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# FREE LUNCH FOR LARGE FINANCIAL MARKETS WITH CONTINUOUS PRICE PROCESSES

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A large financial market is described by a sequence of traditional market models with finite numbers of assets. There are various concepts in the spirit of no asymptotic arbitrage related to the contiguity of a sequence of equivalent martingale measures with respect to the sequence of historical probabilities. In this article, I show that in the case of continuous price processes, the existence of a bicontiguous sequence of martingale measures is equivalent to the property of no asymptotic free lunch with bounded risk.

**1. Introduction.** A large financial market is a sequence of traditional market models, each of them based on a finite number of assets. Under the assumption that there is no kind of arbitrage on any of the finite markets, Kabanov and Kramkov (1994) introduced the notions of no asymptotic arbitrage of first (NAA1) and second kind (NAA2). The notion NAA1 happens to be equivalent to the contiguity of the sequence of the objective probabilities with respect to the sequence of the upper envelopes of the equivalent local martingale measures, whereas NAA2 is equivalent to the contiguity of the sequence of the lower envelopes of the local martingale measures with respect to the sequence of the objective probabilities; compare Kabanov and Kramkov (1998). These results can be formulated in another way to show that the results on NAA1 and NAA2 are not symmetric, but are of a different nature; see Klein and Schachermayer (1996a, b). In any case, NAA1 and NAA2 could be related only to one-sided contiguity properties of sequences of equivalent local martingale measures, which is the analogue of absolute continuity of measures in the case of a sequence of probability spaces.

However, the direct analogue to the existence of an equivalent martingale measure for the case of a large financial market seems to be the existence of a sequence of martingale measures which is contiguous with respect to the sequence of the objective probabilities and vice versa (i.e., *bicontiguous*). So it was natural to ask whether the symmetric property of bicontiguity has an economic interpretation. The general answer to this question was given by Klein (2000), where the notion of no asymptotic free lunch (NAFL) was introduced. This led to a general version of the fundamental theorem of asset pricing for large financial markets. The condition NAFL is rather technical and, unfortunately, cannot be replaced by a weaker condition in general. The aim of the present note is to

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show that for *continuous* processes, the bicontiguity property is equivalent to no asymptotic free lunch with bounded risk (NAFLBR), which has an intuitive interpretation.

Let me conclude the Introduction with a remark on the history of the theory of large financial markets. The idea of asymptotic arbitrage appeared first in Ross (1976) and in the important note of Huberman (1982). The theory of large financial markets can be seen as a "modern" version of arbitrage pricing theory (APT). However, the classic APT does not touch the problem of the existence of an equivalent martingale measure, so it does not provide a version of the fundamental theorem of asset pricing for a large financial market.

**2. Definitions and notations.** I slightly adapt the setting of Kabanov (1997) so that it is appropriate for our presentation. Let $\mathcal{S}$ be the space of semimartingales $X$ defined on the infinite time interval $[0, \infty)$ and starting from zero; $\mathcal{S}$ is a Frechet space with the quasinorm

$$D(X) = \sup\left\{\sum_{n\geq 1} 2^{-n}\mathbb{E}(1 \wedge |H \cdot X_n|) : H \text{ predictable}, |H| \leq 1\right\}.$$

As usual $H \cdot X_t$ is the stochastic integral of $H$ with respect to $X$ on the interval $(0, t]$. We fix in $\mathcal{S}$ a closed convex subset $\mathcal{X}^1$ of *continuous* processes $X$ such that $|X| \leq 1$, which contains 0 and satisfies the following condition: If $X, Y \in \mathcal{X}^1$ and $H, G \geq 0$ are bounded predictable processes, $HG = 0$ and $|Z| \leq 1$, where $Z = H \cdot X + G \cdot Y$, then $Z \in \mathcal{X}^1$. Let $\mathcal{X} = \bigcup_{\alpha>0} \alpha\mathcal{X}^1$. Note that the concatenation property is not identical to that in Kabanov (1997). Indeed, there the processes were only assumed to be bounded *below* by $-1$. I consider only continuous processes, so I can, without loss of generality, assume that the processes are bounded. Indeed, continuous processes can always be bounded above as well. Define convex sets $\mathbf{K}_\alpha = \{X_\infty : X \in \alpha\mathcal{X}^1\}$ and $\mathbf{K} = \{X_\infty : X \in \mathcal{X}\} = \bigcup_{\alpha>0} \mathbf{K}_\alpha$. Moreover, define a set of equivalent probability measures $\mathbf{M}^e(\mathcal{X}) = \{\mathbb{Q} \sim \mathbb{P} : \mathbb{E}_\mathbb{Q} X_\infty \leq 0 \text{ for all } X \in \mathcal{X}\}$.

