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STOCHASTIC EULER EQUATIONS ON THE TORUS

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Existence of solutions for stochastic Euler equations is proved for the two-dimensional case. The laws of solutions of stochastic Navier–Stokes equations are shown to be relatively compact and all limit points (as the viscosity converges to zero) are laws of solutions to stochastic Euler equations.

1. Introduction. We consider the stochastic Euler equation

$$du = -\langle u, \nabla\rangle u\, dt + f(t, u)\, dt + g(t, u)\, dw_t \tag{1.1}$$

in two dimensions with periodic boundary condition, with the incompressibility condition $\operatorname{div} u = 0$. This equation is the limiting case ($\nu = 0$) of the stochastic Navier–Stokes equation

$$du = \nu\Delta u\, dt - \langle u, \nabla\rangle u\, dt + f(t, u)\, dt + g(t, u)\, dw_t. \tag{1.2}$$

Due to the presence of the Laplace operator, the latter is easier to tackle. Using the crucial orthogonality property of the nonlinear term $(\langle u, \nabla\rangle u, u) = 0$ one can get an (a priori) estimate of the L^2-norm of the gradient of u, which is fundamental in the proofs of the existence result. To have such an estimate for Euler, we restrict ourselves to periodic boundary conditions in two dimensions where an additional property of the nonlinear term can be exploited [see (2.2) below].

The main result (Theorem 5.1) is the existence of a solution to (1.1) under quite general conditions on the force and noise terms f, g. The solution lives on a Loeb space $\mathbf{\Omega}$ with a prescribed Wiener process w (constructed using nonstandard analysis) and the techniques are extensions of those developed in [2].

From the main theorem, the existence of statistical solutions follows easily (Theorem 6.2).

In the final section we discuss the limiting behavior of the solutions u^ν of the stochastic Navier–Stokes equations with viscosity ν, as $\nu \to 0$. If f, g are suitably Lipschitz, then for each $\nu > 0$ there is a unique solution u^ν; we show that the laws of solutions with $1 \geq \nu > 0$ are relatively compact and each limit point as $\nu \to 0$ is the law of a solution to the stochastic Euler equation (1.1) living on $\mathbf{\Omega}$ with the same driving Wiener process w.

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2. Preliminaries. We say that a function u: $[0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies *periodic boundary conditions* if there is an $L > 0$ such that $u(t, x_1 + kL, x_2 + mL) = u(t, x_1, x_2)$, for all $x_1, x_2 \in \mathbb{R}$, $t \in [0, T]$, $k, m \in \mathbb{Z}$. We take $D = [0, L] \times [0, L]$ and follow the usual construction: let $\mathbf{H}$ be the closure of

$$\mathscr{V} = \{u|D: u \in C^\infty(\mathbb{R}^2, \mathbb{R}^2): \ u \text{ periodic and } \operatorname{div} u = 0\}$$

in the $L^2(D)$ norm. Then $\mathbf{H}$ is a separable Hilbert space with the scalar product

$$(u, v) = \sum_{i=1}^{2} \int_D u^i(x) v^i(x)\,\mathrm{d}x$$

and the norm $|u|^2 = (u, u)$. The Laplace operator $\Delta = \sum_{i=1}^{2}(\partial^2/\partial x_i^2)$ is densely defined in $\mathbf{H}$ and we take A to be the self-adjoint extension of $-\Delta$ in $\mathbf{H}$. It is a nonnegative operator which gives an orthonormal basis of $\mathbf{H}$ consisting of eigenvectors $\{e_k\}$ of A with eigenvalues $0 < \lambda_k \nearrow \infty$. Let $\mathbf{V}$ be the domain of $A^{1/2}$. We equip $\mathbf{V}$ with the scalar product

$$((u, v)) = \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)$$

and associated norm $\|u\|$.

We define a trilinear form b by

$$b(u, v, z) = \sum_{i,j=1}^{2} \int_D u^i(x) \frac{\partial v^j(x)}{\partial x_i} z^j(x)\,\mathrm{d}x.$$

It satisfies, for $u, v, w \in \mathbf{V}$, the fundamental relation $b(u, v, z) = -b(u, z, v)$ which implies $b(u, v, v) = 0$, and the inequality

$$b(u, v, z) \le c|u|_\alpha |v|_\beta |z|_\gamma,$$

where $|u|_\rho = |A^{\rho/2} u|$, $\alpha + \beta + \gamma > 2$, $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 0$. In addition,

$$b(u, v, z) \le c\|u\|^{1/2}|u|^{1/2}\,\|v\|^{1/2}|v|^{1/2}\,\|z\|. \tag{2.1}$$

For any $u, v \in \mathbf{V}$, the mapping $z \to b(u, v, z)$ defines a continuous functional on $\mathbf{V}$ which gives rise to the bilinear operator B: $\mathbf{V} \times \mathbf{V} \to \mathbf{V}'$ determined by

$$(B(u, v), z) = b(u, v, z).$$

In the two-dimensional space-periodic case, the form b has the additional property,

$$b(u, u, Au) = 0 \tag{2.2}$$

for $u \in D(A)$. See [4] for details.

3. Nonstandard preliminaries. We gather some preliminary facts we shall need in what follows. The proofs are omitted and can be found in [2]. For an introduction to nonstandard methods, see also [1], [3].

Take the nonstandard extension ${}^*\mathbf{H}$ of $\mathbf{H}$. Denote the *extension of $\{e_k\}_{k\in\mathbb{N}}$ by $\{E_k\}_{k\in{}^*\mathbb{N}}$. Fix $N \in {}^*\mathbb{N}\setminus\mathbb{N}$ and let $\mathbf{H}_N$ be the subspace of ${}^*\mathbf{H}$ spanned by $\{E_1,\dots,E_N\}$. By convention, elements of $\mathbf{H}_N$ will be denoted by capital letters U, V, and the projection from ${}^*\mathbf{H}$ **to** $\mathbf{H}_N$ is denoted by Pr_N.

For $U \in \mathbf{H}_N$ we write $U = \sum_{k=1}^{N} U_k E_k$, so that $|U|^2 = \sum_{k=1}^{N} U_k^2$. If $|U|$ is finite we can define $u = {}^\circ U$ by putting $u_k = \mathrm{st}\, U_k$ for finite k and then $u = \sum_{k=1}^{\infty} u_k e_k$. It is easy to see that $u \in \mathbf{H}$ and u is the standard part of U in the weak topology of $\mathbf{H}$. This produre for obtaining standard elements of $\mathbf{H}$ (or $\mathbf{V}$, etc.) is at the heart of later proofs, so at the referee's suggestion we include a summary of some of the basic results along these lines. First, define the spectrum of spaces $\mathbf{H}^r$ for $r \geq 0$ by

$$\mathbf{H}^r = \{u \in \mathbf{H}\colon |u|_r < \infty\},$$

where $|u|_r^2 = \sum_{k=1}^{\infty} \lambda_k^r u_k^2$. Then define $\mathbf{H}^{-r}$ to be the dual of $\mathbf{H}^r$. Note that $\mathbf{H}^1 = \mathbf{V}$ and $\mathbf{H}^{-1} = \mathbf{V}'$.

