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ORDERED ADDITIVE COALESCENT AND FRAGMENTATIONS ASSOCIATED TO LEVY PROCESSES WITH NO POSITIVE JUMPS

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Abstract

We study here the fragmentation processes that can be derived from Lévy processes with no positive jumps in the same manner as in the case of a Brownian motion (cf. Bertoin [4]). One of our motivations is that such a representation of fragmentation processes by excursion-type functions induces a particular order on the fragments which is closely related to the additivity of the coalescent kernel. We identify the fragmentation processes obtained this way as a mixing of time-reversed extremal additive coalescents by analogy with the work of Aldous and Pitman [2], and we make its semigroup explicit.

Keywords Additive coalescent, fragmentation, Lévy processes, processes with exchangeable increments.

AMS subject classification 60J25, 60G51

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1 Introduction

This paper is centered on the study of the additive coalescent which has been constructed, as a large class of coalescent Markovian processes, by Evans and Pitman in [12]. They describe the dynamics of a system of clusters with finite total mass, in which pairs of clusters merge into one bigger cluster at a rate given by the sum of the two masses (this is made rigourous by giving the associated Lévy system). They also present a construction of Markovian "eternal" coalescent processes on the whole real line which are starting from some infinitesimal masses.

One standard way to study such processes is to consider the dual fragmentation processes that split each cluster into smaller clusters. Coalescent processes are then obtained by appropriate time-reversal. Aldous and Pitman, successively in [1] and [2], have given a construction of the standard additive coalescent, which is the limit as n tends to ∞ of a coalescent process starting at time $-\frac{1}{2}\log(n)$ with n clusters of masses $1/n$, by time-reversing a fragmentation process obtained by logging Aldous' CRT (*Continuum Random Tree*) by a certain family of Poisson processes on its skeleton. They also contructed more general fragmentation processes by using inhomogeneous generalizations of the CRT, and gave the exact entrance boundary of the additive coalescent. Bertoin [4, 5] gave a different approach to obtain the same fragmentation processes, by using partitions of intervals induced by some bridges with exchangeable increments (the Brownian bridge in the case of the standard additive coalescent, giving the "Brownian fragmentation").

Our goal in this paper is to investigate the fragmentation processes that can be associated to Lévy processes with no positive jumps in a way similar to the Brownian fragmentation. One main motivation for studying this "path representation" approach is that it induces a particular ordering on the coalescing clusters that is not seen directly with the CRT construction. We study this order in section 2 in the most general setting, and construct the so-called *ordered additive coalescent*. Our study is naturally connected to the so-called fragmentation with erosion ([2, 5]), which in turn is related to non-eternal coalescents. Then we study the Lévy fragmentation processes in sections 3 and 4 in the case where the Lévy process has unbounded variation, and we give in particular the law of the process at a fixed time $t \geq 0$, which is interpreted as a conditioned sequence of jumps of some subordinators, generalizing a result in [1]. We also give in section 5 a description of the left-most fragment induced by the ordering on the real line, which is similar to [1] and [4]. Then, we show that the fragmentation processes associated to Lévy processes are up to a proper time-reversal eternal additive coalescents, and we study the mixing of the extremal eternal additive coalescent appearing in the time-reversed process in section 6.

For this, we rely in particular on the ballot theorem (Lemma 2), which Schweinsberg [23] has also used in a similar context. We also use a generalization of Vervaat's Theorem (Proposition 1) for the Brownian bridge, which is proved in section 7.

As a conclusion, we make some comments on the case where the Lévy process has bounded variation in section 8.

2 Ordered additive coalescent

There are various ways to construct additive coalescent processes. Among them, we may recall three rather different approches :

- The construction by Evans and Pitman [12] of the additive coalescent semigroup and Lévy system, when the coalescent takes values either in the set of partitions of $\mathbb{N} = \{1, 2 \dots\}$ or in $S^{\downarrow} = \{x_1 \geq x_2 \geq \dots \geq 0, \sum x_i = 1\}$.
- The description of *eternal* additive coalescent on the whole real line $\mathbb{R}$ by Aldous and Pitman [1, 2], by time-reversing a fragmentation process obtained by splitting the skeleton of some continuum random tree.
- The representation by Bertoin [5] of these fragmentation processes by path transformation on bridges with exchangeable increments and excursion-type functions.

We may also mention the method of Chassaing and Louchard [8], which is quite close to Bertoin's and is based on *parking schemes*.

In this paper, we shall mostly focus on the third method. To begin with, we stress that in contrast to the first two constructions, the third method naturally induces a puzzling natural order on the fragments (which are sub-intervals of $[0, 1]$ with total length 1). Indeed, one may wonder for instance why there should be a fragment (the "left-most fragment", see [4]) that always coalesces by the left? We shall answer this question by constructing a more precise process we call *ordered additive coalescent*, in which the initial coalescing fragments can merge in different ways (and which is *not* the same as the ranked coalescent of [12]). This study is naturally connected to the "fragmentation with erosion".

2.1 Finite-state case

First recall the dynamics of the additive coalescent starting from a finite number of clusters with masses $m_1, m_2, \dots, m_n > 0$ (we sometimes designate the clusters by their masses even if there may be some ambiguity). We stress that the sequence $\mathbf{m} = (m_1, \dots, m_n)$ need not be ranked in decreasing order. For each pair of indices (i, j) with $i < j$ let e_{ij} be an exponential r.v. with parameter $m_i + m_j$ (we say that $\kappa(x, y) = x + y$ is the *additive coalescent kernel*). Suppose that these variables are independent. At time $e = \inf_{i<j} e_{ij}$ the clusters with masses m_{i_0} and m_{j_0} merge into a unique cluster of mass $m_{i_0} + m_{j_0}$ where $i_0 < j_0$ are the a.s. unique integers such that $e = e_{i_0 j_0}$. Then the system evolves in the same way, and it stops when only one cluster with mass 1 remains. In the sequel, we will always suppose for convenience that $m_1 + m_2 + \dots + m_n = 1$. The general case follows under some change of the time scale.

These dynamics do not induce any ordering on the clusters. We now introduce a natural order which is closely related to the *additive* property of the coalescent kernel. Indeed, one can view each variable e_{ij} as the minimum of two independent exponential variables e_{ij}^i and e_{ij}^j with respective parameters m_i and m_j. The first time e of coalescence corresponds to some e_{ij}^k with $k \in \{i, j\}$, meaning that clusters i and j have merged at time e, and then we decide to say that cluster k has absorbed the other one. Alternatively, the system evolves as if we took n exponential variables $e_1, \dots, e_n$ with respective rates $(n-1)m_1, \dots, (n-1)m_n$ and by declaring that, at time $e_{i^*} = \min_{1 \leq i \leq n} e_i$, the cluster labeled i^* absorbs one of the $n - 1$ other clusters which is picked at random uniformly. We are going to make this more precise in the sequel.

First we introduce a more "accurate" space. Call *total order* on a set M a subset O of $M \times M$ which satisfies the following properties

1. If $(i,j) \in O$ and $(j,k) \in O$ then $(i,k) \in O$.

2. $\forall i \in M,\ (i,i) \in O$.

3. $\forall i,j \in M,\ (i,j) \in O$ or $(j,i) \in O$.

4. If $(i,j) \in O$ and $(j,i) \in O$ then $i = j$.

A *partial order* on M is a subset of $M \times M$ that satisfies only properties 1, 2 and 4. For any partial order O on M and any subset $M' \subset M$, we can define the restriction of O to M' which is the intersection of O with $M' \times M'$.

Let $\mathcal{O}_\infty$ (resp. $\mathcal{O}_n$, $n \geq 1$) be the set of all partial orders O on $\mathbb{N}$ (resp. $\mathbb{N}_n = \{1, \ldots, n\}$) such that

$$iRj \iff (i,j) \in O \text{ or } (j,i) \in O$$

defines an equivalence relation. This is equivalent to saying that there exists a partition w of $\mathbb{N}$ (resp. $\mathbb{N}_n$) such that the restrictions of O to the blocks of w are total orders, and that two integers in disjoint blocks of w are not comparable. We may also write O uniquely in the form $(w, (O_I)_{I \in w})$ where w is a partition of $\mathbb{N}$ (resp. $\mathbb{N}_n$) and for each $I \in w$, O_I is a total order on I (the restriction of O to I). We call the O_I's *clusters*, and by abuse of notation, by $\mathcal{I} \in O$ we mean that $\mathcal{I}$ is a cluster of O. For O in $\mathcal{O}_n$ $(1 \leq n \leq \infty)$ and $1 \leq k \leq n$ we call k_O the cluster in which k is appearing.

If O is in $\mathcal{O}_n$ $(n \leq \infty)$, we define for any pair $(\mathcal{I}, \mathcal{J})$ of distinct clusters in O the order $O_{\mathcal{I}\mathcal{J}} \in \mathcal{O}_n$ which has the same clusters as O except that the clusters $\mathcal{I}$ and $\mathcal{J}$ merge into $\mathcal{I}\mathcal{J}$ where

$$(i,j) \in \mathcal{I}\mathcal{J} \iff \begin{cases} (i,j) \in \mathcal{I} \text{ or} \\ (i,j) \in \mathcal{J} \text{ or} \\ i \in \pi(\mathcal{I}) \text{ and } j \in \pi(\mathcal{J}) \end{cases}$$

and π is the projection on the first coordinate axis. Let also

$$S_1^{+,n} = \{\mathbf{m} = (m_1, \ldots, m_n),\ m_i > 0\ \forall 1 \leq i \leq n,\ \sum_{i=1}^{n} m_i = 1\}$$

and

$$S_1^{+,\infty} = \{\mathbf{m} = (m_1, m_2, \ldots),\ m_i > 0\ \forall i \geq 1,\ \sum_{i=1}^{+\infty} m_i = 1\}.$$

Last, for $n \leq \infty$, $\mathbf{m} \in S_1^{+,n}$ and $O \in \mathcal{O}_n$, we call *mass of cluster* $\mathcal{I} \in O$ the number $m_{\mathcal{I}} = \sum_{i \in \pi(\mathcal{I})} m_i$.

For $n \in \mathbb{N}$, we now describe the dynamics of the so-called $\mathcal{O}_n$-additive coalescent with "proto-galaxy masses" $\mathbf{m} = (m_1, \ldots, m_n) \in l_1^{+,n}$. Let $O \in \mathcal{O}_n$ be the current state of the process, and $\#O \geq 2$ the number of clusters. Consider n exponential r.v.'s $(e_k)_{1 \leq k \leq n}$ with respective parameters $(m_k)_{1 \leq k \leq n}$. There is a.s. a unique k^* such that $e_{k^*} = \min_{1 \leq k \leq n} e_k$. At time $e_{k^*}/(\#O - 1)$, which is exponential with parameter $\#O - 1$, cluster k^*_O merges with one of the $\#O - 1$ other clusters $\mathcal{I}^*$ picked at random uniformly. The state of the process then turns to $O_{k^*_O \mathcal{I}^*}$, and the system evolves similarly until only one cluster remains. The dynamics of the process of the ranked sequence of the clusters' masses are then the same as the additive coalescent described above : it is easily seen that two clusters $\mathcal{I}, \mathcal{J} \in O$ merge together at rate $m_{\mathcal{I}} + m_{\mathcal{J}}$. Indeed, $P[k_O^* = \mathcal{I}] = P[k^* \in \mathcal{I}] = m_{\mathcal{I}}$. We call $\mathbb{P}_O^{\mathbf{m}}$ the law of the process with initial state O.

We say that the cluster k_O^* *absorbs* cluster $\mathcal{I}^*$ (more generally we designate the order in every cluster $\mathcal{I}$ by the binary relation "i absorbs j"). When the system stops evolving, it is constituted of a single cluster O_∞ with mass $\sum m_i$, which is a total order on $\mathbb{N}_n$. There is always a left-most fragment (here we call fragment any integer) $\min O_\infty$, which is the fragment that has absorbed all the others. Notice that the process is increasing in the sense of the inclusion of sets.

Remarks. —This construction has to be compared with Construction 3 in [12], but where the system keeps the memory of the orders of coalescence by labeling the edges of the resulting tree in their order of appearance. It would be interesting to give a description of a limit labeled tree in an asymptotic regime such as in [2].

—It is immediate that, when ignoring the ordering, the evolution of the cluster masses starting at $m_1, \ldots, m_n$ is a finite-state additive coalescent evolution (in fact, the ordered coalescent is not so different of the classical coalescent, it only contains an extra information at each of the coalescence times). If we had replaced the time $e_{k^*}/(\#O - 1)$ of first coalescence by e_{k^*} above, the evolution of the clusters' masses would give the *aggregating server* evolution described in [5].

—One may remark that this way of ordering the clusters can be seen as a particular case of Norris [18] who studies coagulation equations by finite-state Markov processes approximation, and where clusters may coagulate in different manners depending on their shapes. In this direction, the "shape" of a cluster is simply its order.

2.2 Bridge representation

We now give a representation of the ordered coalescent process by using aggregative server systems coded by bridges with exchangeable increments as in [5]. Let $n < \infty$ and $\mathbf{m}$ be in $S_1^{+,n}$. Let $b_{\mathbf{m}}$ be the bridge with exchangeable increments on $[0,1]$ defined by

$$b_{\mathbf{m}}(s) = \sum_{i=1}^{n} m_i (1_{\{U_i \leq s\}} - s), \quad 0 \leq s \leq 1, \tag{1}$$

where the $(U_i)_{1 \leq i \leq n}$ are independent uniform r.v.'s on $[0,1]$.

Definition 1 *Let f be a* bridge *in the Skorohod space $\mathbb{D}([0,\ell])$, $\ell > 0$, i.e. $f(0) = f(\ell) = f(\ell-) = 0$. Let $x_{\min}$ be the location of the right-most minimum of f, that is, the largest x such that $f(x-) \wedge f(x) = \inf f$. We call* Vervaat transfom *of f, or* Vervaat excursion *obtained from f, the function $Vf \in \mathbb{D}([0,\ell])$ defined by*

$$Vf(x) = f(x + x_{\min}[mod\ \ell]) - \inf_{[0,\ell]} f, \quad x \in [0,\ell)$$

and $Vf(\ell) = \lim_{x \to \ell^-} Vf(x)$.

