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MAXIMIZING THE PROBABILITY OF A PERFECT HEDGE[1]

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In the framework of continuous-time, Itô processes models for financial markets, we study the problem of maximizing the probability of an agent's wealth at time T being no less than the value C of a contingent claim with expiration time T. The solution to the problem has been known in the context of complete markets and recently also for incomplete markets; we rederive the complete markets solution using a powerful and simple duality method, developed in utility maximization literature. We then show how to modify this approach to solve the problem in a market with partial information, the one in which we have only a prior distribution on the vector of return rates of the risky assets. Finally, the same problem is solved in markets in which the wealth process of the agent has a nonlinear drift. These include the case of different borrowing and lending rates, as well as "large investor" models. We also provide a number of explicitly solved examples.

1. Introduction. In a complete financial market any contingent claim C can be replicated on a finite-time horizon $[0, T]$ starting with initial capital x equal to the "Black–Scholes" price $C(0)$ of the claim. In the case $x < C(0)$, however, it is not a priori clear what strategy should be used to offset the future liability C. One possible criterion is to maximize the probability of a "perfect hedge," $P[X^{x,\pi}(T) \geq C]$ over the set of admissible portfolio processes $\pi(\cdot)$, where $X^{x,\pi}(\cdot)$ is the wealth process of the agent starting with initial capital x and investing according to the investment strategy $\pi(\cdot)$. In a special case of a one-dimensional Brownian model with zero interest rate, volatility 1 and constant claim C, this problem was solved in Kulldorff (1993) and Heath (1993). In Browne (1996) the problem is solved in the context of more general claims, and a general, deterministic-coefficients, multidimensional Brownian motion model, using a PDE approach. The solution in the general case of continuous semimartingales and an arbitrary European claim C is provided in Föllmer and Leukert (1998). The latter paper uses the methodology of testing statistical hypothesis (Neyman–Pearson lemma), first suggested in Heath (1993) [see also Karatzas (1997)]. It also analyzes the problem in the difficult context of incomplete markets.

In this paper we start by introducing the problem and rederiving the solution in the context of a general Itô processes-type model in Section 2. We show that the problem can be solved in an elegant and straightforward way by using the well-known duality approach from the literature on utility maximization [the duality approach to utility maximization problems was implicitly used in

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Pliska (1986), Karatzas, Lehoczky and Shreve (1987), Cox and Huang (1989), in the case of complete markets, and explicitly in He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Xu and Shreve (1992), Cvitanić and Karatzas (1992) for incomplete markets with constraints; see Cvitanić (1997b) or Karatzas (1996) for an overview and further references]. The same approach has been used recently in Cvitanić and Karatzas (1998) for the risk-management problem of minimizing the expected loss $E(C - X^{x,\pi}(T))^+$. We also extend results available in most of the existing literature by considering general margin requirements of the type $X^{x,\pi}(\cdot) \geq A(\cdot)$ for some given process $A(\cdot)$, not necessarily equal to zero [see also Browne (1996)].

In Section 3 we apply this technique to a market with partial observations. Namely, we assume that the vector of the mean return rates of the risky assets is an unobservable random variable with a known a priori distribution that is being updated as the agent observes the asset prices. We describe the optimal solution and calculate it explicitly in the case of normally distributed return rates. This "Bayesian" problem is studied in detail in Karatzas (1997), in the special case of one stock, zero interest rate, volatility 1 and a constant claim C. Lakner (1994), Browne and Whitt (1996) and Karatzas and Zhao (1998) study utility maximization problems under partial observations.

In Section 4 we consider the case of "nonlinear market dynamics" in which the drift of the wealth process $X^{x,\pi}(\cdot)$ is a nonlinear (concave) function of the investment strategy of the agent. This includes the examples of different interest rates for borrowing and lending, as well as a case of a "large investor" whose policy can influence market prices. The approach is again based on the duality methodology developed in utility maximization contexts. In particular, we follow ideas of Cvitanić (1997a) and Cuoco and Cvitanić (1998). We formulate a dual problem and use its optimal solution to construct an optimal solution to the primal problem.

2. A complete market model. We put ourselves in the framework of a financial market $\mathscr{M}$ that consists of one riskless asset, called *bank account*, and several risky assets, called *stocks*. The price processes $S_0(\cdot)$ and $S_1(\cdot), \dots, S_d(\cdot)$ of these assets are modeled by the following dynamics:

$$
\begin{aligned}
& dS_0(t) = S_0(t) r(t)\,dt, \qquad S_0(0) = 1 \\
& dS_i(t) = S_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW^j(t)\right], \qquad (2.1) \\
& \qquad\qquad S_i(0) = s_i > 0;\ i = 1, \dots, d.
\end{aligned}
$$

The standard Brownian motion $W(\cdot) = (W^1(\cdot), \dots, W^d(\cdot))'$ in $\mathbb{R}^d$ is defined on a complete probability space $(\Omega, \mathscr{F}, P)$, endowed with a filtration $\mathbf{F} = \{\mathscr{F}(t)\}_{0 \leq t \leq T}$, the P-augmentation of the filtration $\mathscr{F}^W(t) := \sigma(W(s); 0 \leq s \leq t)$, $0 \leq t \leq T$, generated by $W(\cdot)$. The market coefficients $r(\cdot)$ (interest rate), $b(\cdot) = (b_1(\cdot), \dots, b_d(\cdot))'$ (vector of stock return rates) and $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i,\, j \leq d}$ (volatility matrix) are all assumed to be progressively measurable with respect to $\mathbf{F}$. Moreover, the matrix $\sigma(\cdot)$ is assumed to be invertible, and all processes $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$, $\sigma^{-1}(\cdot)$ are assumed to be bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$.

We introduce the "risk premium" process,

$$\theta_0(t) := \sigma^{-1}(t)\big[b(t) - r(t)\tilde{\mathbf{1}}\big], \qquad 0 \le t \le T, \tag{2.2}$$

where $\tilde{\mathbf{1}} = (1, \dots, 1)' \in \mathbb{R}^d$, that is then also bounded. Consequently, the P-supermartingale,

$$Z_0(t) := \exp\left[-\int_0^t \theta_0'(s)\, dW(s) - \frac{1}{2}\int_0^t \|\theta_0(s)\|^2\, ds\right], \qquad 0 \le t \le T \tag{2.3}$$

is actually a P-martingale, and

$$P_0(\Lambda) := E\big[Z_0(T)1_\Lambda\big], \qquad \Lambda \in \mathscr{F} \tag{2.4}$$

is a probability measure equivalent to P. We also introduce the discount process,

$$\gamma_0(t) := \frac{1}{S_0(t)} = \exp\left(-\int_0^t r(s)\, ds\right), \qquad 0 \le t \le T. \tag{2.5}$$

The discounted stock prices $\gamma_0(\cdot)S_1(\cdot), \dots, \gamma_0(\cdot)S_d(\cdot)$ are martingales under the *equivalent martingale measure* P_0, and the process

$$W_0(t) := W(t) + \int_0^t \theta_0(s)\, ds, \qquad 0 \le t \le T \tag{2.6}$$

is a P_0-Brownian motion, by the Girsanov theorem. This is a standard, continuous-time complete financial market model.

Imagine now a ("small") agent in this market with initial capital x who has to choose, at each time $t \in [0, T]$, which amount $\pi_i(t)$ to invest in each of the stocks $i = 1, \dots, d$. He invests the amount $X(t) - \sum_{i=1}^d \pi_i(t)$ in the bank account, at time t. Here $X(\cdot) \equiv X^{x,\pi}(\cdot)$ denotes his wealth process, which satisfies the equation

$$\begin{aligned} dX(t) &= \left[X(t) - \sum_{i=1}^d \pi_i(t)\right] r(t)\, dt + \sum_{i=1}^d \pi_i(t) \left[b_i(t)\, dt + \sum_{j=1}^d \sigma_{ij}(t)\, dW^j(t)\right] \\ &= r(t)X(t)\, dt + \pi'(t)\sigma(t)\, dW_0(t); \qquad X(0) = x, \end{aligned}$$

or, by Itô's rule, in discounted form,

$$d\big(\gamma_0(t)X(t)\big) = \gamma_0(t)\pi'(t)\sigma(t)\, dW_0(t); \qquad X(0) = x. \tag{2.7}$$

More formally, we have the following definition.

DEFINITION 2.1. (i) Process π: $[0, T] \times \Omega \to \mathbb{R}^d$ is a *portfolio process* if it is $\mathbf{F}$-progressively measurable and satisfies $\int_0^T \|\pi(t)\|^2\, dt < \infty$, a.s.

(ii) For a given portfolio process $\pi(\cdot)$, the process $X(\cdot) \equiv X^{x,\pi}(\cdot)$ defined by (2.7) is called the *wealth process* corresponding to portfolio $\pi(\cdot)$ and initial capital x.

