DOI: 10.1214/EJP.v6-74

Electronic Journal of Probability

Vol. 6 (2001) Paper no. 1, pages 1–10.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol6/paper1.abs.html

A NOTE ON LIMITING BEHAVIOUR OF DISASTROUS ENVIRONMENT EXPONENTS

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Abstract We consider a random walk on the d-dimensional lattice and investigate the asymptotic probability of the walk avoiding a "disaster" (points put down according to a regular Poisson process on space-time). We show that, given the Poisson process points, almost surely, the chance of surviving to time t is like $e^{-\alpha log(\frac{1}{k})t}$, as t tends to infinity if k, the jump rate of the random walk, is small.

Keywords Random walk, disaster point, Poisson process

AMS subject classification 60K35

Submitted to EJP on December 22, 2000. Accepted January 5, 2001.

Introduction

This note concerns a recent work of T. Shiga ([**Shi**]). The following model was considered: We are given a system of independent rate one Poisson processes on $[0, \infty)$, $\underline{N} = \{N_x(t)\}_{x \in \mathbb{Z}^d}$. We are also given an independent simple random walk on $\mathbb{Z}^d, X(t)$, moving at rate k and with, say, $X(0) = \underline{0}$.

Of course simply by integrating out over $\underline{N}, X$ we have (taking $\delta N_{X(s)}(s) = N_{X(s)}(s) - N_{X(s)}(s-)$)

$$\forall t \geq 0 \quad P[\forall\ 0 \leq s \leq t\ \delta N_{X(s)}(s) = 0] \ = \ e^{-t}.$$

The problem becomes non-trivial when considering

$$p(t, N) \ = \ P[\forall\ 0 \leq s \leq t\ \delta N_{X(s)}(s) = 0 \big| \underline{N}] \ =$$

$$P[\forall\ 0 \leq s \leq t\ \delta N_{X(s)}(s) = 0 \big| \underline{N}(s)\ s \leq t].$$

It is non-trivial, but was shown in [**Shi**], that the random quantity $p(t, N)$ satisfies

$$\lim_{t \to \infty} \frac{\log p(t, N)}{t} = -\lambda(d, k)$$

It was shown that as k becomes large λ tends to one in all dimensions and that in dimensions three and higher λ is equal to one for k sufficiently large. The focus of this note is on the other behaviour of $\lambda(d, k)$: the behaviour as $k \to 0$. It was shown in [**Shi**] that there existed two constants $c_1, c_2 \in (0, \infty)$ so that

$$c_1 < \liminf_{k \to 0} \frac{\lambda(d, k)}{\log(\frac{1}{k})} \ \leq \ \limsup_{k \to 0} \frac{\lambda(d, k)}{\log(\frac{1}{k})} \ < \ c_2.$$

We wish to show

Theorem 1.0 *There exists a constant* α *so that* $\lim \frac{\lambda(d,k)}{\log(\frac{1}{k})} \ = \ \alpha.$

The paper is organized as follows: in Section One we consider a "shortest path" problem which is easily and naturally dealt with by Liggett's subadditive ergodic theorem (see [**L**]). This yield a constant α. In Section Two we show (Corollary 2.4) that $\liminf_{k \to 0} \frac{\lambda(d,k)}{\log(\frac{1}{k})} \ \geq \ \alpha$ and in Section Three we show (Corollary 3.1) $\limsup_{k \to 0} \frac{\lambda(d,k)}{\log(\frac{1}{k})} \ \leq \ \alpha$, thus completing the proof of Theorem 1.0.

Both of the last two sections rely heavily on block arguments as popularized in [**D**], [**D1**].

Acknowledgement This paper was conceived during a visit to the Tokyo Institute of Technology in January of this year. It is a pleasure to thank Professor Shiga for his clear exposition of the area and for his generosity as host.

I wish also to thank the referee for a thorough reading which has resulted in a much clearer paper.