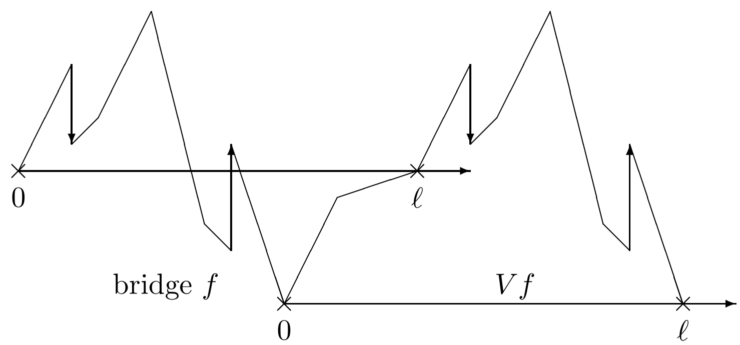

Figure 1: A càdlàg bridge and its Vervaat's transform

Throughout this paper, the functions f we will consider will be sample path of some processes with exchangeable increments that attain a.s. their minimum at a unique location, so that we could have omitted to take the largest location of the minimum in the definition. See [16] for details.

Let $\varepsilon_{\mathbf{m}} = Vb_{\mathbf{m}}$ be the Vervaat excursion obtained from $b_{\mathbf{m}}$, and $s_{\min}$ the location of the infimum of $b_{\mathbf{m}}$, which is a.s. unique by [16]. Let $V_i = U_i - s_{\min}[\text{mod } 1],\ 1 \le i \le n$ be the jump times of $\varepsilon_{\mathbf{m}}$, and remark that $s_{\min}$ is itself one of the U_i's. For $t \ge 0$ and $0 \le s \le 1$, let $\varepsilon^{(t)}_{\mathbf{m}}(s) = -\varepsilon_{\mathbf{m}}(s) + ts$,

$$\overline{\varepsilon}^{(t)}_{\mathbf{m}}(s) := \sup_{0 \le s' \le s} (-\varepsilon_{\mathbf{m}}(s') + ts'), \qquad 0 \le s \le 1,$$

its supremum process, and

$$J(t) := ([a_1(t), b_1(t)], [a_2(t), b_2(t)], \ldots, [a_k(t), b_k(t)])$$

be the sequence of its intervals of constancy ranked in decreasing order of their lengths. Let $\#(t) = k$ their number.

Let

$$F(t) = (1+t).(b_1 - a_1, \ldots, b_{\#(t)} - a_{\#(t)}, 0, 0, \ldots)$$

be the sequence of the corresponding lengths, renormalized by the proper constant so that their sum is 1 (that this constant equals $1+t$ is a consequence of the fact that $\varepsilon^{(t)}_{\mathbf{m}}$ has a slope $1+t$). Also, let $F(\infty) = (m_1, \ldots, m_n)$, which is equal to $F(t)$ for t sufficiently large, a.s. Then by [5] the sequence of the distinct states of $(F(t))_{t \ge 0}$ is equal in law to the time-reversed sequence of the distinct states of the additive coalescent starting at $(m_1, \ldots, m_n)$ (it is in particular easy to observe that the terms in $F(t)$ are constituted of sums of subfamilies extracted from $\mathbf{m}$). On the other hand it is easy to see that every jump time $V_j,\ 1 \le j \le n$ belongs to some $[a_i, b_i)$, and that the left bounds $(a_i(t), 1 \le i \le \#(t))$ are all equal to some V_j. Hence the $(V_j)_{1 \le j \le n}$ induce on the intervals of constancy a random order $O^{\mathbf{m}}_{\varepsilon}(t)$ at time t :

$$(i,j) \in O^{\mathbf{m}}_{\varepsilon}(t) \iff V_i \le V_j \text{ and } V_i, V_j \in [a_k(t), b_k(t)] \text{ for some } k \le \#(t).$$

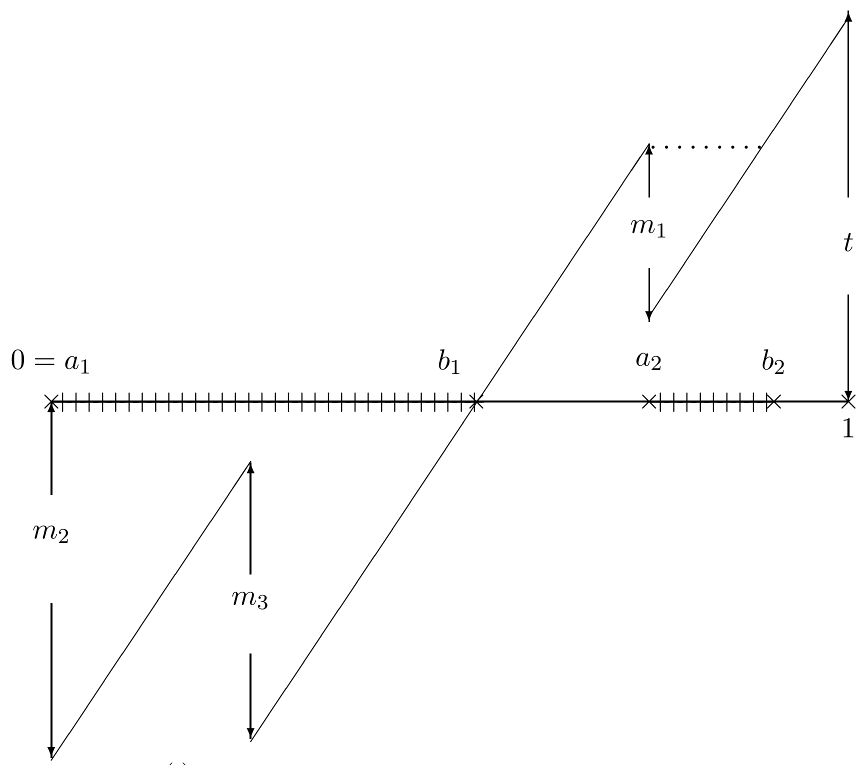

Figure 2: $\varepsilon_{\mathbf{m}}^{(t)}$ and intervals of constancy of its supremum.

We thus deduce a process $(O_\varepsilon^{\mathbf{m}}(t))_{t\geq 0}$ with values in $\mathcal{O}_n$. By convention, let $O_\varepsilon^{\mathbf{m}}(\infty)$ be the element of $\mathcal{O}_n$ constituted of the singletons $\{1\},\ldots,\{n\}$. It is easy to see that it is indeed equal to $O_\varepsilon^{\mathbf{m}}(t)$ for t sufficiently large.

Now we show how to recover the $\mathcal{O}_n$-coalescent with proto-galaxy masses $m_1,\ldots,m_n$ from the bridge $b_{\mathbf{m}}$. In [5] the bridge $b_{\mathbf{m}}$ is defined in a different way : if $(s_i)_{1\leq i\leq n}$ are independent standard exponential r.v.'s, the jump times $U_1',\ldots,U_n'$ of $b_{\mathbf{m}}$ are defined by $U_{k+1}'-U_k'[\text{mod } 1] = s_k/(s_1+\ldots+s_n)$ and U_1' independent uniform on $[0,1]$. At time U_i', the bridge has a positive jump $m_{\sigma(i)}$ where σ is a uniform random permutation on $\mathbb{N}_n$. It is easy to see that the bridge defined in this way has the same law as $b_{\mathbf{m}}$. Let $A_n = s_1+\ldots+s_n$, which is independent of the bridge (since it is independent of the jump times). We then know from Propositions 1 and 2 of [5] that $(F(t^{-1}A_n-1))_{0\leq t<A_n}$ (with $F(+\infty) = (m_1,\ldots,m_n)$) is the aggregative server system described in the second remark of section 2.1.

Therefore, let $T(t)$ be the time-change defined as follows. For $0\leq k\leq n$ let $t_k = \sup\{t\geq 0,\ \#(t) = n-k\}$ with $t_0=\infty,\ t_n = 0$ and

$$A_k = \frac{A_n}{1+t_k}$$

be the first time when $(F(t^{-1}A_n-1))_{0<t<A_n}$ has $n-k$ components. For $0\leq i\leq n-2$ let also

$$I(t) = \frac{t-A_i}{n-i-1} + \sum_{j=0}^{i-1}\frac{A_{j+1}-A_j}{n-j-1} \text{ if } t\in[A_i,A_{i+1}],$$

so that $I(t)$ defines a continuous increasing function on $[0,A_{n-1}]$ whose inverse is denoted by

I^{-1}. Last, for $0 \leq t \leq I(A_{n-1})$ we set

$$T(t) = \frac{A_n}{I^{-1}(t)} - 1, \; T(0) = \infty.$$

We then have the

Lemma 1 *The process*

$$O_t^{\mathbf{m}} = \begin{cases} O_\varepsilon^{\mathbf{m}}(T(t))) & \text{if} \quad 0 \leq t \leq I(A_{n-1}) \\ O_\varepsilon^{\mathbf{m}}(T(I(A_{n-1}))) & \text{if} \quad \quad t > I(A_{n-1}) \end{cases}$$

is an ordered additive coalescent in $\mathcal{O}_n$ with proto-galaxy masses $m_1, \ldots, m_n$.

Proof. The discrete state-evolution has the proper law, since given that two clusters with mass m_i and m_j coalesce, the probability that the mass m_i is at the left of the mass m_j is $m_i/(m_i + m_j)$, which is obtained from the calculation of Proposition 2 in [5]. This proposition also shows that $(A_1, \ldots, A_n)$ are the n first jumps of a Poisson process with intensity 1. Together with the definition of $I(t)$ and $T(t)$, this implies that the time-evolution also has the appropriate law : when the system is constituted of k clusters, the time until the following coalescence is exponential with parameter $k - 1$. □

Remark. In particular, we easily get that the final order $O^{\mathbf{m}}_{I(A_{n-1})} = O^{\mathbf{m}}_\infty$ is defined by $(i, j) \in O^{\mathbf{m}}_{I(A_{n-1})} \iff V_i \leq V_j$, where $(V_j)_{1 \leq j \leq n}$ is a cyclic permutation of the uniformly distributed $(U_j)_{1 \leq j \leq n}$ which is determined by the location of the minimum of the bridge $b_{\mathbf{m}}$.

2.3 The ballot Theorem

Before examining ordered coalescent any further, we present a tool that will be seen to be very useful in the sequel.

The most famous version of the ballot theorem is doubtless its discrete version (see Takács [24]). When we pick the balls one by one without replacement in a box containing a red balls and b green balls, $a > b$, the probability that the red balls are always leading is $(a - b)/(a + b)$. There exists a continuous version, which can be deduced from the discrete one by approximation, as it is done in [24], for non-decreasing processes with derivative a.s. 0, with exchangeable increments. We give an alternative version.

Let $\ell > 0$. From [14] we know that every process b with exchangeable increments on $[0, \ell]$ with bounded variation may be represented in the form

$$b(x) = \alpha x + \sum_{i=1}^{\infty} \beta_i (1_{x \geq U_i} - \frac{x}{\ell}), \quad 0 \leq x \leq \ell \tag{2}$$

where $(U_i)_{i \geq 1}$ is i.i.d. uniform on $[0, \ell]$, and $\alpha, \beta_1, \beta_2, \ldots$ are (not necessary independent) r.v.'s which are independent of the sequence $(U_i)_{i \geq 1}$, and satisfy $\sum_{i=1}^{\infty} |\beta_i| < \infty$ a.s. We call it the *Kallenberg bridge* with drift coefficient α and jumps $\beta_1, \ldots$.

Lemma 2 (ballot Theorem) *Suppose that the jumps* $\beta_1, \ldots$ *are negative. Then we have*

$$\mathbb{P}\Big[b(x) \geq 0 \ \forall x \in [0, \ell] \Big| b(\ell), \beta_1, \beta_2, \ldots\Big] = \max\Big(\frac{b(\ell)}{b(\ell) - \sum_{i=1}^{\infty} \beta_i}, 0\Big)$$

In particular, conditionally on $\sum_{i=1}^{\infty} \beta_i$ *and* $b(\ell)$*, the event* $\{b(x) \geq 0 \ \forall\ 0 \leq x \leq \ell\}$ *is independent of the sequence of the jumps* $(\beta_1, \beta_2, \ldots)$.

Proof. Denote $-\frac{\sum_{i=1}^{\infty} \beta_i}{\ell}$ by Σ. For each i we define the following process on $(0, \ell)$:

$$M_x^i = 1_{U_i \leq x},$$

so that

$$b(x) = (\alpha + \Sigma)x + \sum_{i=1}^{\infty} \beta_i M_x^i$$

Let $(\mathcal{F}_x)_{x \geq 0}$ be the filtration generated by all the processes $M^i_{(\ell - x)-}$ and enlarged with the variables α and β_i's. Then the process

$$\frac{M^i_{(\ell-x)-}}{(\ell - x)}, \qquad 0 \leq x \leq \ell$$

is a martingale with respect to this filtration, and if $\mathbf{M}_x = -\sum_{i=1}^{\infty} \beta_i M_x^i$, so is

$$\frac{\mathbf{M}_{(\ell-x)-}}{\ell - x}, \qquad 0 \leq x \leq \ell$$

which has Σ for starting point. Moreover we remark that it tends to 0 at ℓ with probability 1, as a consequence of Theorem 2.1 (ii) in [15], with $f(t) = t$ (in other words, processes with exchangeable increments with no drift have a.s. a zero derivative at 0). The hypothesis that the jumps β_i are negative enables us to apply the optional sampling theorem which thus gives that, conditionally on the β_i's and α, $\mathbf{M}_x$ stays below $(\alpha + \Sigma)x$ on $(0, \ell)$ with probability $\alpha/(\alpha + \Sigma)$. The second assertion follows. □

As a consequence of this lemma, we obtain that when $\alpha = 0$ a.s., b attains its minimum at a jump time. Indeed, for $i \geq 0$, let $v_i b(x) = b(x + U_i[\text{mod } 1]) - b(U_i)$ the process obtained by splitting the bridge at U_i, and modified at time ℓ so that it is continuous at this time. Since the variables $U_j - U_i$ for $j \neq i$ are also uniform independent, it is easy to see that $v_i b$ is the Kallenberg bridge with jumps β_j, $j \neq i$ and drift coefficient β_i/ℓ. Lemma 2 implies that conditionally on β_i and on $\sum_{i=1}^{\infty} \beta_j$, it is positive (i.e. U_i is the location of the minimum of b, or also that $v_i b$ is equal to Vb) with probability $\beta_i / \sum_{j=1}^{\infty} \beta_j$. Since the sum of these probabilities is 1, we can conclude.

