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INVARIANT WEDGES FOR A TWO-POINT REFLECTING BROWNIAN MOTION AND THE "HOT SPOTS" PROBLEM

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Abstract We consider domains D of $\mathbb{R}^d$, $d \geq 2$ with the property that there is a wedge $V \subset \mathbb{R}^d$ which is left invariant under all tangential projections at smooth portions of ∂D. It is shown that the difference between two solutions of the Skorokhod equation in D with normal reflection, driven by the same Brownian motion, remains in V if it is initially in V. The heat equation on D with Neumann boundary conditions is considered next. It is shown that the cone of elements u of $L^2(D)$ satisfying $u(x) - u(y) \geq 0$ whenever $x - y \in V$ is left invariant by the corresponding heat semigroup. Positivity considerations identify an eigenfunction corresponding to the second Neumann eigenvalue as an element of this cone. For $d = 2$ and under further assumptions, especially convexity of the domain, this eigenvalue is simple.

Keywords Reflecting Brownian motion, Neumann eigenvalue problem, Convex domains.

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1 Introduction

The "hot spots" property of a bounded connected open domain $D \subset \mathbb{R}^d$ refers to the location of the extrema of eigenfunctions corresponding to the second eigenvalue of the Laplacian on D with Neumann boundary conditions. Among the various statements associated with this property [1, 3, 5, 10, 11] let us mention:

(HS) Every eigenfunction corresponding to the second Neumann eigenvalue attains its extrema solely on the boundary.

(HS') There exists an eigenfunction corresponding to the second Neumann eigenvalue which attains its extrema solely on the boundary.

The name comes from considering the heat equation on D:

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta_x u(t,x), & x \in D,\ t > 0, \\ \partial_\nu u(t,x) = 0, & x \in \partial D,\ t > 0, \\ u(0,x) = u_0(x), & x \in D. \end{cases} \tag{1}$$

Under weak assumptions on D an eigenfunction expansion for the solutions of the heat equation is available. Since the eigenfunction corresponding to the first eigenvalue is a constant, the spatial locations of extrema of the solution for a typical initial condition are governed at large values of t by the second eigenspace. The assertion is therefore on the "hottest" and "coldest" spots in D.

Although it has been shown that (HS) holds for special domains by direct calculation, there are large classes of domains about which little is known. For example, for a planar simply connected domain with smooth boundary and no line of symmetry, there is no known criterion (to the author's best knowledge) to check whether (HS) is satisfied, other than calculation when it is possible. In [5] an example was first found of a domain that does not satisfy (HS), and in [3] an example was found of a domain for which the second Neumann eigenvalue (which we denote by μ_2 in what follows), is simple, and the corresponding eigenfunction attains both its strict extrema in the interior. See [1] for a conjecture that (HS) holds for all convex planar domains, and [5] for a conjecture that it holds for planar domains with at most one hole.

Bañuelos and Burdzy [1] were the first to identify rich classes of domains for which (HS) or (HS') hold. They use probabilistic techniques, and in particular, in one of the methods they develop, they consider a coupling of two normally reflecting Brownian motions driven by the same unconstrained Brownian motion. They consider planar domains and show that if all lines tangential to the boundary at its smooth portions form angles with a fixed axis within a range of less then $\pi/2$, then the difference between the processes will form an angle with this axis within this range for all times, if it does initially. They show that as a consequence, solutions to the Neumann heat equation are monotone along all lines within this range of angles when initialized appropriately, and therefore a similar statement holds for at least one eigenfunction corresponding to μ_2. Hence the extrema of this eigenfunction are obtained on the boundary and (HS') holds. Under further assumptions they show that μ_2 is simple hence (HS) must hold. Jerison and Nadirashvili [10] have established the hot spots property for planar domains with two axes of symmetry.

In dimension greater than two, the only domains known to satisfy the hot spots property, other than domains with special symmetry, are those of the form $D' \times [0, a]$ (see Kawohl [11]), or more generally $D' \times D''$ (see [1]). One of our two goals is provide new classes of domains in higher dimension satisfying (HS'). What might naively be expected to be the extension of the result of [1] regarding the coupled Brownian motions, does not hold. For example, the condition that all unit normals to ∂D at its smooth portions form scalar product with a fixed vector within the range of $(-\epsilon, \epsilon)$, does not guarantee that there is an invariant set for the coupled processes, in the sense described above, no matter how small $\epsilon > 0$ is. We show that the approach of [1] can be generalized in a different way. Our assumption on D is that it is piecewise smooth with "convex corners" (see Condition 2.1). We then assume there is a wedge $V \subset \mathbb{R}^d$ (see Definition 2.1) that is left invariant under all projections onto subspaces tangential to ∂D at smooth portions. We prove that two reflecting Brownian motions coupled as described above have difference in V if initialized there. As a result, the Neumann heat semigroup leaves the following cone of $L^2(D)$ invariant:

$$\{u \in L^2(D) : u(x) - u(y) \geq 0, \quad x, y \in D, x - y \in V\}. \tag{2}$$

Invoking positivity considerations the following is shown (see Theorem 3.1).

Assume that for some $\gamma \in \mathbb{R}^d$ one has $\langle v, \gamma \rangle > 0$ for all $v \in V$. Then there is an eigenfunction corresponding to the second Neumann eigenvalue attaining both strict extrema on the boundary.

(HS) follows from (HS') whenever it is known that μ_2 is simple. In a sense, it is typical that μ_2 is simple, as can be seen e.g. in [15]. It is known also that for simply connected planar domains, the multiplicity is at most two [16]. However, for a given domain it is in general hard to determine whether an eigenvalue is simple. An exception is a result that appears in [1], where it is shown that μ_2 is simple for convex planar domains for which the diameter to width ratio exceeds a certain number.

In this paper we identify a new class of planar domains for which μ_2 is simple. It is a subclass of the planar domains for which we prove that (HS') holds (with $V = \mathbb{R}^2_+$). As a result, they satisfy (HS). We show (see Theorem 4.1)

Let D be a planar open domain bounded between the graphs of two C^2 increasing functions, one of which is convex and the other concave, such that D is bounded. Assume that for each corner point Ξ of the domain, $B_r(\Xi) \cap D$ is a sector for some $r > 0$, of angle within $[\pi/4, \pi/2)$. Then the second Neumann eigenvalue on D is simple.

It is conjectured in [1] (p. 5) that μ_2 is simple for all convex planar domains with diameter to width ratio greater than $\sqrt{2}$. The above statement addresses a subclass of this class of domains.

The rest of this paper is organized as follows. In Section 2 we make the assumption of existence of an invariant wedge V, and show that for a pair of solutions to the Skorokhod problem for an arbitrary continuous function, the difference is kept in V. Using results of [14] this is shown to imply a similar statement to semi-martingale reflecting Brownian motions. In Section 3 we exploit general positivity considerations to show that there must exist a second eigenfunction in the cone (2). Section 4 establishes simplicity of μ_2 for a class of planar convex domains. In the appendix we provide examples of three dimensional domains satisfying (HS').

2 An invariance property

In this section we use the results of Lions and Sznitman [14] that guarantee the existence of a unique solution to the Skorokhod problem for arbitrary continuous paths. We prove a certain invariance property in continuous paths space, which then translates to a property of semimartingale reflecting Brownian motions.

We will always consider domains that satisfy the following.

Condition 2.1 *$D \subset \mathbb{R}^d$ is a bounded, connected domain with Lipschitz boundary, and is given by the intersection of a finite collection $\{D_i\}_{i\in I}$ of open sets with C^2 boundary.*

For domains satisfying Condition 2.1 we will consider the vector field n of unit inward normals. The (not necessarily single-valued) vector field n is defined on the boundary ∂D of D as follows. For $x \in \partial D_i$ let $n^{x,i}$ denote the unit inward normal to ∂D_i at x. Then for $x \in \partial D$ we let $I_x = \{i \in I : x \in \partial D_i\}$ and

$$n(x) = \{\nu : \nu = \sum_{i\in I_x} a_i n^{x,i},\ |\nu| = 1,\ a_i \geq 0,\ i \in I_x\}.$$

We remark that the definition of [14] (equation (1) p. 514) of a vector field for a more general class of domains reduces to the above definition for the domains considered here.