(iii) Given a random variable $A \in \mathbf{L}^2(\Omega, \mathscr{F}(T), P)$, a portfolio process $\pi(\cdot)$ is called *admissible for the initial capital* x, and we write $\pi(\cdot) \in \mathscr{A}(x)$, if

$$X^{x,\pi}(t) \geq S_0(t) E_0\big[\gamma_0(T) A \,\big|\, \mathscr{F}(t)\big] =: A(t), \qquad 0 \leq t \leq T \tag{2.8}$$

holds almost surely. Here E_0 denotes expectation with respect to the probability measure P_0 of (2.4). □

REMARK 2.1. Standard results on complete financial markets imply that

$$A(0) := E_0\big[\gamma_0(T) A\big] \leq x \tag{2.9}$$

in (2.8) is the "Black–Scholes price" of the contingent claim A at time $t = 0$; namely, it is equal to the minimum of those values of the initial capital y, for which there exists a "tame" portfolio $\pi(\cdot)$ with $X^{y,\pi}(T) \geq A$, a.s. Accordingly, we call $A(t)$ the "Black–Scholes price" of A at time t, for any given $t \in [0, T]$. The bound of (2.8) can be interpreted as a *margin requirement*: the value $X^{x,\pi}(\cdot)$ of the portfolio $\pi(\cdot)$ is never allowed to fall below the value $A(\cdot) \equiv X^{A(0),\pi_A}(\cdot)$ of the "Black–Scholes hedging portfolio" $\pi_A(\cdot)$ for the contingent claim A, where

$$\begin{aligned} \gamma_0(t) A(t) &= E_0\big[\gamma_0(T) A \big| \mathscr{F}(t)\big] \\ &= A(0) + \int_0^t \gamma_0(u) \pi_A'(u) \sigma(u)\, dW_0(u), \qquad 0 \leq t \leq T. \end{aligned} \tag{2.10}$$

REMARK 2.2. We note from (2.7) and (2.10) that $\gamma_0(t)[X^{x,\pi}(\cdot) - A(\cdot)]$ is a P_0-local martingale, and it is nonnegative by (2.8). It is therefore a P_0-supermartingale. Since (2.10) implies that $\gamma_0(\cdot) A(\cdot)$ is a P_0-martingale, we deduce that $\gamma_0(\cdot) X^{x,\pi}(\cdot)$ is a P_0-supermartingale, and thus

$$E_0\big[\gamma_0(T) X^{x,\pi}(T)\big] \leq x \qquad \forall\, \pi(\cdot) \in \mathscr{A}(x). \tag{2.11}$$

Imagine now that, at time $t = T$, the agent has to deliver a payoff described by a contingent claim C, a random variable in $\mathbf{L}^2(\Omega, \mathscr{F}(T), P)$, with

$$P[C \geq A] = 1 \quad \text{and} \quad P[C > A] > 0. \tag{2.12}$$

We denote the Black–Scholes price of C by

$$C(0) := E_0\big[\gamma_0(T) C\big], \tag{2.13}$$

$$\begin{aligned} C(t) &:= S_0(t) E_0\big[\gamma_0(T) C \big| \mathscr{F}(t)\big] \\ &= S_0(t)\bigg(C(0) + \int_0^t \gamma_0(u) \pi_C'(u) \sigma(u)\, dW_0(u)\bigg), \qquad 0 \leq t \leq T. \end{aligned} \tag{2.14}$$

By Remark 2.1, if $x \geq C(0)$, we have

$$X^{x,\pi}(T) \geq C \quad \text{a.s. for some } \pi(\cdot) \in \mathscr{A}(x). \tag{2.15}$$

In particular, if we have $x = C(0)$ and $\pi(\cdot) \equiv \pi_C(\cdot)$, the Black–Scholes hedging portfolio of the contingent claim C in (2.14), we get $X^{x,\pi}(T) = C$, a.s.

In the case $A(0) \le x < C(0)$, it is no longer possible to have the inequality of (2.15) with probability 1. Instead, we are going to study the problem of maximizing the probability of a perfect hedge,

$$V(x) \equiv V(x;C) := \sup_{\pi(\cdot)\in\mathscr{A}(x)} P\big[X^{x,\pi}(T) \ge C\big]. \tag{2.16}$$

REMARK 2.3. The following is an interesting margin requirement that also turns out to be relatively easy to deal with:

$$X^{x,\pi}(t) \ge C(t) - kS_0(t) \quad \text{for all } 0 \le t \le T \tag{2.17}$$

for some given, fixed $k > 0$. This means that the value of the hedging portfolio $\pi(\cdot)$ is never allowed to fall below the current price $C(\cdot)$ of the contingent claim [as in (2.14)], by more than the value of k dollars invested (at time zero) in the bank account. The requirement (2.17) is the special case of (2.8), if we set

$$A = C - kS_0(T). \tag{2.18}$$

We concentrate now on the stochastic control problem (2.16). If $x \ge C(0)$, it follows from (2.15) that $V(x) = 1$. We therefore analyze only the case $A(0) \le x < C(0)$.

We use a duality approach, familiar from utility maximization literature, and start with the function $U(z) = 1_{\{z\le 0\}}$ and its (random, $\mathscr{F}(T)$-measurable) Legendre–Fenchel transform

$$\begin{aligned}\tilde{U}(\zeta,\omega) &:= \max_{z\le C(\omega)-A(\omega)}\big[1_{\{z\le 0\}} + \zeta z\big]\\ &= \begin{cases}\zeta\big[C(\omega)-A(\omega)\big] & \zeta\big[C(\omega)-A(\omega)\big] \ge 1,\\ 1 & 0 \le \zeta\big[(C(\omega)-A(\omega)\big] < 1,\end{cases}\end{aligned} \tag{2.19}$$

for $0 < \zeta \le \infty$. It is easily seen that the minimum is attained by any random variable of the form

$$I(\zeta,\omega) := \begin{cases} C(\omega)-A(\omega) & \zeta\big[C(\omega)-A(\omega)\big] > 1,\\ 0 & 0 \le \zeta\big[C(\omega)-A(\omega)\big] < 1,\\ \big[C(\omega)-A(\omega)\big]1_E(\omega) & \zeta\big[C(\omega)-A(\omega)\big] = 1,\end{cases} \tag{2.20}$$

for some event $E \in \mathscr{F}(T)$.

Denote

$$H_0(t) := \gamma_0(t)Z_0(t), \qquad 0 \le t \le T. \tag{2.21}$$

We see from (2.19) that, for any initial capital $x \in [A(0), C(0))$ and any $\pi(\cdot) \in \mathscr{A}(x)$, $\zeta > 0$ we have

$$1_{\{C - X^{x,\pi}(T)\le 0\}} \le \tilde{U}(\zeta H_0(T)) - \zeta H_0(T)(C - X^{x,\pi}(T)) \quad \text{a.s.} \tag{2.22}$$

Taking expectations, and recalling (2.11), (2.13) and (2.19), we obtain

$$
\begin{aligned}
P\big[X^{x,\pi}(T) \ge C\big] &\le E\big[\tilde{U}(\zeta H_0(T))\big] - \zeta E\big[H_0(T)(C - X^{x,\pi}(T))\big] \\
&\le E\big[\tilde{U}(\zeta H_0(T))\big] - \zeta(C(0) - x) \\
&= G_0(\zeta) - \zeta\big[(C(0) - x) - K_0(\zeta)\big] =: F_0(\zeta),
\end{aligned}
\tag{2.23}
$$

where we have denoted

$$G_0(\zeta) := P\big[(C - A)\zeta H_0(T) < 1\big], \qquad 0 < \zeta \le \infty, \tag{2.24}$$

$$K_0(\zeta) := E\big[H_0(T)(C - A)1_{\{(C-A)\zeta H_0(T)\ge 1\}}\big], \qquad 0 < \zeta \le \infty. \tag{2.25}$$

Both these functions are right-continuous and monotone, with $G_0(0+) = 1$, $K_0(0+) = 0$ and

$$
\begin{gathered}
G_0(\infty) = P[A = C], \qquad K_0(\infty) = E\big[H_0(T)(C - A)\big] = C(0) - A(0), \\
G_0(\zeta) + \zeta K_0(\zeta) = 1 + E\big(\zeta H_0(T)[C - A] - 1\big)^+ = 1 + \int_0^\zeta K_0(u)\,du,
\end{gathered}
\tag{2.26}
$$

for $0 \le \zeta < \infty$. We see from this that function $F_0(\cdot)$ of (2.23) is convex and that it attains its minimum at $\hat{\zeta} > 0$ given by (2.29) below.

REMARK 2.4. The inequalities of (2.23) become equalities for some $\hat{\pi}(\cdot) \in \mathscr{A}(x)$ and $\hat{\zeta} > 0$, if and only if we have

$$E\big[H_0(T)X^{x,\hat{\pi}}(T)\big] = x \tag{2.27}$$

and

$$
\begin{aligned}
C - X^{x,\hat{\pi}}(T) = {}& [C - A]1_{\{\hat{\zeta}H_0(T)[C-A]>1\}} \\
&+ [C - A]1_{E\cap\{\hat{\zeta}H_0(T)[C-A]=1\}} \quad \text{a.s.}
\end{aligned}
\tag{2.28}
$$

for some set $E \in \mathscr{F}(T)$. In this case, $\hat{\pi}(\cdot)$ is optimal, since the upper bound of (2.23) on the value function $V(x)$ is attained.

Moreover, we shall see that we can take

$$\hat{\zeta} = \inf\big\{\zeta > 0 / K_0(\zeta) \ge C(0) - x\big\}. \tag{2.29}$$

PROPOSITION 2.1. *For every $x \in [A(0), C(0))$ and $\hat{\zeta} \in (0, \infty]$ given by (2.29), there exists a set $E \in \mathscr{F}(T)$ such that the random variable*

$$
\begin{aligned}
\hat{X}(T) := {}& C1_{\{\hat{\zeta}H_0(T)[C-A]\le 1\}} \\
&+ A1_{\{\hat{\zeta}H_0(T)[C-A]>1\}} - (C - A)1_{E\cap\{\hat{\zeta}H_0(T)[C-A]=1\}}
\end{aligned}
\tag{2.30}
$$

satisfies

$$E\big[H_0(T)\hat{X}(T)\big] = x. \tag{2.31}$$

PROOF. From (2.25), (2.13) and $K_0(\hat{\zeta}) \geq C(0) - x$, we see that

$$x \geq E\Big[H_0(T)\big(C\mathbf{1}_{\{\hat{\zeta}H_0(T)[C-A]<1\}} + A\mathbf{1}_{\{\hat{\zeta}H_0(T)[C-A]\geq 1\}}\big)\Big] \tag{2.32}$$

Thus, we see that (2.31) holds for $\hat{X}(T)$ of (2.30) for some set $E \in \mathscr{F}(T)$, if we show that

$$x \leq E\Big[H_0(T)\big(C\mathbf{1}_{\{\hat{\zeta}H_0(T)[C-A]\leq 1\}} + A\mathbf{1}_{\{\hat{\zeta}H_0(T)[C-A]>1\}}\big)\Big]. \tag{2.33}$$

Indeed, the difference between the right-hand sides of (2.33) and (2.32) is equal to

$$E\big[(C-A)H_0(T)\mathbf{1}_{\{\hat{\zeta}H_0(T)[C-A]=1\}}\big] = \frac{1}{\hat{\zeta}}P\big[\hat{\zeta}H_0(T)(C-A)=1\big].$$

Since for any number $0 \leq y \leq (1/\hat{\zeta})P[\hat{\zeta}H_0(T)(C-A)=1]$, we can find a set $E \in \mathscr{F}(T)$ such that

$$y = \frac{1}{\hat{\zeta}}P\big[E \cap \{\hat{\zeta}H_0(T)(C-A)=1\}\big] = E\big[H_0(T)(C-A)\mathbf{1}_{E\cap\{\hat{\zeta}H_0(T)[C-A]=1\}}\big],$$

(2.31) follows from (2.32) and (2.33).