On the other hand, we obtain by a simple time-reversal on $[0, \ell]$ the same result for processes with exchangeable increments and positive jumps. Moreover, if b has **positive** jumps, we have the

Corollary 1 *Conditionally on the* β_i*'s, the first jump of* Vb *is* β_i *with probability* $\beta_i / \sum_{j=1}^{\infty} \beta_j$*, that is, it has the law of a* size biased pick *from the sequence* $\beta_1, \beta_2 \ldots$

Proof. It is immediate from our discussion above when considering $b(\ell - x)$, which has negative jumps. $\square$

An important consequence is that the left-most fragment $\min O_\infty^{\mathbf{m}}$ is a $\mathbf{m}$-size biased pick from $\{1, \dots, n\}$, that is

$$P[\min O_\infty^{\mathbf{m}} = i] = m_i.$$

2.4 Infinite state case : fragmentation with erosion

We now give a generalization of the previous results to additive coalescents with an infinite number of clusters. We will use approximation methods that are very close to [5], with the difference that the processes we are considering have bounded variation, which makes the approximations technically more difficult (in particular the functionals of trajectories such as the Vervaat's transform are not continuous).

It is conceptually easy to generalize the construction by Evans and Pitman [12] of partition valued additive coalescents. We are following the same approach by replacing the set of partitions $\mathcal{P}_\infty$ by the set $\mathcal{O}_\infty$.

We endow $\mathcal{O}_\infty$ with a topology as in [12] : first endow the finite set $\mathcal{O}_n$ with the discrete topology. For $n \leq \alpha \leq \infty$, call π^n the function from $\mathcal{O}_\alpha$ to $\mathcal{O}_n$ corresponding to the restriction to $\mathbb{N}_n$. Then the topology on $\mathcal{O}_\infty$ is that generated by $(\pi^n)_{n\geq 1}$. It is a compact totally disconnected metrizable space, and the distance $d(O, O') = \sup_{n\geq 1} 2^{-n} 1_{\{\pi^n(O) \neq \pi^n(O')\}}$ induces the same topology.

We also denote by $(O_t)_{t\geq 0}$ the canonical process associated to càdlàg functions on $\mathcal{O}_\infty$. Last, for $O \in \mathcal{O}_\infty$ let $\mu^{\mathbf{m}}(O, .)$ be the measure that places, for each pair of distinct clusters $(\mathcal{I}, \mathcal{J})$ mass $m_{\mathcal{I}}$ at $O_{\mathcal{I}\mathcal{J}}$. We wish to show that there exist for each $\mathbf{m} \in S_1^{+,\infty}$, a family of laws $(\mathbb{P}_O^{\mathbf{m}})_{O \in \mathcal{O}_\infty}$ such that

- If $O \in \mathcal{O}_\infty$ contains the cluster $(n, n+1, n+2, \dots)$ with any order on it (for example, the natural order on $\mathbb{N}$) for some $n \geq 1$, then $(\pi^n(O_t))_{t\geq 0}$ is under $\mathbb{P}_O^{\mathbf{m}}$ a $\mathcal{O}_n$-coalescent with proto-galaxy masses $\mathbf{m}^{[n]} = (m_1, \dots, m_{n-1}, \sum_{i=n}^\infty m_i)$ started at $\pi^n(O)$.
- Under $\mathbb{P}_O^{\mathbf{m}}$, the canonical process is a Feller process with Lévy system given by the jump kernel $\mu^{\mathbf{m}}$.
- The map $(\mathbf{m}, O) \mapsto \mathbb{P}_O^{\mathbf{m}}$ from $S_1^{+,\infty} \times \mathcal{O}_\infty$ to the space of measures on $\mathbb{D}(\mathbb{R}_+, \mathcal{O}_\infty)$, is weakly continuous (where we endow $S_1^{+,\infty}$ with the usual l_1 topology).

Again, we could state the result for more general $\mathbf{m}$'s (with finite sum $\neq 1$), but this would only require an easy time-change.

We claim that these properties follow from the same arguments as in [12]. Indeed, Theorem 10, Lemma 14 and Lemma 15 there still hold when $\mathcal{P}_\infty$ (the space of partitions on $\mathbb{N}$) is replaced by the topologically very similar space $\mathcal{O}_\infty$. We have, however, to check that the construction of a "coupled family of coalescents" as in [12] (Definition 12 and Lemmas 14 and 15 there), still exists in our ordered setting. For this we adapt the arguments of Lemma 16 there: we use the same construction with the help of Poisson measures, but we do not neglect the orientation of the edges in the random birthday trees $\mathcal{T}(Y_j^{n,\mathbf{m},O})$ we obtain in a similar way. In this way, we construct from these birthday trees ordered coalescents instead of the ordinary coalescents (see the first remark of section 2.1).

We will give a description of the $\mathcal{O}_\infty$-additive coalescent processes with the help of bridges extending that in section 2.2.

Let now $\mathbf{m}$ be in $S_1^{+,\infty}$, and let $b_{\mathbf{m}}$ be the Kallenberg bridge with jumps $m_1, m_2, \dots$ constructed from an i.i.d. sequence of uniform variables $U_1, U_2, \dots$ in $[0,1]$. Let U^* be the a.s. unique ([16]) location of the minimum of $b_{\mathbf{m}}$. We know from Lemma 2 that it is a jump time for $b_{\mathbf{m}}$, that is, $U^* = U_i$ for some i a.s. Let $\varepsilon_{\mathbf{m}}$ be the associated Vervaat excursion. Last, for $i \in \mathbb{N}$ let $V_i = U_i - U^*[\text{mod } 1]$ be the jump times of $\varepsilon_{\mathbf{m}}$.

Similarly as above let

$$\overline{\varepsilon}_{\mathbf{m}}^{(t)}(s) = \sup_{0 \le s' \le s} (ts - \varepsilon_{\mathbf{m}}(s)), \qquad 0 \le s \le 1.$$

We denote by $([a_i(t), b_i(t)])_{i \ge 1}$ the intervals of constancy of $\overline{\varepsilon}_{\mathbf{m}}^{(t)}$. We may thus construct an $\mathcal{O}_\infty$-valued process $(O_\varepsilon^{\mathbf{m}}(t))_{t \ge 0}$ which consists on the order induced by the V_j's on each of the $[a_i(t), b_i(t))$. Remark that every $a_i(t)$ corresponds to some U_j. Indeed, the bridge $b_{\mathbf{m}}$ leaves its local minima by a jump, otherwise by exchangeability of the increments the minimum would be attained continuously with positive probability. As a consequence, every cluster of $O_\varepsilon^{\mathbf{m}}(t)$ has a minimum, the "left-most fragment". Denote by O_{Sing} the element of $\mathcal{O}_\infty$ constituted of the singletons $\{1\}, \{2\}, \dots$, and let $O_\varepsilon^{\mathbf{m}}(\infty) = O_{\text{Sing}}$. Our claim is that

Theorem 1 *The process*

$$O_t^{\mathbf{m}} = O_\varepsilon^{\mathbf{m}}\Big(\frac{e^{-t}}{1 - e^{-t}}\Big) \qquad t \ge 0$$

has law $\mathbb{P}_{O_{\text{Sing}}}^{\mathbf{m}}$.

Moreover, $O_t^{\mathbf{m}}$ *has a limit* $O_\infty^{\mathbf{m}} = O_\varepsilon^{\mathbf{m}}(0)$ *at* $+\infty$*, and the left-most fragment* $\min O_\infty^{\mathbf{m}}$ *is a* $\mathbf{m}$*-size biased pick from* $\mathbb{N}$*.*

We are going to prove this theorem by using a limit of the bridge representation of the ordered additive coalescent described in section 2.2 and the weak continuity properties of $P_O^{\mathbf{m}}$. Recall that $(U_i)_{i \ge 1}$ are the jump times of $b_{\mathbf{m}}$. Let $b_{\mathbf{m}}^n$ be the bridge defined as in (1) with jumps $m_1/S_n, \dots, m_n/S_n$ where $S_n = \sum_{i=1}^n m_i$ and with jump times $U_1, \dots, U_n$. Let $\varepsilon_{\mathbf{m}}^n$ be the associated Vervaat excursion. Last, let $\overline{\varepsilon}_{\mathbf{m}}^{(n,t)}$ be the supremum process of $(ts - \varepsilon_{\mathbf{m}}^n(s))_{0 \le s \le 1}$. We begin with the

Lemma 3 *Almost surely, we may extract a subsequence of* $(\varepsilon_{\mathbf{m}}^n)_{n \ge 1}$ *which converges uniformly to* $\varepsilon_{\mathbf{m}}$ *as* n *goes to infinity.*

Proof. It is trivial that $b_{\mathbf{m}}^n$ converges uniformly to $b_{\mathbf{m}}$ since the jump times coincide (the bridges are built on the same U_i's). To get the uniform convergence of the Vervaat excursions, it suffices to show that a.s. for n sufficiently large and up to the extration of subsequences, the location of the minimum of the bridge $b_{\mathbf{m}}^n$, which is some jump time U_n^*, remains unchanged.

For $i \ne j \in \mathbb{N}$, consider the probability $p_{i,j}^n = \mathbb{P}[U_n^* = U_i, U^* = U_j]$. Fix i. Then for $j \ne i$, on $\{U^* = U_j\}$, there is a.s. some $\eta > 0$ such that $b_{\mathbf{m}}(U_i-) \ge b_{\mathbf{m}}(U_j-) + \eta$. Since we have uniform convergence of $b_{\mathbf{m}}^n$ to $b_{\mathbf{m}}$, this implies that $p_{i,j}^n$ tends to 0, and also

$$\forall \epsilon > 0, \forall k \in \mathbb{N}, \exists n_{\epsilon,k} \in \mathbb{N}, \forall n \ge n_{\epsilon,k}, \forall j \in \mathbb{N}_k,\ j \ne i,\ p_{i,j}^n \le \frac{\epsilon}{2k}.$$

Next, independently of n let k be sufficiently large so that

$$\sum_{j=k+1}^{\infty} \mathbb{P}[U^* = U_j] = \sum_{j=k+1}^{\infty} m_j < \frac{\epsilon}{2},$$

where the equality is obtained from Corollary 1. In this case we obtain that for $\epsilon > 0$ and n large,

$$\sum_{j \neq i} p_{i,j}^n = \mathbb{P}[U_n^* = U_i \neq U^*] < \epsilon.$$

By dominated convergence, this implies that

$$\lim_{n\to\infty} \mathbb{P}[U_n^* \neq U^*] = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mathbb{P}[U_n^* = U_i \neq U^*] = 0,$$

since

$$\mathbb{P}[U_n^* = U_i \neq U^*] \leq \mathbb{P}[U_n^* = U_i] = \frac{m_i}{m_1 + \ldots + m_n} \leq \frac{m_i}{m_1}.$$

Where the last equality above is also obtained from Corollary 1. From this we deduce that up to extraction of a subsequence, $U_n^* = U^*$ for n sufficiently large. □

Next we associate to each integer n an ordered additive coalescent process $(O_t^n)_{t\geq 0}$ taking values in $\mathcal{O}_\infty$ as follows. Let $([a_i^n(t), b_i^n(t)])_{1\leq i\leq \#(n,t)}$ be the intervals of constancy of $\overline{\varepsilon}_{\mathbf{m}}^{(n,t)}$ (the process defined as $\overline{\varepsilon}_{\mathbf{m}}^{(t)}$, but for the bridge $b_{\mathbf{m}}^n$). We know from section 2.2 how to obtain an ordered coalescent in $\mathcal{O}_n$ from $b_{\mathbf{m}}^n$, with proto-galaxy masses $m_1, \ldots, m_n$, and starting from the singletons $\{1\}, \ldots, \{n\}$ by a proper time-change $T_n(t)$ from the process $(O_\varepsilon^n(t))_{t\geq 0}$, with obvious notations (each T_n requires the choice of a variable $A_n^{(n)}$ with law Gamma$(1, n)$, so we take independent variables $(A_n^{(n)})_{n\in\mathbb{N}}$ with the proper distributions). We now just turn $\{n\}$ into the cluster $(n, n+1, \ldots)$ with the natural order induced by $\mathbb{N}$ in the initial state, and assign mass $m_n/2^{i+1}$ to the integer $n+i$. We thus obtain a $\mathcal{O}_\infty$ coalescent with proto-galaxy masses $m_1, \ldots, m_{n-1}, m_n/2, m_n/4, \ldots$. This last sequence converges in l_1 norm to $\mathbf{m}$. Hence the coalescent converges in law to the ordered coalescent starting from all singletons, with proto-galaxy masses $\mathbf{m}$, in virtue of the weak continuity property for $(\mathbb{P}_O^{\mathbf{m}})_{\mathbf{m}\in S_1^{+,\infty}, O\in\mathcal{O}_\infty}$.

Proof of Theorem 1. We have a.s. that, if $[a, b]$ is an interval of constancy for the process $(\overline{\varepsilon}_{\mathbf{m}}^{(t)}(s))_{0\leq s\leq 1}$, then for every $s \in]a, b[$,

$$\max(\overline{\varepsilon}_{\mathbf{m}}^{(t)}(s-), \overline{\varepsilon}_{\mathbf{m}}^{(t)}(s)) < \overline{\varepsilon}_{\mathbf{m}}^{(t)}(a).$$

This is proved in [5], Lemma 7. This assertion combined with the uniform convergence in Lemma 3 implies, up to extraction of subsequences, the pointwise convergence of $(a_i^n(t))_{i\geq 1}$ (resp. $(b_i^n(t))_{i\geq 1}$) to $(a_i(t))_{i\geq 1}$ (resp. $(b_i(t))_{i\geq 1}$) as n goes to infinity, for every $i \geq 0$. Moreover, this gives that the b_i's are not equal to some of the V_j's (else the process would jump at the end of an interval of constancy of its supremum process, which would be absurd).