Note that $\mathcal{X}^1$ can be taken as the set of all stochastic integrals $H \cdot S$ with respect to one fixed $d$-dimensional continuous semimartingale $S$ based on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ such that $|H \cdot S| \leq 1$. The space $\mathcal{X}^1$ is closed by the theorem of Mémin (1980). This setting describes a financial market with $d$ assets, where the price process is given by $S$. The elements of the set $\mathbf{K}$ are interpreted as final outcomes when trading according to a certain strategy $H$. An easy consideration shows that $\mathbf{M}^e(\mathcal{X}) \neq \varnothing$ is equivalent to the existence of an equivalent probability measure $\mathbb{Q}$ such that $S$ is a local martingale under $\mathbb{Q}$. This implies any condition of no arbitrage type on the market.

In a *large financial market*, consider a sequence of small market models, that is, a sequence of semimartingales $(S^n)$, where $S^n$ is based on $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n), \mathbb{P}^n)$. The interpretation of the superscript $n$ in expressions such as $\mathbf{K}^n$ and $\mathbf{M}^e(\mathcal{X}^n)$ is then obvious. Throughout this article, $\mathbf{M}^e(\mathcal{X}^n) \neq \varnothing$ for all $n \in \mathbb{N}$, which ensures that there is no form of arbitrage opportunity on any of the small markets.

The following notations are used: $(\mathbb{Q}^n) \lhd \rhd (\mathbb{P}^n)$ means that the sequence of probability measures $(\mathbb{Q}^n)$ is contiguous with respect to the sequence of probability measures $(\mathbb{P}^n)$ and vice versa. For $L^p(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, write $L^{p,n}$ where $p = 1$ or $\infty$. Let $\operatorname{supp}\mu$ denote the support of a probability measure $\mu$ on the real line and let $\delta_x$ denote the Dirac measure at the point $x \in \mathbb{R}$. Write $I$ for the interval $[-1, 1]$ and $\alpha I$ for $[-\alpha, \alpha]$.

## 3. Main result.

DEFINITION 3.1. There is an asymptotic free lunch with bounded risk (AFLBR) on a large financial market if there is $c \in (0, 1]$ and a sequence $\xi^{n_k} \in \mathbf{K}_1^{n_k}$ such that:

(i) $\mathbb{P}^{n_k}(\xi^{n_k} \geq c) \geq c$ for all $k \in \mathbb{N}$ and
(ii) $\lim_{k\to\infty} \mathbb{P}^{n_k}(\xi^{n_k} < -\varepsilon) = 0$ for all $\varepsilon > 0$.

We use the notation NAFLBR if there is no AFLBR.

An alternative description of NAFLBR using the distributions of the random variables in all attainability sets $\mathbf{K}^n$ will be presented. It is indeed the continuity assumption that enables a characterization using measures on $\mathbb{R}$ with *bounded* support. In general, the laws of the elements of the attainability sets $\mathbf{K}^n$ would be unbounded on the positive real line [cf. the notion of admissible integrands as in Delbaen and Schachermayer (1994), where the corresponding stochastic integrals were only supposed to be bounded *below*]. However, continuous processes can always be stopped to allow them to be bounded above as well.

Let $\mathcal{P}_b$ be the space of all probability measures on $\mathbb{R}$ with bounded support. Equip $\mathcal{P}_b$ with the topology $\sigma(\mathcal{P}_b, C)$, where $C$ are the continuous functions on $\mathbb{R}$. For $\alpha > 0$, let $\mathcal{P}(\alpha I) = \{\mu \in \mathcal{P}_b : \operatorname{supp}\mu \subseteq \alpha I\}$ and $\mathcal{P}_b(\mathbb{R}_+) = \{\mu \in \mathcal{P}_b : \operatorname{supp}\mu \subseteq [0, +\infty)\}$.

REMARK 3.2. The topology $\sigma(\mathcal{P}_b, C)$ does not coincide with the topology of weak convergence in a probabilistic sense on $\mathcal{P}_b$. Indeed, the set of functionals $C$ is used, not the *bounded* continuous functions $C_b(\mathbb{R})$ as usual for the topology of weak convergence. Consider, for example, $\mu_n = 2^{-n}\delta_{2^n} + (1 - 2^{-n})\delta_0$. This sequence converges weakly to $\delta_0$, but does not converge in $\sigma(\mathcal{P}_b, C)$. However, restricted to $\mathcal{P}(\alpha I)$, the topology of weak convergence and $\sigma(\mathcal{P}_b, C)$ are equal.