The set $N = N(D)$ is defined as

$$N = \{n^{x,i} : i \in I_x,\ x \in \partial D\}. \tag{3}$$

For $n \in N$, the n-projection $\pi_n : \mathbb{R}^d \to \mathbb{R}^d$ is defined as

$$\pi_n v = v - \langle v, n\rangle n.$$

For $x, y \in \mathbb{R}^d$, let $\overline{xy}$ denote the closed line segment $\overline{xy} = \{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$. Let $W = W(D)$ be defined by

$$W = \{x - y : x, y \in \bar{D},\ \overline{xy} \not\subset \bar{D}\}.$$

Definition 2.1 *A proper subset V of $\mathbb{R}^d$ is called a* wedge *if it is closed, convex, has non-empty interior, and if $v \in V$ implies $\alpha v \in V$ for all $\alpha \geq 0$.*

The most significant assumption we make is the following.

Condition 2.2 *There exists a wedge V satisfying $W \subset V \cup -V$, and*

$$v \in V, n \in N \Longrightarrow \pi_n v \in V.$$

We will say that a wedge V is *invariant for D* if it satisfies Condition 2.2. We next formulate an equivalent to Condition 2.2.

Condition 2.2′ *There exists a wedge V satisfying $W \subset V \cup -V$, and*

$$v \in \partial V, n \in N \Longrightarrow \langle m, n\rangle\langle n, v\rangle \leq 0,$$

where m is any inward normal to ∂V at v.

Proposition 2.1 *Conditions 2.2 and 2.2′ are equivalent.*

Proof: Let V satisfy Condition 2.2. Let $v \in \partial V$ and let m be an inward normal to ∂V at v. By assumption, $\pi_n v \in V$ for any $n \in N$, hence by convexity $\langle \pi_n v - v, m\rangle \geq 0$. It follows that $\langle m, n\rangle\langle n, v\rangle \leq 0$, hence Condition 2.2′ holds (with the same set V).

Next, let V satisfy Condition 2.2′. We will show that for $v \in V$ and $n \in N$ one has $\pi_n v \in V$. If we assume the contrary then for some $v \in V$ and $n \in N$ we have $\pi_n v \in V^c$. Since V is convex with non-empty interior, every neighborhood of v contains a point in the interior. Since V^c is open and π_n continuous, there is a point $w \in V^o$ for which $u \doteq \pi_n w \in V^c$. The convexity of V implies that the line segment $\overline{wu}$ intersects ∂V at exactly one point, say z. There must exist an inward normal $\tilde{m}$ to ∂V at z such that $\langle \tilde{m}, w - z\rangle > 0$, and consequently $\langle \tilde{m}, u - z\rangle < 0$. Note however that $\pi_n w = \pi_n z = u$. Hence $\langle \tilde{m}, \pi_n z - z\rangle < 0$ and it follows that $\langle \tilde{m}, n\rangle\langle n, z\rangle > 0$, in contradiction with Condition 2.2′. Therefore Condition 2.2 holds. □

The definition of a solution to the Skorokhod problem for an arbitrary continuous function follows [14]. The notation $|h|_t$ is for the total variation of a function $h : [0, \infty) \to \mathbb{R}^d$ on $[0, t]$ with respect to the Euclidean norm on $\mathbb{R}^d$.

Definition 2.2 *For $w \in C([0, \infty) : \mathbb{R}^d)$ such that $w(0) \in \bar{D}$ we say that the pair (x, l)* solves the Skorokhod problem (w, D, n) *if the conditions listed below are satisfied.*

$$x \in C([0, \infty), \bar{D}), \quad l \in C([0, \infty), \mathbb{R}^d),$$

$$x_t = w_t + l_t, \quad t \geq 0,$$

$$l|_{[0,t]} \in BV(0, t) \text{ for all } t < \infty,$$

$$|l|_t = \int_0^t 1_{x_s \in \partial D} d|l|_s, \quad l_t = \int_0^t \xi_s d|l|_s \text{ with } \xi_s \in n(x_s).$$

The following result is a special case of [14] Theorem 2.2 (see also [14] Remark 2.4 regarding domains with convex corners; the uniform exterior sphere condition obviously holds).

Theorem 2.1 (Lions and Sznitman) *Let $D \subset \mathbb{R}^d$ satisfy Condition 2.1 and let $w \in C([0, \infty), \mathbb{R}^d)$ with $w(0) = \bar{D}$. Then there exists a unique solution (x, l) to the Skorokhod problem (w, D, n).*

Let $w \in C([0, \infty), \mathbb{R}^d)$ be such that $w(0) = 0$ and let $x_0, y_0 \in \bar{D}$ be given. By Theorem 2.1 there exists a unique solution to the Skorokhod problem $(x_0 + w, D, n)$ [resp., $(y_0 + w, D, n)$] which we denote by (x, l^x) [resp., (y, l^y)]. Thus for all $t \geq 0$

$$x_t = x_0 + w_t + \int_0^t \nu_s^x d|l^x|_s, \quad \nu_s^x \in n(x_s),$$

$$y_t = y_0 + w_t + \int_0^t \nu_s^y d|l^y|_s, \quad \nu_s^y \in n(y_s).$$

Define

$$\delta_t = x_t - y_t = \delta_0 + \int_0^t \nu_s^x d|l^x|_s - \int_0^t \nu_s^y d|l^y|_s \tag{4}$$

and note that it is of bounded variation on any bounded interval.

Note that every wedge may occupy no more than a half space i.e., there must exist a $\gamma \in \mathbb{R}^d$ for which $\langle v, \gamma \rangle \geq 0$ for all $v \in V$. The following condition is slightly stronger.

Condition 2.3 *There exists a $\gamma \in \mathbb{R}^d$ such that for every $v \in V \setminus \{0\}$, $\langle v, \gamma \rangle > 0$.*

Theorem 2.2 *Assume Conditions 2.1 and 2.2 hold, let V be an invariant wedge for D and assume that it satisfies Condition 2.3. Then if $\delta_0 \in V$, one has $\delta_t \in V$ for all $t > 0$.*

Let V be some set satisfying Condition 2.2. We define on V^c a vector field m as follows. For each $\epsilon > 0$ let V_ϵ be the open convex subset of $\mathbb{R}^d$ defined by

$$V_\epsilon = \{v \in \mathbb{R}^d : \mathrm{dist}(v, V) < \epsilon\}. \tag{5}$$

For each $v \in V^c$ let $m(v)$ be the unit inward normal to ∂V_ϵ at v, with $\epsilon = \mathrm{dist}(v, V)$. Note that the function is well defined since the normal is unique.

Lemma 2.1 *Let $u : [s, t] \to V^c$ be continuous and of bounded variation. Then*

$$\mathrm{dist}(u(t), V) - \mathrm{dist}(u(s), V) = \int_s^t \langle -m(u(\theta)), du(\theta) \rangle. \tag{6}$$

Proof: We first show that the integral on the right is well defined by showing that $m(u(\theta))$ is in fact continuous in θ. It is enough to show that $m(u)$ is continuous in u. The proof of this fact is elementary and for completeness we have included it in the appendix (see Lemma 5.1).

Define $\psi : V^c \to \mathbb{R}^d$ by $\psi(x) = \mathrm{dist}(x, V)$. It is elementary to show that for any $r \in \mathbb{R}^d$, $|r| = 1$, and $x \in V^c$, the directional derivative of ψ at x satisfies $\lim_{\epsilon \downarrow 0} \epsilon^{-1}(\psi(x+\epsilon r) - \psi(x)) = \langle -m(x), r \rangle$. This shows that $\nabla \psi(x)$ is well defined and is equal to $-m(x)$. Since m is continuous ψ is C_1, and it follows that

$$\psi(u(t)) - \psi(u(s)) = \int_s^t \langle \nabla \psi(u(\theta)), du(\theta) \rangle$$

which proves the lemma. □

Below, we borrow some ideas from the proof of [8] Theorem 2.2.