Let $U \in \mathbf{H}_N$ and $u \in \mathbf{H}^r$. We write $U \approx u$ in $\mathbf{H}^r$ to mean that U is nearstandard to u in the strong topology of $\mathbf{H}^r$, and $U \approx_{\mathrm{w}} u$ in $\mathbf{H}^r$ denotes that U is nearstandard to u in the weak topology of $\mathbf{H}^r$. In each case, u is the *standard part* of U (in the topology concerned), written $u = {}^\circ U$, and is given as above by $u = (u_k)$ where $u_k = \mathrm{st}\, U_k$ for finite k. Proofs of the following may be found in [2].

First, we consider strong topologies.

PROPOSITION 3.1 ([2], Proposition 2.7.1). *Let $U \in \mathbf{H}_N$, $u \in \mathbf{H}^r$ and $r, q \in \mathbb{R}$.*

(a) *$U \approx u$ in $\mathbf{H}^r$ iff*

$$\sum_{k=1}^{N} \lambda_k^r (U_k - {}^*u_k)^2 \approx 0$$

(hence $u_k = {}^\circ U_k$ for finite k if $U \approx u$).

(b) *U is nearstandard in $\mathbf{H}^r$ iff*

$$\sum_{k=1}^{\infty} \lambda_k^r\, {}^\circ U_k^2 \approx \sum_{k=1}^{N} \lambda_k^r U_k^2 < \infty.$$

(c) *If $q < r$ and $|U|_r < \infty$, then U is nearstandard in $\mathbf{H}^q$.*

Note that $|U|_r < \infty$ is not sufficient for U to be strongly $\mathbf{H}^r$-nearstandard (e.g., take $U_N = 1$ and $U_k = 0$ for $k < N$). The situation is different for the weak topology.

PROPOSITION 3.2 ([2], Proposition 2.7.2). *Let* $U \in \mathbf{H}_N$ *and* $u \in \mathbf{H}^r$ (*for* $r \in \mathbb{R}$).

(a) $U \approx_{\mathrm{w}} u$ *in* $\mathbf{H}^r$ *iff* $(U, {}^*v) \approx (u, v)$ *for all* $v \in \mathbf{H}^{-r}$ (*and then* $u_k = {}^\circ U_k$ *for all finite* k).

(b) *If* $|U|_r < \infty$, *then* U *is weakly nearstandard in* $\mathbf{H}^r$ *and* $|{}^\circ(U)|_r \le {}^\circ|U|_r$.

The following lemma gives a property of the nonlinear term which is crucial for the proof of the main theorem.

LEMMA 3.3 ([2], Lemma 2.7.7). *If* $\|U\|$ *and* $\|V\|$ *are finite with* $u = {}^\circ U$, $v = {}^\circ V$ *and* $z \in \operatorname{dom} A$ *then*

$$ {}^*b(U, V, {}^*z) \approx b(u, v, z). $$

The next two results are the key to obtaining standard integral equalities from internal ("nonstandard") ones.

The symbol $\mathrm{d}_{\mathrm{L}}\tau$ denotes Loeb integration with respect to the Loeb measure Λ_{L} (where Λ is *Lebesgue measure) and $\mathrm{d}\tau$ denotes *Lebesgue integration (i.e., with respect to Λ).

PROPOSITION 3.4 ([2], Proposition 3.5.1). *Let* $X\colon {}^*[0, T] \to \mathbf{H}_N$ *with*

$$ \int_0^T |X(\tau)|_r \,\mathrm{d}\tau < \infty $$

for some $r \in \mathbb{R}$, *and define* $x\colon {}^*[0, T] \to \mathbf{H}^r$ *by* $x(\tau) = {}^\circ X(\tau)$.

Then:

(a) x *is Bochner integrable on* ${}^*[0, T]$ (*with respect to* $\mathrm{d}_{\mathrm{L}}\tau$).

(b) *If* $(X(\tau), {}^*v)$ *is* S-*integrable on* ${}^*[0, T]$ *for all* v *from a total subset of* $\mathbf{H}^{-r}$, *then*

$$ \int_0^T x(\tau)\,\mathrm{d}_{\mathrm{L}}\tau = {}^\circ\!\int_0^T X(\tau)\,\mathrm{d}\tau. $$

In particular, this is true if $|X(\tau)|_r$ *is* S-*integrable*.

For our future needs we give two tailor-made "lifting" theorems.

We consider the set $K_m = \{u\colon \|u\| \le m\}$ equipped with the strong topology of $\mathbf{H}$.

PROPOSITION 3.5 ([2], Proposition 3.5.2). *Suppose that* $f\colon [0, \infty) \times \mathbf{V} \to \mathbf{V}'$ *is jointly measurable with*

$$ f(t, \cdot) \in C(K_m, \mathbf{V}'_{\mathrm{weak}}) \quad \textit{for all } m $$

and

$$ |f(t, u)|_{\mathbf{V}'} \le a(t)(1 + \|u\|), \tag{3.1} $$

where $a \in L^1(0, T)$ *for all* T.

Let

$$F(\tau, U) = \mathrm{Pr}_N {}^*f(\tau, U).$$

Then for a.a. finite τ (with respect to $\mathrm{d_L}\tau$), for all U with $\|U\| < \infty$, $F(\tau, U)$ is weakly nearstandard in $\mathbf{V}'$ and

$$^\circ F(\tau, U) = f(^\circ\tau, {}^\circ U).$$

The next result is a variation on Proposition 3.5.3 of [2].

THEOREM 3.6. *Suppose that f and F are as in Proposition 3.5. Suppose further that U: $^*[0, T] \to \mathbf{H}_N$ is an internal measurable function with*

$$\sup_{\tau \le T} \|U(\tau)\| < \infty.$$

Define v: $^*[0, T] \to \mathbf{H}$ by $v(\tau) = {}^\circ U(\tau)$.

Then:

(a) *$v(\tau) \in \mathbf{V}$ for all $\tau \in {}^*[0, T]$.*

(b) *The functions $B(v(\tau), v(\tau))$ and $f(\tau, v(\tau))$ are Bochner integrable in $\mathbf{V}'$ with respect to $\mathrm{d_L}\tau$ on $^*[0, T]$ and*

$$\int_0^T B(v(\tau), v(\tau))\,\mathrm{d_L}\tau = {}^\circ\!\int_0^T B_N(U(\tau), U(\tau))\,\mathrm{d}\tau,$$

$$\int_0^T f(^\circ\tau, v(\tau))\,\mathrm{d_L}\tau = {}^\circ\!\int_0^T F(\tau, U(\tau))\,\mathrm{d}\tau,$$

where $B_N(U, U) = \mathrm{Pr}_N {}^*B(U, U)$.

(c) *If $U_k(\tau)$ is* S-*continuous for each finite k, then $v(\sigma) = v(\tau)$ whenever $\sigma \approx \tau$ and writing $u(t) = v(t)$ for real t, the above equalities take the form*

$$\int_0^T B(u(t), u(t))\,\mathrm{d}t = {}^\circ\!\int_0^T B_N(U(\tau), U(\tau))\,\mathrm{d}\tau,$$

$$\int_0^T f(t, u(t))\,\mathrm{d}t = {}^\circ\!\int_0^T F(\tau, U(\tau))\,\mathrm{d}\tau.$$

(d) *$u \in L^\infty(0, T; \mathbf{V}) \cap C(0, T; \mathbf{H}) \cap C(0, T; \mathbf{V}_{\mathrm{weak}})$.*

PROOF. The proof makes crucial use of Lemma 3.3 and Propositions 3.4, 3.5. We indicate only the points where the proof differs from that of [2], Proposition 3.5.3.

