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A LAW OF LARGE NUMBERS ON RANDOMLY DELETED SETS

BY MARK D. ROTHMANN AND RALPH P. RUSSO

Georgia Institute of Technology and University of Iowa

Consider a system into which units having random magnitude enter at arbitrary times and remain "active" (present in the system) for random periods. Suppose units of high magnitude have stochastically greater lifetimes (tend to stay active for longer periods) than units of low magnitude. Of interest is the process $\{\mu(t)\colon\ t \geq 0\}$, where $\mu(t)$ denotes the average magnitude of all units active at time t. We give conditions which guarantee the convergence of $\mu(t)$ and we determine the form of the limit. Some related processes are also studied.

1. Introduction. Consider a system into which units of varying magnitude enter at fixed points in time. Suppose each is endowed with a lifetime, during which the unit is "active" (alive, in service, etc.), and that unit lifetime and magnitude are positively associated. Then, at any specific point in time, the collection of active units comprises only a portion of those that have entered the system. Moreover, since large units tend to remain active for longer durations than small units, one cannot regard the magnitudes of the active units as a random sample from some underlying distribution of magnitudes. For example, it is reasonable to assume that marriages ending ultimately in divorce possess stochastically shorter lifetimes than those that do not. Thus, the expected fraction of active marriages that will end in divorce is smaller than the probability that a new marriage will end so. Similarly, in business, the time required to complete a task (lifetime) is often positively associated with its complexity (magnitude). An inspector observing the system at a specific point in time may be interested in the distribution (or the mean) complexity of tasks currently not completed, rather than the underlying distribution (or expected value) of task complexity.

Let $(X, L), (X_1, L_1), (X_2, L_2), \dots$ be an independent and identically distributed sequence of random vectors with $P\{L \geq 0\} = 1$ and $P\{X \leq x\} = G(x)$. Suppose $\{t_n, n \geq 1\}$ is a strictly increasing sequence of nonnegative real numbers with $t_n \to \infty$. For $k \geq 1$, let t_k be the time of arrival of some kth unit into a system, let X_k be the magnitude of that unit and let L_k be the time it remains in the system (its lifetime) prior to deletion.

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We call such a unit active at time t if $I_{k,t} = I\{t_k \le t < t_k + L_k\} = 1$. We also define for $t \ge 0$,

$$A_t = \sum_{k \ge 1} I_{k,t}, \qquad \Gamma_t = \sum_{k \ge 1} P\{t_k \le t < t_k + L\},$$

$$S_t = \sum_{k \ge 1} X_k I_{k,t} \quad \text{and} \quad \mu(t) = \begin{cases} S_t/A_t, & \text{if } A_t \ge 1, \\ 0, & \text{if } A_t = 0. \end{cases}$$

Here, A_t represents the number of units active at time t, Γ_t represents the expected number of such units and $\mu(t)$ represents their average magnitude. Our goal is to determine conditions under which the continuous time process of averages $\{\mu(t): t \ge 0\}$ is almost surely convergent, and in these cases to identify the limit.

In Rothmann and Russo (1994) we assumed additionally that X and L are independent. When X possesses a moment generating function in an open interval about the origin, we prove (Theorem 2) that $\mu(t) \to EX$ almost surely if $\Gamma_{t_n}/\log n \to \infty$. When $E|X|^r$ is finite for some r in $[1,2)$, we require the stronger condition $\liminf(n^{-1/r}\Gamma_{t_n}) > 0$. When $r \ge 2$, we require also that Γ_{t_n} be nondecreasing (Theorems 3 and 3a). Furthermore, all results are shown to be "sharp."

Here we consider the more interesting case where unit life is influenced by the corresponding magnitude. For example, "large" units may have stochastically greater lifetimes than "small" units. In this case, the effect of large units on the process tends to be longer lasting than that of small units, and one expects the process limit (when it exists) to exceed EX. In Section 2 we present our main results, in Section 3 some applications and in Section 4 we comment on the case where the interarrivals are independent random variables.

2. The main results. Let S denote the support of X. We assume in this section that $\{R_x(\cdot): x \in S\}$ is a collection of functions for which

$$P\{L > t \mid X\}(\omega) = R_x(t) \quad \text{if } X(\omega) = x,$$

where ω is any point in the basic space, and there exists a distribution function $1 - R^*$ with $R^*(0) > 0$ and

$$R_x(t) \le R^*(t) \quad \text{for } x \in S \text{ and } t \ge 0. \tag{2.1a}$$

One can think of R^* as the survival function of a "uniformly dominant" lifetime variable L^*. For $k \ge 2$ define the increment $a_k = t_k - t_{k-1}$, and for $t \ge 0$ define

$$\Gamma_t^* = \sum_{k \ge 1} R^*(t - t_k) I(t_k \le t) \quad \text{and} \quad \Gamma_t(x) = \sum_{k \ge 1} R_x(t - t_k) I(t_k \le t),$$

where Γ_t^* [resp. $\Gamma_t(x)$] represents the expected number of units active at time t were all units endowed with lifetime L^* (resp., with the lifetime of a unit having magnitude x). We assume further that for all x in S,