Let $\mathcal{L}(\mathbf{K}_\alpha^n) = \{\mathcal{L}(\xi) : \xi \in \mathbf{K}_\alpha^n\}$, where $\mathcal{L}(\xi)$ denotes the law of $\xi$. Define $\mathcal{L}_\alpha$ to be the closure of $\bigcup_{n\geq 1} \mathcal{L}(\mathbf{K}_\alpha^n)$ in $\mathcal{P}(\alpha I)$ with respect to the topology of weak convergence [which equals $\sigma(\mathcal{P}_b, C)$ closure]. Let $\mathcal{L} = \bigcup_{\alpha>0} \mathcal{L}_\alpha$ and let $\overline{\mathcal{L}}$ be the closure of $\mathcal{L}$ with respect to the topology $\sigma(\mathcal{P}_b, C)$.

LEMMA 3.3. NAFLBR $\Leftrightarrow \mathcal{L} \cap \mathcal{P}_b(\mathbb{R}_+) = \{\delta_0\}$.

PROOF. ($\Rightarrow$) If the claim fails, there is $\alpha > 0$ and $\mu \in \mathcal{L}_\alpha$ such that $\operatorname{supp}\mu \subseteq [0, \alpha]$ and $\mu \neq \delta_0$. There is a sequence $\mu_k \in \mathcal{L}(\mathbf{K}_\alpha^{n_k})$, such that $\mu_k \to \mu$ weakly. So the result is $\mu_k[-\alpha, -\varepsilon] \to 0$ for all $\varepsilon > 0$. Because $\mu \neq \delta_0$, there is $c \in (0, \alpha)$ such that $\mu_k[c, \alpha] \geq c$ for all $k$ large enough. The $\xi^k \in \mathbf{K}_\alpha^{n_k}$ with $\mu_k = \mathcal{L}(\xi^k)$ form an AFLBR.

($\Leftarrow$) The laws of the $\xi^{n_k}$ of the AFLBR have a cumulation point $\mu \in \mathcal{L}_1$ (because $\mathcal{L}_1$ is compact for the topology of weak convergence). This yields a contradiction. $\square$

Now let us introduce a variant of the NAFL condition of Klein (2000). Again it is the continuity of the processes that enables a characterization using measures on $\mathbb{R}$ with bounded support. In the following discussion it will become clear that this characterization is equivalent to the original definition of NAFL.

DEFINITION 3.4. There is NAFL$'$ if $\overline{\mathcal{L}} \cap \mathcal{P}_b(\mathbb{R}_+) = \{\delta_0\}$.

The model fulfills *bicontiguity* if there is a sequence $\mathbb{Q}^n \in \mathbf{M}^e(\mathcal{X}^n)$ such that $(\mathbb{Q}^n) \lhd \rhd (\mathbb{P}^n)$. Let us now state the main result of this article:

THEOREM 3.5. *For the model of a large financial market with continuous processes* NAFLBR, NAFL, NAFL$'$ *and bicontiguity are equivalent.*

The proof is organized as follows. In Section 4 it is shown that NAFLBR $\Rightarrow$ NAFL$'$ (immediate from Proposition 4.1). Lemma 4.5 gives bicontiguity $\Rightarrow$ NAFLBR. In the Appendix the definition of NAFL is recalled and NAFL$'$ $\Rightarrow$ NAFL is proved. Moreover, the main theorem of Klein (2000) is recalled which gives NAFL $\Rightarrow$ bicontiguity.

## 4. Proofs.

PROPOSITION 4.1. NAFLBR $\Rightarrow \mathcal{L}$ is $\sigma(\mathcal{P}_b, C)$ closed and so $\mathcal{L} = \overline{\mathcal{L}}$.

The consequence of Proposition 4.1 for the attainability sets is that whenever a measure $\mu$ on the real line is approximated by the laws of some elements of the attainability sets $\mathbf{K}^n$ in the sense of the topology $\sigma(\mathcal{P}_b, C)$, this $\mu$ also can be approximated by the laws of random variables which are, moreover, bounded by some fixed $\alpha > 0$. This is not clear a priori, because without Proposition 4.1, there is no way to know whether it is possible to choose elements of the attainability sets such that the weight of the points "far off" is zero. Heuristically speaking, the reason for this is that *continuous* processes can be stopped in a manner that they are all bounded by a uniform constant.

