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SOME NON-LINEAR S.P.D.E.'S THAT ARE SECOND ORDER IN TIME

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Abstract: We extend J.B. Walsh's theory of martingale measures in order to deal with stochastic partial differential equations that are second order in time, such as the wave equation and the beam equation, and driven by spatially homogeneous Gaussian noise. For such equations, the fundamental solution can be a distribution in the sense of Schwartz, which appears as an integrand in the reformulation of the s.p.d.e. as a stochastic integral equation. Our approach provides an alternative to the Hilbert space integrals of Hilbert-Schmidt operators. We give several examples, including the beam equation and the wave equation, with nonlinear multiplicative noise terms.

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1 Introduction

The study of stochastic partial differential equations (s.p.d.e.'s) began in earnest following the papers of Pardoux [14], [15], [16], and Krylov and Rozovskii [8], [9]. Much of the literature has been concerned with the heat equation, most often driven by space-time white noise, and with related parabolic equations. Such equations are first order in time, and generally second order in the space variables. There has been much less work on s.p.d.e.'s that are second order in time, such as the wave equation and related hyperbolic equations. Some early references are Walsh [22], and Carmona and Nualart [2], [3]. More recent papers are Mueller [12], Dalang and Frangos [6], and Millet and Sanz-Solé [11].

For linear equations, the noise process can be considered as a random Schwartz distribution, and therefore the theory of deterministic p.d.e.'s can in principle be used. However, this yields solutions in the space of Schwartz distributions, rather than in the space of function-valued stochastic processes. For linear s.p.d.e.'s such as the heat and wave equation driven by space-time white noise, this situation is satisfactory, since, in fact, there is no function-valued solution when the spatial dimension is greater than 1. However, since non-linear functions of Schwartz distributions are difficult to define (see however Oberguggenberger and Russo [13]), it is difficult to make sense of non-linear s.p.d.e.'s driven by space-time white noise in dimensions greater than 1.

A reasonable alternative to space-time white noise is Gaussian noise with some spatial correlation, that remains white in time. This approach has been taken by several authors, and the general framework is given in Da Prato and Zabczyk [4]. However, there is again a difference between parabolic and hyperbolic equations: while the Green's function is smooth for the former, for the latter it is less and less regular as the dimension increases. For instance, for the wave equation, the Green's function is a bounded function in dimension 1, is an unbounded function in dimension 2, is a measure in dimension 3, and a Schwartz distribution in dimensions greater than 3.

There are at least two approaches to this issue. One is to extend the theory of stochastic integrals with respect to martingale measures, as developed by Walsh [22], to a more general class of integrands that includes distributions. This approach was taken by Dalang [5]. In the case of the wave equation, this yields a solution to the non-linear equation in dimensions 1, 2 and 3. The solution is a *random field,* that is, it is defined for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Another approach is to consider solutions with values in a function space, typically an L^2-space: for each fixed $t \in \mathbb{R}_+$, the solution is an L^2-function, defined for almost all $x \in \mathbb{R}^d$. This approach has been taken by Peszat and Zabczyk in [18] and Peszat [17]. In the case of the non-linear wave equation, this approach yields function-valued solutions in all dimensions. It should be noted that the notions of *random field* solution and *function-valued* solution are *not* equivalent: see Lévèque [10].

In this paper, we develop a general approach to non-linear s.p.d.e's, with a focus on equations that are second order in time, such as the wave equation and the beam equation. This approach goes in the direction of unifying the two described above, since we begin in Section 2 with an extension of Walsh's martingale measure stochastic integral [22], in such a way as to integrate processes that take values in an L^2-space, with an integral that takes values in the same space. This extension defines stochastic integrals of the form

$$\int_0^t \int_{\mathbb{R}^d} G(s, \cdot - y) Z(s, y)\, M(ds, dy),$$

where G (typically a Green's function) takes values in the space of Schwartz distributions, Z is an adapted process with values in $L^2(\mathbb{R}^d)$, and M is a Gaussian martingale measure with spatially homogeneous covariance.

With this extended stochastic integral, we can study non-linear forms of a wide class of s.p.d.e.'s, that includes the wave and beam equations in all dimensions, namely equations for which the p.d.e. operator is

$$\frac{\partial^2 u}{\partial t^2} + (-1)^k \Delta^k u,$$

where $k \geq 1$ (see Section 3). Indeed, in Section 4 we study the corresponding non-linear s.p.d.e.'s. We only impose the minimal assumptions on the spatial covariance of the noise, that are needed even for the linear form of the s.p.d.e. to have a function-valued solution. The non-linear coefficients must be Lipschitz and vanish at the origin. This last property guarantees that with an initial condition that is in $L^2(\mathbb{R}^d)$, the solution remains in $L^2(\mathbb{R}^d)$ for all time.

In Section 5, we specialize to the wave equation in a weighted L^2-space, and remove the condition that the non-linearity vanishes at the origin. Here, the compact support property of the Green's function of the wave equation is used explicitly. We note that Peszat [17] also uses weighted L^2-spaces, but with a weight that decays exponentially at infinity, whereas here, the weight has polynomial decay.

2 Extensions of the stochastic integral

In this section, we define the class of Gaussian noises that drive the s.p.d.e.'s that we consider, and give our extension of the martingale measure stochastic integral.

Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the topological vector space of functions $\varphi \in C_0^\infty(\mathbb{R}^{d+1})$, the space of infinitely differentiable functions with compact support, with the standard notion of convergence on this space (see Adams [1], page 19). Let Γ be a non-negative and *non-negative definite* (therefore symmetric) tempered measure on $\mathbb{R}^d$. That is,

$$\int_{\mathbb{R}^d} \Gamma(dx)\, (\varphi * \tilde{\varphi})(x) \geq 0, \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d),$$

where $\tilde{\varphi}(x) = \varphi(-x)$, "$*$" denotes convolution, and there exists $r > 0$ such that

$$\int_{\mathbb{R}^d} \Gamma(dx)\, \frac{1}{(1 + |x|^2)^r} < \infty. \tag{2.1}$$

We note that if $\Gamma(dx) = f(x)dx$, then

$$\int_{\mathbb{R}^d} \Gamma(dx)(\varphi * \tilde{\varphi})(x) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(x) f(x-y)\varphi(y)$$

(this was the framework considered in Dalang [5]). Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz space of rapidly decreasing C^∞ test functions, and for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, let $\mathcal{F}\varphi$ denote the Fourier transform of φ:

$$\mathcal{F}\varphi(\eta) = \int_{\mathbb{R}^d} \exp(-i\,\eta \cdot x)\varphi(x)\, dx.$$

According to the Bochner-Schwartz theorem (see Schwartz [20], Chapter VII, Théorème XVII), there is a non-negative tempered measure μ on $\mathbb{R}^d$ such that $\Gamma = \mathcal{F}\mu$, that is

$$\int_{\mathbb{R}^d} \Gamma(dx)\, \varphi(x) = \int_{\mathbb{R}^d} \mu(d\eta)\, \mathcal{F}\varphi(\eta), \qquad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{2.2}$$

Examples. (a) Let δ_0 denote the Dirac functional. Then $\Gamma(dx) = \delta_0(x)\, dx$ satisfies the conditions above.