We now prove (2.33). We know that function $F_0(\cdot)$ attains its minimum at $\hat{\zeta}$, so that for any $-\hat{\zeta} < \varepsilon < 0$ we have $\varepsilon^{-1}(F_0(\hat{\zeta}) - F_0(\hat{\zeta}+\varepsilon) \geq 0$, implying

$$\begin{aligned}
x &\leq \frac{1}{\varepsilon}E\big[((\hat{\zeta}+\varepsilon)[A-C]H_0(T)) \vee 1\big] \\
&\quad - E\big[(\hat{\zeta}[A-C]H_0(T)) \vee 1\big] + E\big[H_0(T)C\big] \\
&= E\big[[A-C]H_0(T)\mathbf{1}_{\{(\hat{\zeta}+\varepsilon)H_0(T)[C-A]>1\}} + H_0(T)C\big] \\
&\quad - \frac{1}{\varepsilon}E\big[(1-\hat{\zeta}[C-A]H_0(T))\mathbf{1}_{\{(1/\hat{\zeta})<H_0(T)[C-A]\leq 1/(\hat{\zeta}+\varepsilon)\}}\big].
\end{aligned}$$

We note that the last term is nonpositive, and we obtain (4.22) by omitting it and letting $\varepsilon \to 0$. □

THEOREM 2.1. *For any given $x \in [A(0), C(0))$, $\hat{\zeta} \in (0, \infty]$ given by* (2.29), *and set E as in Proposition* 2.1, *there exists a portfolio process $\hat{\pi}(\cdot) \in \mathscr{A}(x)$ such that* (2.27) *and* (2.28) *hold and that portfolio is optimal for the problem of* (2.16),

$$V(x) = P\big[X^{x,\hat{\pi}}(T) \geq C\big] = F_0(\hat{\zeta}). \tag{2.34}$$

PROOF. Recall the random variable $\hat{X}(T)$ of (2.30) and define the P_0-martingale,

$$\begin{aligned}
\hat{X}(t)\gamma_0(t) &:= E_0\big[\hat{X}(T)\gamma_0(T) \mid \mathscr{F}(t)\big] \\
&= x + \int_0^t \hat{\pi}'(u)\gamma_0(u)\sigma(u)\,dW_0(u), \qquad 0 \leq t \leq T,
\end{aligned} \tag{2.35}$$

where $\hat{\pi}(\cdot)$ is a portfolio process determined through the martingale representation theorem [see Karatzas and Shreve (1991)]. The process $\hat{X}(\cdot)$ defined by (2.35) clearly satisfies $\hat{X}(0) = x$, $\hat{X}(\cdot) \equiv X^{x,\hat{\pi}}(\cdot)$, as well as (2.28) and (2.27) by Proposition 2.1. Optimality of $\hat{\pi}(\cdot)$ now follows from Remark 2.4. $\square$

We see that the optimal portfolio $\hat{\pi}(\cdot)$ of (2.34) coincides with the hedging portfolio for the contingent claim $\hat{X}(T)$ of (2.30); in the special case $A = 0$ and $P[\hat{\zeta} H_0(T)(C - A) = 1] = 0$, $\hat{X}(T)$ is a "knock-out" option with payoff C, "knocked out" on the event $\{\hat{\zeta} H_0(T)[C - A] > 1\}$. The knock-out option interpretation of the optimal policy in this context was first given in Browne (1996). For $x = A(0)$, the conditions (2.27) and (2.29) are satisfied by $\hat{\zeta} = \infty$ and the optimal portfolio $\hat{\pi}(\cdot)$ of Theorem 2.1 coincides with $\hat{\pi}_A(\cdot)$, the hedging portfolio for the contingent claim A in (2.10).

We state separately the result for the special case when $H_0(T)[C - A]$ is a constant. In particular, this holds in the interesting risk-neutral case with $\theta_0(\cdot) \equiv 0$, $r(\cdot)$ and $C - A$ constant.

PROPOSITION 2.2. *Suppose that $H_0(T)[C - A]$ is a (nonrandom) positive constant. Then we have*

$$V(x) = P[X^{x,\hat{\pi}}(T) \ge C] = \frac{x - A(0)}{H_0(T)[C - A]} \quad \text{for } A(0) \le x < C(0), \tag{2.36}$$

where $\hat{\pi}(\cdot)$ is any portfolio in $\mathscr{A}(x)$ for which (2.27) *holds and such that*

$$X^{x,\hat{\pi}}(T) = C 1_{E^c} + A 1_E \tag{2.37}$$

for some $E \in \mathscr{F}(T)$.

PROOF. We have $K_0(\zeta) = 0$ and $G_0(\zeta) = 1$ for $0 < \zeta < (H_0(T)[C - A])^{-1}$, and $K_0(\zeta) = C(0) - A(0)$, $G_0(\zeta) = 0$ for $\zeta \ge (H_0(T)[C - A])^{-1}$. Therefore, $\hat{\zeta} = (H_0(T)[C - A])^{-1}$ in (2.29). Moreover, (2.23) with $\zeta = \hat{\zeta}$ implies

$$P\big[X^{x,\pi}(T) \ge C\big] \le \frac{x - A(0)}{H_0(T)[C - A]}.$$

But the right-hand side is attained for any portfolio $\hat{\pi} \in \mathscr{A}(x)$ satisfying (2.37) for some $E \in \mathscr{F}(T)$ and (2.27); indeed, we have then

$$E\big[H_0(T) X^{x,\hat{\pi}(T)}\big] = E\big[H_0(T)(C 1_{E^c} + A 1_E)\big] = x$$

and hence

$$P\big[X^{x,\hat{\pi}}(T) \ge C\big] = P[E^c] = \hat{\zeta} E\big[H_0(T)(C - A) 1_{E^c}\big] = \hat{\zeta}(x - A(0)). \qquad \square$$

EXAMPLE 2.1. *Deterministic coefficients.* Consider the case of an arbitrary contingent claim C and the margin requirements in the form $A = C - k S_0(T)$ for some $k > 0$. We also assume that $r(\cdot)$, $\sigma(\cdot)$ and $b(\cdot)$ are deterministic and that $\|\theta_0(t)\| > 0$, for all $t \in [0, T]$. Then we have

$$K_0(\zeta) = kP_0\big[k\zeta Z_0(T) \geq 1\big], \qquad G_0(\zeta) = P\big[k\zeta Z_0(T) < 1\big]$$

and

$$V(x) := \sup_{\pi(\cdot)\in\mathscr{A}(x)} P\big[X^{x,\pi}(T) \geq C\big] = F_0(\hat{\zeta}(x)), \qquad A(0) < x < C(0), \tag{2.38}$$

where $\hat{\zeta}(x) = \hat{\zeta}$ is given by (2.29). From (2.35) (taking $E = \varnothing$), the optimal portfolio $\hat{\pi}(\cdot) \in \mathscr{A}(x)$ and wealth $\hat{X}(\cdot) \equiv X^{x,\hat{\pi}}(\cdot)$ processes for this problem are given by

$$\begin{aligned} \gamma_0(t)\hat{X}(t) &= \gamma_0(t)C(t) - kP_0\big[k\hat{\zeta}Z_0(T) \geq 1 \mid \mathscr{F}(t)\big] \\ &= x + \int_0^t \gamma_0(u)\hat{\pi}'(u)\sigma(u)\,dW_0(u), \qquad 0 \leq t \leq T. \end{aligned} \tag{2.39}$$

It is easily computed that

$$\begin{aligned} &P_0\big[k\hat{\zeta}Z_0(T) \geq 1 \mid \mathscr{F}(t)\big] \\ &\quad = \Phi\left(\frac{\log(k\hat{\zeta}) - \int_0^t \theta_0'(u)\,dW_0(u) + \frac{1}{2}\int_0^T \|\theta_0(u)\|^2\,du}{\sqrt{\int_t^T \|\theta_0(u)\|^2\,du}}\right), \qquad 0 \leq t \leq T, \end{aligned} \tag{2.40}$$

where $\Phi(\cdot)$ is the cumulative standard normal distribution function. In particular, this implies that we shall have $K_0(\hat{\zeta}) = C(0) - x$ if and only if

$$\log(k\hat{\zeta}) + \frac{1}{2}\int_0^T \|\theta_0(u)\|^2\,du = \sqrt{\int_0^T \|\theta_0(u)\|^2\,du}\,\Phi^{-1}\left(\frac{C(0)-x}{k}\right) \tag{2.41}$$

is satisfied. In conjunction with (2.39) and (2.40), we obtain

$$\begin{aligned} &\gamma_0(t)\hat{X}(t) = \gamma_0(t)C(t) - k \\ &\qquad \times \Phi\left(\frac{-\int_0^t \theta_0'(u)\,dW_0(u) + \sqrt{\int_0^T \|\theta_0(u)\|^2\,du}\,\Phi^{-1}((C(0)-x)/k)}{\sqrt{\int_t^T \|\theta_0(u)\|^2\,du}}\right), \\ &\qquad\qquad 0 \leq t \leq T. \end{aligned} \tag{2.42}$$

Next, an application of Itô's rule on the right-hand side of (2.42), together with (2.39) and (2.14), implies [with $\varphi(\cdot)$ denoting the standard normal density function]

$$
\begin{aligned}
\gamma_0(t)\hat{\pi}(t) &= \gamma_0(t)\pi_C(t) + k\frac{(\sigma^{-1}(t))'\theta_0(t)}{\sqrt{\int_t^T \|\theta_0(u)\|^2\,du}} \\
&\qquad \times \varphi\Biggl(\frac{-\int_0^t \theta_0'(u)\,dW_0(u) + \sqrt{\int_0^T \|\theta_0(u)\|^2\,du}\,\Phi^{-1}((C(0)-x)/k)}{\sqrt{\int_t^T \|\theta_0(u)\|^2\,du}}\Biggr) \\
&= \gamma_0(t)\pi_C(t) + k\frac{(\sigma^{-1}(t))'\theta_0(t)}{\sqrt{\int_t^T \|\theta_0(u)\|^2\,du}} \\
&\qquad \times \varphi\biggl(\Phi^{-1}\biggl(\frac{\gamma_0(t)[C(t)-\hat{X}(t)]}{k}\biggr)\biggr), \qquad 0 \le t \le T.
\end{aligned} \tag{2.43}
$$

Note that if $d = 1$ and $\theta_0(\cdot) \equiv \theta_0$ is constant, $\hat{X}(\cdot)$ and $\hat{\pi}(\cdot)$ do not depend on θ_0 (except through its sign), similarly to the case of the standard, Black–Scholes "delta hedging."