Section One

In this section we consider only the Poisson processes N. The random walk will not be directly considered at all, though sometimes it will be implicit, as in the definition of a *path* below:

A path γ is a piecewise constant right continuous function with left limits

$[0,\infty) \to \mathbb{Z}^d$ so that for all t $||\gamma(t) - \gamma(t-)||_1 \leq 1$.

The collection of paths beginning at $x \in \mathbb{Z}^d$ which avoid points in N up to time t will be denoted by $\Gamma^{x,t}$. More formally

$$\Gamma^{x,t} \; = \; \left\{\gamma : \forall\; 0 \leq s \leq t \;\delta N_{\gamma(s)}(s) = 0, \gamma(0) = x\right\}.$$

(Again, consistent with previous notation, $\delta N_{\gamma(s)} \;=\; N_{\gamma(s)}(s) \;-\; N_{\gamma(s)}(s-)$.) For $\gamma \in \Gamma^{x,t}, S^x(\gamma,t) \;=\; \sum_{0 \leq s \leq t} I_{\gamma(s) \neq \gamma(s-)}$ where I is the usual indicator function. In words S counts the number of jumps that γ makes in time interval $[0,t]$. If $x = 0$ we suppress the suffix x.

Finally we define

$$\alpha(t,N) = \min\{S(\gamma,t) : \gamma \in \;\Gamma^t \;=\; \Gamma^{0,t}\}.$$

Proposition 1.1 $\alpha \;=\; \lim_{t\to\infty} \frac{1}{t}\alpha(t,N)$ *exists.*

Proof Define random variables $X_{s,t}$ for $0 \leq s < t < \infty$ by

$$X_{0,t} \;=\; \alpha(t,N)$$

and for $0 < s < t$

$$X_{s,t} = \inf\{S(\gamma,t) - S(\gamma,s) : \gamma \in \Gamma^t, \gamma(s) = x_s\}$$

where $x_s = \min\{x \in \mathbb{Z}^d : \exists\gamma \in \Gamma^s$ so that $S(\gamma,s) \;=\; \alpha(s,N), \gamma(s) = x\}$ under any well ordering of the points $x \in \mathbb{Z}^d$.

Then the random variables satisfy the conditions for Liggett's subadditive ergodic theorem. Given the ergodicity of our Poisson processes we conclude that the a.s. limit of $\frac{1}{t}\alpha(t,N)$ is non random.

$\square$

We now show that the constant α of Proposition 1.1 is strcitly positive. This fact will follow from Theorem 1.0 and the results of **[Shi]**, however we include it for completeness and because the argument given is a precursor to the block argument of Proposition 2.2.

Proposition 1.2 *The constant α is strictly positive.*

Fix $\varepsilon > 0$ small we shall give conditions on the smallness of ε as the proof progresses. Choose integer L so that $L^d e^{-L} \;<\; \varepsilon$.

We divide up space time into cubes

$$V(\underline{n},r) \;=\; [n_1L,(n_1+1)L) \times [n_2L,(n_2+1)L) \times \cdots [n_dL,(n_d+1)L) \times [rL,(r+1)L).$$

We associate 0-1 random variables $\psi(\underline{n},r)$ to these cubes by taking $\psi(\underline{n},r)$ to be 1 if and only if

$$\forall x \in [n_1L,(n_1+1)L) \times [n_2L,(n_2+1)L) \times \cdots [n_dL,(n_d+1)L)$$

$$N_x((r+1)L-) - N_x(rL) \geq 1.$$

We note that the ψ random variables are *i.i.d.* and that, by the choice of L, the probability that $\psi(\underline{n}, r) \neq 1$ is $< \varepsilon$.

To show our result it is sufficient to show that as m tends to infinity $\alpha(mL, N) \geq \frac{m}{2}$ with probability tending to one.