For (b), we can see that $|B_N(U(\tau), U(\tau))|_{\mathbf{V}'}$ is S-integrable since $|B_N(U(\tau), U(\tau))|_{\mathbf{V}'} \le \|U(\tau)\|^2 < c(T) < \infty$, using the property (2.1) of b. The rest follows as before. For F we have

$$|F(\tau, U(\tau))|_{\mathbf{V}'} \le {}^*a(\tau)(1 + \|U(\tau)\|) \le {}^*a(\tau)(1 + c(T))$$

and so $|F(\tau, U(\tau))|_{\mathbf{V}'}$ is S-integrable [since $^*a(\tau)$ is].

For (d), the weak continuity of u in $\mathbf{V}$ follows because $\|u\|$ is bounded and each u_k is continuous. Again using the boundedness of $\|u\|$, this means that u is strongly continuous in $\mathbf{H}$ (in fact in $\mathbf{H}^r$ for any $r < 1$). □

We recall the construction of a Wiener process on $\mathbf{H}$ and the stochastic integral that was given in [2], Section 3.6.

Let Q be a fixed nuclear operator and let (d_i) be an orthonormal family of eigenvectors of Q with eigenvalues γ_i, so

$$\operatorname{tr} Q = \sum_{i=1}^{\infty} \gamma_i < \infty.$$

For each n let $Q_n = \operatorname{Pr}_n Q \operatorname{Pr}_n$. It is routine to construct a standard Wiener process in $\mathbf{H}_n$ with covariance Q_n; simply take

$$w^{(n)} = \sum_{i=1}^{\infty} \beta_i \operatorname{Pr}_n d_i,$$

where β_i are independent real Wiener processes with $E(\beta_i^2) = \gamma_i t$.

Now take any nonstandard (internal) filtered probability space

$$\mathbf{\Omega}_0 = (\Omega, \mathscr{A}, (\mathscr{A}_\tau)_{\tau \geq 0}, \Pi)$$

carrying an internal Wiener process $W(\tau) \in \mathbf{H}_N$ adapted to $(\mathscr{A}_\tau)_{\tau \geq 0}$, with covariance Q_N. [A canonical candidate is provided by taking $\Omega = {}^*C(0, \infty; \mathbf{H}_N)$ and the Wiener measure induced by $W^{(N)}$, the process with covariance Q_N; in this case we would have $\mathscr{A} = {}^*\mathscr{B}(\Omega)$ and $\mathscr{A}_\tau = {}^*\sigma\{W(\tau')\colon \tau' \leq \tau\}$]. In the following, however, we do not assume that we are working with the canonical Wiener space.)

We now show how the Loeb space $\mathbf{\Omega} = (\Omega, \mathrm{L}(\mathscr{A}), \Pi_{\mathrm{L}})$ can be equipped with a standard filtration and $W(\tau)$ induces a Wiener process in $\mathbf{H}$ with covariance Q.

Let $P = \Pi_{\mathrm{L}}$ and $\mathscr{F} = \mathrm{L}(\mathscr{A})$. Denote by $\mathscr{N}$ the family of P-null sets. A right continuous filtration is defined by

$$\mathscr{F}_t = \bigcap_{t <^\circ \tau} \sigma(\mathscr{A}_\tau) \vee \mathscr{N}.$$

Then we have the following theorem

THEOREM 3.7 ([2], Theorem 3.6.1). *For P-a.a. ω, $W(\tau)$ is $\mathbf{H}$-near-standard and* S-*continuous in $|\cdot|$ for all finite τ. The formula $w(t) = \operatorname{st}_{\mathbf{H}} W(\tau)$ (for $\tau \approx t$) defines a standard Wiener process on $\mathbf{\Omega}$ with values in* $\mathbf{H}$, *adapted to* $\mathscr{F}_t$, *and with covariance Q.*

For the stochastic integral, we have Theorem 3.8.

THEOREM 3.8 ([2] Theorem 3.6.2). *Suppose that $T < \infty$ and*

$$G\colon {}^*[0, T] \times \Omega \to {}^*L(\mathbf{H}_N, \mathbf{H}_N)$$

is internal, **measurable*, *and adapted to the internal filtration* $(\mathscr{A}_\tau)_{\tau\geq 0}$. *Assume that*:

(i) $E\left(\int_0^T |G(\tau)|^2_{\mathbf{H}_N,\mathbf{H}_N}\,\mathrm{d}\tau\right) < \infty$;

(ii) *for a.a.* ω, $|G(\cdot,\omega)|^2_{\mathbf{H}_N,\mathbf{H}_N}$ *is* S-*integrable on* $^*[0,T]$.

Put

$$Y(\tau) = \int_0^\tau G(\sigma)\,\mathrm{d}W(\sigma),$$

the internal Itô integral in $\mathbf{H}_N$.

Then:

(a) $|Y(\tau)|$ *is nearstandard in* $\mathbf{H}$ *for all* $\tau \in {}^*[0,T]$, *for a.a.* ω;

(b) $^\circ Y(\tau) = L^2\!-\!\lim {}^\circ Y^{(m)}(\tau)$ *uniformly in* τ (*the limit involving the* $\mathbf{H}$ *norm*) *where*

$$Y^{(m)}(\tau) = \int_0^\tau G(\sigma)\,\mathrm{d}W^{(m)}(\sigma);$$

(c) $Y(\tau)$ *is* S-*continuous in* $|\cdot|$ *for a.a.* ω.

For G and Y as in the above theorem, we can put $y(t) = {}^\circ Y(\tau)$ for $\tau \approx t$ and y is a continuous $\mathbf{H}$-valued process.

For our applications we will have $G(\tau,\omega)$ that is a lifting of a standard process $g\colon [0,T]\times\Omega \to L(\mathbf{H},\mathbf{H})$ in the following sense.

DEFINITION 3.9. The internal process $G(\tau,\omega)$ as above is a *lifting* of g if $^\circ G(\tau,\omega) = g(^\circ\tau,\omega)$ for a.a. (τ,ω), the standard part taken in the sense described below.

DEFINITION 3.10. Suppose that $G \in {}^*L(\mathbf{H}_N,\mathbf{H}_N)$. This means that $G\mathrm{Pr}_N \in {}^*L(\mathbf{H},\mathbf{H})$. Using the well-defined notion of weak nearstandardness ($\approx_{\mathrm{w}}$) for $^*L(\mathbf{H},\mathbf{H})$, if $g \in L(\mathbf{H},\mathbf{H})$ we define

$$G \approx_{\mathrm{w}} g \iff G\,\mathrm{Pr}_N \approx_{\mathrm{w}} g.$$

We write $^\circ G = g$ and we say that G is *weakly nearstandard* to $g \in L(\mathbf{H},\mathbf{H})$.

We have the following proposition.