LEMMA 4.2. *Let $\alpha > \beta$. Then $\mathcal{L}_\alpha \cap \mathcal{P}(\beta I) = \mathcal{L}_\beta$.*

Lemma 4.2 is very similar to Lemma 5.3 of Delbaen (1992) and its proof is adapted to the present setting. The following lemmas are analogues of Lemmas 5.1 and 5.2 there.

LEMMA 4.3. *Let $(\xi^k)$ be a sequence in $\bigcup_{n\geq 1} \mathbf{K}_1^n$ such that $\mathbb{P}^{n_k}(\xi^{k-} \geq \varepsilon) \to 0$ for all $\varepsilon > 0$. Then $\mathbb{P}^{n_k}(\sup_t |X_t^{n_k}| \geq \varepsilon) \to 0$ for all $\varepsilon > 0$, where $X_\infty^{n_k} = \xi^k$.*

PROOF. Review the proof of Lemma 5.1 of Delbaen (1992). Instead of convergence in probability on one fixed probability space, use the following convergence for $(\xi^n): \xi^n \to 0$ if $\mathbb{P}^n(\xi^n \geq \varepsilon) \to 0$ for each $\varepsilon > 0$. Proceed as in Delbaen (1992). $\square$

LEMMA 4.4. *Let $\xi^k = x^k + X_\infty^{n_k}$, where $x_k \in \mathbb{R}_+$ and $X_\infty^{n_k} \in \mathbf{K}^{n_k}$ for some $n_k$. Suppose that $|\xi^k| \leq 1$ for all $k \in \mathbb{N}$ and $\mathbb{P}^{n_k}(\xi^{k-} \geq \varepsilon) \to 0$ for all $\varepsilon > 0$. Then $\mathbb{P}^{n_k}(\sup_t (x^k + X_t^{n_k})^- \geq \varepsilon) \to 0$ for all $\varepsilon > 0$.*

PROOF. Adapt the proof of Lemma 5.2 of Delbaen (1992) using Lemma 4.3. $\square$

PROOF OF LEMMA 4.2. The proof proceeds in a similar way as the proof of Lemma 5.3 in Delbaen (1992). Let $\beta < \alpha$ and $\mu \in \mathcal{L}_\alpha \cap \mathcal{P}(\beta I)$. There is a sequence $(\mu_k)$ in $\bigcup_{n\geq 1} \mathcal{L}(\mathbf{K}_\alpha^n)$ such that $\mu_k \to \mu$ weakly. Each $\mu_k$ is the law of a random variable $\xi^k \in \mathbf{K}_\alpha^{n_k}$. It is necessary to find a sequence $(\eta^k)$ with $\eta^k \in \bigcup_{n\geq 1} \mathbf{K}_\beta^n$ such that the corresponding laws $\nu^k = \mathcal{L}(\eta^k)$ converge weakly to $\mu$.

I claim that the sequence $\xi^k + \beta$ satisfies the assumptions of Lemma 4.4. Indeed, $|\xi^k + \beta| \leq \alpha + \beta$. Moreover, for all $\varepsilon > 0$ and $k \to \infty$,

$$\begin{aligned} \lim \mathbb{P}^{n_k}(\xi^k + \beta \leq -\varepsilon) &= \lim \mathbb{P}^{n_k}\big(\xi^k \leq -(\beta + \varepsilon)\big) \\ &= \lim \mu_k[-\alpha, -(\beta + \varepsilon)] = 0. \end{aligned}$$

Lemma 4.4 yields, for $X^{n_k}$ with $X_\infty^{n_k} = \xi^k$, that $\mathbb{P}^{n_k}(\sup_t (\beta + X_t^{n_k})^- \geq \varepsilon) \to 0$ for all $\varepsilon > 0$. Equivalently, $\mathbb{P}^{n_k}(\sup_t (\beta - X_t^{n_k})^- \geq \varepsilon) \to 0$ is shown. Define stopping times $T_\varepsilon^k = \inf\{t : |X_t^{n_k}| \geq \beta + \varepsilon\}$. Then $\mathbb{P}^{n_k}(T_\varepsilon^k < \infty) \to 0$ for all $\varepsilon > 0$. Let $\varepsilon_l \to 0$ be a strictly positive sequence and let $(k_l)$ be a subsequence such that

$$\mathbb{P}^{n_{k_l}}\big[T_{\varepsilon_l}^{k_l} < \infty\big] \to 0. \tag{4.1}$$