$$\Gamma_{t_n}(x)/\Gamma^*_{t_n} = \sum_{1\le k\le n} R_x(t_n - t_k) \Big/ \sum_{1\le k\le n} R^*(t_n - t_k)$$

$$= \sum_{1\le k\le n} P\{L > t_n - t_k \mid X = x\} \Big/ \sum_{1\le k\le n} P\{L^* > t_n - t_k\} \tag{2.1b}$$

$$\to \phi(x)$$

for some measurable function ϕ having $E\phi(x) > 0$, and

$$\Gamma^*_{t_n}/\log n \to \infty. \tag{2.1c}$$

Conditions (2.1b) and (2.1c) insure that active units accumulate quickly enough for convergence to occur. Since $R^*(0) > 0$, (2.1c) holds if $a_n \log n \to 0$ [see the proof of Corollary 2.1 in Rothmann and Russo (1994)]. Later in this section, we remark on the "reasonableness" of condition (2.1b).

THEOREM 2.1. *Suppose* (2.1) *holds and* $E\exp\{tX\} < \infty$ *for all* t *in an open interval containing the origin. Then* $\mu(t_n) \to EX\phi(X)/E\phi(X)$ *almost surely. If, in addition,* $\liminf(\Gamma^*_{t_{n+1}}/\Gamma^*_{t_n}) \ge 1$, *then* $\mu(t) \to EX\phi(X)/E\phi(X)$ *almost surely.*

REMARKS. The conclusion of the theorem holds if we simultaneously weaken the condition on X and strengthen condition (2.1c), so that (1) $E|X|^r < \infty$ for some r in $[1, 2)$ and $\liminf(n^{-1/r}\Gamma^*_{t_n}) > 0$ or (2) $E|X|^r < \infty$ for some $r \ge 2$, $\Gamma^*_{t_n}$ is nondecreasing and $\liminf n^{-1/r}\Gamma_{t_n} > 0$. This is evident from our proof of Theorem 2.1 [see Theorems 3 and 3a of Rothmann and Russo (1994)]. When a_n is a nonincreasing sequence, $\Gamma^*_{t_n}$ is a nondecreasing sequence and $\liminf(\Gamma^*_{t_{n+1}}/\Gamma^*_{t_n}) \ge 1$. If $R_x(t)$ is nondecreasing in x for each real t, then EX is a lower bound to the almost sure limit [by Proposition 7.1.5 of Ross (1983), since ϕ is nondecreasing]. We illustrate this last point with two examples:

EXAMPLE 1 (Random detection). Suppose X is $U(0, 1)$, $a_n \log n \downarrow 0$ and $R_x(t) = (1-t)x$ for $0 \le t \le 1$ and $0 \le x \le 1$. Let $R_x = R_0$ for $x < 0$ and $R_x = R_1$ for $x > 1$. Then, if $X_k = x$, L_k is zero with probability $1 - x$ (the kth unit is "undetected"), but is otherwise uniform on $(0, 1)$. We set $R^* = R_1$, so that $\phi(x) = x$ for $0 \le x \le 1$ and $\Gamma^*_{t_n}/\log n \to \infty$. Since a_n is nonincreasing, $\mu(t) \to 2/3$ almost surely.

EXAMPLE 2 (The limiting proportion of types). Suppose $a_n \equiv \delta > 0$, $P\{X = 0\} = p = 1 - P\{X = 1\}$, $R_0(t)/R_1(t) \to \alpha$, some $0 \le \alpha \le 1$, and $tR_1(t) \to \infty$ as $t \to \infty$. [For a concrete example, take $R_0(t) = \alpha(t+1)^{-1/2}$ and $R_1(t) = (t+1)^{-1/2}$ for $t > 0$.] If $X_k = i$ we will say that the kth entering unit is "type i," $i = 0, 1$. In the limit, p is the proportion of entering units which are type 0. However, if $0 \le \alpha < 1$, such units tend to have shorter lifetimes than type 1 units, and are thus expected to comprise a smaller (than p) proportion of the active population. At time t, we call this proportion π_t and observe that $\pi_t = 1 - \mu(t)$. Set $R^* = R_1$. From a routine calculation we have $\phi(0) = \alpha$ and $\phi(1) = 1$. We have also

$$\begin{aligned}\Gamma^*_{t_n}/\log n &= \sum_{1 \le k \le n} R^*(t_n - t_k)/\log n \\ &= \sum_{1 \le k \le n} (1/n-k)(n-k)R_1((n-k)\delta)/\log n \to \infty.\end{aligned}$$

Since a_n is nonincreasing, $\mu(t) \to (1-p)/(\alpha p + (1-p))$ almost surely. Thus, $\pi_t \to 1 - (1-p)/(\alpha p + (1-p)) = \alpha p/(\alpha p + 1 - p)$ almost surely.

In Rothmann and Russo (1997) we prove the following results which show that condition (2.1b) is not nearly as restrictive as it looks.