Proof of Theorem 2.2: Let $u \in \partial V_\epsilon$ and $m = m(u)$. We first show that for all $n \in N$

$$\langle m, n \rangle \langle n, u \rangle \leq -\epsilon \langle m, n \rangle^2. \tag{7}$$

Let $v = u + \epsilon m$. Then it is easy to see that $v \in \partial V$ and that m is an inward normal to ∂V at v. By Proposition 2.1, Condition 2.2$'$ holds. Hence $\langle m, n \rangle \langle n, v \rangle \leq 0$, $n \in N$ and the estimate (7) follows.

Assume that the conclusion of the theorem does not hold. Note that $\delta_s = 0$ implies $\delta_t = 0$, $t > s$. Note also that by Condition 2.3 there is a γ such that for $v \in V$ either $\langle v, \gamma \rangle > 0$ or $v = 0$. Hence there must exist s, t such that for all $\theta \in [s, t]$, $\delta_\theta \in V^c \cap \{u : \langle u, \gamma \rangle > 0\}$ and such that $\mathrm{dist}(\delta_t, V) > \mathrm{dist}(\delta_s, V)$. It follows from Lemma 2.1 that

$$\int_s^t \langle m(\delta_\theta), \nu_\theta^x \rangle d|l^x|_\theta < \int_s^t \langle m(\delta_\theta), \nu_\theta^y \rangle d|l^y|_\theta,$$

where $\nu_\theta^x \in n(x_\theta), \nu_\theta^y \in n(y_\theta), \theta \in [s,t]$. Therefore there must exist a $\theta \in [s,t]$ such that either

$$\text{(a)} \qquad \langle m(\delta_\theta), \nu_\theta^x \rangle < 0 \text{ and } x_\theta \in \partial D;$$

or

$$\text{(b)} \qquad \langle m(\delta_\theta), \nu_\theta^y \rangle > 0 \text{ and } y_\theta \in \partial D.$$

Since the argument is similar in both cases, we only consider case (a).

On one hand, one has in case (a) that $\langle m(\delta_\theta), \nu \rangle < 0$ for $\nu = n^{x_\theta, i}$ and some $i \in I_{x_\theta}$. Hence by (7), $\langle \nu, \delta_\theta \rangle > 0$. On the other hand, since $W \subset V \cup -V$ and $\langle \delta_\theta, \gamma \rangle > 0$ one has that $\delta_\theta \notin W$. It follows that $\overline{x_\theta y_\theta} \subset \bar{D}$. Therefore $\langle \delta_\theta, \nu \rangle \leq 0$ for every $\nu \in n(x_\theta)$. This is a contradiction. □

We close this section by showing an implication to reflecting Brownian motion. On a complete probability space (Ω, F, P) with an increasing family of sub σ-fields $(F_t)_{t \geq 0}$ let $(W_t)_{t \geq 0}$ be a d-dimensional F_t-Brownian motion.

The following result is proved in [14] (condition [14](9) holds by Remark [14] 3.9).

Theorem 2.3 (Lions and Sznitman) *Let D satisfy Condition 2.1 and let $x_0 \in \bar{D}$ be given. Then there exists a unique continuous F_t-semimartingale $(X_t)_{t \geq 0} = (X_t^{x_0})_{t \geq 0}$ satisfying:*

$$\begin{aligned}
&\textit{There exists an } \mathbb{R}^d\textit{-valued continuous bounded variation process } L_t \\
&\textit{such that for all } t \geq 0 \textit{ a.s.} \quad X_t \in \bar{D}, \\
&X_t = x_0 + W_t + L_t, \\
&|L|_t = \int_0^t 1_{X_s \in \partial D} d|L|_s, \\
&L_t = \int_0^t \xi_s d|L|_s \textit{ with } \xi_s \in n(X_s).
\end{aligned} \tag{8}$$

The conclusion of Theorem 2.3 is equivalent to the statement that for a.e. $\omega \in \Omega$, (X, L) solves the Skorokhod problem $(x_0 + W, D, n)$ (see e.g., Remark 3.2 of [14]).

Let $x_0, y_0 \in \bar{D}$. A two-point semimartingale reflecting Brownian motion in $\bar{D}$ started at (x_0, y_0) is a pair (X_t, Y_t) of F_t-semimartingales as in Theorem 2.3 with $(X_t) = (X_t^{x_0})$ and $(Y_t) = (X_t^{y_0})$. Note that by definition X and Y are driven by the same process W, but have different initial conditions, x_0 and y_0.

From Theorems 2.2 and 2.3 we obtain the following.

Corollary 2.1 *Let (X_t, Y_t) be a two-point semimartingale reflecting Brownian motion in $\bar{D}$ started at $(x_0, y_0) \in \bar{D} \times \bar{D}$. Let the assumptions of Theorem 2.2 hold. If $x_0 - y_0 \in V$ then a.s., $X_t - Y_t \in V$ for all $t > 0$.*

In what follows we will denote by $P_{x,y}$ the probability measure on $(\Omega, F, (F_t))$ for which $P_{x,y}[(X_0, Y_0) = (x, y)] = 1$. P_x will denote the restriction of $P_{x,y}$ to the σ-field generated by X.. $E_{x,y}$ and respectively, E_x will denote expectation with respect to $P_{x,y}$ and P_x.

Remark: In [8] Lipschitz continuity of the Skorokhod map in path space is shown to follow from the existence of a certain convex set. Condition 2.2 is in a sense analogous to Assumption 2.1 of [8] and the condition in Lemma 2.1 there. Also, the equivalence between Conditions 2.2 and 2.2′ is a reminiscent of the results of Section 2.5 of [9].

3 Positivity considerations for the heat semigroup

Consider the heat equation with Neumann boundary conditions (1), the corresponding heat kernel $p_t(x,y)$, and the semigroup T_t that it generates. Under Condition 2.1, $p_1(x,y)$ is uniformly bounded, T_t maps $L^2(D)$ into $L^\infty(D)$ and has a discrete spectrum on $L^2(D)$ (see e.g., [1]). Let $\phi_1^1,\ldots,\phi_1^{k_1},\phi_2^1,\ldots,\phi_2^{k_2},\ldots$ denote an orthonormal basis for $L^2(D)$ of eigenfunctions with eigenvalues $\mu_1<\mu_2<\mu_3<\ldots$, where μ_i corresponds to ϕ_i^j. Recall that $k_1=1$, $\mu_1=0$ and $\phi_1^1=1/\mathrm{vol}(D)$. In this section we prove the following result.

Theorem 3.1 *Assume Conditions 2.1 and 2.2 hold, let V be an invariant wedge for D and assume that it satisfies Condition 2.3. Then there is an eigenfunction ϕ_2 corresponding to the second Neumann eigenvalue μ_2, satisfying $\phi_2(x)\geq\phi_2(y)$ whenever $x,y\in D$ and $x-y\in V$. Moreover, for all $y\in D$*

$$\inf_{x\in\partial D}\phi_2(x)<\phi_2(y)<\sup_{x\in\partial D}\phi_2(x). \tag{9}$$

What provides the link between the heat equation and the reflecting Brownian motion is that if u solves (1) then

$$u(t,x)=E_x u_0(X_t) \tag{10}$$

holds true whenever $u_0\in C(D)\cap L^2(D)$. The proof of this fact follows that of [2] Theorem 4.14 where Dirichlet boundary conditions are considered. The estimate needed there on the eigenvalues easily follows from e.g. [1] p. 8. Finally, the boundary conditions are satisfied since by [4] Lemma 4.3(a) the transition density satisfies them.

Let now U denote the orthogonal complement to the first eigenspace in $L^2(D)$ i.e.,

$$U=\{u\in L^2(D):\int_D u(x)dx=0\}.$$

Let the restriction of T_t to U be denoted by $\hat{T}_t$. Define

$$S=\{u\in U: u(x)\geq u(y),\ x,y\in D, x-y\in V\}.$$

The following argument was introduced in [1] (in a slightly different context). As a result of Corollary 2.1 and equation (10), we have that $u(t,x)-u(t,y)=E_{x,y}(u_0(X_t)-u_0(Y_t))$ hence $\hat{T}_t u_0\in S$ whenever $u_0\in S$ is continuous. Since S is closed in U and $\hat{T}_t$ continuous we have, in fact, that $\hat{T}_t S\subset S$. Note also that the spectral radius of $\hat{T}_t$ is $e^{-\mu_2 t}$.