More precisely, we have that $a_i^n(t) \leq a_i(t)$ for n sufficiently large, for every t. Indeed, the process $(ts - \varepsilon_{\mathbf{m}}(s)))_{0\leq s\leq 1}$ jumps at $a_i(t) = V_j$ for some j and $a_i^n(t)$ tends to $a_i(t)$ so that if $a_i^n(t)$ was greater than $a_i(t)$ for arbitrarily large n, $a_i^n(t)$ would not be the beginning of an interval of constancy for $\overline{\varepsilon}_{\mathbf{m}}^n$, for some $n \geq j$.

From this we deduce that (still up to extraction) $(O_\varepsilon^n(t))_{t\geq 0}$ converges in $\mathcal{O}_\infty$ to $(O_\varepsilon^{\mathbf{m}}(t))_{t\geq 0}$ in the sense of finite-dimensional distributions. Indeed, it suffices to show that for every $m \geq 0$ the restriction of the order $O_\varepsilon^n(t)$ to $\mathbb{N}_m$ is equal to the restriction of the order $O_\varepsilon^{\mathbf{m}}(t)$ for n sufficiently large. For this it suffices to choose n such that $[a_i^n(t), a_i(t))$ does not contain a V_j with label $j \leq m$, and that such V_j's does not fall between any $b_i^n(t)$ and $b_i(t)$. This is possible according to the above remarks. For such n the orders induced by the V_j's in each $[a_i(t), b_i(t)]$, and restricted to $\mathbb{N}_m$ are the same.

Similarly, we have that the process $(O_\varepsilon^{\mathbf{m}}(t))_{t\geq 0}$ is continuous in probability at every time t. For this we use the fact that for every i, $a_i(t') = a_i(t)$ a.s. for t' close to t, which is a consequence of [15] Theorem 2.1 (ii) for $f(t) = t$. Indeed, this theorem shows that the bridge $b^{\mathbf{m}}$ has a derivative equal to -1 at 0, and hence by exchangeability $b^{\mathbf{m}}$ has a left derivative equal to -1 at any jump time since we would not lose the exchangeability by suppressing the corresponding jump.

Last, it is easily seen that the time-change $T_n(t)$ converges in probability to $e^{-t}/(1-e^{-t})$. Together with the above, this ends the proof of the first assertion of the theorem. Together with Corollary 1, we get the second one. □

Remark also that the mass of the i-th heavier cluster is given by

$$\frac{b_i(s) - a_i(s)}{1 - e^{-t}} \qquad \text{for } s = \frac{e^{-t}}{1 - e^{-t}} \tag{3}$$

and can be read directly on the intervals of constancy of $\overline{\varepsilon}_{\mathbf{m}}$. In the following studies, we will turn our study to these lengths of constancy intervals.

We conclude this section with a remark concerning the so-called *fragmentation with erosion* ([5, 2]) that typically appears in such "bounded variation" settings as the one in this part ($b_{\mathbf{m}}$ has bounded variation a.s.). The "erosion" comes from the fact that the total sum of the intervals of constancy of $\overline{\varepsilon}_{\mathbf{m}}$ is less than 1. Yet, in our study, we have shown that the erosion is deterministic, and that when we compensate it by the proper multiplicative constant, we obtain after appropriate time-changing an additive coalescent starting at time 0 (at the opposite of eternal coalescent obtained in the infinite variation case that starts from time $-\infty$). We have not pursued it, but we believe that in the ICRT context in [2], the ICRT obtained in the equivalent "bounded variation" setting ($\sum \theta_i^2 = 1$ and $\sum \theta_i < \infty$ with the notations therein) is somehow equivalent to the birthday tree with probabilities $m_1, m_2 \ldots$ so that Poisson logging on its skeleton just gives a process which is somehow isomorphic to a non-eternal additive coalescent.

3 The Lévy fragmentation

We now turn to the study of fragmentation processes associated to Lévy processes. We are motivated by the fact that it is known from [4] and [5] how to obtain eternal coalescent processes from certain types of bridges with exchangeable increments. Following a remark of Doney we may use the methods described in [4] in much more general context. For example, it is natural to wonder what kind of processes can be obtained in the same way from Lévy bridges, which are important examples of bridges with exchangeable increments. Moreover, the following is to be read at the light of the preceding section, which gives an interpretation of the ordering naturally induced on the fragments by their respective places on the real line. Our goal is to make "explicit" the law of the fragmentation process at a fixed time in terms of the hitting times process of the Lévy process.

We begin by giving the setting of our study, and by recalling some properties on Lévy processes with no positive jumps and the excursions of their reflected processes. Most of them can be found in [3]. From now on in this paper, X designates a Lévy process with no positive jumps.

3.1 Lévy processes with no positive jumps, bridges and reflected process

Let ν be the Lévy measure of X. We will make the following assumptions :

- X has no positive jumps.
- X does not drift to $-\infty$, i.e. (together with the above hypothesis) X has first moments and $E[X_1] \geq 0$.
- X has a.s. unbounded variation.

We set

$$X_s^{(t)} = X_s + ts,$$

and

$$\overline{X}_s^{(t)} = \sup_{0 \leq s' \leq s} X_{s'}^{(t)}$$

its supremum process (sometimes called "supremum" by a slight abuse). Let $T_x^{(t)}$ the first hitting time of x by $X^{(t)}$. Recall that the fact that $X^{(t)}$ has no positive jumps implies that the process of first hitting times defined by $T_x^{(t)} = \inf\{u \geq 0, X_u^{(t)} > x\}$ is a subordinator, and the fact that X, and then $X^{(t)}$, does not drift to $-\infty$ implies that $T^{(t)}$ is not killed. Moreover, $T^{(t)}$ is pure jump since $X^{(t)}$ has infinite variation. When $t = 0$ we will drop the exponent $^{(0)}$ in the notation.

The reason why we impose infinite variation is that we are going to study eternal coalescent processes. Nevertheless, we make some comments on the bounded variation case in section 8.

Last, we suppose for technical reasons that for all $s > 0$, the law of X_s has a continuous density w.r.t. Lebesgue measure. We call its density $q_s(x) = \mathbb{P}[X_s \in dx]/dx$. For comfort, we suppose that $q_s(x)$ is bi-continuous in $s > 0$ and $x \in \mathbb{R}$. This is not the weakest hypothesis that we can assume, but it makes some definitions clearer, and some proofs simpler (Vervaat's Theorem,...). In particular, for every $\ell > 0$ we may define the law of the *bridge of* X *from* 0 *to* 0 *with length* ℓ as the limit

$$P_{0,0}^{\ell}(\cdot) = \lim_{\varepsilon \to 0} P^{\ell}[\,\cdot\, \| X_\ell| < \varepsilon], \tag{4}$$

where P^ℓ is the law of the process X stopped at time ℓ, see [13], [16].

Let us now present some facts about reflected processes and excursion theory.

We call reflected process (below its supremum) of X the process $\overline{X} - X$. The general theory of Lévy processes gives that this process is Markov, and since X has no positive jumps, the process $\overline{X}$ is a local time at 0 for the reflected process, and T is the inverse local time process. This local time enable us to apply the Itô excursion theory to the excursions of the reflected process away from 0. The excursion at level $x \geq 0$ is the process

$$\varepsilon^x = (\overline{X}_{T_{x-}+u} - X_{T_{x-}+u}, 0 \leq u \leq T_x - T_{x-}).$$

Itô theory implies that the excursion process is a Poisson point process: there exists a σ-finite measure n, called the excursion measure, which satisfies the Master Formula

$$\mathbb{E}\Big(\sum_{x>0} H(T_{x-}(\omega), \omega, \varepsilon_x(\omega)) \Big) = \mathbb{E}\Big(\int_0^\infty d\overline{X}_s(\omega) \int n(d\varepsilon) H(s, \omega, \varepsilon) \Big)$$

where H is a positive functional which is jointly predictable with respect to its first two components and measurable with respect to the third.

Let $D = \inf\{u > 0 : \varepsilon(u) = 0\}$ be the death time of an excursion ε. We are going to give a "good" representation of the excursion measure conditioned by the duration with the help of the following generalization of Vervaat's Theorem, that we will still call "Vervaat's Theorem" :

Proposition 1 (Vervaat's Theorem) *Let b_X^ℓ be the bridge of X on $[0, \ell]$ from 0 to 0. Let Γ_ℓ be the law of the process $(V b_X^\ell(\ell - x))_{0 \le x \le \ell}$. Then the family $(\Gamma_\ell)_{\ell > 0}$ is a regular version for the "conditional law" $n(\cdot|D)$ in the sense that*

$$n(d\varepsilon) = \int_0^\infty n(D \in d\ell) \Gamma_\ell(d\varepsilon) = \int_0^\infty \frac{d\ell}{\ell} q_\ell(0) \Gamma_\ell(d\varepsilon)$$

Hence, whe will always refer to the time-reversed Vervaat Transform of the bridge of X from 0 to 0 with length $\ell > 0$ as the *excursion of X below its supremum, with duration ℓ.* The explanation for the second equality above is in the first assertion of Lemma 7, and the proof of this proposition is postponed to section 7.

3.2 The fragmentation property

We now state a definition of what we call an (inhomogeneous) fragmentation process, following [4]. Let $S^\downarrow$ be the space of all decreasing positive real sequences with finite sum, and for all $\ell > 0$, let $S_\ell^\downarrow \subset S^\downarrow$ be the space of the elements of $S^\downarrow$ with sum ℓ. Then, for $0 \le t \le t'$, consider for each ℓ an "elementary" probability measure $\kappa_{t,t'}(\ell)$ on $S_\ell^\downarrow$. Next, for all $L = (\ell_1, \ell_2, \ldots) \in S^\downarrow$, let $L_1, L_2, \ldots$ be independent sequences with respective laws $\kappa_{t,t'}(\ell_2)$, $\kappa_{t,t'}(\ell_2), \ldots$ and define $\kappa_{t,t'}(L, .)$ as the law of the decreasing arrangement of the elements of $L_1, L_2, \ldots$.

Definition 2 *We call fragmentation process a (*a priori *not homogeneous in time) Markov process with transition kernel $(\kappa_{t,t'}(L, dL'))_{t<t', L \in S^\downarrow}$.*

Informally, conditionally on the current state of the fragmentation, a fragment then splits in a way depending only on its length, independently of the others, a property which is usually refered to as the *fragmentation property.*

As in the Brownian case developed in [4], the process that we are now defining is a fragmentation process of the **random** interval $[0, T_1]$.

Definition 3 *We call* Lévy fragmentation *associated to X, and we denote it by $(F^X(t))_{t \ge 0}$, the process such that for each t, $F^X(t)$ is the decreasing sequence of the lengths of the interval components of the complementary of the support of the measure $1_{[0,T_1]} d\overline{X}^{(t)}$ (in other terms, of constancy intervals of the supremum of $X^{(t)}$ before T_1).*

Since the support of the measure $d\overline{X}^{(t)}$ is precisely the range of the subordinator $T^{(t)}$, we see that the sum of the "fragments" at a time $t > 0$ is T_1. Indeed, since $X^{(t)}$ has infinite variation, 0 is regular for itself (see Corollary VII,5 in [3]), and it easily follows that the closure of the range of $T^{(t)}$ has zero Lebesgue measure.

The purpose of this section is to prove that the Lévy fragmentation is indeed a fragmentation in the sense of Definition 2. For any $\ell > 0$ we consider the transition kernel $\varphi_{t,t'}(\ell)$ defined as follows:

Definition 4 *For any $t' > t \geq 0$, let $(\varepsilon_\ell^{(t)}(s), 0 \leq s \leq \ell)$ be the generic excursion with duration ℓ of the reflected process $\overline{X}^{(t)} - X^{(t)}$. We denote by $\varphi_{t,t'}(\ell)$ the law of the sequence of the lengths of the constancy intervals for the supremum process of $(s(t'-t) - \varepsilon^{(t)}(s), 0 \leq s \leq \ell)$, arranged in decreasing order.*

By convention let $\varphi_{t,t'}(0)$ be the Dirac mass on $(0, 0, \ldots)$.

Remark. In fact one could define a fragmentation-type process from any excursion-type function f defined on $[0, \ell]$ (that is, which is positive, null at 0 and ℓ and with only positive jumps), deterministic or not, by declaring that $F^f(t)$ is the decreasing sequence of the lengths of the intervals of constancy of the supremum process of $(st - f(s), 0 \leq s \leq \ell)$. We will sometimes refer to it as the fragmentation process associated to f.

Proposition 2 *The process $(F^X(t))_{t\geq 0}$ is a fragmentation process with kernels $\varphi_{t,t'}(L, dL')$ $(0 \leq t < t',\ L \in S^{\downarrow})$. In other words, for $t' > t \geq 0$, conditionally on $F^X(t) = (\ell_1, \ell_2, \ldots)$, if we consider a sequence of independent random sequences $F_1, F_2, \ldots$ with respective laws $\varphi_{t,t'}(\ell_1), \varphi_{t,t'}(\ell_2), \ldots$, then the law of $F^X(t')$ is the one of the sequence obtained by rearranging the elements of $F_1, F_2, \ldots$ in decreasing order.*

Remarks. From the definition of F^X we can see that it is a fragmentation beginning at a *random* state which corresponds to the sequence of the jumps of (T_x) for $x \leq 1$. But we can also define the Lévy fragmentation beginning at $(\ell, 0, \ldots)$ by applying the transition mechanism explained in Proposition 2 to this sequence. We will denote the derived Markov process by F^{ε_ℓ}, the fragmentation beginning from fragment ℓ, which is equal in law to the fragmentation process associated to ε_ℓ by virtue of Proposition 2. Even if the definition of F^X is simpler than that of F^{ε_ℓ}, we will rather study the latter in the sequel since the behaviour of F^X can be deduced from that of the F^{ε_ℓ}'s as we will see at the beginning of section 4.

