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MARKOV PROCESSES WITH IDENTICAL BRIDGES

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Abstract: Let X and Y be time-homogeneous Markov processes with common state space E, and assume that the transition kernels of X and Y admit densities with respect to suitable reference measures. We show that if there is a time $t > 0$ such that, for each $x \in E$, the conditional distribution of $(X_s)_{0\le s\le t}$, given $X_0 = x = X_t$, coincides with the conditional distribution of $(Y_s)_{0\le s\le t}$, given $Y_0 = x = Y_t$, then the infinitesimal generators of X and Y are related by $L^Y f = \psi^{-1}L^X(\psi f) - \lambda f$, where ψ is an eigenfunction of L^X with eigenvalue $\lambda \in \mathbf{R}$. Under an additional continuity hypothesis, the same conclusion obtains assuming merely that X and Y share a "bridge" law for one triple (x, t, y). Our work extends and clarifies a recent result of I. Benjamini and S. Lee.

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1. Introduction

Let $X = (X_t, P^x)$ and $Y = (Y_t, Q^x)$ be non-explosive regular Markov diffusion processes in $\mathbf{R}$. Let $P_t^{x,y}$ denote the conditional law of $(X_s)_{0\le s\le t}$ given $X_0 = x$, $X_t = y$. Let $Q_t^{x,y}$ denote the analogous "bridge" law for Y. Recently, Benjamini & Lee [**BL97**] proved the following result.

(1.1) Theorem. *Suppose that X is standard Brownian motion and that Y is a weak solution of the stochastic differential equation*

$$dY_t = dB_t + \mu(Y_t)\,dt, \tag{1.2}$$

where B is standard Brownian motion and the drift μ is bounded and twice continuously differentiable. If $Q_t^{x,x} = P_t^{x,x}$ for all $x \in \mathbf{R}$ and all $t > 0$, then either (i) $\mu(x) \equiv k$ or (ii) $\mu(x) = k\tanh(kx + c)$, for some real constants k and c.

Our aim in this paper is to generalize this theorem in two ways.

Firstly, we allow X and Y to be general strong Markov processes with values in an abstract state space E. We require that X and Y have dual processes with respect to suitable reference measures, and that X and Y admit transition densities with respect to these reference measures. (These conditions are met by all regular 1-dimensional diffusions without absorbing boundary points.)

Secondly, under an additional continuity condition, we show that the equality of $Q_t^{x,y}$ and $P_t^{x,y}$ for a *single* choice of the triple (x, t, y) is enough to imply that $Q_t^{x,y} = P_t^{x,y}$ for *all* $(x, t, y) \in E\times]0,\infty[\times E$. We provide a simple example illustrating what can go wrong when the continuity condition fails to hold.

The conclusion of Theorem (1.1) is more transparently stated as follows. Given a drift μ define $\psi(x) := \exp\int_0^x \mu(y)\,dy$. Then μ satisfies the conclusion of Theorem (1.1) if and only if

$$\tfrac{1}{2}\psi''(x) = \lambda\,\psi(x), \qquad \forall\, x \in \mathbf{R},$$

where $\lambda := k^2/2$. Thus, Theorem (1.1) can be stated as follows: If X is Brownian motion and if Y is "Brownian motion with drift μ," then X and Y have common bridge laws if and only if μ is the logarithmic derivative of a strictly positive eigenfunction of the local infinitesimal generator of X, in which case the laws of X and Y are related by

$$\left.\frac{dQ^x}{dP^x}\right|_{\mathcal{F}_t} = e^{-\lambda t}\frac{\psi(X_t)}{\psi(X_0)}. \tag{1.3}$$

Theorem (1.1) and our extensions of it depend crucially on the existence of a "reference" measure dominating the transition probabilities of X and Y. This fact is amply demonstrated by the work of H. Föllmer in [**F90**]. Let E be the Banach space of continuous maps of $[0,1]$ into $\mathbf{R}$ that vanish at 0, and let m denote Wiener measure on the Borel subsets of E. Let $X = (X_t, P^x)$ be the associated Brownian motion in E; that is, the E-valued diffusion with transition semigroup given by

$$P_t(x, f) := \int_E f(x + \sqrt{t}y)\,m(dy).$$

This semigroup admits no reference measure; indeed $P_t(x,\cdot\,) \perp P_t(y,\cdot\,)$ unless $x - y$ is an element of the Cameron-Martin space H, consisting of those elements of E that are absolutely continuous and possess a square-integrable derivative. Now given $z \in E$, let $Y = (Y_t, Q^x)$ be Brownian motion in E with drift z. By this we mean the E-valued diffusion with transition semigroup