(b) Let $0 < \alpha < d$ and set $f_\alpha(x) = |x|^{-\alpha}$, $x \in \mathbb{R}^d$. Then $f_\alpha = c_\alpha \mathcal{F} f_{d-\alpha}$ (see Stein [21], Chapter V §1, Lemma 2(a)), so $\Gamma(dx) = f_\alpha(x)\, dx$ also satisfies the conditions above.

Let $F = (F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1}))$ be an $L^2(\Omega, \mathcal{G}, P)$-valued mean zero Gaussian process with covariance functional

$$(\varphi, \psi) \mapsto E(F(\varphi)F(\psi)) = \int_{\mathbb{R}^d} \Gamma(dx)\, (\varphi * \tilde{\psi})(x).$$

As in Dalang and Frangos [6] and Dalang [5], $\varphi \mapsto F(\varphi)$ extends to a worthy martingale measure $(t, A) \mapsto M_t(A)$ (in the sense of Walsh [22], pages 289-290) with covariance measure

$$Q([0,t] \times A \times B) = \langle M(A), M(B)\rangle_t = t \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, 1_A(y)\, 1_B(x+y)$$

and dominating measure $K \equiv Q$, such that

$$F(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \varphi(t,x) M(dt, dx), \qquad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^{d+1}).$$

The underlying filtration is $(\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{N},\ t \geq 0)$, where

$$\mathcal{F}_t^0 = \sigma(M_s(A),\ s \leq t,\ A \in \mathcal{B}_b(\mathbb{R}^d)),$$

$\mathcal{N}$ is the σ-field generated by P-null sets and $\mathcal{B}_b(\mathbb{R}^d)$ denotes the bounded Borel subsets of $\mathbb{R}^d$.

Recall [22] that a function $(s, x, \omega) \mapsto g(s, x; \omega)$ is termed *elementary* if it is of the form

$$g(s, x; \omega) = 1_{]a,b]}(s) 1_A(x) X(\omega),$$

where $0 \leq a < b$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and X is a bounded and $\mathcal{F}_a$-measurable random variable. The σ-field on $\mathbb{R}_+ \times \mathbb{R}^d \times \Omega$ generated by elementary functions is termed the *predictable* σ-field.

Fix $T > 0$. Let $\mathcal{P}_+$ denote the set of predictable functions $(s, x; \omega) \mapsto g(s, x; \omega)$ such that $\|g\|_+ < \infty$, where

$$\|g\|_+^2 = E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, |g(s,y) g(s, x+y)|\right).$$

Recall [22] that $\mathcal{P}_+$ is the completion of the set of elementary functions for the norm $\|\cdot\|_+$.

For $g \in \mathcal{P}_+$, Walsh's stochastic integral

$$M_t^g(A) = \int_0^t ds \int_A g(s,x) M(ds, dx)$$

is well-defined and is a worthy martingale measure with covariation measure

$$Q_g([0,t] \times A \times B) = \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, 1_A(y) 1_B(x+y)\, g(s,y) g(s,x+y)$$

and dominating measure

$$K_g([0,t] \times A \times B) = \int_0^t ds \int_{\mathbb{R}^d} \Gamma(dx) \int_{\mathbb{R}^d} dy\, 1_A(y) 1_B(x+y)\, |g(x,y) g(s,x+y)|.$$

For a deterministic real-valued function $(s, x) \mapsto g(s, x)$ and a real-valued stochastic process $(Z(t, x),\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d)$, consider the following hypotheses ($T > 0$ is fixed).

(G1) For $0 \leq s \leq T$, $g(s, \cdot) \in C^\infty(\mathbb{R}^d)$, $g(s, \cdot)$ is bounded uniformly in s, and $\mathcal{F}g(s, \cdot)$ is a function.

(G2) For $0 \leq s \leq T$, $Z(s, \cdot) \in C_0^\infty(\mathbb{R}^d)$ a.s., $Z(s, \cdot)$ is $\mathcal{F}_s$-measurable, and in addition, there is a compact set $K \subset \mathbb{R}^d$ such that supp $Z(s, \cdot) \subset K$, for $0 \leq s \leq T$. Further, $s \mapsto Z(s, \cdot)$ is mean-square continuous from $[0, T]$ into $L^2(\mathbb{R}^d)$, that is, for $s \in [0, T]$,

$$\lim_{t \to s} E\left(\|Z(t, \cdot) - Z(s, \cdot)\|_{L^2(\mathbb{R}^d)}^2\right) = 0.$$

(G3) $I_{g,Z} < \infty$, where

$$I_{g,Z} = \int_0^T ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s, \cdot)(\xi)|^2) \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}g(s, \cdot)(\xi - \eta)|^2.$$

Lemma 1. *Under hypotheses (G1), (G2) and (G3), for all $x \in \mathbb{R}^d$, the function defined by $(s, y; \omega) \mapsto g(s, x - y) Z(s, y; \omega)$ belongs to $\mathcal{P}_+$, and so*

$$v_{g,Z}(x) = \int_0^T \int_{\mathbb{R}^d} g(s, x-y) Z(s,y) M(ds, dy)$$

is well-defined as a (Walsh-) stochastic integral. Further, a.s., $x \mapsto v_{g,Z}(x)$ belongs to $L^2(\mathbb{R}^d)$, and

$$E\left(\|v_{g,Z}\|_{L^2(\mathbb{R}^d)}^2\right) = I_{g,Z}. \tag{2.3}$$

Proof. Observe that $\|g(\cdot, x-\cdot)Z(\cdot,\cdot)\|_+^2$ is equal to

$$E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy\, |g(s,x-y)Z(s,y)g(s,x-y-z)Z(s,y+z)|\right).$$

Because $g(s,\cdot)$ is bounded uniformly in s by (G1), this expression is bounded by a constant times

$$\begin{aligned}
&E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy\, |Z(s,y)Z(s,y+z)|\right)\\
&\quad = E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dz)\, (|Z(s,\cdot)| * |\tilde{Z}(s,\cdot)|)(-z)\right).
\end{aligned}$$

By (G2), the inner integral can be taken over $K - K = \{z - y : z \in K,\ y \in K\}$, and the sup-norm of the convolution is bounded by $\|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}$, so this is

$$\leq E\left(\int_0^T ds\, \|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)} \Gamma(K-K)\right) = \int_0^T ds\, \|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}\, \Gamma(K-K) < \infty,$$

by (2.1) and the fact that $s \mapsto E(\|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)})$ is continuous by (G2). Therefore, $v_{g,Z}(x)$ will be well-defined provided we show that $(s,y,\omega) \mapsto g(s,x-y)Z(s,y;\omega)$ is predictable, or equivalently, that $(s,y,\omega) \mapsto Z(s,y;\omega)$ is predictable.