In order to prove Proposition 2 we will use, as mentioned above, the same methods as in [4]. Since the proofs are almost the same, we will be a bit sketchy.

First we remark that a "Skorohod-like formula" holds for the supremum processes $\overline{X}^{(t)}$ and $\overline{X}^{(t')}$. This formula is at the heart of the fragmentation property.

Lemma 4 *For any $t' \geq t \geq 0$ we have*

$$\overline{X}_u^{(t)} = \sup_{0 \leq v \leq u} (\overline{X}_v^{(t')} - (t'-t)v) \tag{5}$$

This property holds for any process X which has a.s. no positive jumps; the proof of [4] applies without change.

Remark. In fact we have that

$$\overline{X}_u^{(t)} = \sup_{g \le v \le u} (\overline{X}_v^{(t')} - (t'-t)v) \tag{6}$$

where u belongs to the interval of constancy $[g, d]$ of $\overline{X}^{(t)}$ (in other terms, g is the time in $[0, u]$ where $X^{(t)}$ is maximal).

Proof of Proposition 2. Following [4] we deduce from Lemma 4 that if $\mathcal{G}_t$ is the σ-field generated by the process $\overline{X}^{(t)}$, then $(\mathcal{G}_t)_{t\ge0}$ is a filtration. Indeed, the Skorohod-like formula shows that $\overline{X}^{(t)}$ is measurable w.r.t. $\overline{X}^{(t')}$ for any $t' > t$.

Now suppose that K is a $\mathcal{G}_t$-measurable positive r.v., and let us denote by $\varepsilon_{1,K}^{(t)}, \varepsilon_{2,K}^{(t)}, \dots$ the sequence of the excursions accomplished by $X^{(t)}$ below its supremum, ranked by decreasing order of duration (we call $\ell_{1,K}^{(t)}, \ell_{2,K}^{(t)}, \dots$ the sequence of their respective durations), before time $T_K^{(t)}$. If $n^{(t)}$ is the corresponding excursion measure, and if $n^{(t)}(\ell)$ is the law of the excursion of X below its supremum with duration ℓ, we have the analoguous for Lemma 4 in [4]: conditionally on $\mathcal{G}_t$, the excursions $\varepsilon_{1,K}^{(t)}, \varepsilon_{2,K}^{(t)}, \dots$ are independent random processes with respective distributions $n^{(t)}(\ell_{1,K}^{(t)}), n^{(t)}(\ell_{2,K}^{(t)}), \dots$.

Again, the proof is identical to [4], with the only difference that $n^{(t)}(\ell)$ cannot be replaced by $n(\ell)$, the law of the excursion of X below its supremum with duration ℓ (which stems from Girsanov's theorem in the case of Brownian motion).

Applying this result to $K = \overline{X}_{T_1}^{(t)}$ and using the forthcoming Lemma 5 which will show that $T_K^{(t)} = T_1$, it is now easy to see that $(F(t), t \ge 0)$ has the desired fragmentation property and transition kernels. □

4 The fragmentation semigroup

Our next task is to characterize the semigroup of the fragmentation process at a fixed time. In this direction, it suffices to characterize the semigroup of $F^{\varepsilon_\ell}(t)$ for fixed $t > 0$ and $\ell > 0$, since conditionally on the jumps $\ell_1 > \ell_2 > \dots$ of T before level 1, the fragmentation $F^X(t)$ at time t comes from the independent fragmentations $F^{\varepsilon_{\ell_1}}, F^{\varepsilon_{\ell_2}}, \dots$ at time t.

Our main result is Theorem 2, a generalization of the result of Aldous and Pitman [1] for the Brownian fragmentation. The conditioning mentioned in the statement is explained immediately below (equations (9) and (10)). Recall that $q_t(\cdot)$ is the density of X_t.

Theorem 2 *The following assertions hold :*

(i) *The process $(\Delta T_x^{(t)})_{0\le x\le t\ell}$ of the jumps of $T^{(t)}$ before the level $t\ell$ is a Poisson point process on $(0,\infty)$ with intensity measure $z^{-1}q_z(-tz)dz$.*

(ii) *For any $t > 0$, the law of $F^{\varepsilon_\ell}(t)$ is that of the decreasing sequence of the jumps of $T^{(t)}$ before time $t\ell$, conditioned on $T_{t\ell}^{(t)} = \ell$.*

The first assertion is well-known, and will be recalled in Lemma 7. We essentially focus on the second assertion.

4.1 Densities for the jumps of a subordinator

We recall some results on the law of the jumps of a subordinator that can be found in Perman [19]. From now on in this paper, we will often have to use them.

We consider a subordinator T with no drift, and infinite Lévy measure $\pi(dz)$. We assume that the Lévy measure is absolutely continuous with density $h(z) = \pi(dz)/dz$ that is continuous on $(0,\infty)$. It is then known in particular that for each level x, T_x has a density f, which is characterized by its Laplace transform

$$\int_0^{+\infty} e^{-\lambda u} f(u)du = \exp\Big(-x\int_0^{\infty}(1-e^{-\lambda z})h(z)dz\Big). \tag{7}$$

Next, for all $v > 0$, let $f_v(x)$ denote the density at level x (which is known to exist) of the subordinator T^v which Lévy measure is $h(z)1_{z<v}dz$. It is characterized by its Laplace transform

$$\int_0^{+\infty} e^{-\lambda u} f_v(u)du = \exp\Big(-x\int_0^{v}(1-e^{-\lambda z})h(z)dz\Big) \tag{8}$$

We also denote by $(\Delta_i)_{i\geq 1}$ the decreasing sequence of the jumps of $(T_y)_{0\leq y\leq x}$.

By [19], the k-tuple $(\Delta_1,\ldots,\Delta_k)$ admits for every k a density:

$$\frac{\mathbb{P}[\Delta_1\in du_1,\ldots,\Delta_k\in du_k]}{du_1\ldots du_k} = x^k h(u_1)\ldots h(u_k)\exp\Big(-x\int_{u_k}^{\infty}h(z)dz\Big) \tag{9}$$

which we denote by $p(u_1,\ldots,u_k)$.

Besides, the $k+1$-tuple $(\Delta_1,\ldots,\Delta_k,T_x)$ has density

$$\frac{\mathbb{P}[\Delta_1\in du_1,\ldots,\Delta_k\in du_k, T_x\in ds]}{du_1\ldots du_k ds} = p(u_1,\ldots,u_k)f_{u_k}(s-u_1-\ldots-u_k). \tag{10}$$

The proof relies on the general fact for Poisson measures that, conditionally on the k largest jumps $(\Delta_1,\ldots,\Delta_k)$, the sequence $(\Delta_i)_{i\geq k+1}$ is equal in law to the decreasing sequence of the jumps of the subordinator T^{Δ_k} before time x, that is, the atoms of a Poisson point process with intensity $h(z)1_{\{z<v\}}dz$ at time x. Since T_x has a density, these formulas give the conditional law of the jumps of T before time x given $T_x = s$ for every $s > 0$.

Remark. In the case of a Lévy measure π with finite mass a (compound Poisson case) admitting a density, first condition by the fact that there are k jumps in $[0,x]$ (the probability is $e^{-a}a^k/k!$), then by the size of the jumps, which are independent with law π/a.

4.2 A useful process

Now to prove Theorem 2, we introduce a process with bounded variation that is containing in a practical way all the information on the fragmentation before a fixed time t. In [4] as well as in [6], Bertoin uses the process $(x - tT_x^{(t)})_{x\geq 0}$, that we will denote here by $Y^{(t)}$, for any $t > 0$.

It is clear that $Y^{(t)}$ is a Lévy process with bounded variation and with no positive jumps. As the subordinator $T^{(t)}$ can be recovered from $Y^{(t)}$, the sigma-field generated by the latter coincides with $\mathcal{G}_t$, and in particular it should be possible to deduce $(F^X(s))_{0\leq s\leq t}$ from it.

We begin with a lemma which is related to Lemma 7 in [4]. We denote by $\sigma^{(t)}$ the inverse of $Y^{(t)}$, in particular $\sigma^{(t)}$ is a subordinator.

Lemma 5 *For any $t > 0$, $F^X(t)$ has the law of the decreasing sequence of the jumps of $(T_x^{(t)})$ for $x \leq \overline{X}_{T_1}^{(t)} = 1 + tT_1$, that is, the jumps of $(-Y_x^{(t)}/t)$ for $x \leq \sigma_1^{(t)}$.*
Moreover we have for any y,

$$\sigma_y^{(t)} = y + tT_y \tag{11}$$

Proof. We prove the second assertion, the first one being a straightforward consequence. We have

$$\begin{aligned}
\sigma_x^{(t)} &= \inf\{z \geq 0 : z - tT_z^{(t)} > y\} \\
&= \inf\{\overline{X}_u^{(t)} : \overline{X}_u^{(t)} - tu > y\} \\
&= \overline{X}^{(t)}(\inf\{u \geq 0 : \overline{X}_u^{(t)} - tu > y\}) \\
&= \overline{X}^{(t)}(\inf\{u \geq 0 : \sup_{0\leq v\leq u}(\overline{X}_v^{(t)} - tv) > y\}) \\
&= \overline{X}^{(t)}(\inf\{u \geq 0 : \overline{X}_u > y\}) \\
&= \overline{X}_{T_y}^{(t)}
\end{aligned}$$

where we used the fact that $\overline{X}_u^{(t)} - tu$ is non-increasing on a constancy interval of $\overline{X}_u^{(t)}$, the continuity of $\overline{X}^{(t)}$ which follows from the fact that X has no positive jumps, and formula (5).

We then note that

$$\sigma_{\overline{X}_u}^{(t)} = \overline{X}_{T_{\overline{X}_u}}^{(t)} = \overline{X}_u + tT_{\overline{X}_u}$$

which follows from the fact that $X_s \leq \overline{X}_u$ for $s \leq T_{\overline{X}_u}$, and $X_s^{(t)} \leq \overline{X}_u + tu \leq \overline{X}_u + tT_{\overline{X}_u} = X_{T_{\overline{X}_u}}^{(t)}$.
Applying this for $u = T_y$ we have

$$\sigma_y^{(t)} = \overline{X}_{T_y}^{(t)} = y + tT_y$$

□

Remark that this last result implies that the process of first hitting times of $Y^{(t)}$ is not killed, so that $Y^{(t)}$ is oscillating or drifting to ∞. Moreover, from the fact that the Laplace exponents of $X^{(t)}$ and $T^{(t)}$ are inverse functions, we obtain that $\mathbb{E}[T_1^{(t)}] = 1/\mathbb{E}[X_1^{(t)}] = 1/(\mathbb{E}[X_1] + t)$ so that

$$\mathbb{E}[Y_1^{(t)}] = \frac{\mathbb{E}[X_1]}{\mathbb{E}[X_1] + t},$$

and $Y^{(t)}$ oscillates if and only if X does so.

Lemma 5 also shows that the information of $F^X(t)$ for fixed t is (very simply) connected to the process $Y^{(t)}$, but also gives us a tool for studying the law of $F^{\varepsilon_\ell}(t)$. Indeed, we know that

$(\overline{X}_u)_{u\geq 0} = (T_u^{-1})_{u\geq 0}$ is a local time for the reflected process of X, so that the previous lemma implies that, up to a multiplicative constant and a drift, X and $Y^{(t)}$ share for all $t > 0$ the same inverse local time processes.

Recall that n is the excursion measure of $\overline{X} - X$ and that V is the lifetime of the canonical process. If ε is an excursion-type function we denote by $\overline{\varepsilon}^{(t)}$ the supremum process of $ts - \varepsilon_s$. We are now able to state the

Lemma 6 *The "law" under $n(d\varepsilon)$ of the decreasing lengths of the constancy intervals of $\overline{\varepsilon}^{(t)}$ is the same as the "law" under the excursion measure of $\overline{Y}^{(t)} - Y^{(t)}$ of the jumps of the canonical process, multiplied by $1/t$ and ranked in the decreasing order. The same holds for the conditioned law $n(d\varepsilon|V = \ell)$ and the corresponding law of the excursion of $\overline{Y}^{(t)} - Y^{(t)}$ with duration $t\ell$.*

Proof. From the above remark, to the excursion ε^x of the reflected process of X at level x (that is, the excursion of X below its supremum and starting at T_x, $\varepsilon^x(u) = x - X_{T_{x-}+u}$ for $0 \leq u \leq T_x - T_{x-}$) we can associate the excursion $\gamma_x^{(t)}$ of $Y^{(t)}$ below its supremum, at level x, given by

$$\gamma_x^{(t)}(u) = x - Y^{(t)}_{u+\sigma^{(t)}_{x-}} = x - u - (x + tT_{x-}) + tT^{(t)}_{u+x+tT_{x-}} \tag{12}$$

for $0 \leq u \leq \sigma_x^{(t)} - \sigma_{x-}^{(t)}$. We underline that the random times $-tT_{x-} + T^{(t)}_{u+x+tT_{x-}}$ appearing in the formula only depend of the process $x - X$ between the times T_{x-} and T_x, that is, of ε^x. Indeed, Lemma 5 implies that

$$\overline{X}^{(t)}_{T_{x-}} = x + tT_{x-},\ \overline{X}^{(t)}_{T_x} = x + tT_x$$

so that the values taken by the jumps of $T^{(t)}$ between T_{x-} and T_x are exactly the length of the constancy intervals of the supremum of $(-\varepsilon_x(u) + tu)_{0\leq u\leq T_x - T_{x-}}$. The proof then follows. □

Remarks. —More precisely, if we call V_k the location of the k-th largest jump of $\gamma_x^{(t)}$, then we have that the i-th largest constancy interval of the supremum process of $(ts - \varepsilon^x(s))_{0\leq s\leq T_x - T_{x-}}$ is at the left of the j-th one if and only if $V_i < V_j$. This is a straightforward consequence of elementary sample path properties of X. In particular, there a.s. exists a left-most interval of constancy of the supremum process of $(ts - \varepsilon_\ell(s))_{0\leq s\leq \ell}$ if and only if the excursion of $Y^{(t)}$ with length $t\ell$ a.s. begins by a jump. This is to be related, of course, with section 2, but also with the forthcoming section 5

—The last proof also implies the fact (that could easily be guessed on a drawing) that if ε is the excursion of X below its supremum with duration 1 and if for $0 \leq x \leq t$, $T'_x = \inf\{u \geq 0,\ -\varepsilon_u + tu > x\}$, then $x - tT'_x,\ 0 \leq x \leq t$ has the law of the excursion of $Y^{(t)}$ below its supremum with duration t.

