega_k}}{c(x)}\tilde{a}(x,\omega_k)\,dB_x - \frac{\omega_k^2(\alpha_n+\alpha_m)}{c(x)^2}\tilde{a}(x,\omega_k)\,dx,$$

$$\tilde{a}(0,\omega_k) = 1$$

and so

$$\mathbb{E}[\hat{a}(L,\omega_1)\cdots\hat{a}(L,\omega_n)] = \mathbb{E}[\tilde{a}(L,\omega_1)\cdots\tilde{a}(L,\omega_n)].$$

Furthermore, we can solve (29) explicitly:

$$\tilde{a}(x,\omega_k) = \exp\left(\iota\omega_k\sqrt{2\alpha_m}\int_0^x\frac{dB_y}{c(y)} - \omega_k^2\alpha_n\int_0^x\frac{dy}{c(y)^2}\right).$$

We can now conclude: $Z_L = \sqrt{2\alpha_m}\int_0^L dBy/(c(y))$ is a centered Gaussian variable with the correct variance (16) and

$$\begin{aligned}
&\frac{1}{2\pi}\int\exp(-\iota\omega\sigma)\hat{f}(\omega)\tilde{a}(L,\omega)\,d\omega\\
&\quad=\int f(t)\frac{1}{\sqrt{2\pi}}\int\exp(\iota\omega(t-\sigma+Z_L))\exp(-\omega^2\alpha_n\theta_L)\,d\omega\,dt\\
&\quad=\frac{1}{\sqrt{4\pi\alpha_n T_L}}\int f(t)\exp\left(-\frac{(t-\sigma+Z_L)^2}{4\alpha_n\theta_L}\right)dt\\
&\quad=f*G_L(\sigma+Z_L).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}\left[A(L,\tau(L)+\varepsilon\sigma_1)^{p_1}\cdots A(L,\tau(L)+\varepsilon\sigma_k)^{p_k}\right]\\
&\qquad=\mathbb{E}\left[f*G_L(\sigma_1+Z_L)^{p_1}\cdots f*G_L(\sigma_k+Z_L)^{p^k}\right],
\end{aligned}$$

which is enough to conclude the proof of Theorem 3.1 because we know already that the process is tight.

The description of the shape of the transmitted pulse is now complete: There is a deterministic spreading due to the convolution by the Gaussian function G_L and a stochastic translation by a Gaussian variable which does not affect the shape.

REMARK 3.6. We have in fact proved a stronger result because it is easy to see that the tightness of the process $(Y_x^\varepsilon)_{x\in[0,L]}$ implies that of the process $A(x,\tau(x)+\varepsilon\sigma)_{x\in[0,L]}$ and so the field $A(x,\tau(x)+\varepsilon\sigma)_{\sigma\in[-T,T],\,x\in[0,L]}$ converges in distribution to $f * G_x(\sigma + Z_x)$, where G_x is the Gaussian kernel and Z_x is the diffusion process defined by

$$Z_x = \int_0^x \frac{\sqrt{2\alpha_m}}{c(y)}\, dB_y.$$

REMARK 3.7. With the chosen normalization of the incident pulse, the only observable signal is the coherent transmission whose amplitude is of order 1. Indeed, if we observe the transmitted signal far from the mean arrival time of the pulse, that is, if we look at

$$A(L, \tau(L) + t) = \frac{1}{2\pi}\int \exp\left(-\iota\frac{\omega t}{\varepsilon}\right)\hat{f}(\omega)\hat{a}^{\varepsilon}(L, \omega)\, d\omega$$

for t strictly nonnegative, then

$$\begin{aligned}
&\mathbb{E}A(L, \tau(L) + t)^n \\
&= \frac{1}{(2\pi)^n}\int \exp\left(-\iota\frac{\sum_j \omega_j t}{\varepsilon}\right)\prod_j \hat{f}(\omega_j)\mathbb{E}\prod_j \hat{a}(L, \omega_j)\, d\omega_1 \cdots d\omega_n \\
&\quad + \frac{1}{(2\pi)^n}\int \exp\left(-\iota\frac{\sum_j \omega_j t}{\varepsilon}\right)\prod_j \hat{f}(\omega_j) \\
&\qquad \times \mathbb{E}\prod_j \left(\hat{a}^{\varepsilon}(L, \omega_j) - \hat{a}(L, \omega_i)\right) d\omega_1 \cdots d\omega_n.
\end{aligned}$$