To simplify notation, assume that (4.1) holds for the whole sequence $(k)$. Set $\tau^k = T_{\varepsilon_k}^k$ and define $\eta^k = \beta/(\beta + \varepsilon_k) X_{\infty \wedge \tau_k}^{n_k}$. Then $|\eta^k| \leq \beta$ by continuity. Moreover, by definition, $\eta^k \in \mathbf{K}^{n_k}$. I claim that the laws $\nu_k = \mathcal{L}(\eta^k)$ converge to $\mu$ weakly which implies $\mu \in \mathcal{L}_\beta$. Indeed, let $w \in C(\alpha I)$. Then

$$\begin{aligned}|\langle \nu_k - \mu_k, w\rangle| &= \left|\mathbb{E}^{n_k} w(\eta^k) - \mathbb{E}^{n_k} w(\xi^k)\right| \\ &\le \left|\mathbb{E}^{n_k}\left[\left(w\left(\frac{\beta}{\beta+\varepsilon_k}\xi^k\right) - w(\xi^k)\right)\mathbb{1}_{\{\tau^k=\infty\}}\right]\right| \\ &\quad + \left|\mathbb{E}^{n_k}\left[(w(\eta^k) - w(\xi^k))\mathbb{1}_{\{\tau^k<\infty\}}\right]\right|.\end{aligned}$$

The $\mathbb{1}_{\{\tau_k=\infty\}}$ part converges to 0 because $|\xi^k| \le \alpha$ and $w \in C(\alpha I)$. The $\mathbb{1}_{\{\tau^k<\infty\}}$ part is bounded above by $2\|w\|_\infty \mathbb{P}^{n_k}(\tau^k < \infty)$, which converges to 0 by (4.1). So $|\langle \nu_k - \mu, w\rangle| \le |\langle \nu_k - \mu_k, w\rangle| + |\langle \mu_k - \mu, w\rangle| \to 0$ for each continuous $w$. □

PROOF OF PROPOSITION 4.1. Suppose the statement of the proposition is not true. Then there is $\mu_0 \in \overline{\mathcal{L}}$ such that $\mu_0 \notin \mathcal{L}$. Without loss of generality, $\operatorname{supp}\mu_0 \subseteq I$. In particular, $\mu_0 \notin \mathcal{L}_1$. Hence there exist $w_1^1, w_2^1, \dots, w_N^1 \in C(I)$ and $\varepsilon_0 > 0$ such that for all $\mu \in \mathcal{L}_1$, there is $i = i(\mu) \le N$ with

$$\langle \mu_0, w_i^1\rangle > \langle \mu, w_i^1\rangle + \varepsilon_0. \tag{4.2}$$

It is shown that there are $w_1^2, w_2^2, \dots, w_N^2 \in C(2I)$ such that $w_j^2|_I = w_i^1$ for some $i \le N$ and (4.2) holds for all $\mu \in \mathcal{L}_2$ with the functions $w_j^2$ instead of the $w_i^1$.

CLAIM. Let $\mu \in \mathcal{L}_2$. Then there exists $w_\mu \in C(2I)$ such that $w_\mu|_I = w_i^1$ for some $i \le N$ and such that $\langle \mu_0, w_\mu\rangle > \langle \mu, w_\mu\rangle + \varepsilon_0$.

Indeed, either $\mu$ is already in $\mathcal{L}_1$ and then (4.2) holds anyway or $\mu \in \mathcal{L}_2 \setminus \mathcal{L}_1$ and then $\delta > 0$ exists such that $\mu((1+\delta)I) < 1 - \delta$, because otherwise $\mu \in \mathcal{L}_1$ by Lemma 4.2. Take $w_1^1$ and define $w_\mu \in C(2I)$ as

$$w_\mu = w_1^1 \mathbb{1}_I + v\mathbb{1}_{(1+\delta)I\setminus I} - c\mathbb{1}_{2I\setminus(1+\delta)I},$$

where $c = (2\varepsilon_0 - \langle \mu_0, w_1^1\rangle + \|w_1^1\|_\infty)/\delta$ and $v : (1+\delta)I \setminus I \to \mathbb{R}$ is chosen such that $w_\mu$ is continuous and $\|v^+\|_\infty \le \|w_1^1\|_\infty$, where $v^+ = v \vee 0$. Observe that

$$\langle \mu_0, w_\mu\rangle - \langle \mu, w_\mu\rangle \ge \langle \mu_0, w_\mu\rangle - \|w_1^1\|_\infty + 2\varepsilon_0 - \langle \mu_0, w_\mu\rangle + \|w_1^1\|_\infty > \varepsilon_0,$$

which proves the claim.