PROPOSITION 3.11 ([2], Proposition 2.7.6). *Let* $G \in {}^*L(\mathbf{H}_N,\mathbf{H}_N)$ *with* $|G|_{\mathbf{H}_N,\mathbf{H}_N} < \infty$. *Then* G *is weakly nearstandard and*

$$|^\circ G|_{\mathbf{H}_N,\mathbf{H}_N} \leq {}^\circ|G|_{\mathbf{H}_N,\mathbf{H}_N}.$$

THEOREM 3.12 ([2], Theorem 3.6.4). *Suppose that* G *and* $Y = \int G\,\mathrm{d}W$ *are as in Theorem* 3.8, *and* $y = {}^\circ Y$ *as above. If* G *is a lifting of an adapted process* $g\colon [0,T]\times\Omega \to L(\mathbf{H},\mathbf{H})$, *then*

$$y(t) = \int_0^t g(s)\,\mathrm{d}w(s).$$

4. Solutions of the deterministic Euler equations. We give here a proof of existence of the deterministic Euler equation to illustrate, in a simple situation, the method used in the main existence theorem proved in the next section.

The Euler equation understood as an evolution equation in $\mathbf{V}'$ takes the form

$$\frac{\partial u}{\partial t} + B(u, u) = f(t, u). \tag{4.1}$$

THEOREM 4.1. *Suppose that the forcing term* $f\colon [0, \infty) \times \mathbf{V} \to \mathbf{V}'$ *is jointly measurable with*

$$f(t, \cdot) \in C(K_m, \mathbf{V}'_{\mathrm{weak}}) \quad \textit{for all } m$$

and

$$|f(t, u)|_{\mathbf{V}'} \le a(t)(1 + \|u\|), \tag{4.2}$$

where $a \in L^1(0, T)$. *Then for each* $u_0 \in \mathbf{V}$ *there exists a solution to* (4.1) *satisfying*

$$\sup_{t \le T} \|u(t)\| < c(T),$$

where $c(T)$ *is a finite constant depending on* T (*and* f).

PROOF. The corresponding Galerkin equation for $U(\tau)$ in $\mathbf{H}_N$ has the form

$$\begin{aligned} \mathrm{d}U(\tau) &= [-B_N(U(\tau), U(\tau)) + F(\tau, U(\tau))]\,\mathrm{d}\tau, \\ U(0) &= \mathrm{Pr}_N{}^*u_0, \end{aligned} \tag{4.3}$$

where $B_N(U, V) = \mathrm{Pr}_N{}^*B(U, V)$ and $F = \mathrm{Pr}_N{}^*f$. Then we have, using $b(u, u, Au) = 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\|U(\tau)\|^2 &= ((F(\tau, U(\tau)), U(\tau))) \\ &\le {}^*a(\tau)(1 + \|U(\tau)\|)\|U(\tau)\| \\ &\le 2{}^*a(\tau)(1 + \|U(\tau)\|^2). \end{aligned}$$

From this, Gronwall's lemma gives

$$\|U(\tau)\| \le c(T), \qquad 0 \le \tau \le T$$

with $c(T)$ finite for finite T. So $U(\tau)$ is nearstandard in the strong topology of $\mathbf{H}$ and the weak topology of $\mathbf{V}$, which gives ${}^\circ U(\tau) \in \mathbf{V}$ for all finite τ.

We can now apply Theorem 3.6 above. Writing (4.3) in integral form it is easily seen that the function $U(\tau)$ is S-continuous in $\mathbf{H}$. So define a standard function

$$u\colon [0, \infty) \to \mathbf{H}$$

by $u(t) = {}^\circ U(\tau)$ for every $\tau \approx t$, and then $u(t)$ is strongly continuous in $\mathbf{H}$ and weakly continuous in $\mathbf{V}$.

Theorem 3.6 now gives immediately that

$$u(s) = u_0 - \int_0^s B(u(t), u(t))\,dt + \int_0^s f(t, u(t))\,dt$$

for all $s \le T$. □

5. Stochastic Euler equations. The following is the main existence theorem for the stochastic Euler equation. Recall that the set K_m occuring in the statement was defined above by $K_m = \{v\colon \|v\| \le m\} \subseteq \mathbf{V}$ with the strong topology of $\mathbf{H}$. (Note that continuity on each K_m is weaker than continuity on $\mathbf{V}$ in either the $\mathbf{H}$-norm or the weak topology of $\mathbf{V}$.)

The property (2.2) that holds for periodic boundary conditions is crucial in the estimate of $\|U(\tau)\|^2$ in the proof since it allows us to eliminate the form b.

THEOREM 5.1. *Assume $n = 2$ and periodic boundary conditions. Suppose $u_0 \in \mathbf{V}$ and*

$$f\colon [0, \infty) \times \mathbf{V} \to \mathbf{V}, \qquad g\colon [0, \infty) \times \mathbf{V} \to L(\mathbf{H}, \mathbf{V})$$

are jointly measurable with:

(i) $f(t, \cdot) \in C(K_m, \mathbf{V}'_{\text{weak}})$ *for all* m;
(ii) $g(t, \cdot) \in C(K_m, L(\mathbf{H}, \mathbf{H})_{\text{weak}})$ *for all* m;
(iii) $\|f(t, u)\| \le a_1(t)(1 + \|u\|)$, *where* $a_1 \in L^1(0, T)$ *for all* T;
(iv) $|g(t, u)|_{\mathbf{H},\mathbf{V}} \le a_2(t)(1 + \|u\|)$ *where* $a_2 \in L^2(0, T)$ *for all* T.

Then the equation

$$(5.1)\qquad \begin{aligned}(u(t), v) = (u(0), v) &- \int_0^t b(u(t), u(t), v)\,\mathrm{d}t \\ &+ \int_0^t (f(t, u(t)), v)\,\mathrm{d}t + \int_0^t (g(t, u(t)), v)\,\mathrm{d}w_t,\end{aligned}$$

for all $v \in \mathbf{V}$, which is (1.1) *in integral form, has a solution u with a.a. trajectories in $C(0, T; \mathbf{V}_{\text{weak}}) \cap L^\infty(0, T; \mathbf{V})$ for all T, and satisfying*

$$(5.2)\qquad E\Big(\sup_{\tau \le T} \|u(\tau)\|^2\Big) < \infty.$$

PROOF. Let $U(\tau)$ be an internal solution to

$$(5.3)\qquad \begin{aligned}&\mathrm{d}U(\tau) = [-\overline{B}(U(\tau), U(\tau)) + F(\tau, U(\tau))]\,\mathrm{d}\tau + G(\tau, U(\tau))\,\mathrm{d}W(\tau),\\ &U(0) = \mathrm{Pr}_N {}^*u_0,\end{aligned}$$

where $\overline{B}$ is given by

$$\overline{B}(U, V) = \begin{cases} B_N(U, V), & \text{if } |U|^2 + |V|^2 \le K^2, \\ K^2(|U|^2 + |V|^2)^{-1} B_N(U, V), & \text{otherwise} \end{cases}$$

for some infinite K, and F, G are given by

$$F(\tau, U) = \mathrm{Pr}_N {}^*f(\tau, U),$$
$$G(\tau, U)V = \mathrm{Pr}_N {}^*g(\tau, U)V.$$

The setting of (5.3) is that described in Section 3, namely an internal filtered space $\mathbf{\Omega}_0 = (\Omega, \mathscr{A}, (\mathscr{A}_\tau)_{\tau \geq 0}, \Pi)$ carrying an internal Wiener process W with covariance Q_N. The growth and continuity conditions on f, g and the boundedness of $\overline{B}$ ensure (using the transfer of the standard theory of SDEs) that there is such a space carrying an internal solution $U(\tau, \omega)$ to (5.3), defined for all $\tau \in {}^*[0, \infty)$, with U adapted to $(\mathscr{A}_\tau)_{\tau \geq 0}$. This space need not be the canonical Wiener space.