$$Q_t(x, f) := \int_E f(x + tz + \sqrt{t}y)\, m(dy).$$

Given $(x,t,y) \in E \times]0,\infty[\times E$, let $P_t^{x,y}$ be the P^0-distribution of the process $\{x + X_s + (s/t)(y - x - X_t) : 0 \le s \le t\}$. Evidently, (i) $(x, y) \mapsto P_t^{x,y}$ is weakly continuous, (ii) $P_t^{x,y}(X_t = y) = 1$, and (iii) $\{P_t^{x,y} : y \in E\}$ is a regular version of the family of conditional distributions $Q^x(\{X_s; 0 \le s \le t\} \in \cdot \mid X_t = y)$, regardless of the choice of $z \in E$. In other words, X and Y have common bridge laws. However, the laws of X and Y are mutually absolutely continuous (as in (1.3)) if and only if $z \in H$.

Before stating our results we describe the context in which we shall be working. Let $X = (X_t, P^x)$ now denote a strong Markov process with cadlag paths and infinite lifetime. We assume that the state space E is homeomorphic to a Borel subset of some compact metric space, and that the transition semigroup $(P_t)_{t\ge 0}$ of X preserves Borel measurability and is without branch points. In other words, X is a Borel right processes with cadlag paths and infinite lifetime; see [**G75, S88**]. The process X is realized as the coordinate process $X_t : \omega \mapsto \omega(t)$ on the sample space Ω of all cadlag paths from $[0,\infty[$ to E. The probability measure P^x is the law of X under the initial condition $X_0 = x$. We write $(\mathcal{F}_t)_{t\ge 0}$ for the natural (uncompleted) filtration of $(X_t)_{t\ge 0}$ and $(\theta_t)_{t\ge 0}$ for the shift operators on Ω: $X_s \circ \theta_t = X_{s+t}$.

In addition, we assume the existence of transition densities with respect to a reference measure and (for technical reasons) the existence of a dual process. (The duality hyothesis (1.4) can be replaced by conditions ensuring the existence of a nice Martin exit boundary for the space-time process $(X_t, r+t)_{t\ge 0}$; see [**KW65**].)

Let $\mathcal{E}$ denote the Borel σ-algebra on E.

(1.4) Hypothesis. (Duality) There is a σ-finite measure m^X on $(E, \mathcal{E})$ and a second E-valued Borel right Markov process $\hat{X}$, with cadlag paths and infinite lifetime, such that the semigroup $(\hat{P}_t)$ of $\hat{X}$ is in duality with (P_t) relative to m^X:

$$\int_E f(x) P_t g(x)\, m^X(dx) = \int_E \hat{P}_t f(x) g(x)\, m^X(dx), \tag{1.5}$$

for all $t > 0$ and all positive $\mathcal{E}$-measurable functions f and g.

(1.6) Hypothesis. (Transition densities) There is an $\mathcal{E} \otimes \mathcal{B}_{]0,\infty[} \otimes \mathcal{E}$-measurable function $(x,t,y) \mapsto p_t(x,y) \in]0,\infty[$ such that

$$P^x(f(X_t)) = P_t f(x) = \int_E p_t(x,y)\, f(y)\, m^X(dy), \qquad \forall\, t > 0, \tag{1.7}$$

and

$$\hat{P}^x(f(X_t)) = \hat{P}_t f(x) = \int_E p_t(y,x)\, f(y)\, m^X(dy), \qquad \forall\, t > 0, \tag{1.8}$$

for any bounded $\mathcal{E}$-measurable function f. Furthermore, we assume that the Chapman-Kolmogorov identity holds:

$$p_{t+s}(x,y) = \int_E p_t(x,z) p_s(z,y)\, m^X(dz), \qquad \forall\, s,t > 0, x, y \in E. \tag{1.9}$$

Hypothesis (1.6) implies that $m^X(U) > 0$ for every non-empty finely open subset of E.

When (1.4) is in force, the existence and uniqueness of a (jointly measurable) transition density function $p_t(x,y)$ such that (1.7)–(1.9) hold is guaranteed by the apparently weaker condition: $P_t(x,\cdot) \ll m^X$, $\hat{P}_t(x,\cdot) \ll m^X$ for all $x \in E$, $t > 0$. See, for example, [**D80, W86, Y88**]. For more discussion of processes with "dual transition densities," see [**GS82**; §3].