We recall an abstract result on linear operators that leave a cone invariant. Let E be a real normed space. A closed set $K\subset E$ is called a *cone* if for all $a,b\in K$, $\alpha\geq 0$ one has $a+b,\alpha a\in K$, and if $K\cap -K$ contains only the zero vector. If $a,b\in E$ and $b-a\in K$, one writes $a\leq b$. The following is from [12] Theorem 9.2.

Theorem 3.2 *Let A be an operator in a real Banach space E that leaves a cone K invariant (i.e., $AK\subset K$). Assume*

i. $\overline{K-K}=E$,

ii. *A is compact and its spectral radius satisfies $r(A)>0$.*

Then $r(A)$ is an eigenvalue of A with a corresponding eigenvector in K.

Proof of Theorem 3.1: We verify the hypotheses of Theorem 3.2. The fact that $S \cap -S = \{0\}$ follows easily from the assumption that V has a nonempty interior. Hence S is obviously a cone in the Banach space U with the norm of $L^2(D)$. To show that $\overline{S-S} = U$ it is enough to show that every $u \in U$ that satisfies a global Lipschitz condition can be written as the difference between two elements of S. Assume then that

$$|u(x) - u(y)| \leq \kappa |x-y|, \quad x, y \in D.$$

By Condition 2.3 there must be γ and $\epsilon > 0$ such that $\langle v, \gamma \rangle \geq \epsilon |v|$ for all $v \in V$. Let x_0 be such that $\int_D \langle x - x_0, \gamma \rangle dx = 0$ and let $z(x) = \langle x - x_0, \gamma \rangle$. Then

$$u(x) - u(y) + \kappa \epsilon^{-1} \langle \gamma, x-y \rangle \geq 0 \tag{11}$$

for all $x, y \in D$, $x - y \in V$. That is, $u + \kappa\epsilon^{-1} z \in S$. Obviously $z \in S$, hence we obtain that $\overline{S-S} = U$. As discussed before, $\hat{T}_t$ leaves S invariant. Moreover, $r(\hat{T}_t) = e^{-\mu_2 t} > 0$. Hence by Theorem 3.2 there is an eigenfunction ϕ_2 in S, corresponding to μ_2.

To conclude (9) we proceed as in the proof of Theorem 3.3 of [1]. Every eigenfunction must be real analytic in D [1] hence cannot be constant in an open set unless it is identically zero. However, if y is any interior point then $\phi_2(y) \leq \phi_2(x)$ if x belongs to the set $(y+V) \cap D$, which has non-empty interior. Consequently, ϕ_2 cannot attain its maximum at y. □

4 Simplicity in dimension two

In this section we deal solely with planar domains. It is known since [16] that the multiplicity of μ_2 for simply connected planar domains is at most two. For a certain family of planar domains we prove that μ_2 is simple.

Theorem 4.1 *Let D be a bounded domain of the form*

$$D = \{x \in \mathbb{R}^2 : g(x_1) < x_2 < h(x_1)\},$$

where g and h are non-decreasing C^2 functions, g is convex and h concave. Let $(\xi_1, g(\xi_1))$ and $(\xi_2, g(\xi_2))$ denote the two corner points (i.e., ξ_1 and ξ_2 are the only two solutions ξ to the equation $g(\xi) = h(\xi)$). Assume that for $i = 1, 2$, both g and h are affine at some neighborhood of ξ_i and the angle between the graphs of g and h near ξ_i is greater than or equal to $\pi/4$. Then the second Neumann eigenvalue on D is simple.

Before proving the above result, let us show that Theorem 3.1 can be applied to domains in $\mathbb{R}^2$ to recover the following result of [1], section 3, specialized to domains with convex corners.

Theorem 4.2 (Bañuelos and Burdzy) *Let $g, h : [0, a] \to \mathbb{R}$ be nondecreasing and such that both g and $-h$ are given by the maximum over a finite collection of continuous functions on $[0, a]$ that are C_2 on $(0, a)$. Assume that $g < h$ on $(0, a)$ while $g = h$ on $\{0, a\}$, and that $g(0^+) < h(0^+)$, $g(a^-) < h(a^-)$. Let*

$$D = \{x \in \mathbb{R}^2 : g(x_1) < x_2 < h(x_1),\ 0 < x_1 < a\}.$$

Then there is an eigenfunction corresponding to the second Neumann eigenvalue on D such that for all $y \in D$

$$\inf_{x\in\partial D} \phi_2(x) < \phi_2(y) < \sup_{x\in\partial D} \phi_2(x).$$

Proof: We will show that the assumptions of Theorem 3.1 are satisfied. Condition 2.1 trivially holds. Let $V = \mathbb{R}^2_+$. Then it is obvious that V is a wedge and that $W \subset V \cup -V$. By monotonicity, the inward normal at smooth portions of the boundary always satisfies

$$\langle n, e_1\rangle\langle n, e_2\rangle \leq 0.$$

Note that whenever $v \in \partial V \setminus \{0\}$ and m is an inward normal to ∂V at v then either v is a positive multiple of e_1 and $m = e_2$ or v is a positive multiple of e_2 and $m = e_1$. Hence in view of the last display, Condition 2.2$'$ holds, and V is an invariant wedge for D. Condition 2.3 holds trivially. The conclusions of Theorem 3.1 are therefore valid. $\square$

Remark: Let $m_g = \sup g'$ and $m_h = \sup h'$ where the supremum is over all smooth points. Then it is easy to see that the set $\{x : 0 \leq x_2 \leq \max(m_g, m_h)x_1\}$ is an invariant wedge for D as well.

We introduce some notation. Consider the Banach space $\tilde{U}$ of functions on D that satisfy a global Lipschitz condition as well as

$$\int_D u(x)dx = 0,$$

equipped with the norm

$$\|u\|_\sim = \sup_{x\in D} |u(x)| + \sup_{x,y\in D: x\neq y} \frac{|u(x) - u(y)|}{|x - y|}.$$

Similarly to the definition of S, let

$$\tilde{S} = \{u \in \tilde{U} : u(x) \geq u(y),\ x, y \in D, x - y \in V\}.$$

The proof of Theorem 4.2 shows that the conclusions of Theorem 3.1 are valid with $V = \mathbb{R}^2_+$. We fix $V = \mathbb{R}^2_+$ in what follows. Setting

$$M(u; x) = \min[\langle\nabla u(x), e_1\rangle, \langle\nabla u(x), e_2\rangle], \quad x \in D,$$

we define the subset S' of $\tilde{S}$ by

$$S' = \{u \in \tilde{S} : M(u; x) > 0, x \in D\}.$$

Let $B_\rho(x)$ denote the open disc of radius ρ centered at x and let $\sigma_\rho^{\alpha,\beta}$ denote the sector given by

$$\sigma_\rho^{\alpha,\beta} = \{(\rho_1 \cos\gamma, \rho_1 \sin\gamma) : \rho_1 < \rho, \alpha < \gamma < \beta\}.$$

For $i = 1, 2$, let Ξ_i denote the corner point $(\xi_i, g(\xi_i))$. Let also $B_\rho(\Xi) = \cup_{i=1,2} B_\rho(\Xi_i)$. For $\epsilon > 0$ let

$$D^{-\epsilon} = \{x \in D : \mathrm{dist}(x, \partial D) > \epsilon\}.$$

Recall that by assumption g and h are affine near the corner points and let r be some fixed (small enough) number such that $B_r(\Xi_i) \cap D$, $i = 1, 2$ are sectors. Set

$$D_\epsilon = D^{-\epsilon} \setminus B_{\sqrt{\epsilon}}(\Xi), \quad \epsilon > 0,$$

and

$$D'_\epsilon = D_{3\epsilon/4} \setminus D_{5\epsilon/4}, \quad \epsilon > 0.$$

In what follows, c denotes a positive constant whose value may change from line to line. We state three lemmas and prove them in the end of this section.