1. If $na_n \to 0$, then $\phi(x) = R_x(0)/R^*(0)$, in which case $EX\phi(X)/E\phi(X) = E(X \mid L > 0)$.
2. If $na_n \to c \in (0, \infty)$, then $\phi(x) = (1 - E(e^{-L/c} \mid X = x))/(1 - E(e^{-L^*/c}))$, in which case
$$EX\phi(X)/E\phi(X) = E\big(X\big(1 - E(e^{-L/c} \mid X)\big)\big)/\big(1 - E(e^{-L/c})\big).$$
3. If $na_n \uparrow \infty$, $a_n \downarrow 0$ and $EL^* < \infty$, then $\phi(x) = E(L \mid X = x)/EL^*$, in which case $EX\phi(X)/E\phi(X) = E(XE(L \mid X))/EL$.
4. If $na_n \uparrow \infty$, $a_n \downarrow \gamma \ge 0$ and $EL = \infty$, then $\phi(x) = \lim_{t \to \infty} R_x(t)/R^*(t)$, provided this limit exists.

It can be shown that in case 1, $(\log A_t)/t \to \infty$ a.s. (faster than exponential growth of the active population); in case 2, $(\log A_t)/t \to 1/c$ a.s. (exponential growth); and in cases 3 and 4, $(\log A_t)/t \to 0$ a.s. (slower than exponential growth). Case 2 gives applicable results for a population that grows exponentially; for example, a population of biological units or a population of carriers of a contagious disease.

In cases 1–4, if the conditional life distributions are Laplace transform ordered in x [see Shaked and Shantikumar (1994)], then

$$E(X\phi(X))/E\phi(X) \ge E(X).$$

Note that Laplace transform ordering is a weaker condition than stochastic ordering. For $0 \le c \le \infty$, let $m(c)$ denote the almost sure limit of $\mu(t)$ as given in cases 1–4. If the hazard rate functions of the conditional life distributions are nonincreasing in x, then $m(c)$ is nonincreasing in c [see Rothmann and Russo (1997)]. Larger c's correspond to slower rates of entry (slower growth of the active population) and a larger limiting mean.

The following result is applicable to "short" interarrivals and requires neither that X possess a moment generating function nor that a uniformly dominant life exists.

THEOREM 2.2. *Suppose* $E|X| < \infty$. *If* $\Gamma_{t_n}(x)/n \to \phi(x)$ *for some measurable function* ϕ *with* $E\phi(X) > 0$, *then* $\mu(t) \to E(X\phi(X))/E\phi(X)$ *almost surely.*

Suppose $R_x(0) > 0$ for x in S. If $\limsup na_n < \infty$, then $\delta_x < \Gamma_{t_n}(x)/n \le 1$ for some $\delta_x > 0$. If $na_n \to c \in [0, \infty)$, then $\Gamma_{t_n}(x)/n \to \phi(x) = R_x(0)$ or $1 - E(e^{-L/c} \mid X = x)$ according as $c = 0$ or $c > 0$. The resulting limits of $\mu(t)$ are as in cases 1 and 2 above.

EXAMPLE 3. Suppose X has d.f. $G(x) = 1 - 1/x^2$ for $x \ge 1$, $na_n \downarrow 1$ and the conditional life distribution of L, given $X = x$, is exponential with mean x. Then $\phi(x) = 1 - E(e^{-L} \mid X = x) = x/1 + x$ and thus $\mu(t) \to E(X^2/1 + X)/E(X/1 + X) = \log 2/(1 - \log 2)$ a.s.

The proof of Theorem 2.1 will require three lemmas. Define $t_{nk} = t_n - t_k$ and let ε denote an arbitrary constant in $(0, 1)$. If Z is a random variable, $r(Z)$ denotes the ratio Z/EZ. All statements involving random denominators are asserted to hold on the set where those denominators are nonzero.

LEMMA 1. *For $n \ge 1$, let T_n denote the row sum of a rowwise independent, triangular array of indicator variables. If $ET_n/\log n \to \infty$, then $r(T_n) \to 1$ almost surely.*

PROOF. We have $\operatorname{var}(T_n) \le ET_n$ by independence. By Bernstein's inequality [see either Serfling (1980) or Uspensky (1937)], for $0 < \delta < 1$ and all large n we have $P\{|T_n - ET_n| > \delta ET_n\} \le 2\exp\{-\delta^2(ET_n)^2/(2ET_n + \delta ET_n)\} \le n^{-2}$. $\square$

LEMMA 2. *Suppose $\mathscr{C}$ is a Borel set for which $E(\phi(X)I(X \in \mathscr{C})) > 0$. If* (2.1) *holds, then*

$$r\Bigg(\sum_{1 \le k \le n} I(X_k \in \mathscr{C}, L_k > t_{nk})\Bigg) \to 1 \quad \textit{almost surely}.$$

PROOF. We have

$$P\{X_k \in \mathscr{C}, L_k > t_{nk}\} = \int_{\mathscr{C}} P\{L > t_{nk} \mid X = x\}\, dG(x)$$

so that for large n, by the dominated convergence theorem,

$$\begin{aligned}
&\sum_{1 \le k \le n} P\{X_k \in \mathscr{C}, L_k > t_{nk}\}/\log n \\
&\quad = \int_{\mathscr{C}} \sum_{1 \le k \le n} P\{L > t_{nk} \mid X = x\}\, dG(x)/\log n \\
&\quad = \big(\Gamma^*_{t_n}/\log n\big) \int_{\mathscr{C}} \big(\Gamma_{t_n}(x)/\Gamma^*_{t_n}\big)\, dG(x) \\
&\quad \ge \big(\Gamma^*_{t_n}/\log n\big) E(\phi(X) I(X \in \mathscr{C}))/2.
\end{aligned}$$