For this, set $t_j^n = jT2^{-n}$ and

$$Z_n(s,y) = \sum_{j=0}^{2^n-1} Z(t_j^n, x)\, 1_{]t_j^n, t_{j+1}^n]}(s).$$

Observe that

$$\begin{aligned}
\|Z_n\|_+^2 &= E\left(\sum_{j=0}^{2^n-1} \int_{t_j^n}^{t_{j+1}^n} ds \int_{\mathbb{R}^d} \Gamma(dx)\, (|Z(t_j^n,\cdot)| * |\tilde{Z}(t_j^n,\cdot)|)(-x)\right)\\
&\leq T2^{-n} \sum_{j=0}^{2^n-1} E(\|Z(t_j^n,\cdot)\|^2_{L^2(\mathbb{R}^d)})\, \Gamma(K-K)\\
&< \infty.
\end{aligned}$$

Therefore, $Z_n \in \mathcal{P}_+$, since this process, which is adapted, continuous in x and left-continuous in s, is clearly predictable. Further,

$$\begin{aligned}
&E\left(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dx)\, (|Z(s,\cdot) - Z_n(s,\cdot)|) * (|\tilde{Z}(s,\cdot) - \tilde{Z}_n(s,\cdot)|)(-x)\right)\\
&\qquad \leq \int_0^T ds\, E(\|Z(s,\cdot) - Z_n(s,\cdot)\|^2_{L^2(\mathbb{R}^d)})\, \Gamma(K-K).
\end{aligned}$$

The integrand converges to 0 and is uniformly bounded over $[0,T]$ by (G2), so this expression converges to 0 as $n \to \infty$. Therefore, Z is predictable.

Finally, we prove (2.3). Clearly, $E(\|v_{g,Z}\|^2_{L^2(\mathbb{R}^d)})$ is equal to

$$E\left(\int_{\mathbb{R}^d} dx \left(\int_0^T \int_{\mathbb{R}^d} g(s,x-y)Z(s,y)M(ds,dy)\right)^2\right).$$

Since the covariation measure of M is Q, this equals

$$E\left(\int_{\mathbb{R}^d} dx \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dz) \int_{\mathbb{R}^d} dy\, g(s,x-y)Z(s,y)g(s,x-z-y)Z(s,z+y)\right). \tag{2.4}$$

The inner integral is equal to $(g(s,x-\cdot)Z(s,\cdot)) * (\tilde{g}(s,x-\cdot)\tilde{Z}(s,\cdot))(-z)$, and since this function belongs to $\mathcal{S}(\mathbb{R}^d)$ by (G1) and (G2), (2.4) equals

$$E\left(\int_{\mathbb{R}^d} dx \int_0^T ds \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}(g(s,x-\cdot)Z(s,\cdot))(\eta)|^2\right),$$

by (2.2). Because the Fourier transform takes products to convolutions,

$$\mathcal{F}(g(s,x-\cdot)Z(s,\cdot))(\eta) = \int_{\mathbb{R}^d} d\xi' e^{i\xi'\cdot x}\mathcal{F}g(s,\cdot)(-\xi')\mathcal{F}Z(s,\cdot)(\eta-\xi'),$$

so, by Plancherel's theorem,

$$\int_{\mathbb{R}^d} dx\, |\mathcal{F}(g(s,x-\cdot)Z(s,\cdot))(\eta)|^2 = \int_{\mathbb{R}^d} d\xi'\, |\mathcal{F}g(s,\cdot)(-\xi')\mathcal{F}Z(s,\cdot)(\eta-\xi')|^2. \tag{2.5}$$

The minus can be changed to plus, and using the change of variables $\xi = \eta + \xi'$ (η fixed), we find that (2.3) holds. ∎

Remark 2. *An alternative expression for $I_{g,Z}$ is*

$$I_{g,Z} = E\left(\int_0^T ds \int_{\mathbb{R}^d} \mu(d\eta)\, \|g(s,\cdot) * (\chi_\eta(\cdot)Z(s,\cdot))\|^2_{L^2(\mathbb{R}^d)}\right),$$

where $\chi_\eta(x) = e^{i\eta\cdot x}$. Indeed, notice that (2.5) is equal to

$$\int d\xi'\, |\mathcal{F}g(s,\cdot)(\xi')\mathcal{F}(\chi_\eta(\cdot)Z(s,\cdot))(\xi')|^2,$$

which, by Plancherel's theorem, is equal to

$$\|g(s,\cdot) * (\chi_\eta(\cdot)Z(s,\cdot))\|^2_{L^2(\mathbb{R}^d)}.$$

Fix $(s,x) \mapsto g(s,x)$ such that (G1) holds. Consider the further hypotheses:

(G4) $\int_0^T ds\, \sup_\xi \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}g(s,\cdot)(\xi-\eta)|^2 < \infty$,

(G5) For $0 \le s \le T$, $Z(s,\cdot) \in L^2(\mathbb{R}^d)$ a.s., $Z(s,\cdot)$ is $\mathcal{F}_s$-measurable, and $s \mapsto Z(s,\cdot)$ is mean-square continuous from $[0,T]$ into $L^2(\mathbb{R}^d)$.

Fix g such that (G1) and (G4) hold. Set

$$\mathcal{P} = \{Z : \text{(G5) holds}\}.$$

Define a norm $\|\cdot\|_g$ on $\mathcal{P}$ by

$$\|Z\|_g^2 = I_{g,Z}.$$

We observe that by (G4) and (G5) (and Plancherel's theorem), $I_{g,Z} \le \tilde{I}_{g,Z} < \infty$, where

$$\tilde{I}_{g,Z} = \int_0^T ds\, E(\|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}) \left(\sup_\xi \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}g(s,\cdot)(\xi-\eta)|^2 \right).$$

Let

$$\mathcal{E} = \{Z \in \mathcal{P} : \text{ (G2) holds}\}.$$

By Lemma 1, $Z \mapsto v_{g,Z}$ defines an isometry from $(\mathcal{E}, \|\cdot\|_g)$ into $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$. Therefore, this isometry extends to the closure of $(\mathcal{E}, \|\cdot\|_g)$ in $\mathcal{P}$, which we now identify.