4.3 Proof of Theorem 2

The last step before the proof is a lemma that gives explicit densities for the characteristics of $T^{(t)}$.

Lemma 7 *The Lévy measure $\pi^{(t)}(dz)$ of $T^{(t)}$ is absolutely continuous w.r.t. Lebesgue measure, with density*

$$h^{(t)}(z) = \frac{1}{z} q_z(-tz) 1_{z>0}. \tag{13}$$

Moreover, for every $x > 0$, $\mathbb{P}[T_x^{(t)} \in ds]$ has density

$$\mathbb{P}[T_x^{(t)} \in ds]/ds = \frac{x}{s} q_s(x - st) \tag{14}$$

Proof. Following from the fact that $X^{(t)}$ is a Lévy process with no positive jumps, we have the well-known result $(x, s \in \mathbb{R}_+)$

$$x\mathbb{P}[X_s^{(t)} \in dx]ds = s\mathbb{P}[T_x^{(t)} \in ds]dx, \tag{15}$$

see Corollary VII,3 in [3] for example. From this we deduce (14), as $q_s(x - st)$ is the density of $X^{(t)}$.

Next, we know from Corollary 8.8 page 45 in [22] that the Lévy measure of $T^{(t)}$ is on (a, ∞) the weak limit of $(1/\varepsilon)\mathbb{P}[T_\varepsilon^{(t)} \in ds]$ for any $a > 0$. We thus obtain (13). □

Proof of Theorem 2.

From Lemma 6 we know that the law of $F^{\varepsilon_\ell}(t)$ is equal to the law of the decreasing sequence of sizes of the jumps of the excursion of $(\overline{Y}^{(t)} - Y^{(t)})/t$ with length ℓ. But Vervaat's Theorem (Proposition 1) implies that this excursion has the same law as $Vy_{t\ell}^{(t)}(\ell - \cdot)$ where $(y_{t\ell}^{(t)}(x))_{0\leq x\leq \ell}$ is the bridge with length $t\ell$ of $Y^{(t)}$ from 0 to 0. Since $T_{t\ell}^{(t)}$, and hence $Y_{t\ell}^{(t)}$, has a continuous density by Lemma 7, the law of $y_{t\ell}^{(t)}$ is defined as the limit as $\epsilon \to 0$ of the law of $Y^{(t)}$ before time $t\ell$ conditioned on $|t\ell - tT_{t\ell}^{(t)}| < \epsilon$. Now since the Lévy measure of $T^{(t)}$ also has a continuous density by Lemma 7, we get that under this limit probability, the jumps of the canonical process are the same as the jumps of $T^{(t)}/t$ conditioned on $T_{t\ell}^{(t)} = \ell$ in the sense of Perman [19]. This concludes the proof. □

Notice that formulas (10), (13) and (14) make the densities of the first k terms of $F^{\varepsilon_\ell}(t)$ $(k \in \mathbb{N})$ explicit in terms of $(q_t(x), t > 0, x \in \mathbb{R})$.

5 The left-most fragment

In the two preceding sections, we did not consider specifically the sample path properties of the excursion-type functions ε_ℓ that we used to describe the Lévy fragmentation. As a link with section 2 we are now studying some properties of the order induced by $[0, \ell]$ on the constancy intervals of the supremum process of $(st - \varepsilon_\ell(s))_{0\leq s\leq \ell}$.

We can be a bit more precise than in the proof of Theorem 2 in our description of the bridge of $Y^{(t)}$. The Lévy-Itô decomposition of subordinators imply that $(T_x^{(t)})_{0\leq x\leq t\ell}$ may be written in the form

$$T_x^{(t)} = \sum_{i=0}^{\infty} \Delta_i 1_{x\geq U_i}$$

where $(\Delta_i)_{i\geq 1}$ is the sequence of the jumps of $T^{(t)}$ before time $t\ell$ ranked in decreasing order, and $(U_i)_{i\geq 1}$ is an i.i.d. sequence of r.v.'s uniform on $[0, t\ell]$, independent of $(\Delta_i)_{i\geq 1}$. Thus the bridge of $Y^{(t)}$ with length $t\ell$ from 0 to 0 may be written in the form $x - t\sum_{i=0}^{\infty} \delta_i 1_{x\geq U_i}$ where $(\delta_i)_{i\geq 1}$ has the conditional law of $(\Delta_i)_{i\geq 1}$ given $T^{(t)}_{t\ell} = \ell$. We recognize a Kallenberg bridge with exchangeable increments, zero drift coefficient, finite variation and only negative random jumps $(t\delta_i)_{i\geq 0}$.

Recall from Lemma 2 that bridges with exchangeable increments, finite variation and only positive jumps a.s. attain their minimum at a jump, so that their Vervaat's Transforms begin with a jump. Hence Vervaat's Theorem applied to $Y^{(t)}$ combined with the discussion after Lemma 2 gives that the excursions of $Y^{(t)}$ below its supremum a.s. start by a jump, which is a size-biased pick from the variables $(\delta_i)_{i\geq 1}$. In fact this is proved by other means to give an alternative proof of Vervaat's Theorem itself in the bounded variation case, see 7.2 below. Nonetheless, it is clearer in our setting to present it rather as a consequence of Vervaat's Theorem. We denote this first jump by $tH^\ell(t)$, and according to the remark in section 4.2, $H^\ell(t)$ is equal in law to the left-most constancy interval of the supremum process of $(ts - \varepsilon_\ell(s))_{0\leq s\leq \ell}$.

As a consequence of Corollary 1, we also have that $tH^\ell(t)$ has the law of a size-biased pick from the jumps of the bridge of $Y^{(t)}$ with length $t\ell$ from 0 to 0. Equivalently, $H^\ell(t)$ has the law of a size-biased pick from the jumps of $T^{(t)}$ before time $t\ell$ conditioned on $T^{(t)}_{t\ell} = \ell$. It has been already proved by Schweinsberg in [23] in the Brownian case, by similar methods. As noticed in this article, remark that $H^\ell(t)$ has the law of a size-biased without being a size-biased itself. Generalizing a result of Bertoin in the Brownian case, we can do even better, that is, to show that the *process* $(H^\ell(t))_{t\geq 0}$ has the law of a size-biased marked fragment in a sense we precise here.

Let U be uniform on $[0, \ell)$, independent of ε_ℓ, and let $H^\ell_*(t)$ the length of the constancy interval of the supremum process of $(ts - \varepsilon_\ell(s))_{0\leq s\leq \ell}$ that contains U. At every time $t > 0$, $H^\ell_*(t)$ has the law of a size-biased pick from the elements of $F^{\varepsilon_\ell}(t)$.

Theorem 3 *The processes H^ℓ and H^ℓ_* have the same law; they are both (in general time-inhomogeneous) Markov processes with transition kernel Q given by (for $t \leq t'$, $h' \leq h$):*

$$Q_{t,t'}(h, dh') = \frac{(t'-t)hq_{h'}(-t'h')q_{h-h'}(t'h' - th)}{(h-h')q_h(-th)}dh'. \tag{16}$$

Proof. Recall from section 3.2 that conditionally on the length $H^\ell(t) = h$ of the left-most fragment, at time t, the law of the fragmentation starting with this fragment is $\varphi_{t,t'}(h)$. Remark also that the left-most fragment at time t' comes from the fragmentation of the left-most fragment at time t. By the fact that $H^\ell(t)$ is a size-biased pick from the jumps of $T^{(t)}$ before $t\ell$ conditioned by $T^{(t)}_{t\ell} = \ell$, and by replacing time 0 by time t, time t by time $t'-t$, and X by $X^{(t)}$, we obtain that the left-most fragment at time t' given its value h at time t has the law of a size-biased of the jumps of $T^{(t')}$ before time $(t'-t)h$ conditionally on $T^{(t')}_{(t'-t)h} = h$

Now recall from [21] the explicit law $\digamma(dz)$ of a size-biased pick from jumps of a subordinator T before a fixed time x, conditionally on the value s of the subordinator at this time :

$$\digamma(dz) = \frac{zxh(z)f(s-z)}{sf(s)}dz \tag{17}$$

where h and f are respectively the (continuous) density of the Lévy measure of T, and the (continuous) density of the law of T_x. Thanks to this formula and the expressions of the density of $T^{(t')}$ and its Lévy measure in terms of q in Lemma 7 we obtain (16).

Next, it is trivial that the initial states for H^ℓ and H_*^ℓ are the same (namely ℓ). It just remains to prove that H_*^ℓ is a Markov process with the same transition function as H^ℓ.

For this, let us condition on $H_*^\ell(t) = h$ at time t. It means that U belongs to an interval of excursion of $X^{(t)}$ below its supremum with length h. But then, U is uniform on this interval and independent of this excursion conditionally on its length $H_*^\ell(t)$. The value at time t' of the process H_*^ℓ is thus by definition a size-biased from the fragments of the Lévy fragmentation process with initial state $(h, 0, \ldots)$. It thus has the same law that $H^\ell(t')$ given $H^\ell(t) = h$, that is, precisely $Q_{t,t'}(h, dh')$. □

Remark. The fact that $H^\ell(t)$ is non zero a.s. gives in particular an interesting property for the excursions : The excursions of the reflected process of X out of 0 (under the excursion measure, or with fixed duration) have at 0 an "infinite slope" in the sense that

$$\frac{\varepsilon(s)}{s} \underset{s \longrightarrow 0}{\longrightarrow} +\infty$$

Indeed, the only other possibility is that they begin with a jump ($\varepsilon(0) > 0$), but it is never the case when X has infinite variation. This result could be also be deduced from Millar [17] who shows that X "moves away" from a local maximum faster than does the supremum process from 0 at time 0.

6 The mixing of extremal additive coalescents

In this section, we associate by time-reversal a coalescent process to each Lévy fragmentation. We will show that it is a ranked eternal additive coalescent as described by Evans and Pitman [12] and Aldous and Pitman [2], where the initial random data (at time $-\infty$) depend on X. It is thus a mixing of the extremal coalescents of [2], and we will give the exact law of this mixing.

The most natural way to identify the mixing is to use the representation of the extremal coalescents by Bertoin [5], with the help of Vervaat's Transforms of some bridges with exchangeable increments and deterministic jumps, by noticing that the bridge of X with length 1 is such a bridge, but with random jumps. We will focus on the case where the total mass is 1, so that we do not have to introduce too many time-changes.

Definition 5 *Recall the definition of F^{ε_1} from section 3. Call* Lévy coalescent *derived from X the process defined on the whole real line by*

$$C^{\varepsilon_1}(t) = F^{\varepsilon_1}(e^{-t}), \qquad t \in \mathbb{R} \tag{18}$$

More generally, if $(F(t), t \geq 0)$ is some fragmentation-type process, we will call $(F(e^{-t}), t \in \mathbb{R})$ the "associated coalescent".

We first recall the results of [5]. Let $l_2^\downarrow$ be the set of (non-negative) decreasing l_2 sequences. For $a \geq 0$ and $\theta \in l_2^\downarrow$ define the bridge

$$b_{a,\theta} = ab_s + \sum_{i=1}^{\infty} \theta_i (1_{U_i \leq s} - s)$$

where b is a standard Brownian bridge on $[0,1]$ and the U_i are as usual independant uniform r.v. on $[0,1]$. We call $b_{a,\theta}$ the *Kallenberg bridge* with jumps θ and Brownian bridge component ab (it is a particular case of the general representation for bridges with exchangeable increments, see [14]).

Let (ϑ_i) be a decreasing positive sequence such that $\sum \vartheta_i^2 \leq 1$ (we call the corresponding space $l_2^{1,\downarrow}$), and

$$\varsigma = \sqrt{1 - \sum_{i=1}^{\infty} \vartheta_i^2}. \tag{19}$$

Consider the fragmentation $F^{\vartheta}(t)$ associated to the excursion $Vb_{\varsigma,\vartheta}$ (it consists on the lengths of the intervals of constancy of the supremum process of $ts - Vb_{\varsigma,\vartheta}(s)$). Let $C^{\vartheta}(t) = F^{\vartheta}(e^{-t})$ be the associated coalescent process. Then ([5], Theorem 1 and [2], Theorems 10 and 15) it is an extreme eternal additive coalescent process, the mapping

$$\vartheta \in l_2^{1,\downarrow} \longmapsto C^{\vartheta}$$

is one-to-one, and every extreme eternal additive coalescent (where the total mass of the clusters is 1) can be represented in this way up to a deterministic time-translation. We call $\mathbb{P}^{\vartheta}$ its law, and for $t_0 \in \mathbb{R}$ we denote by $\mathbb{P}^{\vartheta,t_0}$ the law of the time translated coalescent $(C^{\vartheta}(t-t_0), t \in \mathbb{R})$. In this way, the law of any extreme eternal additive coalescent is of this form.

We now denote by σ the Gaussian component of X. For $\theta \in l_2^{\downarrow}$, let $k(\theta) \geq 0$ be such that

$$k(\theta)^2\sigma^2 = 1 - k(\theta)^2 \sum \theta_i^2,$$

that is

$$k(\theta) = \frac{1}{\sqrt{\sigma^2 + \sum_{i=1}^{\infty} \theta_i^2}}.$$

Remark that $k(\theta).\theta$ is in $l_2^{1,\downarrow}$ and that $k(\theta)\sigma = \sqrt{1 - \sum_{i=1}^{\infty}(k(\theta)\theta_i)^2}$ is the corresponding "ς".