It is equally straightforward to calculate

$$
\begin{aligned}
V(x) &= P\bigl[k\hat{\zeta}Z_0(T) < 1\bigr] \\
&= \Phi\biggl(-\Phi^{-1}\biggl(\frac{C(0)-x}{k}\biggr) + \sqrt{\int_0^T \|\theta_0(u)\|^2\,du}\biggr).
\end{aligned} \tag{2.44}
$$

This recovers and generalizes results of Kulldorff (1993), Heath (1993) and Browne (1996).

3. A market model with partial information. In this section we study the following variation on the stochastic control problem of Section 2: given a contingent claim C we want to maximize the probability of a perfect hedge $P[X^{x,\pi}(T) \ge C]$ over a class of portfolio processes $\pi(\cdot)$ which are adapted to the natural filtration generated by the stock prices; in the formulation of the adaptive stochastic control problem studied here, we assume that the vector of stock appreciation rates b is not directly observable, so that, as the underlying price process evolves, the investor observes the outcomes and thus obtains information about the true value of b. The case when $d = 1$, $C \equiv 1$, $r(\cdot) \equiv 0$ and $\sigma(\cdot) \equiv 1$ was studied by Karatzas (1997). We study a more general case by modifying the duality approach presented in Section 2. We prove the existence of an optimal control process $\hat{\pi}(\cdot)$ and provide an example in which the value of the problem and the optimal portfolio is found explicitly.

We start with a given probability space $(\Omega, \mathscr{F}, P_0)$. Let $W_0(\cdot) = \{W_0(t), 0 \le t \le T\}$ be an $\mathbb{R}^d$-valued Brownian motion on this probability space on the finite time horizon $[0, T]$, and $B\colon \Omega \mapsto \mathbb{R}^d$ be a random vector independent of the process $W_0(\cdot)$ under the probability measure P_0, and with known distribution

$\mu(\Lambda) = P_0[B \in \Lambda]$, $\Lambda \in \mathscr{B}(\mathbb{R}^d)$ that satisfies

$$\int_{\mathbb{R}^d} \|b\| \mu(db) < \infty. \tag{3.1}$$

We will denote by $\mathbf{F} = \{\mathscr{F}(t), 0 \le t \le T\}$ the augmentation of the filtration

$$\mathscr{F}(t) := \sigma\big(W_0(s), 0 \le s \le t\big) \tag{3.2}$$

generated by the process $W_0(\cdot)$, and by $\mathbf{G} = \{\mathscr{G}(t),\ 0 \le t \le T\}$ the augmentation of the enlarged filtration

$$\mathscr{G}^{B, W_0}(t) := \sigma\big(B, W_0(s), 0 \le s \le t\big) \tag{3.3}$$

generated by process $W_0(\cdot)$ and random variable B.

We introduce the interest rate process $r(\cdot)$, which is assumed to be a bounded $\mathbf{F}$-progressively measurable scalar process, and the volatility process $\sigma(\cdot)$, which is a bounded, $\mathbf{F}$-progressively measurable process with values in the space of full-rank $d \times d$ matrices with bounded inverse. Furthermore, we assume that $r(\cdot)$ and $\sigma(\cdot)$ are functions of past and present stock prices defined by (3.7) below; more precisely, of the form $r(t) = R(t, S(\cdot))$, $0 \le t \le T$ and $\sigma_{ij}(t) = \Sigma_{ij}(t, S(\cdot))$, $0 \le t \le T$, $1 \le i,\ j \le d$, where R: $[0, T] \times C([0, T]; \mathbb{R}^d_+) \mapsto \mathbb{R}$ and Σ_{ij}: $[0, T] \times C([0, T]; \mathbb{R}^d_+) \mapsto \mathbb{R}$ are progressively measurable functionals [see Karatzas and Shreve (1991), Definition 3.5.15].

As before, we introduce the risk premium $\theta_0(t)$: $= \sigma^{-1}(t)(B - r(t)\tilde{\mathbf{1}})$, $0 \le t \le T$.

The following two lemmas are straightforward to prove. The proof is similar to that in Karatzas and Zhao (1998), Lemmas 2.1 and 2.2 [see also Lakner (1994)].

LEMMA 3.1. *$W_0(\cdot)$ is a $(\mathbf{G}, P_0)$-Brownian motion, and the exponential process*

$$Z(t) := \exp\bigg(\int_0^t \theta_0'(s)\, dW_0(s) - \tfrac{1}{2}\int_0^t \big\|\theta_0(s)\big\|^2 s\, ds\bigg), \qquad 0 \le t \le T \tag{3.4}$$

is a $(\mathbf{G}, P_0)$-martingale.

We can now define a new probability measure P by

$$P[\Sigma] := E_0[Z(T)1_\Sigma], \qquad \Sigma \in \mathscr{G}(T), \tag{3.5}$$

where E_0 is the expectation operator under the measure P_0. The two probability measures P and P_0 are equivalent on $\mathscr{G}(T)$.

LEMMA 3.2. *Under the probability measure P of* (3.5), *the process*

$$W(t) := W_0(t) - \int_0^t \theta_0(s)\, ds, \mathscr{G}(t), \qquad 0 \le t \le T \tag{3.6}$$

is a standard d-dimensional Brownian motion, independent of the random variable B. Furthermore, we have

$$P[B \in \Lambda] = P_0[B \in \Lambda] = \mu(\Lambda) \qquad \forall\, \Lambda \in \mathscr{B}(\mathbb{R}^d).$$

Consider now a financial market $\mathscr{M}$ with a bank account and d stocks. The price processes $S_0(\cdot)$ and $S_1(\cdot), \dots, S_d(\cdot)$ of these assets evolve according to the equations,

$$
\begin{aligned}
dS_0(t) &= S_0(t) r(t)\, dt, \qquad S_0(0) = 1,\\
dS_i(t) &= S_i(t)\Big[B_i\, dt + \sum_{j=1}^{d} \sigma_{ij}(t)\, dW^j(t)\Big]\\
&= S_i(t)\Big[r(t)\, dt + \sum_{j=1}^{d} \sigma_{ij}(t)\, dW_0^j(t)\Big],\\
&\qquad\qquad S_i(0) = s_i > 0;\ i = 1, \dots, d.
\end{aligned}
\tag{3.7}
$$

We assume that the functionals R and Σ_{ij} are such that the last equation has a unique, $\mathbf{F}$-adapted solution [see Protter (1990) for sufficient conditions].

REMARK 3.1. Filtration $\mathbf{F}$ of (3.2) is generated by the stock-price vector $S(\cdot) := (S_1(\cdot), \dots, S_d(\cdot))$; namely, $\mathscr{F}(t) = \sigma(S(u), 0 \le u \le t) =: \mathscr{F}'(t)$. Indeed, the inclusion $\mathscr{F}(t) \subset \mathscr{F}'(t)$ is valid since (3.7) can be solved for $W_0(t)$, thanks to invertability of $\sigma(\cdot)$, which, together with $\mathscr{F}'(t)$-measurability of $r(t)$ and $\sigma(t)$, shows that $W_0(t)$ is $\mathscr{F}'(t)$-measurable. The reverse inclusion follows from the assumption that (3.7) has a unique, $\mathbf{F}$-adapted solution $S(t)$.

We now define a wealth process, a portfolio process and admissible portfolio processes as in Definition 2.1, but we emphasize here that $\mathscr{F}$ is the filtration of the "available information" and that neither the stock drift B nor the Brownian motion $W(\cdot)$ is adapted to it, so that the investor's portfolio choices should be affected by the information contained in the stock prices only. We define contingent claims C and A as in Section 2, with the interpretation of $\mathscr{F}_T$-measurability of C as the requirement that the random payoff made at T be independent of any information other than the stock prices up to (and including) time T.

LEMMA 3.3. *The price $C(t)$ of a contingent claim C at any time t, $0 \le t \le T$, is given by* (2.14). *Furthermore, the "hedging portfolio" $\pi_C(\cdot)$ is admissible; in particular, it is adapted to the filtration $\mathbf{F}$ generated by the stock price process $S(\cdot)$.*

The proof is shown as usual, using the martingale representation theorem and dynamics (2.7), which are still valid, due to (3.7).

We now study the stochastic control problem (2.16) in the market with restricted information and we suppose that the investor's initial wealth x satisfies $A(0) \le x < C(0)$. We start by introducing the $(\mathbf{F}, P_0)$-martingale,

$$
\begin{aligned}
\hat{Z}(t) &:= E_0\big[Z(T) \mid \mathscr{F}(t)\big] = E_0\big[E_0\big[Z(T) \mid \mathscr{G}(t)\big] \mid \mathscr{F}(t)\big]\\
&= E_0\big[Z(t) \mid \mathscr{F}(t)\big].
\end{aligned}
\tag{3.8}
$$

Denote $Z_0(t) := (Z(t))^{-1}$ and $\hat{Z}_0(t) := (\hat{Z}(t))^{-1}$. Note that by the "Bayes rule" we have

$$E\big[Z_0(t) \mid \mathscr{F}(t)\big] = \frac{E_0\big[Z_0(t)Z(t) \mid \mathscr{F}(t)\big]}{E_0\big[Z(t) \mid \mathscr{F}(t)\big]} = (\hat{Z}(t))^{-1} = \hat{Z}_0(t). \tag{3.9}$$

To use the duality approach, we introduce the functions U, $\tilde{U}$ and I as in Section 2; see, in particular, (2.19) and (2.20). Denote

$$\hat{H}_0(t) := \gamma_0(t)\hat{Z}_0(t), \tag{3.10}$$

the analogue of the process $H_0(\cdot)$ of (2.21). We recall (2.22), with $H_0(T)$ replaced by $\hat{H}_0(T)$. Furthermore, since for any admissible π the random variable $X^{x,\pi}(T)$ is $\mathscr{F}_T$-measurable, we have, from (3.9) and the analogue of (2.11),

$$\begin{aligned} E\big[\hat{H}_0(T)X^{x,\pi}(T)\big] &= E\big[\gamma_0(T)(\hat{Z}(T))^{-1}X^{x,\pi}(T)\big] \\ &= E_0\big[\gamma(T)X^{x,\pi}(T)\big] \le x. \end{aligned} \tag{3.11}$$

Taking expectations in the analogue of (2.22) and using (3.11) and (2.19), we see that the analogue of (2.23) holds. Moreover, the same proofs as before show that

(3.12) Proposition 2.1 and Theorem 2.1 remain valid, with $H_0(T)$ replaced by $\hat{H}_0(T)$.