Let $\mathbf{\Omega}$ be the Loeb space with Wiener process $w = {}^\circ W$ given by Theorem 3.7.

The internal Itô formula for $\|U(\tau)\|^2$ gives (using $B(U, U, AU) = 0$)

$$\begin{aligned}\|U(\tau)\|^2 = \|U(0)\|^2 &+ 2\int_0^\tau ((F(\sigma, U(\sigma)), U(\sigma)))\,\mathrm{d}\sigma \\ &+ \int_0^\tau \mathrm{tr}[{}^*A^{1/2} G(\sigma, U(\sigma)) Q_N G(\sigma, U(\sigma))^{\mathrm{T}} {}^*A^{1/2}]\,\mathrm{d}\sigma \\ &+ 2\int_0^\tau ((U(\sigma), G(\sigma, U(\sigma))))\,\mathrm{d}W(\sigma),\end{aligned} \tag{5.4}$$

where $((U, G)) \in \mathbf{H}_N$ is given by $(((U, G)), V) = ((U, GV))$.

Now by the growth condition on f we have, as in the proof of Theorem 4.1,

$$|((F(\sigma, U), U))| \leq 2{}^*a_1^2(\sigma)(1 + \|U\|^2). \tag{5.5}$$

Similarly, using the condition on g,

$$\begin{aligned}\mathrm{tr}\big({}^*A^{1/2} G(\sigma, U) Q_N G(\sigma, U)^{\mathrm{T}}\, {}^*A^{1/2}\big) &\leq \mathrm{tr}\, Q_N \cdot |G(\sigma, U)|^2_{\mathbf{H},\mathbf{V}} \\ &\leq 2\mathrm{tr}\, Q_N\, {}^*a_2^2(\sigma)(1 + \|U\|^2)\end{aligned} \tag{5.6}$$

So, substituting (5.6), (5.5) in (5.4) we have, for any τ,

$$\begin{aligned}\sup_{\sigma \leq \tau} \|U(\sigma)\|^2 \leq \|U(0)\|^2 &+ \int_0^\tau {}^*a(\sigma)(1 + \|U(\sigma)\|^2)\,\mathrm{d}\sigma \\ &+ 2\sup_{\sigma \leq \tau} |M(\sigma)|,\end{aligned} \tag{5.7}$$

where $a(t) = 2(a_1(t) + a_2^2(t)\mathrm{tr}\, Q)$, and $M(\tau)$ is the internal martingale

$$M(\tau) = \int_0^\tau ((U(\sigma), G(\sigma, U(\sigma))))\,\mathrm{d}W(\sigma).$$

Now M has quadratic variation

$$[M](\tau) = \int_0^\tau \mathrm{tr}[V(\sigma)^{\mathrm{T}} Q_N V(\sigma)]\,\mathrm{d}\sigma \leq \mathrm{tr}\, Q_N \int_0^\tau |V(\sigma)|^2\,\mathrm{d}\sigma,$$

where $V(\sigma) = ((U(\sigma), G(\sigma, U(\sigma))))$ and we have

$$|V(\sigma)| \leq \|U(\sigma)\|\, |G(\sigma, U(\sigma))|_{\mathbf{H},\mathbf{V}} \leq {}^*a_2(\sigma)\|U(\sigma)\|\,(1 + \|U(\sigma)\|) \tag{5.8}$$

so

$$[M](\tau) \leq \sup_{\sigma \leq \tau} \|U(\sigma)\|^2 \int_0^\tau {}^*a(\sigma)(1 + \|U(\sigma)\|^2)\,\mathrm{d}\sigma. \tag{5.9}$$

Now the Burkholder–Davis–Gundy inequality gives

$$E\Big(\sup_{\sigma \leq \tau} |M(\sigma)|\Big) \leq \kappa E([M](\tau)^{1/2})$$

for some finite constant κ. From (5.9) we have, using Young's inequality,

$$[M](\tau)^{1/2} \leq \frac{1}{4\kappa} \sup_{\sigma \leq \tau} \|U(\sigma)\|^2 + c_1 \int_0^\tau {}^*a(\sigma)(1 + \|U(\sigma)\|^2)\,\mathrm{d}\sigma,$$

and so

$$\begin{aligned} E\Big(\sup_{\sigma \leq \tau} |M(\sigma)|\Big) \leq \tfrac{1}{4} E\Big(\sup_{\sigma \leq \tau} \|U(\sigma)\|^2\Big) \\ + c_2 E\Big(\int_0^\tau {}^*a(\sigma)(1 + \|U(\sigma)\|^2)\,\mathrm{d}\sigma\Big). \end{aligned} \tag{5.10}$$

Substituting in (5.7) we obtain

$$E\Big(\sup_{\sigma \leq \tau} \|U(\sigma)\|^2\Big) \leq c\Big(\|U(0)\|^2 + E\int_0^\tau {}^*a(\sigma)(1 + \|U(\sigma)\|^2)\,\mathrm{d}\sigma\Big) \tag{5.11}$$

for all τ. Now applying Gronwall's lemma we get

$$E\|U(\tau)\|^2 \leq c(T) < \infty$$

for finite $\tau \leq T$. Inserting this in (5.11) we obtain

$$E\Big(\sup_{\tau \leq T} \|U(\tau)\|^2\Big) < \infty \tag{5.12}$$

for all finite T. Now we wish to apply Theorem 3.8 to the function

$$\Psi(\sigma, \omega) = G(\sigma, U(\sigma, \omega)).$$

We have, putting $a_3(t) = \lambda_1^{-1} a_2(t)$,

$$|G(\sigma, U(\sigma, \omega))|_{\mathbf{H}_N, \mathbf{H}_N} \leq {}^*a_3(\sigma)(1 + \|U(\sigma, \omega)\|).$$

The function ${}^*a_3(\tau)$ is SL^2, so, together with (5.12), this shows that

$$\sigma \mapsto |\Psi(\sigma, \omega)|^2_{\mathbf{H}_N, \mathbf{H}_N}$$

is S-integrable for a.a. ω, and

$$\begin{aligned} E\Big(\int_0^\tau |\Psi(\sigma)|^2_{\mathbf{H}_N, \mathbf{H}_N}\,\mathrm{d}\sigma\Big) &\leq E\Big(\int_0^\tau {}^*a_3^2(\sigma)(1 + \|U(\sigma)\|)^2\,\mathrm{d}\sigma\Big) \\ &\leq E\Big(\sup_{\sigma \leq \tau}(1 + \|U(\sigma)\|)^2 \int_0^\tau {}^*a_3^2(\sigma)\,\mathrm{d}\sigma\Big) \\ &< \infty \end{aligned}$$

by (5.12). So by Theorem 3.8,

$$\int_0^\tau G(\sigma, U(\sigma))\, \mathrm{d}W(\sigma)$$

is S-continuous in $|\cdot|$ for a.a. ω.