For $u \in C(2I)$ such that $u|_I = w_i^1$ for some $i \le N$, define an open set in $\mathcal{P}(2I)$ by $V_u = \{\nu \in \mathcal{P}(2I) : \langle \mu_0 - \nu, u\rangle > \varepsilon_0\}$. By the above considerations, the sets $V_u$ form an open covering of the set $\mathcal{L}_2$, which is compact with respect to the topology of weak convergence on $\mathcal{P}(2I)$. So there is a finite subcovering of $\mathcal{L}_2$, that is, functions $\tilde{w}_1, \dots, \tilde{w}_k$ such that $\mathcal{L}_2 \subseteq \bigcup_{j=1}^k V_{\tilde{w}_j}$. (Without loss of generality, take

$k \geq N$ and assume that for each $i \leq N$, there is at least one $j \leq k$ such that $\tilde{w}_j|_{[-1,1]} = w_i^1$.) For $i \leq N$, let

$$w_i^2 = \min_{\{j : \tilde{w}_j|_I = w_i^1\}} \tilde{w}_j.$$

Thus $w_1^2, \dots, w_N^2 \in C(2I)$ is found such that (4.2) holds for all $\mu \in \mathcal{L}_2$. By induction, find for each $n \in \mathbb{N}$ functions, $w_1^n, \dots, w_N^n \in C(nI)$ such that $w_i^n|_{(n-1)I} = w_i^{n-1}$ for $i \leq N$ and (4.2) holds for all $\mu \in \mathcal{K}_n$. Hence there are $w_1, \dots, w_N \in C$ defined by $w_i(x) = \lim_{n\to\infty} w_i^n(x)$, such that for each $\mu \in \mathcal{L}$ there exists an $i \leq N$ with $\langle \mu_0, w_i\rangle > \langle \mu, w_i\rangle + \varepsilon_0$. This shows $\mu_0 \notin \overline{\mathcal{L}}$, a contradiction. $\square$

LEMMA 4.5. *Bicontiguity* $\Rightarrow$ NAFLBR.

PROOF. Suppose there is an AFLBR. The contiguity $(\mathbb{P}^n) \vartriangleleft (\mathbb{Q}^n)$ implies that there is $\delta > 0$ with $\mathbb{Q}^{n_k}(\xi^{n_k} \geq c) \geq \delta$ for all $k$. Moreover, $(\mathbb{Q}^n) \vartriangleleft (\mathbb{P}^n)$ gives $\mathbb{Q}^{n_k}(\xi^{n_k} < -\varepsilon) \to 0$ for all $\varepsilon > 0$. Taking $\varepsilon$ small and then $n_k$ large gives $\mathbb{E}_{\mathbb{Q}^{n_k}} \xi^{n_k} > 0$, a contradiction to the definition of $\mathbf{M}^e(\mathcal{X}^n)$. $\square$

## APPENDIX: ABOUT ASYMPTOTIC FREE LUNCH

Recall the definition of NAFL in Klein (2000). Let $\mathbf{C} = \mathbf{K} - L_+^\infty$ and define $D^\varepsilon = \{\eta \in L^\infty : 0 \leq \eta \leq 1, \mathbb{E}\eta \geq \varepsilon\}$. Let $(a_j)$ and $(b_j)$ be strictly positive sequences and define $V^{a,b} = \mathrm{conv}_{j\geq 1}[B_{a_j}(L^1) \cap B_{b_j}(L^\infty)]$, where $B_r(L)$ denotes the closed ball with radius $r$ of the Banach space $L$. Recall that the superscript $n$ denotes the respective sets for the $n$th small market.

DEFINITION A.1. There is NAFL if, for each $\varepsilon > 0$, there exist a strictly positive, decreasing sequence $(a_j)$ with $a_j \to 0$ and a strictly positive, increasing sequence $(b_j)$ with $b_j \to \infty$ such that $\mathbf{C}^n \cap (D^{\varepsilon,n} + V^{a,b,n}) = \varnothing$ for all $n \in \mathbb{N}$.

By Definition A.1 it is clear that the set $\mathbf{C}^n$ is separated from $D^{\varepsilon,n}$ for each $\varepsilon > 0$ by some Mackey neighborhood $V^{a,b,n}$. This separation is a direct translation of the concept of no free lunch given by Kreps (1981) to a sequence of probability spaces. Indeed, if NAFL holds, it is not possible to approximate a strictly positive gain by elements of the sequence of sets $(\mathbf{C}^n)$ in a Mackey (or, equivalently, weak star) sense.

PROPOSITION A.2. NAFL$'$ $\Rightarrow$ NAFL.