The *trace* of a path $\gamma \in \Gamma^{mL}$ is the sequence of points in $\mathbb{Z}^d$, $\underline{n}_i$ $0 \leq i \leq m$ so that for $0 \leq i \leq m$,

$$(\gamma(iL), iL) \ \in \ V(\underline{n}_i, iL).$$

The crucial observation is that for such $\gamma, \underline{n}_i$,

$$S^0(\gamma, mL) \ \geq \ \sum_{i=0}^{m-1} \psi(\underline{n}_i, i)| + \ L \sum_{i=0}^{m-1} \left(||\underline{n}_{i+1} - \underline{n}_i||_\infty - 1\right)_+$$

since if $\psi(\underline{n}_i, i) \ = \ 1$ then γ must make at least one jump in the time interval $[iR, (i+1)R)$ and if, furthermore $\left(||\underline{n}_{i+1} - \underline{n}_i||_\infty - 1\right)_+ \ = \ f$, then in this time interval γ must make more than fL jumps.

Thus to show that $\alpha(mL, N) \geq \frac{m}{2}$ it suffices to show that for all $\{\underline{n}_i\}$ with

$$\sum_{i=0}^{m-1} \left(||\underline{n}_{i+1} - \underline{n}_i||_\infty - 1\right)_+ \leq \frac{m}{2L} \tag{1}$$

it is the case that

$$\{\sum_{i=0}^{m-1} \psi(\underline{n}_i, i)| \geq \frac{m}{2}\}.$$

By simple large deviations arguments the probability that for any given $\{\underline{n}_i\}$, $\{\sum_{i=0}^{m-1} \psi(\underline{n}_i, i)| \geq \frac{m}{2}\}$ is less than $2^m(\varepsilon)^{\frac{m}{2}}$. Thus it remains only to count the number of $\{\underline{n}\}$ satisfying (1).

We write (for positive integer g_i) $A(g_1, g_2, \cdots, g_m)$ for the set of $(\underline{n}_1, \underline{n}_2, \cdots \underline{n}_m)$ so that for $1 \leq i \leq m$, $\left(||\underline{n}_i - \underline{n}_{i-1}||_\infty - 1\right)_+ \ = \ g_i$. We first give a crude bound on the cardinality of $A(g_1, g_2, \cdots g_n)$: $\underline{n}_0$ is required to be $\underline{0}$, after having "chosen" $\underline{n}_0, \underline{n}_1 \cdots \underline{n}_{i-1}$ we have 3^d choices for $\underline{n}_i$ if $g_i = 0$, otherwise we have at most $2d(2g_i+3)^{d-1}$ choices for $\underline{n}_i$. Thus (using $2d \leq 3^d$)

$$|A(g_1, g_2, \cdots g_m)| \ \leq \ 3^{md} \prod (2g_i + 3)^{d-1}.$$

We may find K so that for all g, $(2g+3)^{d-1} \leq K2^g$; we conclude that

$$|A(g_1, g_2, \cdots g_m)| \ \leq \ 3^{md} K^m 2^{\sum_{i=1}^m g_i} \ \leq \ C^m$$

for some universal C not depending on d, ε, if $\sum g_i \leq \frac{m}{2L}$.

By elementary combinatorics the number of $(g_1, g_2, \cdots g_m)$ so that $\sum_{i=1}^m g_i \ = \ r$ is $\binom{m+r-1}{r}$, thus the number of $(g_1, g_2, \cdots, g_m)$ so that $\sum_{i=1}^m g_i \ \leq \ \frac{m}{2L}$ is less than 2^{2m}. We conclude that the number of $\{\underline{n}_i\}$ satisfying (1) is bounded by $(4C)^m$. Thus the probability that $\alpha(mL, N)$ exceeds $\frac{m}{2}$ is at least $1 - (4C)^m 2^m (\varepsilon)^{\frac{m}{2}}$. This tends to one as m tends to infinity provided that ε was fixed sufficiently small.