Let $Y = (Y_t, Q^x)$ be a second E-valued Borel right Markov process with cadlag paths and infinite lifetime. The process Y is assumed to satisfy all of the conditions imposed on X above. In particular, we can (and do) assume that Y is realized as the coordinate process on Ω. The transition semigroup of Y is denoted $(Q_t)_{t\ge 0}$ and we use m^Y and $q_t(x,y)$ to denote the reference measure and transition density function for Y. (The bridge laws $P_t^{x,y}$ and $Q_t^{x,y}$ for X and Y will be discussed in more detail in section 2.)

In what follows, the prefix "co-" refers to the dual process $\hat{X}$ (or $\hat{Y}$).

(1.10) Theorem. *Let X and Y be strong Markov processes as described above, satisfying Hypotheses (1.4) and (1.6). Suppose there exists $t_0 > 0$ such that $Q_{t_0}^{x,x} = P_{t_0}^{x,x}$ for all $x \in E$. Then*

(a) $P^x|_{\mathcal{F}_t} \sim Q^x|_{\mathcal{F}_t}$ and $\hat{P}^y|_{\mathcal{F}_t} \sim \hat{Q}^y|_{\mathcal{F}_t}$, for all $x \in E$, $y \in E$, and $t > 0$;

(b) There exist a constant $\lambda \in \mathbf{R}$, a Borel finely continuous function $\psi : E \to]0,\infty[$, and a Borel co-finely continuous function $\hat{\psi} : E \to]0,\infty[$ such that for all $t > 0$,

$$P_t\psi(x) = e^{\lambda t}\,\psi(x), \qquad \forall\, x \in E, \tag{1.11}$$

$$\hat{P}_t\hat{\psi}(x) = e^{\lambda t}\,\hat{\psi}(x), \qquad \forall\, x \in E, \tag{1.12}$$

$$Q^x|_{\mathcal{F}_t} = e^{-\lambda t}\frac{\psi(X_t)}{\psi(X_0)}\,P^x|_{\mathcal{F}_t}, \qquad \forall\, x \in E, \tag{1.13}$$

$$\hat{Q}^x|_{\mathcal{F}_t} = e^{-\lambda t}\frac{\hat{\psi}(X_t)}{\hat{\psi}(X_0)}\,\hat{P}^x|_{\mathcal{F}_t}, \qquad \forall\, x \in E. \tag{1.14}$$

The function $\psi\hat{\psi}$ is a Borel version of the Radon-Nikodym derivative dm^Y/dm^X.

(c) $Q_t^{x,y} = P_t^{x,y}$ for all $(x,t,y) \in E\times]0,\infty[\times E$;

(1.15) Remarks.

(i) Given functions ψ and $\hat{\psi}$ as in (1.11) and (1.12), the right sides of (1.13) and (1.14) determine the laws of Borel right Markov processes Y^* and $\hat{Y}^*$ on E. It is easy to check that Y^* and $\hat{Y}^*$ are in duality with respect to the measure $\psi\hat{\psi}\cdot m^X$, that Hypotheses (1.4) and (1.6) are satisfied, and that Y^* (resp. $\hat{Y}^*$) has the same bridge laws as X (resp. $\hat{X}$).

(ii) As noted earlier, any one-dimensional regular diffusion without absorbing boundaries satisfies Hypotheses (1.4) and (1.6). Such a diffusion is self-dual with respect to its speed measure, which serves as the reference measure. Moreover, the transition density function of such a diffusion is jointly continuous in (x,t,y). See [**IM**; pp. 149–158].

(1.16) Theorem. *Let X and Y be right Markov processes as described before the statement of Theorem (1.10). Suppose, in addition to (1.4) and (1.6), that for each $t > 0$ the transition density functions $p_t(x,y)$ and*

$q_t(x,y)$ are separately continuous in the spatial variables x and y. If there is a triple $(x_0,t_0,y_0)\in E\times]0,\infty[\times E$ such that $P^{x_0,y_0}_{t_0}=Q^{x_0,y_0}_{t_0}$, then the conclusions (a), (b), and (c) of Theorem (1.10) remain true.