LEMMA A.3. *Let $(a_j)$ and $(b_j)$ be sequences as in Definition* A.1 *and let $\zeta \in V^{a,b,n}$. Then $\mathbb{P}^n(|\zeta| \geq 2b_k) < a_k/b_k$ for all $k \in \mathbb{N}$.*

PROOF. Begin with $\zeta = \sum_{i=1}^{\infty} \lambda_i \zeta_i$, where $\zeta_i \in B_{a_i}(L^{1,n}) \cap B_{b_i}(L^{\infty,n})$ and $\sum_{i=1}^{\infty} \lambda_i = 1$. For fixed $k \in \mathbb{N}$, $|\zeta_i| \leq b_i \leq b_k$ for $i \leq k$ and $\|\zeta_i\|_{L^{1,n}} \leq a_i \leq a_k$ for $i > k$. Let $\widetilde{\zeta}_1 = \sum_{i=1}^{k} \lambda_i \zeta_i$ and $\widetilde{\zeta}_2 = \sum_{i=k+1}^{\infty} \lambda_i \zeta_i$. Then

$$\mathbb{P}^n(|\widetilde{\zeta}_2| \geq b_k) \leq \frac{1}{b_k} \mathbb{E}^n |\widetilde{\zeta}_2| \leq \frac{1}{b_k} \sum_{i=k+1}^{\infty} \lambda_i \mathbb{E}^n |\zeta_i| < \frac{a_k}{b_k}$$

and, because $|\widetilde{\zeta}_1| < b_k$, $\mathbb{P}^n(|\zeta| \geq 2b_k) < \mathbb{P}^n(|\widetilde{\zeta}_2| \geq b_k)$. $\square$

LEMMA A.4. *There is an* AFL $\Leftrightarrow$ *there is* $c > 0$ *such that for all sequences* $(a_j)$ *and* $(b_j)$ *as in Definition* A.1, *there exists* $n = n(a, b)$ *and* $X_\infty^n \in \mathbf{K}^n \cap (D^{c,n} + V^{a,b,n})$ *such that* $X_t^n \leq 1$ *for all* $t \in \mathbb{R}_+$.

PROOF. ($\Rightarrow$) Indeed, for each pair of sequences $(a, b)$, let $\xi^n \in \mathbf{C}^n$ be the random variables that form an AFL, where $n = n(a, b)$. There is $X^n \in \mathcal{X}^n$ and $\eta^n \in L_+^{\infty,n}$ such that $X_\infty^n = \xi^n + \eta^n$. Define the stopping time $T^n = \inf\{t \geq 0 : X_t^n \geq 1\}$. Because $X^n$ is continuous, clearly $X_{t\wedge T^n}^n \leq 1$ for all $t \in \mathbb{R}_+$ and by the definition of $\mathcal{X}^n$, $(X_{t\wedge T^n}^n) \in \mathcal{X}^n$. Clearly $X_{\infty\wedge T^n}^n \in \mathbf{K}^n \cap (D^{c,n} + V^{a,b,n})$.

($\Leftarrow$) is obvious. $\square$

PROOF OF PROPOSITION A.2. Suppose there is an AFL. Then there is $c > 0$, and for all $(a_j)$ and $(b_j)$ as in Definition A.1, there is $n = n(a, b)$ and $X_\infty^n$ as in Lemma A.4. Define a partial order on the pairs of sequences by $((a^1), (b^1)) \leq ((a^2), (b^2))$ if $a^2(j) \leq a^1(j)$ and $b^2(j) \leq b^1(j)$ for all $j$. Let $\mu^{n(a,b)} = \mathcal{L}_{X_\infty^n}$.

CLAIM. The net $(\mu^{n(a,b)})$ in $\mathcal{L}$ has a $\sigma(\mathcal{P}_b, C)$ cumulation point $\mu \in \overline{\mathcal{L}}$ such that $\operatorname{supp} \mu \subseteq [0, 1]$ and $\mu \neq \delta_0$ (which is a contradiction).