EXAMPLE 3.1. Consider an arbitrary contingent claim C and the margin requirement of the form $A = C - kS_0(T)$ for some $k > 0$. We assume that $d = 1$ and that the random variable B has a normal distribution with mean f and variance l^2. From (3.8) we have

$$\hat{Z}(t) = \frac{\sigma}{\sqrt{l^2 t + \sigma^2}} \exp\left[-\frac{(f-r)^2}{2l^2} + \frac{(l^2 W_0(t) + (f-r)\sigma)^2}{2l^2(l^2 t + \sigma^2)}\right]. \tag{3.13}$$

The function $K_0(\cdot)$ is given by

$$\begin{aligned} K_0(\zeta) &= kP_0\big[k\zeta\hat{Z}_0(T) \ge 1\big] \\ &= \begin{cases} k\Phi\left(\dfrac{D(\zeta) - (f-r)\sigma}{l^2\sqrt{T}}\right) - k\Phi\left(\dfrac{-D(\zeta) - (f-r)\sigma}{l^2\sqrt{T}}\right), \\ \qquad \text{if } \zeta > \dfrac{\sigma}{k\sqrt{l^2 T + \sigma^2}} \exp\left(-\dfrac{(f-r)^2}{2l^2}\right), \\ 0, \ \text{if } 0 \le \zeta \le \dfrac{\sigma}{k\sqrt{l^2 T + \sigma^2}} \exp\left(-\dfrac{(f-r)^2}{2l^2}\right), \end{cases} \end{aligned} \tag{3.14}$$

where

$$D(\zeta) = \sqrt{\left\{2l^2(l^2 T + \sigma^2)\left[\frac{(f-r)^2}{2l^2} + \log\left(\frac{k\zeta\sqrt{l^2 T + \sigma^2}}{\sigma}\right)\right]\right\} \vee 0}.$$

Therefore, the requirement $K_0(\hat{\zeta}) = C(0) - x$ is equivalent to

$$\Phi\left(\frac{D(\hat{\zeta}) + (f-r)\sigma}{l^2\sqrt{T}}\right) + \Phi\left(\frac{D(\hat{\zeta}) - (f-r)\sigma}{l^2\sqrt{T}}\right) = \frac{C(0)-x}{k} + 1. \tag{3.15}$$

For any real a and b, let

$$J(a;b) := \Phi(a+b) + \Phi(a-b),$$

which is an increasing function in its first argument, and let $J^{-1}(\cdot, b)$ be an inverse of $J(\cdot, b)$ for any real b. Then

$$D(\hat{\zeta}) = l^2\sqrt{T}J^{-1}\left(\frac{C(0)-x}{k} + 1, \frac{(f-r)\sigma}{l^2\sqrt{T}}\right). \tag{3.16}$$

Using the fact that, under the measure P, the random variable $W_0(T) = W(T) + \theta_0 T$ has a normal distribution with the mean $((f-r)/\sigma)T$ and the variance $T + T^2 l^2/\sigma^2$, we can find

$$\begin{aligned} G_0(\zeta) &= P\big[k\zeta\hat{Z}_0(T) < 1\big] = P\big[\big|l^2 W_0(T) + (f-r)\sigma\big| > D(\zeta)\big] \\ &= \Phi\left(-\frac{\sigma D(\zeta)}{l^2\sqrt{T(l^2T+\sigma^2)}} - \frac{(f-r)\sqrt{l^2T+\sigma^2}}{l^2\sqrt{T}}\right) \\ &\quad + \Phi\left(-\frac{\sigma D(\zeta)}{l^2\sqrt{T(l^2T+\sigma^2)}} + \frac{(f-r)\sqrt{l^2T+\sigma^2}}{l^2\sqrt{T}}\right). \end{aligned} \tag{3.17}$$

Therefore, the value function is

$$\begin{aligned} V(x) &= G_0(\hat{\zeta}) \\ &= \Phi\left(\frac{-\sigma J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big)}{\sqrt{l^2T+\sigma^2}} - \frac{(f-r)\sqrt{l^2T+\sigma^2}}{l^2\sqrt{T}}\right) \\ &\quad + \Phi\left(\frac{-\sigma J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big)}{\sqrt{l^2T+\sigma^2}} + \frac{(f-r)\sqrt{l^2T+\sigma^2}}{l^2\sqrt{T}}\right). \end{aligned} \tag{3.18}$$

In particular, if $f = r$, we have

$$V(x) = 2\Phi\left(-\frac{\sigma}{\sqrt{l^2T+\sigma^2}}\Phi^{-1}\left(\frac{C(0)-x}{2k} + \frac{1}{2}\right)\right)$$

This is somewhat surprising at the first glance: the value function is an increasing function of the variance l^2 of the drift, and when $l^2 \to \infty$, the value function tends to one. It becomes less surprising if we recall (2.44), which shows that, for constant drift, the value function is an increasing function of the drift's absolute value. Note that if $f = r$ and $l = 0$, we get $V(x) = 1 + (x - C(0))/k$, in accordance with Proposition 2.2.

Optimal wealth $\hat{X}(\cdot)$ and optimal portfolio $\hat{\pi}(\cdot)$ can be found as in (2.39). We have

$$
\begin{aligned}
&P_0\big[k\zeta \hat{Z}_0(T) \geq 1 | \mathscr{F}(t)\big] \\
&\quad = P_0\big[\big|l^2 W_0(T) + (f-r)\sigma\big| < D(\zeta) \mid \mathscr{F}(t)\big] \\
&\quad = \Phi\bigg(\frac{D(\hat{\zeta}) - (f-r)\sigma - l^2 W_0(t)}{l^2\sqrt{T-t}}\bigg) \\
&\qquad - \Phi\bigg(\frac{-D(\hat{\zeta}) - (f-r)\sigma - l^2 W_0(t)}{l^2\sqrt{T-t}}\bigg) \\
&\quad = \Phi\bigg(\frac{\sqrt{T} J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big) - W_0(t)}{\sqrt{T-t}} \\
&\qquad\qquad - \frac{(f-r)\sigma}{l^2\sqrt{T-t}}\bigg) \\
&\qquad - \Phi\bigg(\frac{-\sqrt{T} J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big) - W_0(t)}{\sqrt{T-t}} \\
&\qquad\qquad - \frac{(f-r)\sigma}{l^2\sqrt{T-t}}\bigg),
\end{aligned}
\tag{3.19}
$$

so that

$$
\begin{aligned}
\hat{X}(t) = C(t) &- ke^{rt}\Phi\bigg(\frac{\sqrt{T} J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big) - W_0(t)}{\sqrt{T-t}} \\
&\qquad - \frac{(f-r)\sigma}{l^2\sqrt{T-t}}\bigg) \\
&+ ke^{rt}\Phi\bigg(\frac{-\sqrt{T} J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big) - W_0(t)}{\sqrt{T-t}} \\
&\qquad - \frac{(f-r)\sigma}{l^2\sqrt{T-t}}\bigg).
\end{aligned}
\tag{3.20}
$$

An application of Itô's rule to the right-hand side of (3.20) together with (2.39) and (2.14), implies

$$
\begin{aligned}
\hat{\pi}(t) \;=\; &\pi_C(t) + \frac{ke^{rt}}{\sigma\sqrt{T-t}} \\
&\times \varphi\bigg[\frac{\sqrt{T} J^{-1}\big(((C(0)-x)/k) + 1, (f-r)\sigma/l^2\sqrt{T}\big) - W_0(t)}{\sqrt{T-t}} - \frac{(f-r)\sigma}{l^2\sqrt{T-t}}\bigg]
\end{aligned}
\tag{3.21}
$$

$$-\frac{ke^{rt}}{\sigma\sqrt{T-t}}\varphi$$
$$\times\left[\frac{-\sqrt{T}J^{-1}\big(((C(0)-x)/k)+1, (f-r)\sigma/l^2\sqrt{T}\big)-W_0(t)}{\sqrt{T-t}}-\frac{(f-r)\sigma}{l^2\sqrt{T-t}}\right].$$

Note that $S(t)=S(0)\exp\{\sigma W_0(t)+(r-\sigma^2/2)t\}$ so that $W_0(t)$ is known if $S(t)$ is observed, and the last expression depends only on the model parameters and the observed price of the stock.

4. The case of a concave drift. In this section we generalize the standard model by allowing the drift of the wealth process to be nonlinear. This allows, for example, the model with different interest rates for borrowing and for lending, as well as some "large investor" models [see, for example, Cuoco and Cvitanić (1998)]. More precisely, we now assume the following dynamics of the wealth process:

$$dX^{x,\pi}(t)=g\big(t,\pi(t),X^{x,\pi}(t)\big)\,dt+\pi'(t)\sigma(t)\,dW(t); \qquad X^{x,\pi}(0)=x. \tag{4.1}$$

Here, we impose the following assumption.