By (5.12) again, $\overline{B}(U(\sigma), U(\sigma)) = B_N(U(\sigma), U(\sigma))$ for all finite σ (for a.a. ω), and so by Theorem 3.6

$$\int_0^\tau [-\overline{B}(U(\sigma), U(\sigma)) + F(\sigma, U(\sigma))]\, \mathrm{d}\sigma$$

is a.s. S-continuous in $\mathbf{V}'$. Hence $U(\tau)$ is a.s. S-continuous in $\mathbf{V}'$ for finite τ and we may define

$$u\colon [0, \infty) \times \Omega \to \mathbf{H}$$

by

$$u(t, \omega) = {}^\circ U(\tau, \omega)$$

for any $\tau \approx t$.

We will see that this u is the required solution. The estimate (5.12) gives condition (5.2) of the theorem.

To see that u satisfies equation (5.1), we have from Theorem 3.6,

$$\begin{aligned} {}^\circ\!\int_0^\tau [-B_N(U(\sigma), U(\sigma)) + F(\sigma, U(\sigma))]\, \mathrm{d}\sigma \\ = \int_0^t [-B(u(s), u(s)) + f(s, u(s))]\, \mathrm{d}s. \end{aligned} \tag{5.13}$$

where $t = {}^\circ\tau$. Finally, the growth condition (iii) together with (5.12) and Proposition 3.11 shows that $G(\sigma, U(\sigma, \omega))$ is weakly nearstandard for a.a. (ω, σ) (for finite σ). Anderson's Luzin theorem ([2], Theorem 3.3.13) shows that for a.a. finite σ we have

$${}^\circ G_{ij}(\sigma, U) = g_{ij}({}^\circ\sigma, {}^\circ U),$$

all finite i, j, whenever $\|U\| < \infty$ and so for a.a. ω, for a.a. finite σ,

$${}^\circ G(\sigma, U(\sigma, \omega)) = g({}^\circ\sigma, u({}^\circ\sigma, \omega)),$$

that is, $\Psi(\tau, \omega) = G(\tau, U(\tau, \omega))$ lifts $g(t, u(t, \omega))$. Now by Theorem 3.12, we have for a.a. ω,

$${}^\circ\!\int_0^\tau G(\sigma, U(\sigma, \omega))\, \mathrm{d}W(\sigma) = \int_0^\tau g(s, u(s, \omega))\, \mathrm{d}w(s) \tag{5.14}$$

for all $t < \infty$. Putting (5.13) and (5.14) together we see that u solves (5.1). □

REMARK. There are two other possible variations of this proof at the stage where we take the internal solution U to (5.3). First, for $\mathbf{\Omega}_0$ we could take a single internal space carrying a Wiener process W, on which every equation of the form (5.3) has a solution. It is known that such a space exists in the standard setting (for example a universal space, constructed using NSA). Then by transfer there is a *universal space for finite-dimensional SDE's, giving a prescribed Wiener process for all such equations.

The other possibility is to take smooth approximations $\bar{F}$ and $\bar{G}$ to F and G with $\bar{F} \approx F$ and $\bar{G} \approx G$ in an appropriate sense. Then it would be possible to work with the canonical Wiener space for $\mathbf{\Omega}_0$, which carries a unique solution to (5.3) with $\bar{F}, \bar{G}$ in place of F, G. Since ${}^\circ\bar{F} = {}^\circ F$ and similarly for $\bar{G}, G$, the rest should be clear.

Note that we can extend the above existence result to the case where the initial condition u_0 is random and independent of the Wiener process w, distributed according to a probability measure μ on $\mathbf{V}$ with $\int_{\mathbf{V}} \|u\|^2 \, \mathrm{d}\mu(u) < \infty$. In this case we adapt the above proof, taking $U(0) \in \mathbf{H}_N$ to be an internal random variable distributed according to $\mu_N = {}^*\mu \circ \mathrm{Pr}_N^{-1}$, independent of the internal Wiener process W. Then we have

$$E\Big(\sup_{\sigma \le \tau} \|U(\sigma)\|^2\Big) \le c\bigg(\int_{\mathbf{H}_N} \|U\|^2 \, \mathrm{d}\mu_N(U) + E\int_0^\tau {}^*a(\sigma)(1 + \|U(\sigma)\|^2) \, \mathrm{d}\sigma\bigg)$$

in place of (5.11). Gronwall's lemma gives

$$E\Big(\sup_{t \le T} \|u(t)\|^2\Big) < \infty$$

as before, since

$$\int_{\mathbf{H}_N} |U|^2 \, \mathrm{d}\mu_N(U) = \int_{{}^*\mathbf{V}} |\mathrm{Pr}_N v|^2 \, \mathrm{d}^*\mu(v) \le \int_{\mathbf{V}} \|u\|^2 \, \mathrm{d}\mu(u).$$

6. Statistical solutions. The notion of a statistical solution of a differential equation is concerned with the evolution of the probability distribution of the solution, given that the initial condition is random. The equation governing the evolving measure was derived by Foias, building on ideas due to Hopf. The derivation of the Foias equation [(6.1) below] assumes the existence of a unique solution to the underlying equation, but even when this is not the case (or is unknown), the Foias equation makes sense and we can search for solutions, which are called *statistical solutions*.

First we give the definition of a statistical solution for the stochastic system (1.1).

A function θ: $\mathbf{V} \to \mathbb{C}$ is a *test functional* if it is of the form $\theta(u) = e^{i(u,v)}$ for some $v \in \mathbf{V}$. We may directly compute the derivatives: $\theta'(u) = ive^{i(u,v)}$, $(\theta''(u)v_1, v_2) = -(v, v_1)(v, v_2)e^{i(u,v)}$ and for $v_1 \in \mathbf{V}'$ we have $\theta''(u)v_1 \in \mathbf{V}$. Moreover, $\|\theta'(u)\| = \|v\|$ and $|\theta''(u)|_{\mathbf{H},\mathbf{H}} = |v|^2$.

DEFINITION 6.1. A family μ_t of Borel probability measures on $\mathbf{V}$ is called a *statistical solution* of the stochastic Euler equations (1.1) if:

(i) The function $t \mapsto \int_{\mathbf{V}} \|u\|^2 \, \mathrm{d}\mu_t(u)$ is $L^\infty(0, T)$ for all $T < \infty$.

(ii) For any test functional θ: $\mathbf{V} \to \mathbb{R}$, and for any $t > 0$,

$$\begin{aligned} &\int_{\mathbf{V}} \theta(u) \, \mathrm{d}\mu_t(u) - \int_{\mathbf{V}} \theta(u) \, \mathrm{d}\mu_0(u) \\ &\quad = \int_0^t \int_{\mathbf{V}} \Big(-b(u, u, \theta'(u)) + (f(s, u), \theta'(u)) \\ &\qquad\qquad + \tfrac{1}{2} \mathrm{tr}(Q g^{\mathrm{T}}(s, u) \theta''(u) g(s, u)) \Big) \, \mathrm{d}\mu_s(u) \, \mathrm{d}s, \end{aligned} \tag{6.1}$$

where Q is the covariance of w and g^{T} denotes the adjoint operator.