$\square$

Section Two

Fix $\varepsilon > 0$, arbitrarily small. Given $c > 0$ fixed, we say that a cube $[-cR, cR]^d$ is *good* if $\forall x \in [-cR, cR]^d$

$$\inf_{\gamma \in \Gamma^{x,R}} S^x(\gamma, R) \geq R(\alpha - \varepsilon).$$

Lemma 2.1 *Given $\delta, c > 0$, there exists $R_0 = R_0(c, \delta)$ so that for all $R \geq R_0$,*

$$P\Big[[-cR, cR]^d \textit{ is good }\Big] \geq 1 - \delta$$

Proof Given ε, c, there exists k so that for any R, we can pick points $x_1^R, x_2^R \cdots x_{k/\varepsilon^d}^R \in [-cR, cR]^d$ so that every point of $[-cR, cR]^d$ is within $R\varepsilon/10$ of x_j^R for at least one j. Given this property it is clear that event

$$\{\inf_{x \in [-cR,cR]^d} \inf_{\gamma \in \Gamma^{x,R}} S(\gamma, R) \ < \ R(\alpha - \varepsilon)\}$$

is contained in

$$\{\inf_{x_j^R} \inf_{\gamma \in \Gamma^{x_j^R,R}} S(\gamma, R) \ < \ R(\alpha - \varepsilon/2)\}$$

Thus we have

$$P\Big[[-cR, cR]^d \text{ is good }\Big] \geq 1 - \frac{k}{\varepsilon^d} P[\alpha(R, N) < R(\alpha - \varepsilon/2)].$$

This last term is greater than $1 - \delta$ if R is sufficiently large. $\square$.

We have not fully specified how small we require δ to be but, conditional on this we will fix R at a level so large that the conclusions of Lemma 2.1 hold for δ and also so that $\gg \frac{1}{\varepsilon}$.

Lemma 2.2 *Given c and $R \geq R_0$ fixed, there exists $k_0 > 0$ so that if $0 < k \leq k_0$ and cube $[-cR, cR]^d$ is good then for any random walk $X(t)$ starting in the cube, the chance of survival to time R is bounded above by $k^{R(\alpha - 2\varepsilon)}$. More generally given c, $R \geq R_0$ we have for $k \leq k_0$ that the chance that the random walk makes $\geq f\alpha R$ jumps in time R is bounded above by $k^{fR(\alpha-\varepsilon)}$.*

Proof Let the starting point of X be x. By definition of $\alpha(R, N)$ and a cube being good we have

$$P[X(\cdot) \in \Gamma^{x,R}] \leq P[S(X(.), R) \geq R(\alpha - \varepsilon)] \leq (Rk)^{R(\alpha-\varepsilon)}.$$

This latter term is less than $k^{R(\alpha-2\varepsilon)}$ if k is sufficiently small. $\square$

We choose c to equal $10(\alpha + 1)$ and divide up the lattice into cubes $C(\underline{n}) = 2cR\underline{n} + [-cR, cR]^d$. We divide up space time into cubes $D(\underline{n}, i) = C(\underline{n}) \times [iR, (i + 1)R]$. We say that $D(\underline{n}, i)$ is good if $[-cR, cR]^d$ is good (in the old sense) after translating Poisson system $(\underline{N})$ spatially by $2cR\underline{n}$ and temporally by iR.

We define random variables $\psi(\underline{n}, i)$ taking values 0 or 1 by

$$\psi(\underline{n}, i) \;=\; 1 \text{ if } D(\underline{n}, i) \text{ is good.}$$

The random variables $\psi(\underline{n}, i)$ are not independent, but it should be noted that random variables $\psi(\underline{n}_1, i_1), \psi(\underline{n}_2, i_2) \cdots \psi(\underline{n}_j, i_j)$ are independent if the i_h s are all distinct.

A v-chain β is a sequence (β_j, j) $j = 0, 1, \cdots v-1$. We do not require that $|\beta_{j+1} - \beta_j|_1$ be less than or equal to 1.

An $(r-v)$-chain is a sequence (β_j, j) $j = r, r+1, \cdots v-1$.