(1.17) Remark. Let us suppose that X is a real-valued regular diffusion on its natural scale, and that its speed measure m^X admits a strictly positive density ρ with respect to Lebesgue measure. Let L^X denote the local infinitesimal generator of X. Then (1.11) implies $L^X\psi=\lambda\psi$, or more explicitly

$$\frac{1}{\rho(x)}\psi''(x)=\lambda\psi(x).$$

Moreover, (1.13) means that the transition semigroups of X and Y are related by

$$Q_t(x,dy)=\exp(-\lambda t)[\psi(y)/\psi(x)]P_t(x,dy).$$

From this it follows that the (local) infinitesimal generators of X and Y are related by

$$L^Y f(x)=L^X f(x)+\frac{2\,\mu(x)}{\rho(x)}\cdot f'(x), \tag{1.18}$$

where $\mu:=(\log\psi)'$. When X is standard Brownian motion (so that $\rho(x)\equiv 2$), the right side of (1.18) is the infinitesimal generator of any weak solution of (1.2). By Remark (1.15)(ii), the additional condition imposed in Theorem (1.16) is met in the present situation. Consequently, Theorem (1.16) implies that the conclusion of Theorem (1.1) is true once we know that the (x_0,t_0,y_0)-bridge law of Y is a Brownian bridge, for *one* triple (x_0,t_0,y_0)

Without some sort of additional condition as in Theorem (1.16), there may be an exceptional set in the conclusions (a)–(c). Recall that a Borel set $N\subset E$ is X-polar if and only if $P^x(X_t\in N$ for some $t>0)=0$ for all $x\in E$.

(1.19) Example. The state space in this example will be the real line $\mathbf{R}$. Let $Z=(Z_t,R^x)$ be a 3-dimensional Bessel process, with state space $[0,\infty[$. (Under R^x, $(Z_t)_{t\ge 0}$ has the same law as the radial part of a standard 3-dimensional Brownian motion started at $(x,0,0)$.) We assume that the probability space on which Z is realized is rich enough to support an independent unit-rate Poisson process $(N(t))_{t\ge 0}$. The process X is presented (non-canonically) as follows:

$$X_t:=\begin{cases}(-1)^{N(t)}Z_t, & \text{if } X_0\ge 0;\\ (-1)^{N(t)+1}Z_t, & \text{if } X_0<0,\end{cases}$$

whereas Y is presented as

$$Y_t:=\begin{cases}(-1)^{N(t)}Z_t, & \text{if } Y_0> 0;\\ (-1)^{N(t)+1}Z_t, & \text{if } Y_0\le 0.\end{cases}$$

Both X and Y are Borel right Markov processes satisfying (1.4) and (1.6); indeed, both processes are self-dual with respect to the reference measure $m(dx):=x^2\,dx$. The singleton $\{0\}$ is a polar set for both processes. If neither x nor y is equal to 0, then $P^{x,y}_t=Q^{x,y}_t$ for all $t>0$. However, $P^{0,y}_t$ and $Q^{0,y}_t$ are different for all $y\in\mathbf{R}$ and $t>0$, because

$$P^{0,y}_t(X_s>0\text{ for all small } s)=Q^{0,y}_t(X_s<0\text{ for all small } s)=1.$$

The reader will have no trouble finding explicit expressions for the transition densities $p_t(x,y)$ and $q_t(x,y)$, thereby verifying that for $t>0$, $y>0$,

$$\begin{aligned} p_t(0+,y) = q_t(0-,y) &= \frac{1+e^{-2t}}{\sqrt{2\pi t^3}} e^{-y^2/2t} \\ &> \frac{1-e^{-2t}}{\sqrt{2\pi t^3}} e^{-y^2/2t} = p_t(0-,y) = q_t(0+,y), \end{aligned}$$

which is consistent with Theorem (1.16).

This example is typical of what can go wrong when the hypothesis $[P_{t_0}^{x,x} = Q_{t_0}^{x,x}, \forall\, x]$ of Theorem (1.10) is weakened to $P_{t_0}^{x_0,y_0} = Q_{t_0}^{x_0,y_0}$. In general, under this latter condition, there is a set $N \in \mathcal{E}$ that is both X-polar and Y-polar and a set $\hat{N} \in \mathcal{E}$ that is both $\hat{X}$-polar and $\hat{Y}$-polar, such that the conclusions drawn in Theorem (1.10) remain true *provided* one substitutes "$x \in E \setminus N$" for "$x \in E$" and "$y \in E \setminus \hat{N}$" for "$y \in E$" throughout. (Actually, the functions ψ and $\hat{\psi}$ can be defined so that (1.11) and (1.12) hold on all of E; these functions will be strictly positive on E, but their finiteness can be guaranteed only off N and $\hat{N}$, respectively.) Since the proof of this assertions is quite close to that of Theorem (1.10), it is omitted.