The result now follows by (2.1c) and Lemma 1. $\square$

The following lemma is a consequence of Theorem 1 of Rothmann and Russo (1997):

LEMMA 3. *Suppose* $P\{X = 1\} = p = 1 - P\{X = 0\}$. *If* $\Gamma_{t_n}(1)/\Gamma_{t_n} \to \gamma$, $\Gamma_{t_n}/\log n \to \infty$ *and* $\liminf(\Gamma_{t_{n+1}}/\Gamma_{t_n}) \geq 1$, *then* $\mu(t) \to p\gamma$ *almost surely*.

PROOF OF THEOREM 2.1. We prove Theorem 2.1 in the continuous time case only. The proof in discrete time is essentially the same. We first consider the case where X is bounded. The result is trivial if X is degenerate, so we assume that $-\infty < \operatorname{ess\,inf} X < \operatorname{ess\,sup} X < \infty$. By Lemma 2,

$$r(A_{t_n}) \to 1 \quad \text{a.s.} \tag{2.2}$$

By the dominated convergence theorem (DCT), upon integration of (2.1b) we have

$$\Gamma_{t_n}/\Gamma^*_{t_n} \to E\phi(X). \tag{2.3}$$

We observe that the following inequalities hold for $t_n \leq t \leq t_{n+1}$:

$$\begin{gathered} A_{t_{n+1}} - 1 \leq A_t \leq A_{t_n}, \qquad \Gamma_{t_{n+1}} - 1 \leq \Gamma_t \leq \Gamma_{t_n}, \\ \Gamma_{t_{n+1}}(x) - 1 \leq \Gamma_t(x) \leq \Gamma_{t_n}(x) \quad \text{and} \quad \Gamma^*_{t_{n+1}} - 1 \leq \Gamma^*_t \leq \Gamma^*_{t_n}. \end{gathered} \tag{2.4a}$$

For $t_n \leq t \leq t_{n+1}$ we have also

$$\begin{aligned} 1 \geq \Gamma_t/\Gamma_{t_n} &= (\Gamma_t/\Gamma_{t_{n+1}})(\Gamma_{t_{n+1}}/\Gamma^*_{t_{n+1}})(\Gamma^*_{t_{n+1}}/\Gamma^*_{t_n})(\Gamma^*_{t_n}/\Gamma_{t_n}), \\ 1 \geq A_t/A_{t_n} &= r(A_{t_{n+1}})(\Gamma_{t_{n+1}}/\Gamma^*_{t_{n+1}})(\Gamma^*_{t_n}/\Gamma_{t_n}) \\ &\quad \times (\Gamma^*_{t_{n+1}}/\Gamma^*_{t_n})(A_t/A_{t_{n+1}})/r(A_{t_n}), \end{aligned} \tag{2.4b}$$

so that by (2.2), (2.3), (2.4) and our lim inf condition,

$$A_t/A_{t_n} \to 1 \quad \text{a.s.} \quad \text{and} \quad \Gamma_t/\Gamma_{t_n} \to 1 \quad \text{uniformly in } t_n \leq t \leq t_{n+1}. \tag{2.5}$$

Now, by (2.2) and (2.5),

$$r(A_t) \to 1 \quad \text{a.s.} \tag{2.6}$$

The following inequalities hold for $t_n \leq t \leq t_{n+1}$ and follow from (2.4a):

$$\begin{gathered} \big((\Gamma_{t_{n+1}}(x) - 1)/\Gamma^*_{t_{n+1}}\big)(\Gamma^*_{t_{n+1}}/\Gamma^*_{t_n}) \leq \Gamma_t(x)/\Gamma^*_t \leq (\Gamma_{t_n}(x)/\Gamma^*_{t_n})\big(\Gamma^*_{t_n}/(\Gamma^*_{t_{n+1}} - 1)\big), \\ \big((\Gamma_{t_{n+1}} - 1)/\Gamma^*_{t_{n+1}}\big)(\Gamma^*_{t_{n+1}}/\Gamma^*_{t_n}) \leq \Gamma_t/\Gamma^*_t \leq (\Gamma_{t_n}/\Gamma^*_{t_n})\big(\Gamma^*_{t_n}/(\Gamma^*_{t_{n+1}} - 1)\big). \end{gathered}$$

These inequalities, when combined with (2.1b), (2.1c), (2.3) and our lim inf condition, yield

$$\Gamma_t(x)/\Gamma^*_t \to \phi(x) \quad \text{and} \quad \Gamma_t/\Gamma^*_t \to E\phi(X). \tag{2.7}$$