Given ψ we associate a score to a $(r-v)$-chain β by

$$J_v(\beta) \;=\; \sum_{j=r}^{j=v-1} \psi(\beta_j, j) \;+\; 9 \sum_{j=r}^{j=v-2} (|\beta_{j+1} - \beta_j|_\infty - 1)_+.$$

Proposition 2.1 *For a random walk starting at time rR in cube $C(\underline{n})$, the chance that it survives until time vR is bounded above by*

$$2^{v-r-1} \exp\left(R(\alpha - 2\varepsilon) \ln(k) \min_\beta J_v(\beta) \right)$$

where the minimum is taken over all (r-v)-chains β with $\beta_r = \underline{n}$.

Proof In the proof we regard v as fixed and use induction on $k = v - r$. The proof follows from induction on k. It is clearly true for $k = 1$ (or $r = v - 1$) and all $\underline{n}$ by Lemma 2.2. Suppose that it is true for $k - 1$ (and all possible $\underline{n}$) and suppose further that X^k is a random walk starting at time $R(v-k)$ in cube $C(\underline{n})$. We consider the random walk over time interval $[(v-k)R, (v-k+1)R]$.

$$P[X^k \text{ survives up to } vR] \;=$$

$$\sum_{\underline{m}} P[X^k \text{ survives up to } (v-k+1)R,$$

$$X^k(v-k+1)R \in C(\underline{m}), X^k \text{ survives up to } vR].$$

By the Markov property for X^k and induction this summation is bounded by

$$\sum_{\underline{m}} P[X^k \text{ survives up to } (v-k+1)R,$$

$$X^k(v-k+1)R \in C(\underline{m})](2^{k-2}) \exp(R(\alpha - 2\varepsilon) \ln(k) J_v^{\underline{m}, k-1, v})$$

where $J_v^{\underline{m}, k-1, v}$ is the minimum of $J_v(\beta)$ over $(v-k+1) - v$-chains β with $\beta_{v-k+1} = \underline{m}$. This in turn is majorized by

$$\sum_{f=2} P\big[X^k \text{ survives up to } (v-k+1)R, X^k((v-k+1)R) \in C(\underline{m})$$

$$\text{with } ||\underline{n}-\underline{m}||_\infty = f\Big](2^{k-2})\exp(R(\alpha-2\varepsilon)\ln(k)J_v^{f,k-1,v})$$

$$+\sum_{||\underline{n}-\underline{n}'||_\infty\le 1} P[X^r \text{ survives up to } (v-k+1)R, X^k((v-k+1)R)\in C(\underline{n}')]$$

$$(2^{k-2})\exp(R(\alpha-2\varepsilon)\ln(k)J_v^{\underline{n}',k-1,v})$$

for $J_v^{f,k-1,v}$ the minimum of $J_v^{\underline{m},k-1,v}$ over $||\underline{n}-\underline{m}||_\infty = f$. By Lemma 2.2 these two summations are bounded by

$$(2^{k-2})\exp\Big((\alpha-2\varepsilon)R\ln(k)(\psi(\underline{n},r)+J_v^{1,k-1,v})\Big) \ +$$

$$(2^{k-2})\sum_{f=2}^{\infty}\exp\Big((f-1)10\alpha R\ln(k) \ + \ R(\alpha-2\varepsilon)\ln(k)J_v^{f,k-1,v}\Big)$$

where $J_v^{1,k-1,v}$ is the minimum of $J_v^{\underline{m},k-1,v}$ over $||\underline{n}-\underline{m}||_\infty \le 1$ (a slightly different definition from that of $J_v^{f,k-1,v}$ for higher f).

If R was chosen sufficiently large this is bounded by

$$2^{k-1}\exp\left(R(\alpha-2\varepsilon)\ln(k)\min_\beta J_v(\beta)\right)$$

where the minimum is taken over all (r-v)-chains β with $\beta_r = \underline{n}$. □

It remains to show that as v tends to infinity $J_v(\beta)$ is roughly v. It is time to properly define δ First fix $K \gg 3^d$ and so that for each integer f at least 1, the number of $\underline{m}$ with $||\underline{m}||_\infty = f$ is less than $K2^{f-1}/100$.