Lemma 4.1 *Let D be a convex domain satisfying Condition 2.1. Let $u(t,x)$ be the solution to the heat equation (1) with initial condition $u_0 \in L^2(D)$ and Neumann boundary conditions. Then $u(1,x)$ is globally Lipschitz in D.*

Lemma 4.2 *Let the assumptions of Theorem 4.1 hold. If $\phi \in \tilde{S}$ is an eigenfunction, then $\phi \in S'$.*

Recall that $P_{x,y}$ denotes the probability law under which (X, Y) is a two point reflecting Brownian motion in D started at (x, y).

Lemma 4.3 *Let the assumptions of Theorem 4.1 hold. Let B be some disc in D, away from its boundary. Then there is a $c = c(B) > 0$ such that for all ϵ small enough, the estimate*

$$E_{x,y}|X_1 - Y_1| 1_{X_1 \in B} \geq c\epsilon |x - y|$$

holds for $x, y \in B_r(\Xi) \cap D'_\epsilon$, $x - y \in V$.

We use several times the fact that the process $|X_t - Y_t|$ is nonincreasing in t, $P_{x,y}$-a.s. To see this, use (4) and the notation of Section 2 to write

$$|\delta_t|^2 - |\delta_0|^2 = 2\int_0^t \langle \delta_s, d\delta_s \rangle = 2\int_0^t \langle \delta_s, \nu_s^x \rangle d|l^x|_s - 2\int_0^t \langle \delta_s, \nu_s^y \rangle d|l^y|_s.$$

For any convex domain it holds that when $x_s \in \partial D$, $\langle \delta_s, \nu_s^x \rangle \leq 0$ (recall that $\delta_s = x_s - y_s$ and ν_s^x is an inward normal to ∂D at x_s). The first integral on the right hand side of the last display is therefore nonincreasing in t. A similar argument reveals that the second integral is nondecreasing, and it follows that $|\delta_t|$ is nonincreasing in t. Therefore $|X_t - Y_t|$ is a.s. nonincreasing.

Proof of Theorem 4.1: We assume that μ_2 is not simple and argue by contradiction.

By Theorem 4.2 there is an eigenfunction ϕ corresponding to μ_2 with $\phi \in S$ (and where $V = \mathbb{R}^2_+$). Let $\phi^\perp$ be an eigenfunction in the second eigenspace, orthogonal to ϕ. Both ϕ and $\phi^\perp$ are assumed to have unit L_2 norm. By Lemma 4.1, both ϕ and $\phi^\perp$ are in $\tilde{U}$. It follows that $\phi \in \tilde{S}$. Moreover, since $\phi^\perp$ and $-\phi^\perp$ cannot simultaneously belong to $\tilde{S}$, we assume (without loss) that $\phi^\perp \notin \tilde{S}$. Introduce

$$\phi^a = (1-a)\phi + a\phi^\perp, \quad a \in [0, 1],$$

and let $a^* = \inf\{a \in [0,1] : \phi^a \notin \tilde{S}\}$. Let also $\psi = \phi^{a^*}$. Note that ϕ^a is continuous as a mapping from $[0,1]$ to $\tilde{U}$, and that $\tilde{S}$ is closed in $\tilde{U}$. Therefore the set of $a \in [0,1]$ for which $\phi^a \in \tilde{S}$ is closed, and since this set does not contain 1, it follows that $a^* < 1$. Furthermore, we have that $\psi \in \tilde{S}$ and $\psi \neq 0$.

Define

$$K^a = \{x \in D : M(\phi^a; x) \leq 0\}, \quad a > a^*,$$

and let K be its limit as $a \downarrow a^*$ in the following sense:

$$K = \{x \in \bar{D} : \exists \{x_n\}, \{a_n\}, x_n \to x, a_n \downarrow a^*, x_n \in K^{a_n}\}.$$

Since $a^* < 1$, there must exist a sequence $a_n \downarrow a^*$ such that K^{a_n} are non-empty. Hence

$$K \neq \emptyset. \tag{12}$$

Since $\psi \in \tilde{S}$, by Lemma 4.2 $\psi \in S'$. We claim that for any compact $A \subset D$,

$$M(\phi^a) \geq c_A > 0 \text{ on } A \text{ for all } a - a^* > 0 \text{ small enough.} \tag{13}$$

To see this, note that one can write

$$\phi^a = \psi + (a - a^*)(\phi^\perp - \phi),$$

therefore

$$|M(\phi^a) - M(\psi)| \leq (a - a^*)(|\phi^\perp|_\sim + |\phi|_\sim), \tag{14}$$

and (13) follows. Hence K must be a subset of the boundary. Let

$$\epsilon = \epsilon(a) = \inf\{\epsilon_1 : D_{\epsilon_1} \cap K^a = \emptyset\}, \quad a > a^*. \tag{15}$$

Since K is a subset of the boundary, $\epsilon(a) \to 0$ as $a \downarrow a^*$. Moreover, on a subsequence of $a \downarrow a^*$ one must have $\epsilon(a) > 0$. Therefore, on this subsequence, one must have that $M(\phi^a; x) \leq 0$ for some $x \in D'_\epsilon$. Nevertheless, we claim the following.

CLAIM: *$M(\phi^a) > 0$ everywhere in D'_ϵ for all $a - a^* > 0$ small enough, where $\epsilon = \epsilon(a)$ defined in (15).*

This claim stands in contradiction with the preceding paragraph. Therefore, once it is proved, we may infer that there can be no two orthogonal eigenfunctions corresponding to μ_2. In what follows we prove the claim.

Let $x, y \in D'_\epsilon$ satisfy $|x - y| < \epsilon/4$ and $x - y \in V$. Recalling that $e^{-\mu_2}\phi^a(x) = E_x\phi^a(X_1)$ and denoting $Z^a_t = \phi^a(X_t) - \phi^a(Y_t)$, one has

$$\begin{aligned} e^{-\mu_2}(\phi^a(x) - \phi^a(y)) &= E_{x,y}(\phi^a(X_1) - \phi^a(Y_1)) \\ &= E_{x,y}Z^a_1 1_{X_1 \in D^c_{2\epsilon}} + E_{x,y}Z^a_1 1_{X_1 \in D_{2\epsilon}}. \end{aligned} \tag{16}$$

We estimate the two terms above as follows. First, by the monotonicity property $|X_1 - Y_1| \leq |x - y|$. Therefore

$$\begin{aligned} E_{x,y}Z^a_1 1_{X_1 \in D^c_{2\epsilon}} \geq\ & P_x(X_1 \in D^c_{2\epsilon}) \times \\ & \inf\{\phi^a(x') - \phi^a(y') : x', y' \in D, |x' - y'| \leq |x - y|, x' - y' \in V\}. \end{aligned}$$

It is well known that the density of X_1 started at x is bounded above and below by positive constants that do not depend on $x \in \bar{D}$. We get

$$\begin{aligned} E_{x,y} Z_1^a 1_{X_1 \notin D_{2\epsilon}} &\geq c\mathrm{Vol}(D_{2\epsilon}^c)|x-y| \inf_D M(\phi^a) \\ &\geq c\epsilon |x-y| \inf_D M(\phi^a). \end{aligned} \tag{17}$$

Note that by (14), the (possibly negative) quantity $\inf_D M(\phi^a)$ approaches zero as $a \downarrow a^*$.