ASSUMPTION 4.1. The random field $g\colon \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable (where $\mathscr{P}$ is the σ-algebra of all predictable sets in $\Omega\times[0,T]$), and satisfies

$$g(\omega,t,0,0)=0 \qquad \forall\ (\omega,t)\in\Omega\times[0,T]. \tag{4.2}$$

Moreover, the function $(\pi,x)\mapsto g(\omega,t,\pi,x)$ is concave for all $(\omega,t)\in\Omega\times[0,T]$, and is also Lipschitz, uniformly in $(\omega,t)\in\Omega\times[0,T]$.

We define the convex conjugate

$$\tilde{g}(\omega,t,\nu,\mu):=\sup_{(\pi,x)\in\mathbb{R}^{d+1}}\big[g(\omega,t,\pi,x)+\pi'\nu+x\mu\big]\geq 0 \tag{4.3}$$

on its effective domain

$$\mathscr{D}_{\omega,t}:=\big\{(\nu,\mu)\in\mathbb{R}^{d+1};\ \tilde{g}(\omega,t,\nu,\mu)<\infty\big\} \tag{4.4}$$

for $(t,\omega)\in\Omega\times[0,T]$. As in El Karoui, Peng and Quenez (1997) (hereafter [EPQ]), one can show that $\mathscr{D}_{\omega,t}$ is included in a bounded set $\tilde{R}$ in $\mathbb{R}^{d+1}$, independent of (ω,t). Denote by $\mathscr{D}$ the set of all pairs of progressively measurable processes $(\nu(\cdot),\mu(\cdot))$ which satisfy $(\nu(t),\mu(t))\in\mathscr{D}_{\omega,t}$, a.e.-$\omega\times t$. We note that class $\mathscr{D}$ is not empty since the zero process (in $\mathbb{R}^{d+1}$) is an element of $\mathscr{D}$. Motivated by the $\mathbf{L}^2$ theory of backward stochastic differential equations, hereafter BSDEs (as presented in [EPQ], for example), we somewhat change our definitions of admissibility of portfolio processes by requiring that

$$E\bigg[\int_0^T\|\pi(u)\|^2\,du\bigg]<\infty \quad\text{and}\quad E\bigg[\sup_{0\leq t\leq T}\big|X^{x,\pi}(t)\big|^2\bigg]<\infty \tag{4.5}$$

and

$$X^{x,\pi}(T) \geq A \quad \text{a.s.,} \tag{4.6}$$

where the $\mathscr{F}(T)$-measurable random variable A, as well as our contingent claim C, are elements of the set $\mathbf{L}^2(\Omega, \mathscr{F}(T), P)$. We study the optimization problem of maximizing the probability of a perfect hedge,

$$V(x) \equiv V(x; C) := \sup_{\pi(\cdot)\in\mathscr{A}(x)} P\big[X^{x,\pi}(T) \geq C\big], \tag{4.7}$$

where $\mathscr{A}(x)$ is again the set of admissible portfolios, in the sense of the above definition.

For any given $(\nu, \mu) \in \mathscr{D}$, we now introduce the processes

$$Z_\nu(t) := \exp\biggl[\int_0^t (\sigma^{-1}(s)\nu(s))'\,dW(s) - \frac{1}{2}\int_0^t \big\|\sigma^{-1}(s)\nu(s)\big\|^2\,ds\biggr] \tag{4.8}$$

and

$$\gamma_\mu(t) := \exp\biggl(\int_0^t \mu(s)\,ds\biggr), \tag{4.9}$$

which correspond to changes of measure and discounting in so-called shadow markets associated with the market with dynamics of a wealth process as in (4.1). Denote by E_ν the expectation under the probability measure

$$P^\nu := E[Z_\nu(T)\mathbf{1}_E], \qquad E \in \mathscr{F}(T),$$

under which the process

$$W_\nu(t) := W(t) - \int_0^t \sigma^{-1}(s)\nu(s)\,ds$$

is a Brownian motion (by the Girsanov theorem). Let us also introduce

$$H_{\mu,\nu}(t) := \gamma_\mu(t) Z_\nu(t), \qquad 0 \leq t \leq T.$$

It follows from the theory of BSDEs that there exist admissible portfolios $\pi_C(\cdot)$ and $\pi_A(\cdot)$ and (minimal) initial wealths $C(0)$ and $A(0)$ such that $X^{C(0),\pi_C}(T) = C$ and $X^{A(0),\pi_A}(T) = A$, a.s. We denote $C(\cdot) := X^{C(0),\pi_C}(\cdot)$, $A(\cdot) = X^{A(0),\pi_A}(\cdot)$. It is also known that, for example for claim C, we have

$$C(t) = \sup_{(\nu,\mu)\in\mathscr{D}} E\biggl[\frac{H_{\mu,\nu}(T)}{H_{\mu,\nu}(t)}C - \int_t^T \frac{H_{\mu,\nu}(s)}{H_{\mu,\nu}(t)}\tilde{g}(s, \nu(s), \mu(s))\,ds \,\bigg|\, \mathscr{F}(t)\biggr], \tag{4.10}$$

and similarly for $A(t)$ (see [EPQ] or Cvitanić, Karatzas and Soner (1998)). Moreover, the comparison theorem for BSDEs implies that if $X^{x_1,\pi_1}(T) \geq X^{x_2,\pi_2}(T)$, a.s., for some admissible $\pi_1(\cdot)$, $\pi_2(\cdot)$, then $X^{x_1,\pi_1}(\cdot) \geq X^{x_2,\pi_2}(\cdot)$, a.s., and, in particular, $x_1 \geq x_2$. Because of this, we restrict ourselves to the case $A(0) < x < C(0)$.

We have, by Itô's rule,

$$
\begin{aligned}
H_{\mu,\nu}(t)X^{x,\pi}(t) - \int_0^t H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds \\
= x + \int_0^t H_{\mu,\nu}(s)\pi'(s)\sigma(s)\,dW(s) \\
- \int_0^t H_{\mu,\nu}(s)\big[\tilde{g}(s,\nu(s),\mu(s)) - g(s,\pi(s),X(s)) \\
- \pi'(s)\nu(s) - X(s)\mu(s)\big]\,ds
\end{aligned}
\tag{4.11}
$$

for all $(\nu(\cdot),\mu(\cdot)) \in \mathscr{D}$ and $\pi(\cdot) \in \mathscr{A}(x)$. We notice now that the right-hand side process is a P-supermartingale, since the second term is a nonincreasing process, and the first term is a P-martingale due to the Burkholder–Davis–Gundy inequalities [see Karatzas and Shreve (1991)], because

$$
\begin{aligned}
&E\bigg(\int_0^T \|H_{\mu,\nu}(s)\pi'(s)\sigma(s)\|^2\,ds\bigg)^{1/2} \\
&\quad\le \bigg(E\Big[\sup_{0\le t\le T} H^2_{\mu,\nu}(t)\Big]E\int_0^T \big\|\gamma_\mu(s)\pi'(s)\sigma(s)\big\|^2\,ds\bigg)^{1/2} < \infty.
\end{aligned}
$$

We are using here the boundedness of the processes $\nu(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$, $\sigma^{-1}(\cdot)$, as well as the definition of admissibility of $\pi(\cdot)$.

The consequence of the above supermartingale property is

LEMMA 4.1. *For every $(\nu(\cdot),\mu(\cdot)) \in \mathscr{D}$ and $\pi(\cdot) \in \mathscr{A}(x)$, we have*

$$
E\big[H_{\mu,\nu}(T)X^{x,\pi}(T)\big] \le x + E\int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds.
$$

As in the complete market case, it follows from the previous lemma and (2.19) that, for any initial capital $x \in (A(0), C(0))$ and any $\pi(\cdot) \in \mathscr{A}(x)$, $\zeta > 0$ and $(\nu,\mu) \in \mathscr{D}$, we have

$$
\begin{aligned}
P[X^{x,\pi}(T) \ge C] &\le E\big[\tilde{U}(\zeta H_{\mu,\nu}(T)) - \zeta H_{\mu,\nu}(T)(C - X^{x,\pi}(T))\big] \\
&\le E\bigg[\tilde{U}(\zeta H_{\mu,\nu}(T)) - \zeta H_{\mu,\nu}(T)C \\
&\qquad + \zeta\int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg] + \zeta x.
\end{aligned}
\tag{4.12}
$$

The following remark is the analogue of Remark 2.4.

REMARK 4.1. The inequalities of (4.12) hold, in fact, as equalities for some $\hat{\pi}(\cdot) \in \mathscr{A}(x)$, $\hat{\zeta} > 0$, $(\hat{\nu},\hat{\mu}) \in \mathscr{D}$ if and only if we have

$$
E\bigg[H_{\hat{\mu},\hat{\nu}}(T)X^{x,\hat{\pi}}(T) - \int_0^T H_{\hat{\mu},\hat{\nu}}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\bigg] = x \tag{4.13}
$$

and

$$X^{x,\hat{\pi}}(T) = A1_{\{\hat{\zeta}H_{\hat{\mu},\hat{\nu}}(T)[C-A]>1\}} + C1_{\{\hat{\zeta}H_{\hat{\mu},\hat{\nu}}(T)[C-A]\le 1\}} \tag{4.14}$$
$$-[C-A]1_{E\cap\{\hat{\zeta}H_{\hat{\mu},\hat{\nu}}(T)[C-A]=1\}} \quad \text{a.s.}$$

for some set $E \in \mathscr{F}(T)$. If the preceding is true, then $\hat{\pi}(\cdot)$ is optimal, since it attains the upper bound of (4.12).

The idea now is to consider the dual problem

$$\tilde{V}(\zeta) := \inf_{(\nu,\mu)\in\mathscr{D}} E\bigg[\tilde{U}(\zeta H_{\mu,\nu}(T)) - \zeta H_{\mu,\nu}(T)C \tag{4.15}$$
$$+ \zeta\int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg],$$

show that it has an optimal solution $(\hat{\nu}_\zeta, \hat{\mu}_\zeta) \in \mathscr{D}$ for every $\zeta > 0$ and find $\hat{\pi}(\cdot) \in \mathscr{A}(x)$ and $\hat{\zeta} > 0$ such that the conditions of Remark 4.1 hold. The duality approach to utility maximization problems employed here was implicitly used in Pliska (1986), Karatzas, Lehoczky and Shreve (1987), Cox and Huang (1989) in the case of complete markets, and explicitly in He and Pearson (1991), Karatzas, Lehoczky, Shreve and Xu (1991), Xu and Shreve (1992), Cvitanić and Karatzas (1992) for (incomplete) markets with constraints. We follow Cuoco and Cvitanić (1998), since their methods apply to our dual problem, which is not convex in (ν, μ), but is convex in the (random) variables $H_{\mu,\nu}(T)$, which are bounded in $\mathbf{L}^2(\Omega, \mathscr{F}(T), P)$, uniformly in $(\nu,\mu) \in \mathscr{D}$.