The first term goes to zero as $\varepsilon \to 0$ due to the boundness of $\hat{a}(L, \omega)$ and the regularity of $\hat{f}$, and the second term goes to zero because of the convergence result previously proved. The study of the reflected signal, defined for t positive by

$$B(0, t) = \frac{1}{2\pi}\int \exp\left(-\iota\frac{\omega t}{\varepsilon}\right)\hat{f}(\omega)\hat{b}^{\varepsilon}(0, \omega)\, d\omega$$

can be deduced from the previous analysis. If we want to work with forward stochastic differential equations, we make the change of variable

$$\begin{aligned}
\tilde{a}^{\varepsilon}(x, \omega) &= \hat{a}^{\varepsilon}(L - x, \omega), \\
\tilde{b}^{\varepsilon}(x, \omega) &= \hat{b}^{\varepsilon}(L - x, \omega).
\end{aligned}$$

Then we find an equation satisfied by the limit in distribution of the propagator matrix $\tilde{Y}^{\varepsilon}(x, \omega_1, \ldots, \omega_n)$ for $(\tilde{a}^{\varepsilon}, \tilde{b}^{\varepsilon})$ with distinct frequencies and we find the following stochastic differential equation for $\tilde{b}(x, \omega_k)$:

$$\begin{aligned}
d\tilde{b}(x, \omega_k) &= \frac{\iota 2\sqrt{2\alpha_m \omega_k}}{c(x)}\tilde{b}(x, \omega_k)\, dB_x - \frac{2\omega_k^2(2\alpha_m + \alpha_n)}{c(x)^2}\tilde{b}(x, \omega_k)\, dx \\
&\quad + \tilde{F}\left(\tilde{Y}(x, \omega_k)\right)\left(dB_x^k - \iota\, d\tilde{B}_x^k\right), \\
\tilde{b}(0, \omega_k) &= 0.
\end{aligned}$$

The equation for the moments is then

$$d\mathbb{E}\left[\prod_j \tilde{b}(x, \omega_j)\right]_x = \frac{1}{c(x)^2}\left((2\alpha_m + \alpha_n)\sum_j \omega_j^2 - 8\alpha_m \sum_{j \neq l} \omega_j \omega_l\right)$$

$$\times \mathbb{E}\left[\prod_j \tilde{b}(x, \omega_j)\right] dx,$$

$$\mathbb{E}\left[\prod_j \tilde{b}(0, \omega_j)\right] = 0,$$

so

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\prod_j \hat{b}^{\varepsilon}(L, \omega_j)\right] = 0.$$

Hence $B(0, t)$ converges in probability to zero.

REMARK 3.8. Although the amplitude of the reflected pulse is of order less than 1 and cannot be observed in the limit, its contribution to the outcoming energy is nonzero because the energy of the coherent transmission is less than the total incoming energy [remember the conservation of energy relation (14)]

$$\frac{\int A(L, \tau(L) + t)^2\, dt}{\int f(t/\varepsilon)^2\, dt} = \frac{\int A(L, \tau(L) + \varepsilon t)^2\, dt}{\int f(t)^2\, dt} \xrightarrow[\varepsilon \to 0]{}$$

$$\frac{\int \hat{f}(\omega)^2 \exp(\iota\omega Z_L - \omega^2 \alpha_m \theta_L)\, d\omega}{\int \hat{f}(\omega)^2\, d\omega} < 1.$$

REFERENCES

[1] ASCH, M., KOHLER, W., PAPANICOLAOU, G., POSTEL, M. and WHITE, B. (1991). Frequency content of randomly scattered signals. *SIAM Rev.* **33** 519–625.

[2] BURRIDGE, R., PAPANICOLAOU, G., SHENG, P. and WHITE, B. (1989). Probing a random medium with a pulse. *SIAM J. Appl. Math.* **49** 582–607.

[3] CLOUET, J. F. (1992). Théorie d'O'Doherty–Anstey et approximation-diffusion, Rapport Interne 273, CMAP, Ecole Polytechnique.

[4] FOUQUE, J. P. and MERZBACH, E. (1994). A limit theorem for linear boundary value problems in random media. *Ann. Appl. Probab.* **4** 549–569.

[5] PAPANICOLAOU, G., STROOCK, D. and VARADHAN, S. (1977). Margingale approach to some limit theorems. *Duke Turbulence Conference. Duke. Univ. Math. Series* **3**. Duke Univ. Press.

[6] STROOCK, D. and VARADHAN, S. (1979). *Multidimensional Diffusion Processes*. Springer, Berlin.

CENTRE DE MATHEMATIQUES APPLIQUEES
ECOLE POLYTECHNIQUE
91128 PALAISEAU CEDEX
FRANCE