Lemma 3. *$\mathcal{P}$ is contained in the closure of* $(\mathcal{E}, \|\cdot\|_g)$.

Proof. Fix $\psi \in C_0^\infty(\mathbb{R}^d)$ such that $\psi \ge 0$, the support of ψ is contained in the unit ball of $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} \psi(x)\, dx = 1$. For $n \ge 1$, set

$$\psi_n(x) = n^d \psi(nx).$$

Then $\psi_n \to \delta_0$ in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{F}\psi_n(\xi) = \mathcal{F}\psi(\xi/n)$, therefore $|\mathcal{F}\psi_n(\cdot)|$ is bounded by 1.

Fix $Z \in \mathcal{P}$, and show that Z belongs to the completion of $\mathcal{E}$ in $\|\cdot\|_g$. Set

$$Z_n(s,x) = Z(s,x) 1_{[-n,n]^d}(x) \quad \text{and} \quad Z_{n,m}(s,\cdot) = Z_n(s,\cdot) * \psi_m.$$

We first show that $Z_{n,m} \in \mathcal{E}$, that is, (G2) holds for $Z_{n,m}$. Clearly, $Z_{n,m}(s,\cdot) \in C_0^\infty(\mathbb{R}^d)$, $Z_{n,m}(s,\cdot)$ is $\mathcal{F}_s$-measurable by (G5), and there is a compact set $K_{n,m} \subset \mathbb{R}^d$ such that supp $Z_{n,m}(s,\cdot) \subset K$, for $0 \le s \le T$. Further,

$$\|Z_{n,m}(t,\cdot) - Z_{n,m}(s,\cdot)\|^2_{L^2(\mathbb{R}^d)} \le \|Z_n(t,\cdot) - Z_n(s,\cdot)\|^2_{L^2(\mathbb{R}^d)} \le \|Z(t,\cdot) - Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)},$$

so $s \mapsto Z_{n,m}(s,\cdot)$ is mean-square continuous by (G5). Therefore, $Z_{n,m} \in \mathcal{E}$.

We now show that for n fixed, $\|Z_n - Z_{n,m}\|_g \to 0$ as $m \to \infty$. Clearly,

$$I_{g,Z_n-Z_{n,m}} = \int_0^T ds \int_{\mathbb{R}^d} d\xi \, E(|\mathcal{F}Z_n(s,\cdot)|^2) \; |1 - \mathcal{F}\psi_m(\xi)|^2 \int_{\mathbb{R}^d} \mu(d\eta) \, |\mathcal{F}g(s,\cdot)(\xi-\eta)|^2.$$

Because $|1 - \mathcal{F}\psi_m(\xi)|^2 \leq 4$ and

$$\begin{aligned} I_{g,Z_n} &\leq \int_0^T ds \, E\left(\int_{\mathbb{R}^d} d\xi \, |\mathcal{F}Z_n(s,\cdot)|^2\right) \left(\sup_\xi \int_{\mathbb{R}^d} \mu(d\eta) \, |\mathcal{F}g(s,\xi-n)|^2\right) \\ &= \tilde{I}_{g,Z_n} \leq \tilde{I}_{g,Z} < \infty, \end{aligned}$$

we can apply the Dominated Convergence Theorem to see that for n fixed,

$$\lim_{m\to\infty} \|Z_n - Z_{n,m}\|_g = \lim_{m\to\infty} \sqrt{I_{g,Z_n-Z_{n,m}}} = 0.$$

Therefore, Z_n belongs to the completion of $\mathcal{E}$ in $\|\cdot\|_g$. We now show that $\|Z - Z_n\|_g \to 0$ as $n \to \infty$. Clearly,

$$\begin{aligned} \|Z - Z_n\|_g^2 &= I_{g,Z-Z_n} \leq \tilde{I}_{g,Z-Z_n} \\ &= \int_0^T ds \, E\left(\|(Z - Z_n)(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right) \left(\sup_\xi \int_{\mathbb{R}^d} \mu(d\eta) \, |\mathcal{F}g(s,\xi-n)|^2\right). \end{aligned}$$

Because

$$\|Z - Z_n\|^2_{L^2(\mathbb{R}^d)} \leq (\|Z\|_{L^2(\mathbb{R}^d)} + \|Z_n\|_{L^2(\mathbb{R}^d)})^2 \leq 4\|Z\|^2_{L^2(\mathbb{R}^d)},$$

and $\tilde{I}_{g,Z} < \infty$, the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \|Z - Z_n\|_g = 0,$$

and therefore Z belongs to the completion of $\mathcal{E}$ in $\|\cdot\|_g$. Lemma 3 is proved. ∎

Remark 4. *Lemma 3 allows us to define the stochastic integral $v_{g,Z} = g \cdot M^Z$ provided g satisfies (G1) and (G4), and Z satisfies (G5). The key property of this stochastic integral is that*

$$E\left(\|v_{g,Z}\|^2_{L^2(\mathbb{R}^d)}\right) = I_{g,Z}.$$

We now proceed with a further extension of this stochastic integral, by extending the map $g \mapsto v_{g,Z}$ to a more general class of g.

Fix $Z \in \mathcal{P}$. Given a function $s \mapsto G(s) \in \mathcal{S}'(\mathbb{R}^d)$, consider the two properties:

(G6) For all $s \geq 0$, $\mathcal{F}G(s)$ is a function and

$$\int_0^T ds \sup_\xi \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}G(s, \xi - \eta)|^2 < \infty.$$

(G7) For all $\psi \in C_0^\infty(\mathbb{R}^d)$, $\sup_{0 \leq s \leq T} G(s) * \psi$ is bounded on $\mathbb{R}^d$.

Set

$$\mathcal{G} = \{s \mapsto G(s) : \text{ (G6) and (G7) hold}\},$$

and

$$\mathcal{H} = \{s \mapsto G(s) : G(s) \in C^\infty(\mathbb{R}^d) \text{ and (G1)and (G4) hold}\}.$$

Clearly, $\mathcal{H} \subset \mathcal{G}$. For $G \in \mathcal{G}$, set

$$\|G\|_Z = \sqrt{I_{G,Z}}.$$

Notice that $I_{G,Z} \leq \tilde{I}_{G,Z} < \infty$ by (G5) and (G6). By Remark 4, the map $G \mapsto v_{G,Z}$ is an isometry from $(\mathcal{H}, \|\cdot\|_Z)$ into $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$. Therefore, this isometry extends to the closure of $(\mathcal{H}, \|\cdot\|_Z)$ in $\mathcal{G}$.