After discussing bridge laws in section 2, we turn to the proof of Theorem (1.10) in section 3. Theorem (1.16) is proved in section 4.

2. Bridges

The discussion in this section is phrased in terms of X, but applies equally to Y. The process X is as described in section 1. All of the material in this section, with the exception of Lemmas (2.8) and (2.9), is drawn from [**FPY93**], to which we refer the reader for proofs and further discussion.

The following simple lemma shows that in constructing $P_t^{x,y}$ it matters not whether we condition P^x on the event $\{X_t = x\}$ or on the event $\{X_{t-} = x\}$.

(2.1) Lemma. *$P^x(X_{t-} = X_t) = 1$ for every $x \in E$ and every $t > 0$.*

In what follows, $\mathcal{F}_{t-}$ denotes the σ-algebra generated by $\{X_s, 0 \le s < t\}$.

(2.2) Proposition. *Given $(x,t,y) \in E \times]0,\infty[\times E$ there is a unique probability measure $P_t^{x,y}$ on $(\Omega, \mathcal{F}_{t-})$ such that*

$$P_t^{x,y}(F) = P^x\left(F \cdot \frac{p_{t-s}(X_s,y)}{p_t(x,y)}\right) \tag{2.3}$$

for all positive $\mathcal{F}_s$-measurable functions F on Ω, for all $0 \le s < t$. Under $P_t^{x,y}$ the coordinate process $(X_s)_{0 \le s < t}$ is a non-homogeneous strong Markov process with transition densities

$$p^{(y,t)}(z,s;z',s') = \frac{p_{s'-s}(z,z') p_{t-s'}(z',y)}{p_{t-s}(z,y)}, \qquad 0 < s < s' < t. \tag{2.4}$$

Moreover $P_t^{x,y}(X_0 = x, X_{t-} = y) = 1$. Finally, if $F \ge 0$ is $\mathcal{F}_{t-}$-measurable, and $g \ge 0$ is a Borel function on E, then

$$P^x(F \cdot g(X_{t-})) = \int_E P_t^{x,y}(F)\, g(y)\, p_t(x,y)\, m(dy). \tag{2.5}$$

Thus $(P_t^{x,y})_{y\in E}$ is a regular version of the family of conditional probability distributions $\{P^x(\cdot\,|X_{t-} = y)$, $y \in E\}$; equally so with X_{t-} replaced by X_t, because of Lemma (2.1).

The following corollaries of Proposition (2.2) will be used in the sequel.

(2.6) Corollary. *The $P_t^{x,y}$-law of the time-reversed process $(X_{(t-s)-})_{0\le s<t}$ is $\hat P_t^{y,x}$, the law of a (y,t,x)-bridge for the dual process $\hat X$.*

(2.7) Corollary. *For each $(\mathcal{F}_{t+})$ stopping time T, a $P_t^{x,y}$ regular conditional distribution for $(X_{T+u}, 0\le u<t-T)$ given $\mathcal{F}_{T+}$ on $\{T<t\}$ is provided by $P_{t-T}^{X_T,y}$.*

Continuity properties are useful in trying to minimize the exceptional sets involved in statements concerning bridge laws. The following simple result will be used in the proof of (1.16).

(2.8) Lemma. *Assume that $x\mapsto p_t(x,y)$ is continuous for each fixed pair $(t,y)\in]0,\infty[\times E$. Fix $0<s<t$ and let G be a bounded $\mathcal{F}_{(t-s)-}$-measurable function on Ω. Then for each $y\in E$,*

$$x\mapsto P_t^{x,y}(G\circ\theta_s)$$

is continuous on E.