Fix $x_0 < x_1 < \cdots < x_m$ so that

$$x_0 < \operatorname{ess\,inf} X < \operatorname{ess\,sup} X < x_m$$

and

$$x_{j+1} - x_j < \varepsilon \quad \text{for } 0 \leq j \leq m - 1.$$

Define

$$h_j = [x_{j-1}, x_j), \qquad p_j = P\{X \in h_j\}$$

and

$$A_t(j) = \sum_{\{k : t_k \le t\}} I(X_k \in h_j, L_k > t_n - t_k).$$

Note that $P\{L > t \mid X \in h_j\} = (1/p_j) \int_{h_j} R_x(t)\, dG(x)$ if $p_j > 0$. Thus, by (2.7) and the DCT,

$$\begin{aligned}
&\sum_{\{k : t_k \le t\}} P\{L > t - t_k \mid X \in h_j\}/\Gamma_t \\
&\quad = (1/p_j)(1/\Gamma_t) \int_{h_j} \sum_{\{k : t_k \le t\}} R_x(t - t_k)\, dG(x) \\
&\quad = (1/p_j)(\Gamma_t^*/\Gamma_t) \int_{h_j} \Gamma_t(x)/\Gamma_t^*\, dG(x) \qquad (2.8)\\
&\quad \to \big(1/p_j E\phi(X)\big) \int_{h_j} \phi(x)\, dG(x).
\end{aligned}$$

By Lemma 3, (2.1c), (2.3) and (2.8) we have

$$A_t(j)/A_t \to (1/E\phi(X)) \int_{h_j} \phi(x)\, dG(x) \quad \text{a.s.} \qquad (2.9)$$

Now

$$(1/A_t) \sum_{1 \le j \le m} x_{j-1} A_t(j) \le \mu(t) \le (1/A_t) \sum_{1 \le j \le m} x_j A_t(j) \le u(t) + \varepsilon,$$

so that almost surely, by (2.9),

$$\begin{aligned}
(1/E\phi(X)) \int x \phi(x)\, dG(x) - \varepsilon &\le (1/E\phi(X)) \sum_{1 \le j \le m} \int_{h_j} x_{j-1} A_t(j) \\
&\le \liminf_{n \to \infty} \mu(t) \le \limsup_{n \to \infty} \mu(t) \\
&\le (1/E\phi(X)) \sum_{1 \le j \le m} \int_{h_j} x_j A_t(j) \\
&\le (1/E\phi(X)) \int x \phi(x)\, dG(x) + \varepsilon.
\end{aligned}$$

Thus, since ε is arbitrary, we conclude that $\mu(t) \to E(X\phi(X))/E\phi(X)$ almost surely.

We now remove the boundedness assumption. Fix $m_0 > 0$ so that $E(\phi(X)I(-m_0 < X < m_0)) > 0$. Fix $m > m_0$ and define

$$\mathscr{A}_{t,m} = \sum_{k \geq 1} I(-m \leq X_k \leq m) I_{k,t},$$

$$W_{n,m} = X_n I(X_n > m) \quad \text{and} \quad V_{n,m} = X_n I(X_n < -m).$$

We have

$$\begin{aligned} S_t/A_t &= \sum_{k \geq 1} X_k I(-m \leq X_k \leq m) I_{k,t}/A_t + \sum_{k \geq 1} W_{k,m} I_{k,t}/A_t \\ &\quad + \sum_{k \geq 1} V_{k,m} I_{k,t}/A_t \\ &= B_{t,m,1} + B_{t,m,2} + B_{t,m,3}. \end{aligned} \tag{2.10}$$

We define a new process where the kth entering unit has respective magnitude and lifetime

$$Y_{k,m} = \begin{cases} X_k, & \text{if } X_k \in [-m, m], \\ -2m, & \text{otherwise} \end{cases}$$

and

$$\lambda_{k,m} = \begin{cases} L_k, & \text{if } X_k \in [-m, m], \\ 0, & \text{otherwise.} \end{cases}$$

We also define $\chi_{k,t} = I(t_k \leq t < t_k + \lambda_{k,m})$, so that

$$B_{t,m,1} = \frac{\sum_{k \geq 1} Y_{k,m} \chi_{k,t} / \sum_{k \geq 1} \chi_{k,t}}{\mathscr{A}_{t,m} r(A_t) \Gamma_t / r(\mathscr{A}_{t,m}) E\mathscr{A}_{t,m} \sum_{k \geq 1} \chi_{k,t}}. \tag{2.11}$$

By (2.6) and Lemma 2 (using similar arguments for $\mathscr{A}_{t,m}$ as for A_t), $r(A_t)/r(\mathscr{A}_{t,m}) \to 1$ a.s. Now, by (2.7),

$$\begin{aligned} \lim_{t \to \infty} (\Gamma_t / E\mathscr{A}_{t,m}) &= \lim_{t \to \infty} (\Gamma_t/\Gamma_t^*) \Big/ \lim_{t \to \infty} \int_{[-m,m]} (\Gamma_t(x)/\Gamma_t^*) \, dG(x) \\ &= E\phi(X) \Big/ \int_{[-m,m]} \phi(x) \, dG(x). \end{aligned}$$

By Theorem 1 applied to the bounded $Y_{k,m}$'s,

$$\begin{aligned} &\lim_{t \to \infty} \sum_{k \geq 1} Y_{k,m} \chi_{k,t} \Big/ \sum_{k \geq 1} \chi_{k,t} \\ &\quad = \int_{[-m,m]} x\phi(x) \, dG(x) \Big/ \int_{[-m,m]} \phi(x) \, dG(x) \quad \text{a.s.} \end{aligned}$$