Lemma 5. *$\mathcal{G}$ is contained in the closure of $(\mathcal{H}, \|\cdot\|_Z)$.*

Proof. Fix $s \mapsto G(s)$ in $\mathcal{G}$. Let ψ_n be as in the proof of Lemma 3. Set

$$G_n(s, \cdot) = G(s) * \psi_n(\cdot).$$

Then $G_n(s, \cdot) \in C^\infty(\mathbb{R}^d)$ by [20], Chap.VI, Thm.11 p.166. By (G6), $\mathcal{F}G_n(s, \cdot) = \mathcal{F}G(s) \cdot \mathcal{F}\psi_n$ is a function, and so by (G7), (G1) holds for G_n. Because $|\mathcal{F}\psi_n| \leq 1$, (G4) holds for G_n because it holds for G by (G6). Therefore, $G_n \in \mathcal{H}$.

Observe that

$$\begin{aligned}
\|G - G_n\|_Z^2 &= I_{G - G_n, Z} \\
&= \int_0^T ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s, \cdot)(\xi)|^2 \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}G(s, \cdot)(\xi - \eta)|^2 |1 - \mathcal{F}\psi_n(\xi - \eta)|^2
\end{aligned}$$

The last factor is bounded by 4, has limit 0 as $n \to \infty$, and $I_{G,Z} < \infty$, so the Dominated Convergence Theorem implies that

$$\lim_{n \to \infty} \|G - G_n\|_Z = 0.$$

This proves the lemma. ∎

Theorem 6. *Fix Z such that (G5) holds, and $s \mapsto G(s)$ such that (G6) and (G7) hold. Then the stochastic integral $v_{G,Z} = G \cdot M^Z$ is well-defined, with the isometry property*

$$E\left(\|v_{G,Z}\|^2_{L^2(\mathbb{R}^d)}\right) = I_{G,Z}.$$

It is natural to use the notation

$$v_{G,Z} = \int_0^T ds \int_{\mathbb{R}^d} G(s, \cdot - y) Z(s,y) M(ds, dy),$$

and we shall do this in the sequel.

Proof of Theorem 6. The statement is an immediate consequence of Lemma 5. ∎

Remark 7. *Fix a deterministic function $\psi \in L^2(\mathbb{R}^d)$ and set*

$$X_t = \left\langle \psi, \int_0^t \int_{\mathbb{R}^d} G(s, \cdot - y) Z(s,y)\, M(ds, dy) \right\rangle_{L^2(\mathbb{R}^d)}.$$

It is not difficult to check that $(X_t,\ 0 \le t \le T)$ is a (real-valued) martingale.

3 Examples

In this section, we give a class of examples to which Theorem 6 applies. Fix an integer $k \ge 1$ and let G be the Green's function of the p.d.e.

$$\frac{\partial^2 u}{\partial t^2} + (-1)^k \Delta^{(k)} u = 0. \tag{3.1}$$

As in [5], Section 3, $\mathcal{F}G(t)(\xi)$ is easily computed, and one finds

$$\mathcal{F}G(t)(\xi) = \frac{\sin(t|\xi|^k)}{|\xi|^k}.$$

According to [5], Theorem 11 (see also Remark 12 in that paper), the linear s.p.d.e

$$\frac{\partial^2 u}{\partial t^2} + (-1)^k \Delta^{(k)} u = \dot{F}(t,x) \tag{3.2}$$

with vanishing initial conditions has a process solution if and only if

$$\int_0^T ds \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}G(s)|^2 < \infty,$$

or equivalently,

$$\int_{\mathbb{R}^d} \mu(d\xi)\, \frac{1}{(1+|\xi|^2)^k} < \infty. \tag{3.3}$$

It is therefore natural to assume this condition in order to study non-linear forms of (3.2).

In order to be able to use Theorem 6, we need the following fact.

Lemma 8. *Suppose (3.3) holds. Then the Green's function G of equation (3.1) satisfies conditions (G6) and (G7).*

Proof. We begin with (G7). For $\psi \in C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} \|G(s) * \psi\|_{L^\infty(\mathbb{R}^d)} &\leq \|\mathcal{F}(G(s) * \psi)\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \frac{|\sin(s|\xi|^k)|}{|\xi|^k} |\mathcal{F}\psi(\xi)|\, d\xi \\ &\leq s \int_{\mathbb{R}^d} |\mathcal{F}\psi(\xi)|\, d\xi < \infty, \end{aligned}$$

so (G7) holds.

Turning to (G6), we first show that

$$\langle \chi_\xi G_{d,k}, \Gamma\rangle = \left\langle \frac{1}{(1+|\xi - \cdot|^2)^k}, \mu \right\rangle, \tag{3.4}$$

where

$$0 \leq G_{d,k}(x) = \frac{1}{\gamma(k)} \int_0^\infty e^{-u} u^{k-1} p(u,x)\, du,$$

$\gamma(\cdot)$ is Euler's Gamma function and $p(u,x)$ is the density of a $N(0,uI)-$ random vector (see [19], Section 5). In particular,

$$\mathcal{F}G_{d,k}(\xi) = \frac{1}{(1+|\xi|^2)^k},$$

and it is shown in [7] and [19] that

$$\langle G_{d,k}, \Gamma\rangle = \langle (1+|\cdot|^2)^{-k}, \mu\rangle, \tag{3.5}$$

and the right-hand side is finite by (3.3). However, the proofs in [19] and [7] use monotone convergence, which is not applicable in presence of the oscillating function χ_ξ. As in [7], because $e^{-t|\cdot|^2}$ has rapid decrease,

$$\left\langle \frac{e^{-t|\cdot|^2}}{(1+|\xi-\cdot|^2)^k}, \mu \right\rangle = \left\langle \mathcal{F}\left(\frac{e^{-t|\cdot|^2}}{(1+|\xi-\cdot|^2)^k}\right), \mathcal{F}\mu \right\rangle = \langle p(t,\cdot) * (\chi_\xi G_{d,k}), \Gamma\rangle.$$