Proof. By Corollary (2.7),

$$P_t^{x,y}(G\circ\theta_s)=\int_E\frac{p_s(x,z)p_{t-s}(z,y)}{p_t(x,y)}\,P_{t-s}^{z,y}(G)\,m^X(dz). \tag{2.9}$$

The ratio on the right side of (2.9) (call it $f_x(z)$) is a probability density with respect to $m^X(dz)$, and the mapping $x\mapsto f_x(z)$ is continuous by hypothesis. It therefore follows from Scheffé's Theorem [**B68**; p. 224] that $x\mapsto f_x$ is a continuous mapping of E into $L^1(m^X)$. ▯

The *backward space-time process* associated with X is the (Borel right) process

$$\overline{X}_t(\omega,r):=(X_t(\omega),r-t),$$

realized on the sample space $\Omega\times\mathbf{R}$ equipped with the laws $P^x\otimes\epsilon_r$. A (universally measurable) function $f:E\times\mathbf{R}\to[0,\infty]$ is $\overline{X}$-excessive if and only if

$$t\mapsto\int_E p_t(x,y)f(y,r-t)\,m^X(dy)$$

is decreasing and right-continuous on $[0,\infty[$ for each $(x,r)\in E\times\mathbf{R}$. For example, if $(y,s)\in E\times\mathbf{R}$ is fixed, then $(x,r)\mapsto 1_{]s,\infty[}(r)\,p_{r-s}(x,y)$ is $\overline{X}$-excessive. A Borel function $f:E\times\mathbf{R}\to\overline{\mathbf{R}}$ is finely continuous with respect to $\overline{X}$ if and only if $t\mapsto f(X_t,r-t)$ is right-continuous $P^x\otimes\epsilon_r$-a.s. for every $(x,r)\in E\times\mathbf{R}$. Since $\overline{X}$ is a right process [**S88**; §16], $\overline{X}$-excessive functions are finely continuous. Because of Hypotheses (1.4) and (1.6), the measure $m^X\otimes\text{Leb}$ on $E\times\mathbf{R}$ is a reference measure for $\overline{X}$. Thus, if two finely continuous functions of $\overline{X}$ agree $m^X\otimes$ Leb-a.e., then they agree on all of $E\times\mathbf{R}$.

(2.10) Lemma. *Fix $n\in\mathbf{N}$ and let $f_1,f_2,\dots f_n$ be bounded real-valued Borel functions on $E\times[0,\infty[$. Then for each $y\in E$, the function*

$$(x,t)\mapsto 1_{]0,\infty[}(t)\,P_t^{x,y}\left(\prod_{i=1}^n\int_0^t f_i(X_s,t-s)\,ds\right) \tag{2.11}$$

is finely continuous with respect to the backward space-time process $(X_t, r-t)_{t\ge 0}$.

Proof. Without loss of generality, we assume that $0 < f_i \le 1$ for every i. The expression appearing in (2.11) can be written as the sum of $n!$ terms of the form

$$1_{]0,\infty[}(t) P_t^{x,y} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \prod_{i=1}^n g_i(X_{s_i}, t-s_i), \tag{2.12}$$

where $(g_1, g_2, \ldots, g_n)$ is a permutation of $(f_1, f_2, \ldots, f_n)$. Let $h(x,t)$ denote the expression in (2.12) multiplied by $p_t(x,y)$. Also, let $\tilde{h}(z,u) := p_u(z,y) \cdot P_u^{z,y}(J_u)$, where

$$J_u := \int_0^u du_2 \int_{u_2}^u du_3 \cdots \int_{u_{n-1}}^u du_n \prod_{i=2}^n g_i(X_{u_i}, u-u_i).$$

For $t > 0$, the Markov property (2.7) yields

$$\begin{aligned} h(x,t) &= p_t(x,y) \cdot P_t^{x,y} \int_0^t g_1(X_{s_1}, t-s_1) J_{t-s_1} \circ \theta_{s_1} \, ds_1 \\ &= p_t(x,y) \cdot P_t^{x,y} \int_0^t g_1(X_{s_1}, t-s_1) P_{t-s_1}^{X(s_1),y}(J_{t-s_1}) \, ds_1 \\ &= \int_E \int_0^t p_{s_1}(x,z) g_1(z, t-s_1) \tilde{h}(z, t-s_1) \, ds_1 \, m^X(dz) \\ &= \int_E \int_0^t p_{t-s}(x,z) g_1(z,s) \tilde{h}(z,s) \, ds \, m^X(dz). \end{aligned} \tag{2.13}$$

The final line in (2.13) exhibits h as a positive linear combination of the space-time excessive functions $(x,t) \mapsto 1_{]s,\infty[}(t) p_{t-s}(x,z)$, showing that h is space-time excessive. Since $(x,t) \mapsto 1_{]0,\infty[}(t) p_t(x,y)$ is also space-time excessive, the function appearing in (2.12) is finely continuous as asserted. ∎

3. Proof of (1.10)

For typographical convenience, throughout this section we assume (without loss of generality) that $t_0 = 2$, so the basic hypothesis under which we are working is that $Q_2^{x,x} = P_2^{x,x}$ for all $x \in E$.