Lemma 2.3 *Given $\varepsilon > 0$ there exists δ so that $0<\delta<\varepsilon/100K$ so that if $X_1, X_2, \cdots X_N$ are i.i.d. Bernoulli δ) random variables for any integer N then*

$$P[\sum_{j=1}^{N} X_j \ \ge \ N\varepsilon + r) \le (\frac{1}{100K})^{N+r}.$$

Proposition 2.2 *With probability one for all v sufficiently large*

$$\inf_{\beta\in J_v} J(\beta) \ge v(1-2\varepsilon)$$

Proof We simply count. Given our definition of $J(\beta)$ we need only consider those $\beta \in J_v$ with $\sum_{j=0}^{v-2}(||\beta_{j+1}-\beta_j||_\infty - 1)_+ \ \le \ v/9$. For $\beta \in J_v$ we say the code of β is the sequence

$$\{(||\beta_1-\beta_0||_\infty-1)_+\cdots(||\beta_{j+1}-\beta_j||_\infty-1)_+\cdots(||\beta_{v-1}-\beta_{v-2}||_\infty-1)_+\}.$$

For fixed code $m_0, m_1\cdots m_{v-2}$ with $\sum m_j \le v/9$ there are (by our choice of K) less than or equal to $K^{v-1}\prod_{j=0}^{j=v-2}2^{m_i-1}$ possible v-chains. For any such β, $J_v(\beta) \ = \ 9\sum m_j \ + \ \sum\psi(\beta_j,j)$ and so

$$P[J_v(\beta)\le v(1-2\varepsilon)] \ \le P[\sum\psi(\beta_j,j)\le v(1-2\varepsilon) \ -9\sum m_j]$$

$$= \; P[\sum(1-\psi(\beta_j,j)) \geq v2\varepsilon \; +9\sum m_j] \leq (\frac{1}{100K})^v(\frac{1}{100K})^{9\sum m_j}.$$

So the probability that for some β with code $m_0, m_1, \cdots m_{v-2}$ $J_v(\beta)$ is less than or equal to $(1-2\varepsilon)v$ is bounded by

$$K^v \prod_{j=0}^{j=v-2} 2^{m_i-1}(\frac{1}{100K})^v(\frac{1}{100K})^{\sum m_j} \;\leq\; (\frac{1}{100})^v.$$

But the number of codes which sum to less than $v/9$ is (assuming w.l.o.g that $v/9$ is an integer) exactly $\sum_{j=0}^{j=v/9}\binom{v+j-1}{v-1} \leq v/9\binom{v+v/9-1}{v-1} \leq 2^v$ for v large. We conclude that $P[\min \; J_v(\beta) \leq v(1-2\varepsilon)] \leq (\frac{1}{50})^v$ for large v. The proposition now follows from the Borel Cantelli Lemma.

□

Corollary 2.4 $\liminf_{k\to 0} \frac{\lambda(d,k)}{\ln(\frac{1}{k})} \geq \alpha(d)$.

Proof By Proposition 2.1 we have that for $k \leq k_0$ that

$$p(vR,N) \leq 2^{v-1}\exp\left(R(\alpha-2\varepsilon)\ln(k)\min J_v(\beta)\right)$$

By Proposition 2.2 we have therefore that for large enough v

$$p(vR,N) \leq 2^{v-1}\exp\left(R(\alpha-2\varepsilon)\ln(k)v(1-2\varepsilon)\right)$$
$$\leq 2^{vR\varepsilon}\exp\left(R(\alpha-2\varepsilon)\ln(k)v(1-2\varepsilon)\right)$$
$$\leq \exp\left(Rv((\alpha-2\varepsilon)\ln(k)(1-2\varepsilon)+\varepsilon)\right)$$

Thus we have that $\lambda(k,d) \geq \ln(\frac{1}{k})(\alpha-2\varepsilon)(1-2\varepsilon) \; - \; \varepsilon$. Since ε is arbitrarily small the Corollary follows.