Since $\sum_{k \geq 1} \chi_{k,t} = \mathscr{A}_{t,m}$, we conclude from (2.11) that

$$\lim_{m \to \infty} \lim_{t \to \infty} B_{t,m,1} = EX\phi(X)/E\phi(X) \quad \text{a.s.} \tag{2.12}$$

Next, we form an image of our process on a new probability space: Attach to the original space a sequence U_k of i.i.d. $U(0,1)$ random variables which are jointly independent of the X_k's. We may redefine the L_k sequence and also define sequences L_k^* and $L_k(x)$ as follows:

$$L_k(x) = \sup\{y: R_x(y) \geq U_k\}, \qquad L_k = L_k(X_k)$$

and

$$L_k^* = \sup\{y: R^*(y) \geq U_k\}.$$

On the new space the joint distributions of the X_k's and L_k's remain the same. However, in addition we have $L_k \leq L_k^*$. Define $I_{k,t}^* = I(t_k \leq t < t_k + L_k^*)$. Now,

$$0 \leq B_{t,m,2} \leq \left(\sum_{k \geq 1} W_{k,m} I_{k,t}^* \Big/ \sum_{k \geq 1} I_{k,t}^*\right) \Big/ \left(A_t \Big/ \sum_{k \geq 1} I_{k,t}^*\right). \tag{2.13}$$

Let $E_{t,m}$ and H_t denote, respectively, the numerator and denominator in (2.13). Since for $|t|$ small, $E\exp\{tW_{k,m}\} < \infty$, we have by (2.1c) and Theorem 2.1 of Rothmann and Russo (1994),

$$\lim_{m \to \infty} \lim_{t \to \infty} E_{t,m} = \lim_{m \to \infty} \int_{(m,\infty)} x\, dG(x) = 0 \quad \text{a.s.} \tag{2.14}$$

Furthermore,

$$H_t = (\Gamma_t/\Gamma_t^*) r(A_t) \Big/ r\left(\sum_{k \geq 1} I_{k,t}^*\right).$$

Thus, since $r(\sum_{k \geq 1} I_{k,t}^*) \to 1$ a.s. (by Lemma 2), we have by (2.6) and (2.7),

$$\lim_{t \to \infty} H_t = E\phi(X) > 0. \tag{2.15}$$

Putting (2.14) and (2.15) together we get

$$\lim_{m \to \infty} \limsup_{t \to \infty} B_{t,m,2} = 0 \quad \text{a.s.} \tag{2.16}$$

Now

$$\left(\sum_{k \geq 1} V_{k,m} I_{k,t}^* \Big/ \sum_{k \geq 1} I_{k,t}^*\right) \Big/ \left(A_t \Big/ \sum_{k \geq 1} I_{k,t}^*\right) \leq B_{t,m,3} \leq 0,$$

so that, by the same argument that yielded (2.16),

$$\lim_{m \to \infty} \liminf_{t \to \infty} B_{t,m,3} = 0 \quad \text{a.s.} \tag{2.17}$$

Statements (2.10), (2.12), (2.16) and (2.17) together yield the final result. □

PROOF OF THEOREM 2.2. The proof is the same as that of Theorem 2.1, with the following changes: Replace Γ_t^* by $\sum_{k \geq 1} I(t_k \leq t)$ and $I_{k,t}^*$ by $I(t_k \leq t)$. Note that (2.14) follows from the ordinary SLLN. □

3. Applications. The pth quantile $(0 < p < 1)$ associated with the units active at time t is given by

$$\xi_{t,p} = \begin{cases} \pi X_t(pA_t) + (1-\pi)X_t(pA_t+1), & \text{if } pA_t = [pA_t], \\ X_t([pA_t]+1), & \text{if } pA_t \neq [pA_t], \end{cases}$$

where $0 \le \pi \le 1$ is fixed, $[y]$ denotes the greatest integer in y and $X_t(i)$ is the ith order statistic among the members of $\{X_k : I_{k,t} = 1\}$.

THEOREM 3.1. *Suppose $\Gamma_{t_n}(x)/\Gamma_{t_n} \to \phi(x)$ for some function ϕ satisfying $\int_{-\infty}^{\infty} \phi(x)\,dG(x) = 1$ and $\Gamma_{t_n}/\log n \to \infty$. If $\psi(p)$ is the unique solution to the equation $p = \int I(x < \psi(p))\phi(x)\,dG(x)$, then $\xi_{t_n,p} \to \psi(p)$ almost surely. If, in addition,* $\liminf(\Gamma_{t_{n+1}}/\Gamma_{t_n}) \ge 1$, *then $\xi_{t,p} \to \psi(p)$ almost surely.*

In Example 1, $\psi(p) = p^{1/2}$. In cases 1–4, if the conditional life distributions are Laplace transform ordered in x, then $G(\psi(p)) \ge p$. If the hazard rate functions of the conditional life distributions are nonincreasing in x, then $\psi(p)$ is nondecreasing in c [see Rothmann and Russo (1997)].