Proof of (1.10)(a). Given $x \in E$ and $t \in]0,2[$, the mutual absolute continuity of $P^x|_{\mathcal{F}_t}$ and $Q^x|_{\mathcal{F}_t}$ follows immediately from the hypothesis $Q_2^{x,x} = P_2^{x,x}$ because of (2.3). Let us now show that if $P^x|_{\mathcal{F}_t} \sim Q^x|_{\mathcal{F}_t}$ for all x, then $P^x|_{\mathcal{F}_{2t}} \sim Q^x|_{\mathcal{F}_{2t}}$ for all x; an obvious induction will then complete the proof. By an application of the monotone class theorem, given a bounded $\mathcal{F}_{2t}$-measurable function F, there is a bounded $\mathcal{F}_t \otimes \mathcal{F}_t$-measurable function G such that $F(\omega) = G(\omega, \theta_t \omega)$ for all $\omega \in \Omega$. Consequently,

$$P^x(F) = \int_\Omega \int_\Omega P^x(d\omega) \, P^{\omega(t)}(G(\omega, \cdot))$$

and

$$Q^x(F) = \int_\Omega \int_\Omega Q^x(d\omega) \, Q^{\omega(t)}(G(\omega, \cdot))$$

so the equivalence of P^x and Q^x on $\mathcal{F}_{2t}$ follows from their equivalence on $\mathcal{F}_t$, as desired. The dual assertion can be proved in the same way once we notice that $\hat{Q}_2^{x,x} = \hat{P}_2^{x,x}$ for all $x \in E$, because of Corollary (2.6). ∎

An important consequence of the equivalence just proved is that X and Y have the same fine topologies, as do their space-time processes. Of course, the same can be said of $\hat{X}$ and $\hat{Y}$.

Proof of (1.10)(b). The argument is broken into several steps.

Step 1: $m^X \sim m^Y$. Indeed, because the transition densities are strictly positive and finite by hypothesis, m^X is equivalent to the $P_2^{x,x}$-distribution of X_1, while m^Y is equivalent to the $Q_2^{x,x}$-distribution of Y_1 (for any fixed $x \in E$).

Step 2. For each $(x,t) \in E \times]0,2[$, $Q_t^{x,y} = P_t^{x,y}$ *for* m^X*-a.e.* $y \in E$. Fix $(x,t) \in E \times]0,2[$. Then by (2.6) and (2.7), the $P_2^{x,x}$-conditional distribution of $(X_s)_{0\le s<t}$, given $X_{t-} = y$, is $P_t^{x,y}$ (for m^X-a.e. $y \in E$). Similarly, the $Q_2^{x,x}$-conditional distribution of $(Y_s)_{0\le s<t}$, given $Y_{t-} = y$, is $Q_t^{x,y}$ (for m^Y-a.e. $y \in E$). The assertion therefore follows from the basic hypothesis ($Q_2^{x,x} = P_2^{x,x}$, $\forall x$) because of Step 1.

Step 3. There exists $b \in E$ *such that* $Q_t^{x,b} = P_t^{x,b}$ *for all* $x \in E$ *and all* $t \in]0,2[$. By Step 2 and Fubini's theorem there exists $b \in E$ such that $P_t^{x,b} = Q_t^{x,b}$ for $m^X \otimes \text{Leb}$-a.e. $(x,t) \in E \times]0,2[$. Let $\mathcal{I}$ denote the class of processes I of the form

$$I_t := \prod_{i=1}^n \int_0^t f_i(X_s, t-s)\, ds, \qquad t \ge 0,$$

where $n \in \mathbf{N}$ and each f_i is a bounded real-valued Borel function on $E \times [0,\infty[$. It is easy to see that for each fixed $t > 0$, the family $\{I_t : I \in \mathcal{I}\}$ is measure-determining on $(\Omega, \mathcal{F}_{t-})$. Therefore, it suffices to show that

$$P_t^{x,b}(I_t) = Q_t^{x,b}(I_t) \tag{3.1}$$

for all $x \in E$, $t \in]0,2[$, and $I \in \mathcal{I}$. But by Lemma (2.10) and the remark made following the proof of (1.10)(a), the two sides of (3.1) are finely-continuous (with respect to the space-time processes $(X_t, r-t)_{t\ge 0}$ and $(Y_t, r-t)_{t\ge 0}$) on all of $E \times]0,\infty[$, as functions of (x,t). By the choice of b these functions agree $m^X \otimes$Leb-a.e. on the (space-time) finely open set $E \times]0,2[$; consequently, they agree everywhere on $E \times]0,2[$, because $m^X \otimes \text{Leb}$ is a reference measure for the space-time processes.