Let B be some disc in D, away from the boundary. The estimate on the second term of (16) is obtained separately for $x, y \in D'_\epsilon \setminus B_{r/2}(\Xi)$ and for $x, y \in D'_\epsilon \cap B_r(\Xi)$. The former case is treated as follows. Recall that on D_ϵ, $M(\phi^a) \geq 0$ and that $|X_s - Y_s| \leq |x - y| \leq \epsilon/4$, $s > 0$. If $X_1 \in D_{2\epsilon}$ then $Y_1 \in D_\epsilon$ and therefore $Z_1^a 1_{X_1 \in D_{2\epsilon}}$ is a.s. nonnegative. It follows that

$$E_{x,y} Z_1^a 1_{X_1 \in D_{2\epsilon}} \geq E_{x,y} Z_1^a 1_{X_s \in D^{-\epsilon/4}, s \in [0,1], X_1 \in B}.$$

On the event in the above indicator, both X_s and Y_s are in D for $s \in [0, 1]$ and therefore $X_1 - Y_1 = x - y$. It follows that the last display is

$$\geq |x-y| \inf_{B_\epsilon} M(\phi^a) \left[\inf_{x \in D^{-3\epsilon/4} \setminus B_{r/2}(\Xi)} P_x(X_s \in D^{-\epsilon/4}, s \in [0,1], X_1 \in B) \right],$$

where B_ϵ is a disc concentered with B and of radius $\mathrm{rad}(B) + \epsilon/4$. The factor in square brackets is $\geq c\epsilon$ for ϵ small. This is a consequence of the relation between the density of a Brownian motion killed at the boundary and the Dirichlet problem and e.g., Theorem 4.2.5 with Lemma 4.6.1 in Davies [7], that together establish a lower bound on the density of the order of the distance $\mathrm{dist}(x, \partial D)$ to the boundary (recall that the boundary is C^2 away from the corners). Hence by (13),

$$E_{x,y} Z_1^a 1_{X_1 \in D_{2\epsilon}} \geq c|x-y|\epsilon. \tag{18}$$

In fact, (18) holds for $x, y \in B_r(\Xi) \cap D'_\epsilon$ as well. Indeed, combining the fact that $Z_1^a 1_{X_1 \in D_{2\epsilon}} \geq 0$ with (13) and Lemma 4.3,

$$\begin{aligned} E_{x,y} Z_1^a 1_{X_1 \in D_{2\epsilon}} &\geq E_{x,y} Z_1^a 1_{X_1 \in B} \\ &\geq c E_{x,y} |X_1 - Y_1| 1_{X_1 \in B} \\ &\geq c\epsilon |x-y| \end{aligned}$$

and (18) follows.

Combining (16), (17) and (18) we obtain that

$$\inf_{D'_\epsilon} M(\phi^a) \geq c\epsilon(a) \inf_D M(\phi^a) + c\epsilon(a).$$

As noted before, $\inf_D M(\phi^a)$ approaches zero as $a \downarrow a^*$. Hence for all $a - a^*$ small the right hand side must be positive. The claim is therefore proved. This concludes the proof of the theorem. $\square$

Proof of Lemma 4.1: Consider the Sobolev space $W^{1,2}(D)$ with the norm

$$\|u\|' = \|u\| + \sum_{i=1}^{d} \|\partial_{x_i} u\|,$$

where $\|\cdot\|$ is the norm in $L^2(D)$. It is well known (e.g. from [13] Theorem III.5.1) that for domains with Lipschitz boundary, $T_{1/2}$ maps $L^2(D)$ into $W^{1,2}(D)$. Hence it suffices to show that $T_{1/2}u$ is Lipschitz whenever $u \in W^{1,2}(D)$. Fix $u \in W^{1,2}(D)$ and let u^k be a sequence of globally Lipschitz functions on D converging to u in $W^{1,2}(D)$. Assume without loss that $\|u^k\|' \le 2\|u\|'$. We show below that for each k, $T_{1/2}u^k$ is globally Lipschitz with constant $c(D)\|u^k\|'$, where $c(D)$ depends only on D. Since T_t is continuous on $W^{1,2}$ and the set of functions with Lipschitz constant $2c(D)\|u\|'$ is closed in $W^{1,2}$, $T_{1/2}u$ is Lipschitz, and the result follows.

Indeed, let y and x^j, $j = 1, 2, \ldots$ be in D such that x^j converge to y and $x^j \neq y$, $j = 1, 2, \ldots$. Consider the probability space $(\Omega, F, (F_t), P)$ of Section 2 and the F_t-Brownian motion W. Let E denote expectation with respect to P. For each j let X^j denote the solution to the Skorokhod problem $(x^j + W, D, n)$ and as before let Y denote the solution to $(y + W, D, n)$. Recall that the processes just defined satisfy (8) with the corresponding initial conditions, hence (10) is applicable to each of them. Using also the fact that $|X_t - Y_t|$ is nonicreasing a.s. it follows that for each j (we have k and t fixed in what follows)

$$\begin{aligned}
&|x^j - y|^{-1}|T_t u^k(x^j) - T_t u^k(y)| \\
&\qquad = |x^j - y|^{-1}|E[(u^k(X^j_t) - u^k(Y_t)); X^j_t \neq Y_t]| \\
&\qquad \le EZ^j,
\end{aligned}$$

where

$$Z^j = \begin{cases} |X^j_t - Y_t|^{-1}|u^k(X^j_t) - u^k(Y_t)| & X^j_t \neq Y_t, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Using a.s. monotonicity again, X^j_t converges to Y_t a.s. and therefore a.s.,

$$\limsup_{j\to\infty} Z^j \le |\nabla u^k(Y_t)|.$$

However, u^k is Lipschitz hence Z^j are uniformly bounded. We therefore apply Fatou's lemma and have

$$\begin{aligned}
\limsup_{j\to\infty} |x^j - y|^{-1}|T_t u^k(x^j) - T_t u^k(y)| &\le E|\nabla u^k(Y_t)| \\
&\le (E|\nabla u^k(Y_t)|^2)^{1/2} \\
&\le c(D)\|u^k\|',
\end{aligned}$$

where the last inequality follows from the well known fact that the density of Y_t is uniformly bounded for $y \in D$ (and t fixed) (see e.g. [1] p. 6). $\square$

Proof of Lemma 4.2: Recall that any eigenfunction is real analytic in D (cf. [1]). Let $\phi \in \tilde{S}$ be an eigenfunction. We first show there must be a disc B where $M(\phi) \ge c_B > 0$. Assume this is not the case. Then for all $x \in D$, $\langle \nabla\phi(x), e_i \rangle = 0$ for either $i = 1$ or $i = 2$. Then either there is a disc in D where $\nabla\phi(x) = |\nabla\phi(x)|e_1$, or there is a disc in D where $\nabla\phi(x) = |\nabla\phi(x)|e_2$. Assume then that B' is a disc where $\nabla\phi(x) = |\nabla\phi(x)|e_1$ (the other case is treated similarly). Let A be a (relatively open) subset of the boundary ∂D where the inward normal n satisfies $\langle n(z), e_1 \rangle > 0$ and $\langle n(z), e_2 \rangle < 0$, for $z \in A$. (It is obvious that there must exist such A.) Now let $x, y \in B'$ be such that $x - y = \alpha e_2$, where $\alpha > 0$. Recall that under $P_{x,y}$, (X, Y) denotes a two-point reflecting Brownian motion in $\bar{D}$, started at (x,y). Consider the event η that (1) X hits ∂D at A before time 1 without hitting ∂D outside A till time 1; (2) $|L^x|_1 > 0$; (3) Y never hits ∂D before time 1; and (4) $X_1, Y_1 \in B'$. This event has positive $P_{x,y}$-probability, as can be seen e.g., as follows. First, construct a C_1 path $w_\cdot$, for which the solutions $\tilde{x}_\cdot, \tilde{y}_\cdot$ to the Skorokhod problems $(x+w_\cdot, D, n)$ and respectively, $(y+w_\cdot, D, n)$ satisfy the conditions (1)-(4) above. Then recall that for convex domains the Skorokhod map $w_\cdot \to \tilde{x}_\cdot$ is continuous in the uniform topology (see [14], Theorem 1.1). Therefore these conditions are also satisfied when $w_\cdot$ is replaced by any element of the tube

$$\{\beta \in C([0,1] : \mathbb{R}^2) : |\beta_s - w_s| < \delta, s \in [0,1]\},$$

provided δ is small enough. But the Wiener measure assigns positive probability to such tubes. Therefore η has positive probability. Now, recall that

$$X_t - Y_t = x - y + \int_0^t n(X_s) d|L^x|_s - \int_0^t n(Y_s) d|L^y|_s.$$

Hence on η one has that $\langle X_1 - Y_1, e_1 \rangle > 0$. Thus

$$\begin{aligned} T_1\phi(x) - T_1\phi(y) &= E_{x,y}(\phi(X_1) - \phi(Y_1)) \\ &\geq E_{x,y}(\phi(X_1) - \phi(Y_1)) 1_\eta \\ &> 0. \end{aligned}$$

Since $T_1\phi = e^{-\mu}\phi$, this implies that $\phi(x) > \phi(y)$. This contradicts the assumption that $\nabla\phi = |\nabla\phi| e_1$ in B', and therefore there must exist a disc B where $M(\phi) \geq c_B > 0$. Let such a disc B be fixed.