Notice that $G_{d,k} \geq 0$, and so

$$|p(t,\cdot) * (\chi_\xi G_{d,k})| \leq p(t,\cdot) * G_{d,k} \leq e^T G_{d,k}$$

by formula (5.5) in [19], so we can use monotone convergence in the first equality below and the Dominated Convergence Theorem in the third equality below to conclude that

$$\begin{aligned} \left\langle \tfrac{1}{(1+|\xi-\cdot|^2)^k}, \mu \right\rangle &= \lim_{t\downarrow 0} \left\langle \tfrac{e^{-t|\cdot|^2}}{(1+|\xi-\cdot|^2)^k}, \mu \right\rangle = \lim_{t\downarrow 0} \langle p(t,\cdot) * (\chi_\xi G_{d,k})), \Gamma\rangle \\ &= \langle \lim_{t\downarrow 0} (p(t,\cdot) * (\chi_\xi G_{d,k})), \Gamma\rangle = \langle \chi_\xi G_{d,k}, \Gamma\rangle, \end{aligned}$$

which proves (3.4). Because $G_{d,k} \geq 0$,

$$\sup_\xi \langle \chi_\xi G_{d,k}, \Gamma\rangle \leq \langle G_{d,k}, \Gamma\rangle < \infty \tag{3.6}$$

by (3.5) and (3.3). The lemma is proved. ∎

4 A non-linear s.p.d.e

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function such that $\alpha(0) = 0$, so that there is a constant $K > 0$ such that for $u, u_1, u_2 \in \mathbb{R}$,

$$|\alpha(u)| \leq K|u| \qquad \text{and} \qquad |\alpha(u_1) - \alpha(u_2)| \leq K|u_1 - u_2|. \tag{4.1}$$

Examples of such functions are $\alpha(u) = u$, $\alpha(u) = \sin(u)$, or $\alpha(u) = 1 - e^{-u}$.

Consider the non-linear s.p.d.e.

$$\frac{\partial^2}{\partial t^2} u(t,x) + (-1)^k \Delta^{(k)} u(t,x) = \alpha(u(t,x))\dot{F}(t,x), \tag{4.2}$$

$$u(0,x) = v_0(x), \qquad \frac{\partial}{\partial t} u(0,x) = \tilde{v}_0(x)$$

where $v_0 \in L^2(\mathbb{R}^d)$ and $\tilde{v}_0 \in H^{-k}(\mathbb{R}^d)$, the Sobolev space of distributions such that

$$\|\tilde{v}_0\|^2_{H^{-k}(\mathbb{R}^d)} \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} d\xi \, \frac{1}{(1+|\xi|^2)^k} |\mathcal{F}\tilde{v}_0(\xi)|^2 < \infty.$$

We say that a process $(u(t,\cdot),\ 0 \leq t \leq T)$ with values in $L^2(\mathbb{R}^d)$ is a solution of (4.2) if, for all $t \geq 0$, a.s.,

$$u(t,\cdot) = \frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 + \int_0^t \int_{\mathbb{R}^d} G(t-s, \cdot - y)\alpha(u(s,y)) M(ds, dy), \tag{4.3}$$

where G is the Green's function of (3.1). The third term is interpreted as the stochastic integral from Theorem 6, so $(u(s,\cdot))$ must be adapted and mean-square continuous from $[0,T]$ into $L^2(\mathbb{R}^d)$.

Theorem 9. *Suppose that (3.3) holds. Then equation (4.2) has a unique solution $(u(t,\cdot),\ 0 \leq t \leq T)$. This solution is adapted and mean-square continuous.*

Proof. We will follow a standard Picard iteration scheme. Set

$$u_0(t,\cdot) = \frac{d}{dt}G(t) * v_0 + G(t) * \tilde{v}_0.$$

Notice that $v_0(t,\cdot) \in L^2(\mathbb{R}^d)$. Indeed,

$$\begin{aligned} \left\| \frac{d}{dt}G(t) * v_0 \right\|_{L^2(\mathbb{R}^d)} &= \left\| \mathcal{F}\frac{d}{dt}G(t) \cdot \mathcal{F}v_0 \right\|_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \sin^2(t|\xi|^k)|\mathcal{F}v_0(\xi)|^2 d\xi \\ &\leq \|v_0\|_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{4.4}$$

and one checks similarly that $\|G(t) * \tilde{v}_0\|_{L^2(\mathbb{R}^d)} \leq \|\tilde{v}\|_{H^{-k}}$. Further, $t \mapsto u_0(t,\cdot)$ from $[0,T]$ into $L^2(\mathbb{R}^d)$ is continuous. Indeed,

$$\lim_{t\to s} \left\| \frac{d}{dt}G(t) * v_0 - \frac{d}{dt}G(s) * v_0 \right\|_{L^2(\mathbb{R}^d} = 0,$$

as is easily seen by proceeding as in (4.4) and using dominated convergence. Similarly,

$$\lim_{t\to s} \|G(t) * \tilde{v}_0 - G(s) * \tilde{v}_0\| = 0.$$

For $n \geq 0$, assume now by induction that we have defined an adapted and mean-square continuous process $(u_n(s,\cdot),\ 0 \leq s \leq T)$ with values in $L^2(\mathbb{R}^d)$, and define

$$u_{n+1}(t,\cdot) = u_0(t,\cdot) + v_{n+1}(t,\cdot), \tag{4.5}$$

where

$$v_{n+1}(t,\cdot) = \int_0^t \int_{\mathbb{R}^d} G(t-s,\cdot - y)\alpha(v_n(s,g))M(ds,dy). \tag{4.6}$$

We note that $(\alpha(u_n(s,\cdot)),\ 0 \leq s \leq T)$ is adapted and mean-square continuous, because by (4.1),

$$\|\alpha(u_n(s,\cdot)) - \alpha(u_n(t,\cdot))\|_{L^2(\mathbb{R}^d)} \leq K\|u_n(s,\cdot) - u_n(t,\cdot)\|_{L^2(\mathbb{R}^d)},$$

so the stochastic integral in (4.6) is well-defined by Lemma 8 and Theorem 6.
Set

$$J(s) = \sup_\xi \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}G(s,\cdot)(\xi - \eta)|^2. \tag{4.7}$$

By (3.3), (3.5) and (3.6), $\sup_{0\leq s\leq T} J(s)$ is bounded by some $C<\infty$, so by Theorem 6 and using (4.1),

$$\begin{aligned} E(\|u_{n+1}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}) &\leq 2\|u_0(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + 2\int_0^t ds\, E(\|\alpha(u_n(s,\cdot))\|^2_{L^2(\mathbb{R}^d)})J(t-s) \\ &\leq 2\|u_0(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + 2KC\int_0^t ds\, E(\|u_n(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}). \end{aligned} \tag{4.8}$$

Therefore, $u_{n+1}(t,\cdot)$ takes its values in $L^2(\mathbb{R}^d)$. By Lemma 10 below, $(u_{n+1}(t,\cdot),\ 0\leq t\leq T)$ is mean-square continuous and this process is adapted, so the sequence $(u_n,\ n\in\mathbb{N})$ is well-defined. By Gronwall's lemma, we have in fact

$$\sup_{0\leq t\leq T}\sup_{n\in\mathbb{N}} E(\|u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}) < \infty.$$