THEOREM 4.1. *For every given $\zeta > 0$, there exists an optimal pair $(\hat{\nu}_\zeta, \hat{\mu}_\zeta) \in \mathscr{D}$ for the dual problem* (4.15).

This theorem is proved exactly as in Cuoco and Cvitanić (1998), Appendix B.

LEMMA 4.2. *The function*

$$\alpha(\zeta) := \tilde{V}(\zeta) + x\zeta, \qquad \zeta \ge 0$$

is Lipschitz-continuous for any given $A(0) < x < C(0)$.

PROOF. Since $D := C - A \in \mathbf{L}^2(\Omega, \mathscr{F}(T), P)$, it follows by BSDEs theory [by analogy with (4.10)] that

$$D(0) := \sup_{(\nu,\mu)\in\mathscr{D}} E\bigg[H_{\mu,\nu}(T)[C-A] - \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg] \tag{4.16}$$
$$< \infty.$$

Note also that

$$\tilde{U}(\zeta H_{\mu,\nu}(T)) = \big(\zeta H_{\mu,\nu}(T)[C-A]\big) \vee 1.$$

We have then, for $\zeta_1 \geq 0$, $\zeta_2 \geq 0$,

$$
\begin{aligned}
E\bigg[&\tilde{U}(\zeta_1 H_{\mu,\nu}(T)) - \zeta_1 H_{\mu,\nu}(T)C + \zeta_1 \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg] \\
&\geq E\bigg[\tilde{U}(\zeta_2 H_{\mu,\nu}(T)) - \zeta_2 H_{\mu,\nu}(T)C + \zeta_2 \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg] \\
&\qquad - E\bigg[H_{\mu,\nu}(T)[2C - A] + \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg]|\zeta_1 - \zeta_2| \\
&\geq \tilde{V}(\zeta_2) - (D(0) + C(0))|\zeta_1 - \zeta_2|.
\end{aligned}
$$

Taking the infimum over $(\nu, \mu) \in \mathscr{D}$ we get $\tilde{V}(\zeta_2) - \tilde{V}(\zeta_1) \leq (D(0) + C(0))|\zeta_1 - \zeta_2|$. Since we can do the same by interchanging the roles of ζ_1 and ζ_2, we are done. □

PROPOSITION 4.1. *For every $A(0) < x < C(0)$, there exists $\hat{\zeta} = \hat{\zeta}_x > 0$ that attains the infimum* $\inf_{\zeta \geq 0} \alpha(\zeta)$.

PROOF. First we show that the infimum cannot be attained at infinity. Suppose that there exists a sequence $\zeta_n \to \infty$ such that $\lim_n \alpha(\zeta_n) \leq 0$. Denote $H_n(T) = H_{\hat{\mu}_{\zeta_n}, \hat{\nu}_{\zeta_n}}(T)$. We have then

$$
\begin{aligned}
x \leq \lim_n E\bigg[&H_n(T)C - \bigg([C - A]H_n(T) \vee \frac{1}{\zeta_n}\bigg) \\
&- \int_0^T H_n(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg].
\end{aligned}
\tag{4.17}
$$

Since

$$
E\bigg[\bigg([C - A]H_n(T) \wedge \frac{1}{\zeta_n}\bigg)\bigg] \leq \frac{1}{\zeta_n} \to 0,
$$

inequality (4.17) implies

$$
x \leq \lim_n E\bigg[H_n(T)A - \int_0^T H_n(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg] \leq A(0),
$$

where the last inequality follows from the representation of $A(0)$ analogous to (4.10). This is in contradiction with $x > A(0)$, and we conclude $\liminf_{\zeta \to \infty} \alpha(\zeta) > 0$. Consequently, being continuous, $\alpha(\zeta)$ either attains its infimum at some $\hat{\zeta} > 0$ or $\alpha(\zeta) \geq \alpha(0) = 0$ for all $\zeta > 0$. Suppose that the latter is true. We have then

$$
\begin{aligned}
x &\geq E\bigg[H_{\mu,\nu}(T)C - \bigg([C - A]H_{\mu,\nu}(T) \vee \frac{1}{\zeta}\bigg) - \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg] \\
&\geq E\bigg[H_{\mu,\nu}(T)C\mathbf{1}_{\{\zeta H_{\mu,\nu}(T)[C-A] \leq 1\}} - \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\bigg]
\end{aligned}
$$

for all $(\nu, \mu) \in \mathscr{D}$. Letting $\zeta \to 0$, we get

$$x \geq E\left[H_{\mu,\nu}(T)C - \int_0^T H_{\mu,\nu}(s)\tilde{g}(s, \nu(s), \mu(s))\,ds\right]$$

and, taking the supremum over $(\nu, \mu) \in \mathscr{D}$,

$$x \geq C(0),$$

a contradiction. □

In the following proposition we identify a candidate for the terminal wealth of an optimal portfolio process, and show that, loosely speaking, its initial wealth in the "optimal shadow market" is equal to x. We denote by Q^C a complement of a set Q.

PROPOSITION 4.2. *Fix an arbitrary $x \in (A(0), C(0))$, and let $\hat{\zeta}$ be the one from Proposition 4.1. Let $(\hat{\nu}, \hat{\mu}) \in \mathscr{D}$ be optimal for the dual problem with value function $\tilde{V}(\hat{\zeta})$, let $\hat{H}(T) = H_{\hat{\mu},\hat{\nu}}(T)$, and introduce the set*

$$Q := \{\hat{\zeta}\hat{H}(T)[C - A] > 1\} \in \mathscr{F}(T). \tag{4.18}$$

There exists then a set $E \in \mathscr{F}(T)$ such that the random variable

$$\hat{X}(T) := C\mathbf{1}_{Q^C} + A\mathbf{1}_Q - (C - A)\mathbf{1}_{E\cap\{\hat{\zeta}\hat{H}(T)[C-A]=1\}} \tag{4.19}$$

satisfies

$$E\left[\hat{H}(T)\hat{X}(T) - \int_0^T \hat{H}(s)\tilde{g}(s, \hat{\nu}(s), \hat{\mu}(s))\,ds\right] = x. \tag{4.20}$$

PROOF. Note that

$$\alpha(\hat{\zeta}) \leq \alpha(\zeta) \leq E\left[\tilde{U}(\zeta\hat{H}(T)) + \zeta\hat{H}(T)C + \zeta\int_0^T \hat{H}(s)\tilde{g}(s, \hat{\nu}(s), \hat{\mu}(s))\right] + \zeta x.$$

It follows that $\hat{\zeta}$ also minimizes the function

$$f(\zeta) := E\left[\tilde{U}(\zeta\hat{H}(T)) + \zeta\hat{H}(T)C + \zeta\int_0^T \hat{H}(s)\tilde{g}(s, \hat{\nu}(s), \hat{\mu}(s))\right] + \zeta x, \qquad \zeta > 0.$$

Take $\varepsilon > 0$. Since $f(\hat{\zeta} + \varepsilon) - f(\hat{\zeta}) \geq 0$, we get

$$x \geq \frac{1}{\varepsilon}E\big[((\hat{\zeta} + \varepsilon)[A - C]\hat{H}(T)) \vee 1\big] - E\big[(\hat{\zeta}[A - C]\hat{H}(T)) \vee 1\big] + E\left[C\hat{H}(T) - \int_0^T \hat{H}(s)\tilde{g}(s, \hat{\nu}(s), \hat{\mu}(s))\,ds\right]$$

$$
\begin{aligned}
&= E\Big[(A-C)\hat{H}(T)\mathbf{1}_{\{\hat{\zeta}\hat{H}(T)[C-A]\ge 1\}} + C\hat{H}(T) \\
&\qquad\qquad - \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\Big] \\
&\quad + \frac{1}{\varepsilon}E\big[([A-C](\hat{\zeta}+\varepsilon)\hat{H}(T)-1)\mathbf{1}_{\{1/(\hat{\zeta}+\varepsilon)<\hat{H}(T)[C-A]\le 1/\hat{\zeta}\}}\Big].
\end{aligned}
$$

By the dominated convergence theorem, the last term tends to 0 as $\varepsilon \to 0$, and we get

$$
\begin{aligned}
x \ge E\Big[&A\hat{H}(T)\mathbf{1}_{\{\hat{\zeta}\hat{H}(T)[C-A]\ge 1\}} + C\hat{H}(T)\mathbf{1}_{\{\hat{\zeta}\hat{H}(T)[C-A]<1\}} \\
&- \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\Big].
\end{aligned}
\tag{4.21}
$$

Similarly, if $\varepsilon < 0$ and close enough to zero, we obtain

$$
\begin{aligned}
x &\le \frac{1}{\varepsilon}E\big[((\hat{\zeta}+\varepsilon)[A-C]\hat{H}(T))\vee 1\big] - E\big[(\hat{\zeta}[A-C]\hat{H}(T))\vee 1\big] \\
&\quad + E\Big[C\hat{H}(T) - \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\Big] \\
&= E\Big[(A-C)\hat{H}(T)\mathbf{1}_{\{(\hat{\zeta}+\varepsilon)\hat{H}(T)[C-A]>1\}} + C\hat{H}(T) \\
&\qquad\qquad - \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\Big] \\
&\quad - \frac{1}{\varepsilon}E\big[(1-\hat{\zeta}[C-A]\hat{H}(T))\mathbf{1}_{\{1/\hat{\zeta}<\hat{H}(T)[C-A]\le 1/(\hat{\zeta}+\varepsilon)\}}\Big] \\
&\le E\Big[(A-C)\hat{H}(T)\mathbf{1}_{\{(\hat{\zeta}+\varepsilon)\hat{H}(T)[C-A]>1\}} + C\hat{H}(T) \\
&\qquad\qquad - \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\Big].
\end{aligned}
$$

Letting $\varepsilon \to 0$, we obtain

$$
\begin{aligned}
x \le E\Big[&A\hat{H}(T)\mathbf{1}_{\{\hat{\zeta}\hat{H}(T)[C-A]>1\}} + C\hat{H}(T)\mathbf{1}_{\{\hat{\zeta}\hat{H}(T)[C-A]\le 1\}} \\
&- \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\Big].
\end{aligned}
\tag{4.22}
$$

The difference between the right-hand sides of (4.22) and (4.21) is equal to

$$
E\big[(C-A)\hat{H}(T)\mathbf{1}_{\{\hat{\zeta}\hat{H}(T)[C-A]=1\}}\big] = \frac{1}{\hat{\zeta}}P\big[\hat{\zeta}\hat{H}(T)(C-A)=1\big].
$$

Since for any number $0 \le y \le (1/\hat{\zeta})P[\hat{\zeta}\hat{H}(T)(C-A)=1]$ we can find a set $E \in \mathscr{F}(T)$ such that

$$y = \frac{1}{\hat{\zeta}} P\big[E \cap \{\hat{\zeta}\hat{H}(T)(C-A)=1\}\big] = E\big[\hat{H}(T)[C-A]\mathbf{1}_{E\cap\{\hat{\zeta}\hat{H}(T)(C-A)=1\}}\big],$$

(4.20) follows from (4.21) and (4.22). □

We now want to show that the random variable of (4.19) can be replicated starting with initial wealth x and using some admissible portfolio $\hat{\pi}(\cdot)$.