Consider now a disc $B^1 = B_\rho(z)$, where $z \in D$. Take any $x, y \in B^1$ with $x - y \in V$. Consider the event that $\text{dist}(X_t, \partial D) > 2\rho$, $t \in [0,1]$ and that $X_1, Y_1 \in B$. Note that on this event Y never hits ∂D before time 1. Also, for ρ small, this event has positive $P_{x,y}$-probability that moreover, is bounded below by a positive constant c that does not depend on x, y (satisfying the condition just stated). Hence $T_1\phi(x) - T_1\phi(y) \geq c c_B |x-y|$ for all such x, y. This implies that $M(\phi; x) \geq c c_B e^\mu > 0$, $x \in B_\rho(z)$, where μ is the corresponding eigenvalue. Since $z \in D$ is arbitrary, $\phi \in S'$. This concludes the proof of the lemma. □

Proof of Lemma 4.3: Let $0 \leq \theta_1 < \theta_2 < \pi/2$ be defined by $\theta_1 = \arctan g'(\xi_1)$, $\theta_2 = \arctan h'(\xi_1)$ and let $\theta = \theta_2 - \theta_1$. Then θ is the angle between the two lines forming the boundary of D in the neighbourhood of the corner Ξ_1. By assumption, $\theta \geq \pi/4$. Note also that θ must satisfy $\theta < \pi/2$. Therefore, by slightly rotating the domain, if necessary, one can obtain that θ_1, θ_2 both lie in the open interval $(0, \pi/2)$, while the rotated domain still satisfies all the assumptions of Theorem 4.1. We assume (without loss) that indeed $\theta_1, \theta_2 \in (0, \pi/2)$.

Consider Ξ_1 as the origin. Let B^1 be some fixed ball in the sector $\sigma_r^{\theta_1,\theta_2}$, away from the boundary of D. Note first that one can replace the expectation of $|X_1 - Y_1| 1_{X_1 \in B}$ by that of $|X_{1/2} - Y_{1/2}| 1_{X_{1/2} \in B^1}$ at the cost of a constant, since

$$\begin{aligned} E_{x,y}[|X_1 - Y_1| 1_{X_1 \in B}] &\geq E_{x,y}[|X_1 - Y_1| 1_{X_{1/2} \in B^1, X_s, Y_s \in D, s \in [1/2,1], X_1 \in B}] \\ &= E_{x,y}[|X_{1/2} - Y_{1/2}| 1_{X_{1/2} \in B^1, X_s, Y_s \in D, s \in [1/2,1], X_1 \in B}] \\ &\geq c_1 E_{x,y}[|X_{1/2} - Y_{1/2}| 1_{X_{1/2} \in B^1}] \end{aligned} \tag{19}$$

where by Markovity of (X,Y), $c_1>0$ can be chosen such that it depends only on B^1 and B.

Let R_1 and R_2 denote the two sides of the sector:

$$R_i=\{(\rho\cos\theta_i,\rho\sin\theta_i):\rho<r\},\quad i=1,2.$$

Let $x,y\in D'_\epsilon\cap B_r(\Xi_1)$ satisfy $x-y\in V$ and $|x-y|\le\sqrt{\epsilon}/10$. We have that either both x and y lie in the subsector $\sigma_r^{\theta_1,\theta_1+2\theta/3}$ (namely, near R_1) or they both lie in $\sigma_r^{\theta_2-2\theta/3,\theta_2}$, and since the argument is similar in both cases, we only consider the first. We point out two straightforward facts: (1) As long as X stays at a distance of at least $\sqrt{\epsilon}/10$ away from R_2, Y does not hit R_2; (2) As long as both X and Y stay away from R_2 (but may hit R_1), the distance $|X-Y|$ is bounded below by $c_2|x-y|$ where $c_2=\min(\sin\theta_1,\cos\theta_1)>0$. Denoting $R_2^b=\{x:\mathrm{dist}(x,R_2)<b\}$ we obtain that the expression in (19) is

$$\begin{aligned}&\ge\quad c_1E_{x,y}[|X_{1/2}-Y_{1/2}|1_{X_s\notin R_2^{\sqrt{\epsilon}/10},s\in[0,1/2],X_{1/2}\in B^1}]\\&\ge\quad c_1c_2|x-y|P_x(X_s\notin R_2^{\sqrt{\epsilon}/10},s\in[0,1/2],X_{1/2}\in B^1).\end{aligned}\tag{20}$$

Since $x\in D'_\epsilon$, the distance from x to $R_2^{\sqrt{\epsilon}/10}$ is at least $c\sqrt{\epsilon}$. The probability in (20) can be estimated by an argument of reflection about R_1: It is equal to the probability of a BM (i.e., a nonreflecting one) started at x not to hit the sides of the sector $\sigma_r^{\theta_1-\theta,\theta_1+\theta}$ till time 1/2, and to end up at $B^1\cup B^2$ at time 1/2, where B^2 is the image of B^1 under the reflection. As in the proof of Theorem 4.1, this can be estimated using the ground state of the Dirichlet problem. In Example 4.6.5 of Davies [7] it is shown that the ground state on the sector $\sigma_r^{\theta_1-\theta,\theta_1+\theta}$ at $x=\rho e^{i\beta}$ equals

$$\tilde R(\rho)\sin(\pi(\beta-(\theta_1-\theta))/\alpha),$$

where

$$\tilde R(\rho)=O(\rho^{\pi/\alpha}),$$

and $\alpha=2\theta$ is the sector's angle. In our case, $\rho\ge3\sqrt{\epsilon}/4$, $\alpha=2\theta\ge\pi/2$, and $(\beta-(\theta_1-\theta))/\alpha$ is bounded away from 0 and 1 (recall that $x\in\sigma_r^{\theta_1,\theta_1+2\theta/3}$). Hence the ground state at x is at least $O(\epsilon)$. Since as in the proof of Theorem 4.1 this establishes a lower bound on the probability in (20), the lemma follows. $\square$

5 Appendix

We provide a few examples of three dimensional domains, where Theorem 3.1 holds. If a domain D is, for example, a convex polyhedron $\cap_i\{x:\langle x-\xi_i,\gamma_i\rangle\le0\}$, then all assumptions we have made in Theorem 3.1 are on the vectors γ_i, and not on ξ_i (other than that the domain is non-empty and bounded). We therefore refer in our examples to the domain only through the set of normals N associated with it (as defined in equation (3)).

In our first example we consider three unit vectors $p_i\in\mathbb{R}^3$, $i=1,2,3$ satisfying $|p_i-p_j|=a<\sqrt{2}$ whenever $i\ne j$. We allow the set N to take the following form (or any subset of it):

$$N=\bigcup_{i,j,k\text{ distinct}}N^{i,j,k},\tag{21}$$

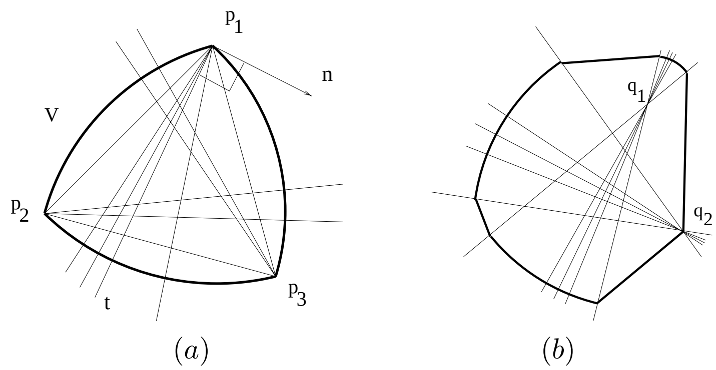


Figure 1: Two examples of invariant wedges and several tangent subspaces as seen on the unit sphere in $\mathbb{R}^3$

where

$$N^{i,j,k} = \{n \in \mathbb{R}^3 : |n| = 1, \langle n, p_i\rangle \le \langle n, p_j\rangle = 0 \le \langle n, p_k\rangle\}.$$

Let

$$V = \text{cone}\{v \in \mathbb{R}^3 : |v| = 1, |v - p_i| \le a, i = 1, 2, 3\}.$$

Figure 1(a) sketches the set V and several subspaces orthogonal to elements of N, intersected with the unit sphere. For example, n and t represent a normal in $N^{2,1,3}$ and its orthogonal subspace. As n varies within $N^{2,1,3}$, t varies over the planes that pass through 0 and p_1 and between p_2 and p_3.