Now the bridge b_X of X from 0 to 0 and length 1 has exchangeable increments, and as such the ranked sequence of its jumps is a random element of $l_2^{\downarrow}$ (see Kallenberg [14]). Let $\Theta_X(d\theta)$ be its law. It is not difficult to see that if θ^* has law $\Theta_X(d\theta)$, then $b_X(1-\cdot)$ has the same law as b_{σ,θ^*}. Let $\widetilde{\Theta}_X(d\vartheta, dt_0)$ be the image of $\Theta_X(d\theta)$ by the mapping

$$\theta \mapsto (\vartheta, t_0) = (k(\theta)\theta, \log k(\theta)).$$

Proposition 3 *The Lévy coalescent C^{ε_1} associated to X is an additive coalescent, and its law is given by the mixing*

$$\int_{(\vartheta,t_0)\in l_2^{1,\downarrow}\times\mathbb{R}} \mathbb{P}^{\vartheta,t_0}(\cdot)\widetilde{\Theta}_X(d\vartheta, dt_0). \tag{20}$$

Proof. Consider the fragmentation F^{ε_1}. It is associated to $Vb_X(1-\cdot)$, which is equal in law to Vb_{σ,θ^*} where θ^* has law Θ_X. Hence for $t \geq 0$, $F^{\varepsilon_1}(t)$ has the law of the intervals of constancy of the supremum process of $(ts - Vb_{\sigma,\theta^*}(s), 0 \leq s \leq 1)$. This last process is equal to

$$\frac{1}{k(\theta^*)}(k(\theta^*)ts - Vb_{k(\theta^*)\sigma,k(\theta^*)\theta^*}(s)), \qquad 0 \leq s \leq 1,$$

so that the supremum processes of $(ts - Vb_{\sigma,\theta^*}(s), 0 \leq s \leq 1)$ and of $(k(\theta^*)ts - Vb_{k(\theta^*)\sigma,k(\theta^*)\theta^*}(s), 0 \leq s \leq 1)$ share the same constancy intervals. Hence, by definition, $F^{\varepsilon_1}(t) = F^{k(\theta^*)\theta^*}(k(\theta^*)t)$, and this means that the associated coalescent is $C^{k(\theta^*)\theta^*,\log k(\theta^*)}$. The law of C^{ε_1} is thus

$$\int_{\theta \in l_2^\downarrow} \mathbb{P}^{k(\theta)\theta,\log k(\theta)}(\cdot)\Theta_X(d\theta) \tag{21}$$

and we conclude by a change of variables. □

Remark. We stress that, whether the Lévy measure of X integrates $|x| \wedge 1$ or not, the typical mixings that appear are not the same : the configurations where $\sum \theta_i = \infty$ have no "weight" in the first case, whereas $\sum \theta_i < \infty$ does not happen in the second. Moreover, under some more hypotheses on X (e.g. that its Lévy measure $\nu(du)$ is absolutely continuous with respect to Lebesgue measure, and that the Lévy process with truncated Lévy measure $1_{u \leq x}\nu(du)$ has densities), one can make more "explicit" the law Θ_X by the same arguments of conditioned Poisson measures as in section 4.1 above.

7 Proof of Vervaat's Theorem

We are going to give two proofs of Proposition 1, the first one being quite technical, and essentially devoted to the unbounded variation case since we have not found how to prove it with simple arguments. Of course this proof applies also in the bounded variation case. The second proof only works for Lévy processes with bounded variation, but uses only tools that are directly connected to this work, such as the ballot theorem.

7.1 Unbounded variation case

Let $(\mathcal{F}_t^0)_{t \geq 0}$ the natural filtration on the space of càdlàg functions on $\mathbb{R}_+$. Let $\widehat{P}$ be the law of the spectrally **positive** Lévy process $\widehat{X} = -X$. Without risk of ambiguity, $\widehat{X}$ will also denote the canonical process on $\mathbb{D}([0,\infty))$. Recall the definition of the law $P_{0,0}^t$ of the bridge of X with length $t > 0$ starting and ending at 0 from (4). Let P^t be the law of the process X killed at time t, and $(\mathcal{F}_t)_{t \geq 0}$ be the P-completed filtration.

Let also $P^J = P^{\tau_{J^c}}$ where $\tau_{J^c} = \inf\{s \geq 0, X_s \notin J\}$ for any interval J. Recall that n is the excursion measure of the reflected process $\overline{X} - X = \widehat{X} - \underline{\widehat{X}}$ where $\underline{\widehat{X}}_t = \inf_{0 \leq s \leq t} \widehat{X}_s$. Since $\widehat{X}$ oscillates or drifts to $-\infty$, every excursion of the process has a finite lifetime D. Let n^u be the measure associated to the excursion killed at time $u \wedge D$. Remark that the measure $n(\cdot, t \leq D)$ is a finite measure, with total mass $\pi((t,\infty))$ where π is the Lévy measure of the subordinator $(\widehat{T}_{-y})_{y \geq 0}$ (where $\widehat{T}_x = \inf\{s \geq 0,\ \widehat{X}_s = x\}$). We already saw that the inverse local time process of $\widehat{X} - \underline{\widehat{X}}$ is $(\widehat{T}_{-y})_{y \geq 0}$, and that it has Lévy measure $q_v(0)dv/v$.

The demonstration that we are giving is close to the method used by Biane [7] for Brownian motion and Chaumont [10] for stable processes. It involves a path decomposition of the trajectories of $\widehat{X}$ under P^t at its minimum. We will first need the following result (see [9]) which is an application of Maisonneuve's formula. Chaumont stated the result only for oscillating Lévy processes, but the proof applies without change to processes drifting to $-\infty$.

Let k_t be the standard killing operator at time t, ζ the life of the canonical process, θ_t be the shift operator and θ'_t be defined by $\widehat{X}_s \circ \theta'_t = \widehat{X}_{s+t} - \widehat{X}_t$. Last, let g_t be the right-most instant at which

$\widehat{X}$ attains its minimum on $[0,t]$. Then, under the measure $\int_0^\infty dtP^t$, the pair $(\widehat{X}\circ k_{g_\zeta}, \widehat{X}\circ\theta'_{g_\zeta})$ has the "law"

$$\int_0^\infty dxP^{(-x,\infty)}\otimes\int_0^\infty dun^u(\cdot, u<D)$$

In other terms, if H and H' are positive measurable functionals, that can be taken of the form

$$H = 1_{\{t_n<\zeta\}}\prod_{i=1}^n f_i(\widehat{X}_{t_i}),\ 0\le t_1\le\ldots\le t_n,$$

then

$$\int_0^\infty dtP^t(H\circ k_{g_\zeta}H'\circ k_{\zeta-g_\zeta}\circ\theta'_{g_\zeta}) = \int_0^\infty dxP^{(-x,\infty)}(H)\int_0^\infty dun^u(H, u<D)$$

Let H and H' be continuous measurable positive bounded functionals (that can be of the form above with the f_i continuous, positive and bounded) such that $H\circ k_u$ is integrable w.r.t. the measure $n(d\omega)du1_{u\le D}$, and f a continuous positive real function with compact support which does not contain 0. We then have by dominated convergence and by definition of $P^t_{0,0}$:

$$\begin{aligned}&\int_0^\infty dtf(t)q_t(0)P^t_{0,0}(H\circ k_{g_\zeta}H'\circ k_{\zeta-g_\zeta}\circ\theta'_{g_\zeta})\\ &= \lim_{\epsilon\downarrow 0}\int_0^\infty dtf(t)q_t(0)P^t(H\circ k_{g_\zeta}H'\circ k_{\zeta-g_\zeta}\circ\theta'_{g_\zeta}||\widehat{X}_\zeta|<\epsilon),\end{aligned}$$

This is thus equal to

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\int_0^\infty dy\int_0^\infty du\int\int dP^{(-y,\infty)}(\omega')dn(\omega)1_{u\le D}H(\omega')\\ &\times\quad 1_{|\omega_u-y|<\epsilon}\frac{H'\circ k_u(\omega)f(\widehat{T}_{-y}(\omega')+u)q_{\widehat{T}_{-y}(\omega')+u}(0)}{\int_{-\epsilon}^{\epsilon}q_{\widehat{T}_{-y}(\omega')+u}(z)dz}\end{aligned}$$

Now remark that the measure $1_{|\omega_u-y|<\epsilon}dy/2\epsilon$ converges weakly to $\delta_{\omega_u}(dy)$ when $\epsilon\downarrow 0$.

Moreover we have that the family of probability measures $(P^{(y,\infty)})_{y<0}$ is weakly continuous in the sense that for any continous bounded functional F, $P^{(y,\infty)}(F)$ is continuous in y. This follows from the a.s. continuity of the subordinator $(\widehat{T}_{-y})_{y\ge 0}$ at a fixed y. The continuity of H and H', $q_x(s)$ and of the killing operator thus implies that the limit we are studying is equal to

$$\int n(d\omega)\int_0^D duH'\circ k_u(\omega)\int dP^{(-\omega_u,\infty)}(\omega')H(\omega')f(\widehat{T}_{-\omega_u}(\omega')+u)$$

or

$$n\Big(\int_0^D duH'\circ k_uE^{(-\omega_u,\infty)}[Hf(\zeta+u)]\Big) \tag{22}$$

In other means, using the Markov property of the excursion measure, we have for H, H' two measurable positive functionals,

$$\begin{aligned}&\int n(d\omega)f(D)\int_0^D duH\circ k_uH'\circ\theta_u\\ &= \int n(d\omega)\int_0^D duH\circ k_u(\omega)\int dP^{(0,\infty)}_{\omega_u}(\omega')H'(\omega')f(u+\widehat{T}_0(\omega'))\end{aligned}$$

so that

$$\int_0^\infty \frac{dv}{v} q_v(0) f(v) n\Big(\int_0^D du H\circ k_u H'\circ\theta_u \Big| D=v\Big)$$
$$= \quad n\Big(\int_0^D du H\circ k_u E^{(0,\infty)}_{\omega_u}[H' f(u+\zeta)]\Big)$$

By comparing this with (22) we obtain the desired result : by interverting and sticking together the pre- and post- minimum processes of the bridge with length t, we obtain a regular version of the conditional "law" $n(\cdot|D)$, such that the sticking point u is uniform on the interval $(0,D)$ (this gives also a way to recover the law of the bridge by splitting the excursion at an independent uniform point). This ends the proof.

Remark. Recently, Chaumont [11] has given a generalization of the Vervaat's Transformation by constructing processes with cyclically exchangeable increments conditioned to spend a fixed time $s>0$ below 0. This conditional law converges when s goes to 0, and would give a (certainly more general!) proof of the Vervaat Theorem if one could precisely and properly identify the limit law ("process conditioned to spend time 0 below 0") as that of the excursion.

7.2 Bounded variation case

There is a more natural way to approach Vervaat's Theorem in the setting of this paper. Let $Y_t = ct - \tau_t$ be a spectrally negative Lévy process with bounded variation, where c is a positive constant and τ is a strict subordinator without drift (which means that it is pure jump and that it is not killed). Without loss of generality we suppose that $c=1$. Let $\overline{Y}$ be the supremum process. We suppose that Y oscillates of drifts to $+\infty$ (that is, $\overline{Y}_\infty = +\infty$ a.s.), which happens if and only if $1 \geq \mathbb{E}[\tau_1]$. We suppose also that the Lévy measure $\kappa(dz)$ of τ and the law of τ_t for any $t>0$ have continuous densities, $\kappa(dz) = h(z)dz$, $\mathbb{P}[\tau_t \in ds]/ds = p_t(s)$ so that we may apply our results on the densities of the jumps of this subordinator.

Let us study the law of the excursions of the reflected process of Y. Let $P_u^\dagger(\cdot)$ be the law of the process $(u - Y_x)_{0\leq x\leq\zeta}$ killed at the time ζ when it reaches 0. Since we know that the process Y oscillates or drifts to $+\infty$, we have $\zeta<\infty$ a.s. The following description then holds:

Proposition 4 *An excursion of Y below its supremum a.s. begins by a jump, and its Itô excursion measure is given by*

$$\widetilde{n}(\cdot) = \int_{\mathbb{R}_+} P_u^\dagger(\cdot) h(u) du \tag{23}$$

In other words, such an excursion begins with a jump with "law" $h(u)du$ and evolves as the dual process of Y killed when it reaches 0.

Proof. We first show that an excursion of Y below its supremum a.s. begins by a jump. From [25] we prove this by time-reversal arguments : if excursions could begin by a jump with positive probability, then an independent exponential time T would belong with positive probability to the interval of life of such an excursion. But the fact that this excursions starts "continuously" would imply that the time-reversed process $(Y_T - Y_{(T-t)-})_{0\leq t\leq T}$ would not jump to its overall infimum with positive probability, which is impossible since for this process, 0 is irregular for $(-\infty, 0)$.

Now we remark that

$$\overline{Y}_s = \int_0^s 1_{\{Y_u = \overline{Y}_u\}} du = \int_0^s 1_{\{Y_{u-} = \overline{Y}_u\}} du$$

which follows from the fact that Y has a slope 1, and where the second equality follows from the fact that the set $\{u : Y_u \neq Y_{u-}\}$ has zero Lebesgue measure a.s. Thus we a.s. have $d\overline{Y}_u = 1_{\{Y_{u-} = \overline{Y}_u\}} du$

For any positive measurable functional F we then have by compensation formula for the Poisson point process of the excursions (recall that $\overline{Y}$ is a local time for the reflected process of Y):

$$\begin{aligned}
\mathbb{E}\Big[\int_0^1 d\overline{Y}_u \int \widetilde{n}(d\gamma) F(\gamma)\Big] &= \mathbb{E}\Big[\sum_{0<x\leq 1} F(\gamma_x)\Big] \\
&= \mathbb{E}\Big[\sum_{0<s} 1_{\{\overline{Y}_s \leq 1\}} 1_{\{Y_{s-} = \overline{Y}_s\}} F\Big((Y_{s-} - Y_{s+u})_{0\leq u \leq \widetilde{T}_{\{0\}}}\Big)\Big]
\end{aligned}$$

where $\widetilde{T}_{\{0\}}$ is the first time when the process $(Y_{s-} - Y_{s+u})_{u\geq 0}$ hits 0. We also remark that the process $\widetilde{Y}$ defined by

$$\widetilde{Y}_u = Y_s - Y_{s+u}$$

has the same law as Y and is independent of $(Y_u)_{0\leq u\leq s}$. As such, it is independent of $\Delta Y_s = Y_s - Y_{s-}$. Noting that

$$Y_{s-} - Y_{s+u} = \Delta Y_s - \widetilde{Y}_s$$

The strong Markov property then gives, since every excursion begins with a jump

$$\mathbb{E}\Big[\sum_{0<x\leq 1} F(\gamma_x)\Big] = \mathbb{E}\Bigg[\sum_{0<s} 1_{\{\overline{Y}_s \leq 1\}} \frac{d\overline{Y}_s}{ds} \widetilde{E}_{-\Delta Y_s}\Big[F\Big((-\widetilde{Y}_u)_{0\leq u\leq \widetilde{T}_{\{0\}}}\Big)\Big]\Bigg]$$

Where $\widetilde{E}_x$ is the expectation with respect to the process $\widetilde{Y}$, starting at $-x$.