We shall call (6.1) the *Foias equation* for the stochastic Euler equation (1.1).

THEOREM 6.2. *Suppose that* f, g *satisfy the conditions of Theorem* 5.1, μ *is a Borel probability measure on* $\mathbf{V}$ *with*

$$\int_{\mathbf{V}} \|u\|^2 \, \mathrm{d}\mu(u) < \infty,$$

and that $u(t)$ *is a solution to the stochastic Euler equation* (1.1) *satisfying* (5.2), *with initial condition* u_0 *random with distribution* μ *on* $\mathbf{V}$. *Then the family of measures*

$$\mu_t(A) = P(u(t) \in A)$$

is a solution to the Foias equation (6.1), *with* $\mu_0 = \mu$.

PROOF. Take any test functional $\theta(u)$. The infinite-dimensional Itô lemma gives

$$\begin{aligned} \mathrm{d}\theta(u(t)) &= \Big[-b(u(t), u(t), \theta'(u(t))) + \big(f(t, u(t)), \theta'(u(t))\big) \\ &\qquad\qquad + \tfrac{1}{2} \mathrm{tr}\big(Q g^{\mathrm{T}}(t, u(t)) \theta''(u(t)) g(t, u(t))\big) \Big] \, \mathrm{d}t \\ &\quad + \big(\theta'(u(t)), g(t, u(t))\big) \, \mathrm{d}w(t) \\ &= \eta(t, u(t)) \, \mathrm{d}t + \xi(t, u(t)) \, \mathrm{d}w(t), \end{aligned}$$

say. We shall prove below that:

(a) $E\Big(\int_0^T |\eta(t, u(t))| \, \mathrm{d}t \Big) < \infty$ for all $T < \infty$;

(b) $E\Big(\int_0^T |\xi(t, u(t))|^2 \, \mathrm{d}t \Big) < \infty$ for all $T < \infty$.

Taking (a) and (b) for granted, we can quickly complete the proof of the theorem,

$$\begin{aligned}\int_{\mathbf{V}} \theta(u)\,\mathrm{d}\mu_t(u) - \int_{\mathbf{V}} \theta(u)\,\mathrm{d}\mu_0(u) &= E\big(\theta(u(t)) - \theta(u(0))\big)\\ &= E\bigg(\int_0^t \eta(s, u(s))\,\mathrm{d}s + \int_0^t \xi(s, u(s))\,\mathrm{d}w(s)\bigg)\\ &= \int_0^t E\eta(s, u(s))\,\mathrm{d}s\end{aligned}$$

using (a) and Fubini for the first term and the zero mean property of the stochastic integral given condition (b). Finally, note that

$$E\eta(s, u(s)) = \int_{\mathbf{V}} \eta(s, u)\,\mathrm{d}\mu_s(u)$$

to see that the Foias equation (6.1) is satisfied. The condition (i) of Definition 6.1 follows from (5.2) automatically.

It remains to check (a) and (b) above. For (a) note first that for any $u \in \mathbf{V}$ [recall that $\theta(u) = e^{i(u,v)}$ for some $v \in \mathbf{V}$] using the growth conditions on f and g we have

$$\begin{aligned}|\eta(t, u)| &\le \big(c\|u\|^2 + |f(t, u)|_{\mathbf{V}'}\big)\|v\| + \tfrac{1}{2}\mathrm{tr}\, Q|g(t, u)|^2_{\mathbf{H},\mathbf{H}}|v|^2\\ &\le c_1 + c_2 a(t)(1 + \|u\|^2)\end{aligned}$$

with $a \in L^1[0, T]$ for all T. Now the property (5.2) of $u(t)$ gives (a). For (b) it is enough to observe that for any $u \in \mathbf{V}$,

$$|\xi(t, u)| \le |\theta'(u)|\,|g(t, u)|_{\mathbf{H},\mathbf{H}} \le c\,|g(t, u)|_{\mathbf{H},\mathbf{H}}$$

and use (5.2) and the growth condition on g again to see that

$$E\bigg(\int_0^T |g(t, u(t))|^2_{\mathbf{H},\mathbf{H}}\,\mathrm{d}t\bigg) < \infty.$$

This completes the proof. □

7. Limit of stochastic Navier–Stokes equations. In this section we shall show that, in an appropriate sense, as the viscosity $\nu \to 0$, limit points of solutions to the stochastic Navier–Stokes equations exist and all are solutions to the stochastic Euler equation.

The stochastic Navier–Stokes equations take the form

$$\mathrm{d}u = (\nu\Delta u - \langle u, \nabla\rangle u + f(t, u))\,\mathrm{d}t + g(t, u)\,\mathrm{d}w_t, \tag{7.1}$$

where $\nu > 0$ is the viscosity and w is the Wiener process on the Loeb space $\mathbf{\Omega}$ as in Section 5. In this section we may take $\mathbf{\Omega}_0$ to be the canonical Wiener space. Suppose that the force terms satisfy the following Lipschitz condition with respect to u:

$$|f(t, u) - f(t, v)|_{\mathbf{V}'} + |g(t, u) - g(t, v)|_{\mathbf{H},\mathbf{H}} \le c|u - v| \tag{7.2}$$

for each t. Then for each $\nu > 0$ there is a unique solution to (7.1) which we denote by u^ν to emphasize the dependence on ν. In fact we have the theorem.

THEOREM 7.1. *Assume the conditions of Theorem* 5.1 *and the Lipschitz condition* (7.2). *Then for each* $\nu > 0$, (7.1) *has a unique solution* u^ν *satisfying*

$$E\Big(\sup_{t\le T}\|u^\nu(t)\|^2 + 2\nu\int_0^T |Au^\nu(t)|^2\,\mathrm{d}t\Big) \le C(T) < \infty \tag{7.3}$$

where $C(T)$ *is independent of* ν.

PROOF. The proof of existence follows that of Theorem 5.1 with (5.3) modified to become

$$\begin{aligned}\mathrm{d}U^\nu(\tau) &= \big[-\overline{B}(U^\nu(\tau),U^\nu(\tau)) - \nu{}^*\!AU^\nu(\tau) + F(\tau,U^\nu(\tau))\big]\,\mathrm{d}\tau\\ &\qquad + G(\tau,U^\nu(\tau))\mathrm{d}W(\tau),\\ U^\nu(0) &= \mathrm{Pr}_N{}^*u_0.\end{aligned} \tag{7.4}$$

The Lipshitz condition on f, g ensures that for any Wiener process W there is a unique internal solution to (7.4).

Compared with the Euler equation, in (5.4) in the case we are now considering there is the additional term $2\nu\int_0^T |{}^*AU^\nu(t)|^2\,\mathrm{d}t$ on the left. The same argument as in Theorem 5.1 leads to

$$E\Big(\sup_{\tau\le T}\|U^\nu(\tau)\|^2 + 2\nu\int_0^T |{}^*AU^\nu(\tau)|^2\,\mathrm{d}\tau\Big) \le C(T) < \infty \tag{7.5}$$

for all finite T with $C(T)$ independent of ν. This gives a solution u^ν satisfying (7.3).