□

Section Three

In this section we will use block/percolation arguments that since **[BG]** may be regarded as standard. Simply to avoid notational encumbrance we will write out the proof for the case $d=1$ but the argument easily extends to all dimensions.

Fix $\varepsilon > 0$. By Proposition 1.1 we have that for R sufficiently large

$$P[\alpha(R,N) \leq R(\alpha+\varepsilon)] > 1-\varepsilon^6.$$

Now note that, by our definition of α, the event $\{\alpha(R,N) \leq R(\alpha+\varepsilon)\}$ is the same as the event $\{\exists\gamma \in \Gamma^R \text{ with } S(\gamma,R) \leq R(\alpha+\varepsilon) \text{ and } |\gamma(R)| \leq R(\alpha+\varepsilon)\}$. Thus for R sufficiently large

$$\{\nexists\gamma \in \Gamma^R \text{ with } S(\gamma,R) \leq R(\alpha+\varepsilon) \text{ and } \gamma(R) \in [0, R(\alpha+\varepsilon)]\}\cap$$

$$\{\nexists\gamma \in \Gamma^R \text{ with } S(\gamma,R) \leq R(\alpha+\varepsilon) \text{ and } \gamma(R) \in [-R(\alpha+\varepsilon), 0]\}$$

has probability less than ε^6. These two events are increasing functions of the Poisson processes and, by symmetry, have equal probabilities, so by the FKG inequalities (as in **[BG]**) we have

$$P[\nexists \gamma \in \Gamma^R \text{ with } S(\gamma, R) \leq R(\alpha+\varepsilon) \text{ and } \gamma(R) \in [0, R(\alpha+\varepsilon)]] \; < \; \varepsilon^3$$

, that is,

$$P[\exists \gamma \in \Gamma^R \text{ with } S(\gamma, R) \leq R(\alpha+\varepsilon) \text{ and } \gamma(R) \in [0, R(\alpha+\varepsilon)]] \; > \; 1-\varepsilon^3$$

and, by symmetry,

$$P[\exists \gamma \in \Gamma^R \text{ with } S(\gamma, R) \leq R(\alpha+\varepsilon) \text{ and } \gamma(R) \in [-R(\alpha+\varepsilon), 0]] \; > \; 1-\varepsilon^3$$

We remark that such paths must be contained in space time rectangle $[-R(\alpha+\varepsilon), R(\alpha+\varepsilon)] \times [0, R]$.

Thus outside probability strictly less than $\frac{1}{\varepsilon}\varepsilon^3 = \varepsilon^2$, we can "navigate" a path $\gamma \in \Gamma^{\frac{R}{\varepsilon}}$ with $S(\gamma, \frac{R}{\varepsilon}) \leq \frac{1}{\varepsilon} R(\alpha+\varepsilon)$, which lies entirely in spacetime rectangle $[-2R(\alpha+\varepsilon), 2R(\alpha+\varepsilon)] \times [0, \frac{R}{\varepsilon}]$ and which has $\gamma(\frac{R}{\varepsilon}) \in [-R(\alpha+\varepsilon), R(\alpha+\varepsilon)]$. Therefore we have with probability at least $1-\varepsilon^2$ there is a path $\gamma \in \Gamma^{\frac{R}{\varepsilon}}$ so that (i) $S(\gamma, \frac{R}{\varepsilon}) \leq R(\alpha+\varepsilon)(1+2\varepsilon)/\varepsilon$

(ii) γ lies entirely within $[-2R(\alpha+\varepsilon), 2R(\alpha+\varepsilon)] \times [0, \frac{R}{\varepsilon}]$.

Now provided that δ is chosen sufficiently small we have also that with probability $> 1-\varepsilon^2$ we have γ satisfying in addition to(i) and (ii) above

(iii) No two jump times of γ are within 2δ of each other or of time 0 or time $\frac{R}{\varepsilon}$. Also the path γ is at all times at least 2δ away from points of N (considered now as a random subset of space time).