We now show that the sequence $(u_n(t,\cdot),\ n\geq 0)$ converges. Let

$$M_n(t) = E(\|u_{n+1}(t,\cdot)-u_n(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}).$$

Using the Lipschitz property of $\alpha(\cdot)$, (4.5) and (4.6), we see that

$$M_n(t) \leq KC\int_0^t ds\, M_{n-1}(s).$$

Because $\sup_{0\leq s\leq T} M_0(s)<\infty$, Gronwall's lemma implies that

$$\sum_{n=0}^{\infty} M_n(t)^{1/2} < \infty.$$

In particular, $(u_n(t,\cdot),\ n\in\mathbb{N})$ converges in $L^2(\Omega\times\mathbb{R}^d, dP\times dx)$, uniformly in $t\in[0,T]$, to a limit $u(t,\cdot)$. Because each u_n is mean-square continuous and the convergence is uniform in t, $(u(t,\cdot),\ 0\leq t\leq T)$ is also mean-square continuous, and is clearly adapted. This process is easily seen to satisfy (4.3), and uniqueness is checked by a standard argument. ∎

The following lemma was used in the proof of Theorem 9.

Lemma 10. *Each of the processes* $(u_n(t,\cdot),\ 0\leq t\leq T)$ *defined in (4.5) is mean-square continuous.*

Proof. Fix $n\geq 0$. It was shown in the proof of Theorem 9 that $t\mapsto u_0(t,\cdot)$ is mean-square continuous, so we establish this property for $t\mapsto v_{n+1}(t,\cdot)$, defined in (4.6). Observe that for $h>0$,

$$E(\|v_{n+1}(t+h,\cdot)-v_{n+1}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}) \leq 2(I_1+I_2),$$

where

$$I_1 = E\left(\left\|\int_t^{t+h}\int_{\mathbb{R}^d} G(t+h-s,\cdot-y)\alpha(u_n(s,y))M(ds,dy)\right\|^2_{L^2(\mathbb{R}^d)}\right),$$

$$I_2 = E\left(\left\|\int_0^t\int_{\mathbb{R}^d} (G(t+h-s,\cdot-y)-G(t-s,\cdot-y))\alpha(u_n(s,y))M(dsdy)\right\|^2_{L^2(\mathbb{R}^d)}\right).$$

Clearly,

$$I_1 \le K^2\int_t^{t+h} ds\, E(\|u_n(s,\cdot)\|^2_{L^2(\mathbb{R}^d)})J(t+h-s),$$

while

$$I_2 = \int_0^t ds \int_{\mathbb{R}^d} d\xi\, |\mathcal{F}(\alpha(u_n(s,\cdot)))(\xi)|^2 \int_{\mathbb{R}^d}\mu(d\eta)\left(\frac{\sin(t+h-s)|\xi-\eta|^k)-\sin((t-s)|\xi-\eta|^k)}{|\xi-\eta|^k}\right)^2.$$

The squared ratio is no greater than

$$4\left(\frac{\sin(h|\xi-\eta|^k)}{|\xi-\eta|^k}\right)^2 \le \frac{C}{(1+|\xi-\eta|^2)^k}.$$

It follows that I_2 converges to 0 as $h \to 0$, by the dominated convergence theorem, and I_1 converges to 0 because the integrand is bounded. This proves that $t \mapsto v_{n+1}(t,\cdot)$ is mean-square right-continuous, and left-continuity is proved in the same way. ∎

5 The wave equation in weighted L^2-spaces

In the case of the wave equation (set $k = 1$ in (4.2)), we can consider a more general class of non-linearities $\alpha(\cdot)$ than in the previous section. This is because of the compact support property of the Green's function of the wave equation.

More generally, in this section, we fix $T > 0$ and consider a function $s \mapsto G(s) \in \mathcal{S}'(\mathbb{R}^d)$ that satisfies (G6), (G7) and, in addition,

(G8) There is $R > 0$ such that for $0 \le s \le T$, supp $G(s) \subset B(0,R)$.

Fix $K > d$ and let $\theta : \mathbb{R}^d \to \mathbb{R}$ be a smooth function for which there are constants $0 < c < C$ such that

$$c(1 \wedge |x|^{-K}) \le \theta(x) \le C(1 \wedge |x|^{-K}).$$

The weighted L^2-space L^2_θ is the set of measurable $f : \mathbb{R}^d \to \mathbb{R}$ such that $\|f\|_{L^2_\theta} < \infty$, where

$$\|f\|^2_{L^2_\theta} = \int_{\mathbb{R}^d} f^2(x)\theta(x)\,dx.$$

Let $H_n = \{x \in \mathbb{R}^d : nR \leq |x| < (n+1)R\}$, set

$$\|f\|_{L^2(H_n)} = \left(\int_{H_n} f^2(x)\,dx\right)^{1/2},$$

and observe that there are positive constants, which we again denote c and C, such that

$$c\sum_{n=0}^{\infty} n^{-K}\|f\|^2_{L^2(H_n)} \leq \|f\|^2_{L^2_\theta} \leq C\sum_{n=0}^{\infty} n^{-K}\|f\|^2_{L^2(H_n)}. \tag{5.1}$$

For a process $(Z(s,\cdot),\ 0 \leq s \leq T)$, consider the following hypothesis:

(G9) For $0 \leq s \leq T$, $Z(s,\cdot) \in L^2_\theta$ a.s., $Z(s,\cdot)$ is $\mathcal{F}_s$-measurable, and $s \mapsto Z(s,\cdot)$ is mean-square continuous from $[0,T]$ into L^2_θ.

Set

$$\begin{aligned}\mathcal{E}_\theta = \{Z : \quad & \text{(G9) holds, and there is } K \subset \mathbb{R}^d \text{ compact such} \\ & \text{that for } 0 \leq s \leq T,\ \operatorname{supp} Z(s,\cdot) \subset K\}.\end{aligned}$$

Notice that for $Z \in \mathcal{E}_\theta$, $Z(s,\cdot) \in L^2(\mathbb{R}^d)$ because $\theta(\cdot)$ is bounded below on K by a positive constant, and for the same reason, $s \mapsto Z(s,\cdot)$ is mean-square continuous from $[0,T]$ into $L^2(\mathbb{R}^d)$. Therefore,

$$v_{G,Z} = \int_0^T \int_{\mathbb{R}^d} G(s,\cdot - y)Z(s,y)M(ds,dy)$$

is well-defined by Theorem 6.