Uniqueness is proved in Theorem 6.6.2 of [2]. □

Note that this gives an existence result stronger than Theorem 6.5.3 of [2] where $C(T)$ depends on ν and in fact $C(T) \nearrow \infty$ as $\nu \to 0$. This is possible due to stronger assumptions on f.

Before we discuss the limiting behavior of the solutions u^ν as $\nu \to 0$, note the following.

THEOREM 7.2. *Suppose that* $\lambda \in {}^*\mathbb{R}$, $\lambda > 0$, *and* λ *is finite. Let* U^λ *be the* (*internal*) *solution to* (7.4) *with viscosity* λ. *Then* $U^\lambda(\tau)$ *is almost surely* S-*continuous for finite* τ. *Writing* $u^\lambda = {}^\circ U^\lambda$ *for all such* λ *we have*:

(a) *If* $\lambda \approx 0$, *then* u^λ *solves the stochastic Euler equation.*

(b) *If* $\lambda \not\approx 0$, *then* u^λ *solves the stochastic Navier–Stokes equations* (7.1), *with viscosity* $\nu = {}^\circ\lambda$ (*i.e.*, $u^\lambda = u^{{}^\circ\lambda}$).

PROOF. Obvious from the proofs of Theorems 5.1 and 7.1. □

Let us now, for convenience, fix the time interval $[0, T]$ and consider the probability laws μ^ν of the solutions u^ν of the stochastic Navier–Stokes equations (7.1), for real $\nu > 0$. From the result above we have a uniform constant $C = C(T)$ such that $E(\sup_{t\le T}\|u^\nu(t)\|^2) \le C$ and almost surely $u^\nu(\cdot,\omega) \in C([0,T],\mathbf{H}) = \mathscr{C}$, say, which we equip with the uniform topology. Denote by $\mathscr{M}$ the set of all Borel probability measures on $\mathscr{C}$ and write

$$\mathscr{M}_0 = \Big\{\mu \in \mathscr{M}\colon E_\mu\Big(\sup_{t\in[0,T]}\|u(t)\|^2\Big) \le C\Big\}.$$

Then each μ^ν belongs to $\mathscr{M}_0$.

Note that if we put $\mathscr{C}_0 = \{u\colon \sup_{[0,T]}\|u(t)\| < \infty\}$, then

$$\mathscr{C}_0 \cap C([0,T],\mathbf{H}) = \mathscr{C}_0 \cap C([0,T],\mathbf{V}_{\text{weak}}) = \mathscr{C}_0 \cap C([0,T],\mathbf{H}_\alpha)$$

for all $\alpha < 1$, so the precise choice of the space $\mathscr{C}$ above is not important.

The main result of this section can now be given.

THEOREM 7.3. (a) *The set* $\{\mu^\nu\colon 0 < \nu \le 1\}$ *is relatively compact in the weak topology of* $\mathscr{M}$.

(b) *If* $\nu_n \to 0$ *and* $\mu^{\nu_n} \to \mu$, *then* μ *is the law of a solution to the stochastic Euler equation* (5.1) *on the Loeb space* $\mathbf{\Omega}$ *with the same driving Wiener process* w.

In fact, for all sufficiently small infinite N, *the process* u^{ν_N} *is a solution to* (5.1) *with law* μ, *where we are using the notation of Theorem* 7.2.

PROOF. As in Theorem 7.2, for each $\lambda \in {}^*(0,1]$ write U^λ for the unique solution to (7.4) with $\nu = \lambda$, and let M^λ be the law of U^λ. Since

$$U^\lambda(\cdot,\omega) \in {}^*C([0,T],\mathbf{H}_N) \subseteq {}^*C([0,T],\mathbf{H}),$$

$M^\lambda \in {}^*\mathscr{M}_0$ by (7.5). Let μ^λ be the law of $u^\lambda = {}^\circ U^\lambda$. (If λ is real, this is consistent with the notation u^ν and μ^ν already introduced for the solutions of the stochastic Navier–Stokes equations and their laws.)

We claim that $M^\lambda \approx \mu^\lambda$ for all $\lambda \in {}^*(0,1]$. To this end take a bounded continuous function $f\colon \mathscr{C} \to \mathbb{R}$ (we are taking the uniform topology on $\mathscr{C}$). Then by Theorem 7.2 we have

$$E_{M^\lambda}({}^*f(U)) = E({}^*f(U^\lambda)) \approx E(f({}^\circ U^\lambda)) = E_{\mu^\lambda}(f(u)).$$

Thus the internal set $S = \{M^\lambda\colon \lambda \in {}^*(0,1]\}$ consists entirely of nearstandard points so the set ${}^\circ S = \{\mu^\lambda\colon \lambda \in {}^*(0,1]\}$ is relatively compact (a basic fact of nonstandard topology; see [2], Proposition 2.6.7, for example). By Theorem 7.2, $\{\mu^\nu\colon 0 < \nu \le 1\} \subseteq {}^\circ S$ and this establishes (a).

For (b), suppose that $\nu_n \to 0$ and $\mu^{\nu_n} \to \mu$ in $\mathscr{M}$. Consider the laws M^{ν_n} of the processes U^{ν_n} for $n \in {}^*\mathbb{N}$. From the above $M^{\nu_n} \approx \mu^{\nu_n}$ for finite n, and so ${}^*d(M^{\nu_n}, {}^*\mu^{\nu_n}) \approx 0$, where d is the Prokhorov metric. Thus, by Robinson's sequential lemma (see, for example, [2], Lemma 2.3.7) there is an infinite N_0 such that ${}^*d(M^{\nu_n}, {}^*\mu^{\nu_n}) \approx 0$ for all $n \le N_0$.

Take $\lambda = \nu_N$, for some infinite $N \le N_0$; then $\lambda \approx 0$. Put $u = u^\lambda$; by Theorem 7.2(b), u solves the stochastic Euler equation. Finally, the law of u is μ^λ (by definition), and we have

$$\begin{aligned}\mu^\lambda &\approx M^\lambda && \text{from the proof of (a)}\\ &\approx {}^*\mu^\lambda && \text{from above and the choice of } N\\ &\approx \mu && \text{by the usual criterion for convergence.}\end{aligned}$$

Thus $\mu = \mu^\lambda$, since both are standard, and the law of u is μ as required. □

REMARK 7.4. Using standard methods one can show that if $\nu_n \to 0$, then the laws of u^{ν_n} are tight [which is also established by Theorem 7.3(a) above]. Taking a subsequence if necessary, we may assume without loss of generality that the laws μ^{ν_n} converge. Then by Skorokhod's theorem, there exists a probability space carrying processes $\bar{u}^{\nu_n}$ with laws μ^{ν_n}, converging in $L^2(\mathscr{C})$ to a solution u to the Euler equation. However, the driving Wiener processes will be different for each of the processes $\bar{u}^{\nu_n}, u$. This contrasts with the result of Theorem 7.3, where the space $\mathbf{\Omega}$ and the driving Wiener process is universal (i.e., the same for each of the solutions u^{ν_n} and u), but the convergence is only in law.

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