We define a 2-dependent oriented percolation scheme on $\{(m,n) : n \geq 0, \; m+n \equiv 0(mod(2))\}$ as follows: We say that the bond from (m,n) to $(m \pm 1, n+1)$ is open if there is a path γ from $(mR(\alpha+\varepsilon), n\frac{R}{\varepsilon})$ to $((m\pm 1)R(\alpha+\varepsilon), (n+1)\frac{R}{\varepsilon})$ that satisfies (i) and

(ii') γ lies entirely within $[(m-2)R(\alpha+\varepsilon), (m+2)R(\alpha+\varepsilon)] \times [n\frac{R}{\varepsilon}, (n+1)\frac{R}{\varepsilon}]$.

(iii') No two jump times of γ are within 2δ of each other or of time $n\frac{R}{\varepsilon}$ or time $(n+1)\frac{R}{\varepsilon}$. Also the path γ is at all times at least 2δ away from points of N

Then we have that (provided ε was chosen sufficiently small) the percolation system is supercritical (see the appendix of **[D2]**, which while formally treating oriented bond percolation, is valid for our bond percolation). That is with probability one there is a point $(0,n)$ with infinitely many "descendents".

Lemma 3.1 *If k is sufficiently small then for all (m,n) if the percolation bond $(m,n) \rightarrow (m \pm 1, n+1)$ is open then with probability at least $k^{\frac{R}{\varepsilon}(\alpha+\varepsilon)(1+3\varepsilon)}$ a random walk started at $mR(\alpha+\varepsilon)$ at time $n\frac{R}{\varepsilon}$ will survive until time $(n+1)\frac{R}{\varepsilon}$ and will be in position $(m\pm 1)R(\alpha+\varepsilon)$ at this time.*

Proof Let a path satisfying (i),(ii') and (iii') be γ. Let its jumps be at times $0 < t_1, t_2, \cdots t_r$ $r \leq R(\alpha+\varepsilon)(1+2\varepsilon)/\varepsilon$. We consider the event that our random walk makes precisely r jumps in the time interval , these jumps occurring within the intervals $(t_i - \delta/3, t_i + \delta/3)$ (one jump in each interval) and the jumps are equal to the corresponding jumps of γ. This event is contained in the event of interest and has probability at least

$$e^{-\frac{R}{\varepsilon}k}\prod_{j=1}^{r}(\frac{2\delta}{3}\frac{k}{2}).$$

This is easily seen to exceed $k^{\frac{R}{\varepsilon}(\alpha+\varepsilon)(1+3\varepsilon)}$ for k small. □

Corollary 3.1 $\frac{\lambda(k,d)}{\ln(\frac{1}{k})} \leq \alpha(d)$.

Proof Given our percolation scheme we have (provided ε was chosen sufficiently small) that there exists n_0 so that $(0, n_0)$ is a point of percolation. That is to say there exists $0 = m_0, m_1, \cdots m_j \cdots$ so that $\forall j \geq 1$, the bond between $(m_{j-1}, n_0 + j - 1)$ and $(m_j, n_0 + j)$ is open.

It follows from induction and Lemma 3.1 that a random walk starting at site 0 at time n_0 has chance at least $k^{\frac{R}{\varepsilon}(\alpha+\varepsilon)(1+3\varepsilon)j}$ of surviving until time $(n_0 + j)\frac{R}{\varepsilon}$ and being at m_j at this time. The chance that a random walk starting at site 0 at time 0 reaches site 0 at time $n_0\frac{R}{\varepsilon}$ is strictly positive ($\underline{N}$ a.s.). So we have for some $c_k(\omega) > 0$ that

$$p((n_0 + j)\frac{R}{\varepsilon}, N) \geq \ c_k(\omega)k^{\frac{R}{\varepsilon}(\alpha+\varepsilon)(1+3\varepsilon)j}$$

for $k \leq k_0$. Thus $\lambda(k, d) \leq \ln(\frac{1}{k})(\alpha + \varepsilon)(1 + 3\varepsilon)$. The corollary follows from the arbitrariness of ε. □

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