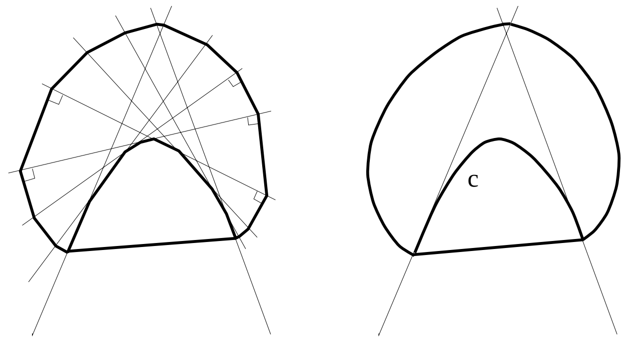
Proposition 5.1 *Let D be any domain satisfying Condition 2.1 with a corresponding set N as in (21), or any subset of it. Then V is an invariant wedge for D, and the conclusions of Theorem 3.1 are valid.*

Proof: The elementary details are omitted. Let v satisfy $|v| = 1$, $|v - p_i| \le a$, $i = 1, 2, 3$ and let n be in $N^{1,2,3}$. We will first show that $\pi_n v \in V$. Let $m_{1,2}$ be a normal to the plane generated by p_1 and p_2 and such that $\langle m_{1,2}, p_3\rangle > 0$. Define similarly $m_{2,3}$ with the condition $\langle m_{2,3}, p_1\rangle > 0$. Then it is easy to show that the following condition is sufficient: (1) $z = \frac{\pi_n v}{|\pi_n v|} - p_2$ satisfies $|z - p_2| \le a$, (2) $\langle m_{1,2}, z\rangle \ge 0$, and (3) $\langle m_{2,3}, z\rangle \ge 0$. This condition can be verified by direct calculation. A similar argument holds for the other sets $N^{i,j,k}$. $\square$

The symmetry of the (p_i) in the last example is not necessary and we have assumed it only for the ease of presentation. We describe more examples without proof. Figure 1(b) shows a set of tangential subspaces and a corresponding invariant wedge. The subspaces all pass through either q_1 or q_2. Figure 2(a) depicts a different structure. Figure 2(b) shows a continuous version of this structure. In particular, any tangent plane to ∂D (at a smooth portion) will be parallel to a plane that is tangential to the curve c and passes though the origin.

In the rest of this section, we prove a result needed in the proof of Lemma 2.1.

Lemma 5.1 *Let $V \subset \mathbb{R}^d$ be a convex closed set and for $\epsilon > 0$ let V_ϵ be defined as in (5). Then the function $u \in V^c \mapsto m \in S^{d-1}$ defined as the unique unit inward normal to ∂V_ϵ at u, where $\epsilon = \text{dist}(u, V)$, is continuous.*



(a) (b)

Figure 2: More examples in dimension three

Proof: By convexity of V, for any $u \in V^c$ there exists a unique $v \in \partial V$ for which $\epsilon = \text{dist}(u, V) = |u - v|$, and moreover $m(u) = (v - u)/\epsilon$.

Let now $u \in V^c$ and $\sigma > 0$ be given. We will show that there exists an $\eta > 0$ such that $u' \in V^c$ and $|u - u'| < \eta$ imply $|m(u) - m(u')| < \sigma$. Let $\epsilon = \text{dist}(u, V)$ and $v \in \partial V$ be such that $|u - v| = \epsilon$. For $u' \in V^c$ let $v' \in \partial V$ be such that $|u' - v'| = \text{dist}(u', V)$. Choose α such that $\tilde{u} = u' + \alpha m(u')$ will satisfy $\text{dist}(\tilde{u}, V) = \epsilon$. Note that $\text{dist}(\tilde{u}, V) = |\tilde{u} - v'|$ and as a consequence $m(\tilde{u}) = m(u')$.

We claim that the following inequality must hold

$$|v - v'| \leq |u - \tilde{u}|. \tag{22}$$

Indeed, the fact that $|u-v| < |u-(\lambda v+(1-\lambda)v')|$ for any $\lambda \in (0,1)$ implies that $\langle u-v, v-v' \rangle \geq 0$. Similarly, $\langle \tilde{u} - v', v' - v \rangle \geq 0$. Hence follows (22).

One now has that

$$\begin{aligned} |m(u) - m(u')| &\leq \frac{|u - \tilde{u}| + |v - v'|}{\epsilon} \\ &\leq \frac{2|u - \tilde{u}|}{\epsilon} \\ &\leq \frac{4|u - u'|}{\epsilon}. \end{aligned}$$

The last inequality follows from $|u' - \tilde{u}| = |\text{dist}(u', V) - \text{dist}(u, V)| \leq |u' - u|$. One may now take $\eta = \sigma\epsilon/4$ and conclude that $|m(u) - m(u')| < \sigma$ if $|u - u'| < \eta$. $\square$

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References

[1] R. Bañuelos and K. Burdzy. *On the "hot spots" conjecture of J. Rauch,* J. Funct. Anal. 164, 1–33 (1999)

[2] R. F. Bass. *Probabilistic Techniques in Analysis,* Series of Prob. Appl. (1995)

[3] R. F. Bass and K. Burdzy. *Fiber Brownian motion and the "hot spots" problem,* Duke Math. J. 105 (2000), no. 1, 25–58.

[4] R. F. Bass and P. Hsu. *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains,* Ann. Prob. Vol. 19 No. 2 486–508 (1991)

[5] K. Burdzy and W. Werner. *A counterexample to the "hot spots" conjecture,* Ann. Math. 149 309–317 (1999)

[6] M. Cranston and Y. Le Jan. *Noncoalescence for the Skorohod equation in a convex domain of* $\mathbb{R}^2$, Probab. Theory Related Fields 87, 241–252 (1990)

[7] E. B. Davies. *Heat Kernels and Spectral Theory,* Cambridge University Press, 1989.

[8] P. Dupuis and H. Ishii. *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications,* Stochastics and Stochastics Reports, Vol. 35, pp. 31–62 (1991)

[9] P. Dupuis and K. Ramanan. *Convex duality and the Skorokhod Problem. I, II,* Probab. Theory Related Fields 115 (1999), no. 2, 153–195, 197–236.

[10] D. Jerison and N. Nadirashvili. *The "hot spots" conjecture for domains with two axes of symmetry,* J. Amer. Math. Soc. 13 (2000), no. 4, 741–772

[11] B. Kawohl. *Rearrangements and Convexity of Level Sets in PDE,* Lecture Notes in Mathematics, Vol. 1150 (1985)

[12] M. A. Krasnosel'skij, Je. A. Lifshits and A. V. Sobolev. *Positive Linear Systes: The Method of Positive Operators,* Sigma Series in Appl. Math. Vol. 5 (1989)

[13] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type* Transl. Math. Monogr., Vol. 23, AMS, Providence (1968)

[14] P. L. Lions and A. S. Sznitman. *Stochastic differential equations with reflecting boundary conditions,* Comm. Pure appl. Math. Vol. 37, 511–537 (1984)

[15] D. Lupo and A. M. Micheletti. *On the persistence of the multiplicity of eigenvalues for some variational elliptic operator depending on the domain,* J. Math. Anal. Appl. 193 no. 3 990–1002 (1995)

[16] N. S. Nadirashvili. *On the multiplicity of the eigenvalues of the Neumann problem,* Soviet Math. Dokl. 33 281–282 (1986)