To conclude, we then apply the compensation formula for the Poisson point process $(-\Delta Y_s)_{s\geq 0}$ which has Lévy measure $h(z)dz$:

$$\begin{aligned}
&E[\overline{Y}_1] \int \widetilde{n}(d\gamma) F(\gamma) = \\
&\qquad \mathbb{E}\Big[\int_0^{+\infty} ds 1_{\{\overline{Y}_s \leq 1\}} \frac{d\overline{Y}_s}{ds} \int h(z) dz E_z\Big[F\Big((-\widetilde{Y}_u)_{0\leq u\leq \widetilde{T}_{\{0\}}}\Big)\Big]\Big]
\end{aligned}$$

which finally gives

$$\int \widetilde{n}(d\gamma) F(\gamma) = \int h(z) dz \widehat{E}_z^{\dagger}\Big[F((\cdot)_{0\leq u\leq \zeta})\Big].$$

□

Let us then try to give more detail about the "law" under $\widetilde{n}$ of the excursion with duration ℓ. Recall that D denotes the lifetime of the excursion, and that the lifetimes are the jumps of a

Poisson point process with intensity $(p_t(t)/t)dt$ according to Lemma 7 (with notations adapted to Y). Hence we have

$$\widetilde{n}(D \in d\ell) = \frac{p_\ell(\ell)}{\ell} d\ell$$

According to the preceding lemma, is seems natural to try to give sense to the conditional probability law $P_u^\dagger(\cdot|\zeta = \ell)$ for every $\ell > 0$. For this we introduce another probability law $P_{u,\ell}$ which is the law of the process $\widetilde{Y} = u - Y$ given that $u - Y_\ell = 0$. This "bridge-type" conditional law is well-defined by the methods described in this article (Kallenberg bridges and joint law of the jumps of a subordinator). It can be rewritten as being the process $u - Y$ before time ℓ with jumps conditioned to have sum $\ell - u$, and it has exchangeable increments. This is also the case for the process obtained by reversing the time at ℓ, which has exchangeable increments, drift coefficient u/ℓ and slope 1, and as such is positive with positive probability u/ℓ from Lemma 2. This allows us to give a "good" regular version for $P_u^\dagger(\cdot|\zeta)$.

Lemma 8 *The probability law $P_{u,\ell}[\cdot|\widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell]]$ is a regular version for $P_u^\dagger(\cdot|\zeta = \ell)$.*

Proof. By Lebesgue's derivation theorem, for every measurable positive functional H, we have that for a.e. $\ell > 0$, $E_u^\dagger[H|\zeta = \ell]$ is the limit of $E_u^\dagger[H|\zeta \in [\ell, \ell+\epsilon]]$ as $\epsilon \to 0$. On the other hand, Markov's property gives that

$$\begin{aligned}
&E_u^\dagger[H, \zeta \in [\ell, \ell+\epsilon]] \\
&= \ E_u[H, \widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell], \ \exists s \in [\ell, \ell+\epsilon], \widetilde{Y}_s = 0] \\
&= \ \int_{w \geq 0} E_u[H, \widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell]|\widetilde{Y}_\ell = w] P_u[\widetilde{Y}_\ell \in dw] P_w^\dagger[\zeta \leq \epsilon] \\
&= \ \int_{w \geq 0} E_u[H|\widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell], \widetilde{Y}_\ell = w] P_u[\widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell]|\widetilde{Y}_\ell = w] \\
&\times \ P_u^\dagger[\widetilde{Y}_\ell \in dw] P_w[\zeta \leq \epsilon]
\end{aligned}$$

Then divide by

$$P_u^\dagger[\zeta \in [\ell, \ell+\epsilon]] = \int_{w \geq 0} P_u[\widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell]|\widetilde{Y}_\ell = w] P_u^\dagger[\widetilde{Y}_\ell \in dw] P_w[\zeta \leq \epsilon],$$

and notice, from the fact that if the process $\widetilde{Y}$ has its first zero at a time in $[z, z+\epsilon]$, it can not be greater than ϵ at time z, that the measure

$$P_u[\widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell]|\widetilde{Y}_\ell = w] P_u^\dagger[\widetilde{Y}_\ell \in dw] P_w[\zeta \leq \epsilon]$$

only charges $[0, \epsilon]$. We then obtain that the limit is $P_u[\,\cdot\,|\widetilde{Y}_s \geq 0 \ \forall s \in [0,\ell], \widetilde{Y}_\ell = 0]$. □

Remark. Notice that formula (15), in the case where X has bounded variation, is an immediate consequence of this lemma and the ballot Theorem.

It is then easy to define the excursion with duration ℓ. Indeed, we obtain from (23) and our last discussion

$$\widetilde{n}(\cdot) \ = \ \int_{\mathbb{R}_+} h(u) du \int_u^{+\infty} P_u^\dagger(\cdot|\zeta = \ell) \frac{u}{\ell} p_\ell(\ell - u) d\ell$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_+} \frac{p_\ell(\ell)}{\ell} d\ell \int_0^\ell \frac{uh(u)p_\ell(\ell-u)}{p_\ell(\ell)} P_u^\dagger(\cdot|\zeta=\ell)du \\
&= \int_{\mathbb{R}_+} \widetilde{n}(D\in d\ell) \int_0^\ell \mu_\ell(du) P_u^\dagger(\cdot|\zeta=\ell),
\end{aligned}
$$

where

$$\mu_\ell(du) = \frac{uh(u)p_\ell(\ell-u)}{p_\ell(\ell)}du.$$

As a consequence,

$$\widetilde{n}(\cdot|D=\ell) = \int_0^\ell \mu_\ell(du) P_u^\dagger(\cdot|\zeta=\ell) \tag{24}$$

which means that the excursion with life ℓ begins with a jump distributed by μ_ℓ and evolves as the dual process of $Y^{(t)}$ to which we added u, conditioned to first hit 0 at time ℓ. Notice from (17) that μ_ℓ is a probability measure, which is the law of a size-biased pick from the jumps of τ before time ℓ conditionally on $\tau_\ell = \ell$.

Let us sum up our study of the excursions of the reflected process of Y with duration ℓ. It begins with a jump u with law μ_ℓ, and then conditionally on this jump it evolves as the Kallenberg bridge on $[0,\ell]$ starting at u, which jumps are that of the subordinator τ before time ℓ given $\tau_\ell = \ell - u$, and conditioned to stay positive on $[0,\ell]$. To complete the proof of Vervaat's Theorem, it suffices to identify this description as that of the Vervaat's Transform of the bridge of Y with length ℓ from 0 to 0.

We recall a lemma that can be found in Pitman-Yor [21]: it states that if (δ_i) is the decreasing sequence of the atoms of a Poisson measure such that $\sum \delta_i < \infty$ a.s. then conditionally on $\sum \delta_i = \ell$, if δ^* is a size-biased from the (δ_i), then given δ^* the law of the other atoms is the law of (δ_i) given $\sum \delta_i = \ell - \delta^*$.

Let $(\delta_i)_{i\geq 1}$ have the law of the decreasing sequence of the jumps of τ before time ℓ given $\tau_\ell = \ell$, so that the bridge y of Y with length ℓ from 0 to 0 is equal in law to the Kallenberg bridge on $[0,\ell]$ with zero drift and jumps $-(\delta_i)_{i\geq 1}$. Recall from Corollary 1 that the final value of Vy has the law of a size-biased pick from the sequence (δ_i). Together with the lemma recalled in the preceding paragraph, we have the following description for the Vervaat's transform of y: conditionally on the value z of a r.v. with law μ_ℓ (a size-biased pick from the $\delta_i' s$), Vy has the law of the bridge of Y starting at 0 and ending at z, conditioned to stay positive. It is thus the time-reversed excursion of Y below its supremum with life ℓ, which ends the proof of Vervaat's Theorem.

We end this section with a slight digression about some interesting properties of the processes $Y^{(t)}$ defined in 4.2, and which could imply Vervaat's Theorem in the general case.

From the bounded variation proof above we know that Vervaat's Theorem holds also for the bridge of the process $(Y_{tx}^{(t)})_{x\geq 0}$.

We have the

Lemma 9 *We have*

$$(Y_{tx}^{(t)})_{x\geq 0} \xrightarrow[t\longrightarrow+\infty]{(\mathcal{L})} X,$$

where this notation means convergence in law in the Skorohod space $\mathbb{D}$.

Proof. First, notice that from the results of section 4, the two reflected processes of $Y^{(t)}_{t\cdot}$ and X share, up to a drift coefficient, the same inverse local time processes, and in particular, the lengths of their excursions are the same.

Then we have $Y^{(t)}_{tx} = t(x - T^{(t)}_{tx})$, where for a fixed x,

$$\begin{aligned} T^{(t)}_{tx} &= \inf\{u \geq 0 : X_u + tu > tx\} \\ &= \inf\{u \geq 0 : X_u > t(x-u)\} \end{aligned}$$

which means that $T^{(t)}_{tx}$ is the infimum abscissa of the intersection points of the trajectory of X and of the line $y = t(u - x)$ (which intuitively tends to $u = x$ when t tends to infinity, which permits to say that $T^{(t)}_{tx}$ should converge to x). The fact that X has no positive jumps implies that these intersection points always exist. Moreover, the a.s. continuity of X at x gives indeed that $T^{(t)}_{tx}$ a.s. converges to x. Then from $X^{(t)}_{T^{(t)}_{tx}} = tx$, we get $X_{T^{(t)}_{tx}} = Y^{(t)}_{tx}$ and a.s. convergence of this last r.v. to X_x, using again the a.s. continuity of X at x.

Since the processes we are considering are Lévy processes, this last convergence implies convergence in law for the processes. □

It seems that this lemma may imply the Vervaat's Theorem for processes with infinite variation, if we could prove that the laws of the bridge and the excursion below its supremum of $(Y^{(t)}_{tx})_{x\geq 0}$ converge weakly to the associated laws for X. But even it is intuitively clear, it seems that it can not be obtained without imposing more regularity properties for the law of X_s, $s > 0$.

8 Concluding remarks

To conclude, we make some comments about the bounded variation case to enlighten the links between section 2 and the rest of the paper. We will give no proof as arguments very closed to the ones already used apply.

Recall that a Lévy process with no positive jumps and bounded variation can be written in the form $Y_s = cs - \tau_s$, where τ is a strict subordinator and $c \in \mathbb{R}$. Moreover, if Y does not drift to $-\infty$ we must have $c \geq E[\tau_1]$, and for convenience (i.e. up to change the time scale in the sequel) we suppose that $c = 1$. Suppose that the Lévy measure of τ is infinite. Let y be the bridge of Y with length 1 from 0 to 0, so that Vy is the time-reversed excursion ε of Y below its supremum with duration 1 (again, we could consider the case where the duration is ℓ by changing the time...). As in section 2.4 let $\overline{\varepsilon}^{(t)}$ be the supremum process of $(ts - \varepsilon(s))_{0\leq s\leq \ell}$ and $F(t)$ be the decreasing sequence of the lengths of its intervals of constancy, multiplied by $1 + t$ so that their sum is 1. Also, the locations of the jump times of ε on $[0, 1]$ induce an order the intervals of constency, which we call $O^Y(t) \in \mathcal{O}_\infty$.

The following assertion should be clear from section 2 :

Proposition 5

$$F\Big(\frac{e^{-t}}{1-e^{-t}}\Big) \qquad t \geq 0$$

is an additive coalescent starting at time 0 from the random element $(\delta_i)_{i\geq 1}$ where the latter has the law of the jumps of τ before time 1, conditionally on $\tau_1 = 1$.

Also, the process $(O^Y(e^{-t}/(1-e^{-t})), t \geq 0)$ is somehow a mixture of ordered additive coalescents starting from O_{Sing}, is Markovian, and must be interpreted as an "ordered additive coalescent with random proto-galaxy masses", determining the order of the coalescences in the process of Proposition 5.

We now make the following remark : with the notations of section 4, fix $t > 0$. We know that the excursions of

$$Y_x^{(t)} = x - tT_x^{(t)} \qquad x \geq 0$$

below its supremum are in a one-to-one correspondence with that of X. On the other hand, the fragmentation process $F^{(t)}$ associated as above to $(Y_{tx}^{(t)}/t)_{x\geq 0}$ (which has slope 1) is, after the appropriate time-reversal, the additive coalescent starting at time 0 from the vector constituted of the decreasing sequence of the jumps of $T^{(t)}$ before time t, conditionally on $T_t^{(t)} = 1$. In other words, it is the additive coalescent C^{ε_1} derived from X (recall the notations of section 6), but starting at time $-\log t$. Moreover, the locations of the jumps of the excursion of $(Y_{tx}^{(t)}/t)_{x\geq 0}$ below its supremum with length 1 can turn this process into the ordered additive coalescent with random proto-galaxy masses $C^{\varepsilon_1}(-\log t)$. Together with the remarks of section 4.2, we obtain an interpretation of the order induced by the position of the intervals of constancy of $\overline{\varepsilon}_1^{(t)}$: it is equal to the order induced by this ordered additive coalescent, that is, the order induced by the location of the jumps of the excursions of $(Y_{tx}^{(t)}/t)_{x\geq 0}$ below its supremum with duration 1, which is obtained by a simple path transformation from the excursions of X below its supremum with duration 1.

Last, if τ is a compound Poisson process, there is a similar result for the processes that start from finite number of fragments, but where the time-change that is appearing is random.

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