Indeed, $X_\infty^n = \eta^n + \zeta^n$, where $\eta^n = X_\infty^n \mathbb{1}_{\{X_\infty^n \geq 0\}}$ and $\zeta^n = X_\infty^n \mathbb{1}_{\{X_\infty^n < 0\}}$. By Lemma A.4, $\eta^n \in D^{c,n}$ and $\zeta^n \in V^{a,b,n}$. First let us prove that $\delta_0$ is a cumulation point of $(\mathcal{L}_{\zeta^{n(a,b)}})$. Let $\varepsilon > 0$ and $w_1, \ldots, w_N \in C$ such that, without loss of generality, $w_i(0) = 0$. Let $(b_j)_{j\geq 2}$ be an arbitrary increasing sequence with $b_j \to \infty$. Put $K_j = \max_{1\leq i\leq N} \|w_i|_{[-2b_{j+1},0]}\|_\infty$ for all $j \in \mathbb{N}$. There is $\delta > 0$ with $\max_{1\leq i\leq N} \|w_i(s)|_{[-2b_2,0]}\|_\infty < \varepsilon/2$ for $|s| < \delta$. Define now $b_1 = \min(b_2, \delta/3)$ and define a decreasing sequence $(a_j)$ with $a_j \to 0$ by $a_j = \varepsilon b_j / 2^{j+1} \kappa_j \wedge a_{j-1}$, where $\kappa_j = \max(K_j, b_j)$. Then, by Lemma A.3,

$$\begin{aligned}
|\langle \delta_0 - \mathcal{L}_{\zeta^n}, w_i \rangle| &= |\mathbb{E}^n w_i(\zeta^n)| \\
&\leq \mathbb{E}^n\bigl(|w_i(\zeta^n)| \mathbb{1}_{A_0}\bigr) + \sum_{j=1}^{\infty} \mathbb{E}^n\bigl(|w_i(\zeta^n)| \mathbb{1}_{A_j}\bigr) \\
&< \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} K_j \frac{a_j}{b_j} < \varepsilon,
\end{aligned}$$

where $n = n(a, b)$ and $A_j = \{-2b_{j+1} \le X_\infty^n < -2b_j\}$ for $j \ge 0$ and $b(0) = 0$. This shows that $\delta_0$ is a $\sigma(\mathcal{P}_b, C)$ cumulation point of the net $(\mathcal{L}_{\zeta^{n(a,b)}})$.

From now on, take a subnet of $n(a, b)$ still denoted by $n(a, b)$, such that $(\mathcal{L}_{\zeta^{n(a,b)}})$ converges to $\delta_0$. By the definition of the random variables $\eta^{n(a,b)}$, it is evident that the net $(\mathcal{L}_{\eta^{n(a,b)}})$ has a cumulation point $\mu \in \mathcal{P}([0, 1])$ such that $\mu[c/3, 1] \ge c/3$. I claim that $\mu$ is a $\sigma(\mathcal{P}_b, C)$ cumulation point of $(\mu^{n(a,b)})$. Indeed, choose a subnet of $(\mu^{n(a,b)})$ such that $(\mathcal{L}_{\eta^{n(a,b)}})$ converges to $\mu$. Choose now $\varepsilon > 0$ and $w_1, \dots, w_N \in C$. There exists $(a^0, b^0)$ such that, for all $(a, b) \ge (a^0, b^0)$, $|\langle \delta_0 - \mathcal{L}_{\zeta^{n(a,b)}}, w_i\rangle| < \varepsilon/2$ and $|\langle \mu - \mathcal{L}_{\eta^{n(a,b)}}, w_i\rangle| < \varepsilon/2$ for all $i \le N$. For the same $n = n(a, b)$ and all $i \le N$,

$$\begin{aligned}
|\langle \mu - \mu^n, w_i\rangle| &= \big|\langle \mu, w_i\rangle - \mathbb{E}^n\big(w_i(\eta^n)\mathbb{1}_{\{H^n\cdot S^n_\infty \ge 0\}}\big) - \mathbb{E}^n\big(w_i(\zeta^n)\mathbb{1}_{\{H^n\cdot S^n_\infty < 0\}}\big)\big| \\
&= |\langle \mu, w_i\rangle - \mathbb{E}^n w_i(\eta^n) - \mathbb{E}^n w_i(\zeta^n) + w_i(0)| \\
&\le |\langle \mu - \mathcal{L}_{\eta^n}, w_i\rangle| + |\langle \delta_0 - \mathcal{L}_{\zeta^n}, w_i\rangle| < \varepsilon,
\end{aligned}$$

so $\mu$ is a $\sigma(\mathcal{P}_b, C)$ cumulation point of $(\mu^{n(a,b)})$; hence $\mu \in \overline{\mathcal{L}}$, proving the claim. $\square$

Finally let us recall the main theorem of Klein (2000). Note that this theorem was proved for general (i.e., not necessarily continuous) semimartingales.

THEOREM A.5. NAFL $\Leftrightarrow$ *Bicontiguity.*

In particular, Theorem A.5 closes the final gap in the proof of Theorem 3.5.

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