PROOF OF THEOREM 3.1. We prove the continuous time result only. Fix $\psi_0 < \psi(p)$ so that $p > \int I(-\infty < x \le \psi_0)\phi(x)\,dG(x)$. Define $T_n = I(X_n \ge \psi_0)$,

$$S_t^* = \sum_{k \ge 1} T_k I_{k,t} \quad \text{and} \quad \gamma = \lim_{n\to\infty} \sum_{1 \le k \le n} P\{L_k \ge t_n - t_k \mid T_k = 1\}/\Gamma_{t_n}.$$

For $1 \le k \le n$,

$$\begin{aligned} &P\{L_k \ge t_{nk} \mid T_k = 1\} \\ &\qquad = \int_{[\psi_0,\infty)} P\{L_k \ge t_{nk}, T_k = 1 \mid X_k = x\}\,dG(x)/P\{T_k = 1\}. \end{aligned}$$

Thus, using Fatou's lemma,

$$\begin{aligned} \gamma &= \lim_{n\to\infty} \int_{[\psi_0,\infty)} \sum_{1\le k \le n} P\{L_k \ge t_{nk} \mid X_k = x\}\,dG(x)/\Gamma_{t_n} P\{T_k = 1\} \\ &= \int_{[\psi_0,\infty)} \phi(x)\,dG(x)/P\{T_k = 1\}. \end{aligned}$$

By Lemma 3,

$$S_t^*/A_t \to \int_{[\psi_0,\infty)} \phi(x)\,dG(x) > 1 - p$$

so that $\liminf \xi_{t,p} \ge \psi(p)$ a.s. Additionally, by a similar argument, $\limsup \xi_{t,p} \le \psi(p)$ a.s. $\square$

An application of Theorem 2.2 and 3.1 to the case where $X = L$ yields a result on the "mean total life" of units active at time t, $\lambda_t = \sum_{k\ge 1} L_k I_{k,t}/A_t$, and the associated pth quantile $\beta_{t,p}$ obtained by replacing X_k by L_k in the definition of $\xi_{t,p}$. One expects units with long life to exert a greater influence on λ_t and $\beta_{t,p}$ than units with short life. An observer inspecting the system at time t sees this as a forward shift in the distribution of L. The result is a variation of the "inspection paradox" [see Feller (1971) or Ross (1983)]:

THEOREM 3.2. (a) *Suppose* $na_n \to c \in [0, \infty)$. *If* $0 < EL < \infty$, *then* $\lambda_t \to E(L \mid L > 0)$ *a.s. when* $c = 0$ *and* $\lambda_t \to E(L(1 - e^{-L/c}))/E(1 - e^{-L/c})$ *a.s. when* $0 < c < \infty$. *If* $c = 0$ *and* $\tau(p)$ *is the unique solution to*

$$P\{L > 0\}p = \int I(x < \tau(p))R_x(0)\, dP\{L \le x\},$$

then $\beta_{t,p} \to \tau(p)$ *almost surely. If* $0 < c < \infty$ *and* $\tau(p)$ *is the unique solution to*

$$pE(1 - e^{-L/c}) = \int I(x < \tau(p))(1 - e^{-x/c})\, dP\{L \le x\},$$

then $\beta_{t,p} \to \tau(p)$ *almost surely.*

(b) *Suppose* a_n *and* $1/na_n$ *are nonincreasing*, $a_n \log n \to 0$ *and* $na_n \to \infty$. *If* $P\{L = 0\} < P\{0 \le L \le m\} = 1$ *for some* m, *then* $\lambda_t \to EL^2/EL$ *almost surely. If, in addition,* $\tau(p)$ *is the unique solution to* $pEL = \int I(x < \tau(p))x\, dP\{L \le x\}$, *then* $\beta_{t,p} \to \tau(p)$ *almost surely.*

In Theorem 3.2, we observe that the almost sure limit of $\lambda(t)$ is bounded below by EL, and that of $\beta_{t,p}$ is bounded below by the pth quantile of L. Thus, the mean eventual life of active units exceeds the expected life of entering units.

PROOF OF THEOREM 3.2. (a) This result is an immediate consequence of Theorem 3.1.

(b) For $x > 0$ let $h(n, x) = \sum_{1 \le k \le n-1} I(t_{nk} \le x)$. Fix $x > 0$ and $0 < \delta < 1$. Since $a_n \downarrow 0$, we have for large n,

$$x(1 - \delta) \le t_n - t_{n-h(n,x)} \le x$$

and

$$a_n h(n, x) \le t_n - t_{n-h(n,x)} \le h(n, x)a_{n-h(n,x)}.$$

Thus,

$$(1 - \delta)x/a_{n-h(n,x)} \le h(n, x) \le x/a_n. \tag{3.1}$$

Define $n(\delta) = n - [\delta n]$. Since $t_n - t_{n(\delta)} \ge a_n[\delta n] \to \infty$, we have that $h(n, x) \le \delta n$ for large n. Furthermore, $n(\delta)a_{n(\delta)} \le na_n$, so that by (3.24), $(1 - \delta)xn(\delta)/na_n \le h(n, x) \le x/a_n$. Since δ is arbitrary, $a_n h(n, x)/x \to 1$. Thus, $h(n, x)/h(n, m) \to x/m = \phi(x)$ for $0 \le x \le m$. The result now follows via an application of Theorems 1 and 2. The condition $a_n \log n \to 0$ guarantees that $\Gamma^*_{t_n}/\log n \to \infty$. □