Lemma 11. *For G as above and $Z \in \mathcal{E}_\theta$, $v_{G,Z} \in L^2_\theta$ a.s. and*

$$E(\|v_{G,Z}\|^2_{L^2_\theta}) \leq \int_0^T ds\, \|Z(s,\cdot)\|^2_{L^2_\theta} J(s),$$

where $J(s)$ is defined in (4.7).

Proof. We assume for simplicity that $R = 1$ and K is the unit ball in $\mathbb{R}^d$. Set $D_0 = H_0 \cup H_1$ and, for $n \geq 1$, set $D_n = H_{n-1} \cup H_n \cup H_{n+1}$ and $Z_n(s,\cdot) = Z(s,\cdot)1_{D_n}(\cdot)$. By (5.1), then (G8),

$$\begin{aligned}\|v_{G,Z}\|^2_{L^2_\theta} &\leq \sum_{n=0}^{\infty} n^{-K}\|v_{G,Z}\|^2_{L^2(H_n)} = \sum_{n=0}^{\infty} n^{-K}\|v_{G,Z_n}\|^2_{L^2(H_n)} \\ &\leq \sum_{n=0}^{\infty} n^{-K}\|v_{G,Z_n}\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Therefore, by Theorem 6,

$$\begin{aligned}
E(\|v_{G,Z}\|^2_{L^2_\theta} &\le \sum_{n=0}^{\infty} n^{-K} \int_0^T ds\, E(\|Z_n(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}) J(s) \\
&= \sum_{n=0}^{\infty} n^{-K} \int_0^T ds\, E(\|Z(s,\cdot)\|^2_{L^2(D_n)}) J(s) \\
&\le C \int_0^T ds \sum_{n=0}^{\infty} n^{-K}\, E(\|Z(s,\cdot)\|^2_{L^2(H_n)}) J(s) \\
&\le C \int_0^T ds\, E(\|Z(s,\cdot)\|^2_{L^2_\theta}) J(s).
\end{aligned}$$

This proves the lemma. ∎

For a process $(Z(s,\cdot))$ satisfying (G9), let $\|Z\|_\theta = (I^\theta_{G,Z})^{\frac{1}{2}}$, where

$$I^\theta_{G,Z} = \int_0^T ds\, \|Z(s,\cdot)\|^2_{L^2_\theta} J(s).$$

Because $s \mapsto \|Z(s,\cdot)\|^2_{L^2_\theta}$ is bounded, $I^\theta_{G,Z} < \infty$ provided (G6) holds. Therefore, $\|Z\|_\theta$ defines a norm, and by Lemma 11, $Z \mapsto v_{G,Z}$ from $\mathcal{E}_\theta$ into $L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx)$ is continuous. Therefore this map extends to the closure of $\mathcal{E}_\theta$ for $\|\cdot\|_\theta$, which we now identify.

Theorem 12. *Consider a function $s \mapsto G(s) \in \mathcal{S}'(\mathbb{R}^d)$ such that (G6), (G7) and (G8) hold. Let $(Z(s,\cdot),\ 0 \le s \le T)$ be an adapted process with values in L^2_θ that is mean-square continuous from $[0,T]$ into L^2_θ. Then Z is in the closure of $\mathcal{E}_\theta$ for $\|\cdot\|_\theta$, and so the stochastic integral $v_{G,Z}$ is well-defined, and*

$$E(\|v_{G,Z}\|^2_{L^2_\theta}) \le I^\theta_{G,Z}. \tag{5.2}$$

Proof. Set $Z_n(s,\cdot) = Z(s,\cdot) 1_{[-n,n]^d}(\cdot)$. Then (Z_n) satisfies (G9) and belongs to $\mathcal{E}_\theta$. Because $\|Z_n(s,\cdot)\|_{L^2_\theta} \le \|Z(s,\cdot)\|_{L^2_\theta}$ and $I^\theta_{G,Z} < \infty$, the dominated convergence theorem implies that $\lim_{n\to\infty} \|Z - Z_n\|_\theta = 0$, so Z is in the closure of $\mathcal{E}_\theta$ for $\|\cdot\|_\theta$, and (5.2) holds by Lemma 11. ∎

We now use this result to obtain a solution to the following stochastic wave equation:

$$\frac{\partial^2}{\partial t^2} u(t,x) - \Delta u(t,x) = \alpha(u(t,x)) \dot{F}(t,y), \tag{5.3}$$

$$u(0,x) = v_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = \tilde{v}_0(x).$$

We say that a process $(u(t,\cdot),\ 0 \le t \le T)$ with values in L^2_θ is a solution of (5.3) if $(u(t,\cdot))$ is adapted, $t \mapsto u(t,\cdot)$ is mean-square continuous from $[0,T]$ into L^2_θ and

$$u(t,\cdot) = \frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 + \int_0^t \int_{\mathbb{R}^d} G(t-s,\cdot - y)\alpha(u(s,y)) M(ds,dy), \tag{5.4}$$

where G is the Green's function of the wave equation. In particular, $\mathcal{F}G(s)(\xi) = |\xi|^{-1}\sin(t|\xi|)$ and (G6), (G7) and (G8) hold provided (3.3) holds with $k=1$. Therefore, the stochastic integral in (5.4) is well-defined by Theorem 12.

Theorem 13. *Suppose*

$$\int_{\mathbb{R}^d} \mu(d\xi)\, \frac{1}{1+|\xi|^2} < \infty,$$

$v_0 \in L^2(\mathbb{R}^d)$, $\tilde{v}_0 \in H^{-1}(\mathbb{R}^d)$, *and* $\alpha(\cdot)$ *is a globally Lipschitz function. Then (5.3) has a unique solution in* L^2_θ.

Proof. The proof follows that of Theorem 9, so we only point out the changes relative to the proof of that theorem. Because $\alpha(\cdot)$ is globally Lipschitz, there is $K > 0$ such that for $u, u_1, u_2 \in \mathbb{R}$,

$$|\alpha(u)| \le K(1+|u|) \qquad \text{and} \qquad |\alpha(u_1) - \alpha(u_2)| \le K|u_1 - u_2|.$$

Using the first of these inequalities, (4.8) is replaced by

$$E(\|u_{n+1}(t,\cdot)\|^2_{L^2_\theta}) \le 2\|u_0(t,\cdot)\|_{L^2(\mathbb{R}^d)} + 2KC \int_0^t ds\, (1 + E(\|u_n(s,\cdot)\|^2_{L^2_\theta})).$$

Therefore $u_{n+1}(t,\cdot)$ takes its values in L^2_θ. The remainder of the proof is unchanged, except that $\|\cdot\|_{L^2(\mathbb{R}^d)}$ must be replaced by $\|\cdot\|_{L^2_\theta}$. This proves Theorem 13. ∎

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