ASSUMPTION 4.2. For any given $x \in (A(0), C(0))$, we have

$$P[\hat{\zeta}\hat{H}(T)(C-A)=1]=0$$

REMARK 4.2. This assumption is introduced for simplicity only. It can be avoided by using nonsmooth optimization techniques, as in Cvitanić (1998).

PROPOSITION 4.3. *Let $x \in (A(0), C(0))$ be given. Under Assumption 4.2, there exists a portfolio $\hat{\pi}(\cdot) \in \mathscr{A}(x)$, such that*

$$X^{x,\hat{\pi}}(T) = \hat{X}(T) = C\mathbf{1}_{Q^C} + A\mathbf{1}_Q,$$

the random variable of (4.19).

PROOF. From [EPQ] and Proposition 4.2, it is sufficient to show that

$$\begin{aligned} &E\left[H_{\mu,\nu}(T)\hat{X}(T) - \int_0^T H_{\mu,\nu}(s)\tilde{g}(s,\nu(s),\mu(s))\,ds\right] \\ &\qquad \le E\left[\hat{H}(T)\hat{X}(T) - \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\right] = x \end{aligned} \tag{4.23}$$

for all $(\nu,\mu) \in \mathscr{D}$ [i.e, that the supremum over $\mathscr{D}$ of the left-hand side is attained at $(\hat{\nu},\hat{\mu})$]. Fix $\varepsilon \in (0,1)$ and $(\nu,\mu) \in \mathscr{D}$. Define processes (suppressing dependence on t),

$$\begin{aligned} &G_\varepsilon := (1-\varepsilon)\hat{H} + \varepsilon H_{\mu,\nu}, \qquad \nu_\varepsilon := G_\varepsilon^{-1}((1-\varepsilon)\hat{\nu}\hat{H} + \varepsilon\nu H_{\mu,\nu}), \\ &\mu_\varepsilon := G_\varepsilon^{-1}((1-\varepsilon)\hat{\mu}\hat{H} + \varepsilon\mu H_{\mu,\nu}). \end{aligned}$$

It is easy to check (by Itô's rule) that

$$G_\varepsilon(\cdot) = H_{\nu_\varepsilon,\mu_\varepsilon}(\cdot).$$

By optimality of $(\hat{\nu},\hat{\mu})$ in the dual problem, since $(\nu_\varepsilon,\mu_\varepsilon) \in \mathscr{D}$, and by convexity of $\tilde{g}(\cdot)$, we get

$$\begin{aligned} 0 \le \frac{1}{\varepsilon} E\left[(\hat{\zeta}G_\varepsilon(T)[C-A]) \vee 1 - \hat{\zeta}G_\varepsilon(T)C + \int_0^T G_\varepsilon(s)\tilde{g}(s,\nu_\varepsilon(s),\mu_\varepsilon(s))\,ds\right] \\ - \frac{1}{\varepsilon} E\left[(\hat{\zeta}\hat{H}(T)[C-A]) \vee 1 - \hat{\zeta}\hat{H}(T)C + \int_0^T \hat{H}(s)\tilde{g}(s,\hat{\nu}(s),\hat{\mu}(s))\,ds\right] \end{aligned}$$

$$
\begin{aligned}
&\le E\big[\hat\zeta C(\hat H(T) - H_{\mu,\nu}(T)) \\
&\qquad\quad + \hat\zeta[C-A](H_{\mu,\nu}(T) - \hat H(T))\mathbf{1}_{\{\hat\zeta\hat H(T)[C-A]>1,\,\hat\zeta G_\varepsilon(T)[C-A]>1\}}\big] \\
&\quad + E\bigg[\hat\zeta\bigg(\int_0^T H_{\mu,\nu}(s)\tilde g(s,\nu(s),\mu(s))\,ds - \int_0^T \hat H(s)\tilde g(s,\hat\nu(s),\hat\mu(s))\,ds\bigg)\bigg] \\
&\quad + \frac{1}{\varepsilon}E\big[(\hat\zeta[C-A]G_\varepsilon(T) - 1)\mathbf{1}_{\{\hat\zeta\hat H(T)[C-A]\le 1,\,\hat\zeta G_\varepsilon(T)[C-A]>1\}}\big] \\
&\quad + \frac{1}{\varepsilon}E\big[(1 - \hat\zeta[C-A]\hat H(T))\mathbf{1}_{\{\hat\zeta\hat H(T)[C-A]>1,\,\hat\zeta G_\varepsilon(T)[C-A]\le 1\}}\big] \\
&\le \text{(since this last term is nonpositive)} \\
&E\big[\hat\zeta C(\hat H(T) - H_{\mu,\nu}(T)) \\
&\qquad + \hat\zeta[C-A](H_{\mu,\nu}(T) - \hat H(T))\mathbf{1}_{\{\hat\zeta\hat H(T)[C-A]>1,\,\hat\zeta G_\varepsilon(T)[C-A]>1\}}\big] \\
&\quad + E\bigg[\hat\zeta\bigg(\int_0^T H_{\mu,\nu}(s)\tilde g(s,\nu(s),\mu(s))\,ds - \int_0^T \hat H(s)\tilde g(s,\hat\nu(s),\hat\mu(s))\,ds\bigg)\bigg] \\
&\quad + E\big[\hat\zeta[C-A](H_{\mu,\nu}(T) - \hat H(T))\mathbf{1}_{\{\hat\zeta\hat H(T)[C-A]\le 1,\,\hat\zeta G_\varepsilon(T)[C-A]>1\}}\big].
\end{aligned}
$$

Letting $\varepsilon \to 0$ and invoking Assumption 4.2, we see that the last term tends to zero, and we complete the proof of (4.23). □

The following theorem is now a consequence of Propositions 4.2 and 4.3 and Remark 4.1.

THEOREM 4.2. *Under Assumption* 4.2 *and given initial wealth* $x \in (A(0), C(0))$, *there exists an optimal portfolio* $\hat\pi \in \mathscr{A}(x)$ *for the problem* (4.7) *under the dynamics* (4.1). *It can be taken as the portfolio that replicates, at time* $t = T$, *the value* $\hat X(T)$ *of* (4.19), *with* $E = \varnothing$.

EXAMPLE 4.1. *Price pressure.* Let $d = 1$, $C = 1$, $A = 0$, $r(\cdot) \equiv 0$, $\sigma(\cdot) \equiv 1$, $b_1(\cdot) = b > 0$ and $0 < \varepsilon < b$. Suppose also that the wealth dynamics are given by

$$
dX(t) = \big(\pi(t)b - |\pi(t)|\varepsilon\big)\,dt + \pi(t)\,dW(t), \qquad X(0) = x \in (0,1). \tag{4.24}
$$

This can be interpreted as a "large investor" model in which buying the risky asset depresses its expected return, while shorting it increases the expected return [see Cuoco and Cvitanić (1998)]. For $\varepsilon = 0$, this example is a special case of Example 2.1 with $k = 1$. It is seen from (2.43) [with $\pi_C(\cdot) \equiv 0$], that the optimal portfolio in that case depends only on the sign of the drift if the drift is constant, and it is then nonnegative if the drift is nonnegative. Since we have $b - \varepsilon > 0$, we should expect from the dynamics (4.24) that the corresponding optimal portfolio will still be given by (2.43), which for our values of the parameters becomes

$$\hat{\pi}(t) = \frac{1}{\sqrt{T-t}}\varphi\big(\Phi^{-1}(\hat{X}(t))\big). \tag{4.25}$$

We justify this rigorously by looking at the dual problem. It is easily seen that

$$\tilde{g}(\omega, t, \nu, \mu) = 0 \quad \text{for } \mu = 0, |\nu + b| \le \varepsilon$$

and $\tilde{g}(\omega, t, \nu, \mu) = \infty$ otherwise. Thus, the dual problem (4.15) becomes

$$\tilde{V}(\zeta) = \inf_{\nu \in \mathscr{D}_\varepsilon} E\big[(1 - \zeta Z_\nu(T))^+\big],$$

where $\mathscr{D}_\varepsilon$ is the set of progressively measurable processes $\nu(\cdot)$ such that $|\nu(\cdot)+b| \le \varepsilon$, a.s. We see that we are minimizing a nonincreasing convex functional of the value at time T of the martingale $Z_\nu(\cdot)$. By Theorem 5.2 in Xu and Shreve (1992) we conclude that the optimal solution is given by the process $\hat{\nu}(\cdot) \equiv \varepsilon - b$. Therefore, the optimal terminal wealth $\hat{X}(T)$ of (4.19) is the same as the optimal terminal wealth for the problem of Example 2.1 with $\theta_0(\cdot) \equiv b - \varepsilon$, and the same feedback form of (2.43), nameley (4.25), is valid.

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