4. Random interarrivals. In this section we show how the results of Section 2 can be extended to cover the case where the interarrivals are independent random variables. We assume that α_n and β_n are fixed positive sequences for which

$$P\{\alpha_n \le a_n \le \beta_n\} = 1 \quad \text{for } n \ge 1,$$

as would be the case when (for example) $a_n \sim U(\alpha_n, \beta_n)$. The following statements correspond to cases 1–3 of Section 2: If $E|X| < \infty$, $\sum_{n\ge 1} \alpha_n = \infty$ and $n\beta_n \to 0$, then

$$\mu(t_n) \to E(X \mid L > 0) \quad \text{a.s.} \tag{4.1}$$

If $E|X| < \infty$ and $\lim_{n\to\infty} n\alpha_n = \lim_{n\to\infty} n\beta_n = c \in (0,\infty)$, then

$$\mu(t_n) \to E\big(X\big(1 - E(e^{-L/c} \mid X)\big)\big)/(1 - Ee^{-L/c}) \quad \text{a.s.} \tag{4.2}$$

If $Ee^{tX} < \infty$ for all t in an open interval containing the origin, condition (2.1a) holds with $EL^* < \infty$, $n\alpha_n \uparrow \infty$, $\beta_n \downarrow 0$, $\alpha_n/\beta_n \to 1$ and $\beta_n \log n \to 0$, then

$$\mu(t_n) \to E(XE(L \mid X))/EL \quad \text{a.s.} \tag{4.3}$$

Proof of (4.1). Define for $n \ge 1$,

$$A_\alpha(n) = \sum_{1\le k\le n} I(L_k > r_{n,k}) \quad \text{and} \quad A_\beta(n) = \sum_{1\le k\le n} I(L_k > s_{n,k}),$$

where $r_{n,k} = \sum_{n-k+1\le j\le n} \alpha_j$ and $s_{n,k} = \sum_{n-k+1\le j\le n} \beta_j$. Define also the process

$$\mu_\beta(s_n) = \sum_{1\le k\le n} X_k I(L_k > s_{n,k})/A_\beta(n).$$

We have

$$\begin{aligned}
\big|\mu(t_n) - \mu_\beta(s_n)\big| &\le \Big|\mu(t_n) - \sum_{1\le k\le n} X_k I(L_k > t_{nk})/A_\beta(n)\Big| \\
&\quad + \Big|\sum_{1\le k\le n} X_k\big(I(L_k > t_{nk}) - I(L_k > s_{n,k})\big)\Big/A_\beta(n)\Big| \\
&\le \big(A_\alpha(r_n)/A_\beta(s_n) - 1\big) \sum_{1\le k\le n} |X_k|\, I(L_k > r_{n,k})/A_\alpha(r_n) \\
&\quad + \big(A_\alpha(r_n)/A_\beta(s_n)\big) \sum_{1\le k\le n} |X_k|\, I(L_k > r_{n,k})/A_\alpha(r_n) \\
&\quad - \sum_{1\le k\le n} |X_k|\, I(L_k > s_{n,k})/A_\beta(s_n).
\end{aligned}$$

Since $\sum_{1\le k\le n} |X_k|\, I(L_k > r_{n,k})/A_\alpha(r_n)$ and $\sum_{1\le k\le n} |X_k|\, I(L_k > s_{n,k})/A_\beta(s_n)$ converge almost surely to the same finite limit and

$$\begin{aligned}
A_\alpha(r_n)/A_\beta(s_n) &= \big(A_\alpha(r_n)/EA_\alpha(r_n)\big)\big(EA_\alpha(r_n)/EA_\beta(s_n)\big) \\
&\quad \times \big(EA_\beta(s_n)/A_\beta(s_n)\big) \to 1 \quad \text{a.s.}
\end{aligned} \tag{4.4}$$

we have that $|\mu(t_n) - \mu_\beta(s_n)| \to 0$ almost surely. In (4.4) we observe that $A_\alpha(r_n)/EA_\alpha(r_n)$ and $EA_\beta(s_n)/A_\beta(s_n)$ are almost surely convergent to 1, by Lemma 2. Also, we have

$$EA_\alpha(r_n)/EA_\beta(s_n) = \left(EA_\alpha(r_n)/n\right)/\left(EA_\beta(s_n)/n\right) \to 1. \tag{4.5}$$

(see the remarks following Theorem 2.2). Since $\mu_\beta(s_n) \to E(X \mid L > 0)$ almost surely by Theorem 2.2 and case 1, statement (4.1) follows. □

The proof of (4.2) is similar to that of (4.1). The proof of (4.3) uses Theorem 2.1 and is also similar to that of (4.1). To obtain (4.5) in this case, one uses an argument like that found in the proof of Theorem 3 of Rothmann and Russo (1995).

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SCHOOL OF MATHEMATICS
GEORGIA INSTITUTE OF TECHNOLOGY
ATLANTA, GEORGIA 30332
E-MAIL: rothmann@math.gatech.edu

DEPARTMENT OF STATISTICS
UNIVERSITY OF IOWA
IOWA CITY, IOWA 52242
E-MAIL: rrusso@stat.uiowa.edu