Step 4. In view of Step 3 there exists $b \in E$ such that $P_1^{x,b} = Q_1^{x,b}$ for all $x \in E$. This b will remain fixed in the following discussion. Recall from (1.10)(a) that the laws P^x and Q^x are (locally) mutually absolutely continuous for each $x \in E$. Let Z_t denote the Radon-Nikodym derivative $dP^x|_{\mathcal{F}_{t+}}/dQ^x|_{\mathcal{F}_{t+}}$. Then Z is a strictly positive right-continuous martingale and a multiplicative functional of X; see, for example, [**K76**; Thm. 5.1]. The term *multiplicative* refers to the identity

$$Z_{t+s} = Z_t \cdot Z_s \circ \theta_t, \qquad P^x\text{-a.s.}, \forall\, x \in E, \forall\, s,t \ge 0.$$

Using (2.3) we see that for any $x \in E$,

$$\begin{aligned} P_1^{x,b}(F) = Q_1^{x,b}(F) &= Q^x\left(F\, \frac{q_{1-s}(X_s,b)}{q_1(x,b)}\right) \\ &= P^x\left(F \cdot Z_s\, \frac{q_{1-s}(X_s,b)}{q_1(x,b)}\right) \\ &= P_1^{x,b}\left(F \cdot Z_s\, \frac{q_{1-s}(X_s,b)}{q_1(x,b)}\, \frac{p_1(x,b)}{p_{1-s}(X_s,b)}\right) \end{aligned}$$

for any $F \in \mathcal{F}_{s+}$, provided $0 < s < 1$. Since Z_s is $\mathcal{F}_{s+}$ measurable, it follows that

$$Z_s = \frac{p_{1-s}(X_s, b)}{q_{1-s}(X_s, b)} \frac{q_1(x, b)}{p_1(x, b)} \qquad P_1^{x,b}\text{-a.s.} \tag{3.2}$$

for all $x \in E$ and $0 < s < 1$. Since P^x and $P_1^{x,b}$ are equivalent on $\mathcal{F}_{s+}$ for $0 < s < 1$, we see that

$$Z_s = \varphi_s(X_0, X_s) \qquad P^x\text{-a.s.}, \forall\, s \in]0,1[, \forall x \in E,$$

where

$$\varphi_s(x, z) := \frac{\psi_s(z)}{\psi_0(x)}$$

and

$$\psi_s(z) := \frac{p_{1-s}(z, b)}{q_{1-s}(z, b)}.$$

The function $(z, s) \mapsto 1_{[0,1[}(s) p_{1-s}(z, b)$ is an excessive function of the *forward* space-time process $(X_t, t + r)_{t \geq 0}$ restricted to $E \times [0, \infty[$; it is therefore space-time finely continuous on $E \times [0, 1[$. In the same way $(z, s) \mapsto q_{1-s}(z, b)$ is finely continuous on $E \times [0, 1[$ with respect to the space-time process $(Y_t, t+r)_{t \geq 0}$. But the fine topology of the latter process is the same as that of $(X, r + t)_{t \geq 0}$ because of the mutual absolute continuity ($P^x|_{\mathcal{F}_t} \sim Q^x|_{\mathcal{F}_t}$, $\forall (x, t)$) already established. It follows that $(z, s) \mapsto \psi_s(z)$ is space-time finely continuous on $E \times [0, 1[$. Now from the multiplicativity of Z and the strict positivity of the transition densities of X we deduce that for all $x \in E$ and all $t, s > 0$ such that $t + s < 1$, there is an $m^X \otimes m^X$-null set $N(x, t, s) \subset E \times E$ such that

$$\varphi_{t+s}(x, y) = \varphi_t(x, z) \cdot \varphi_s(z, y) \tag{3.3}$$

provided $(y, z) \notin N(x, t, s)$. By the preceding discussion, the two sides of (3.3) are space-time finely continuous as functions of (y, s). Moreover, $m^X \otimes \text{Leb}$ is a reference measure for $(X_t, r + t)$; thus, two space-time finely continuous functions equal $m^X \otimes \text{Leb}$-a.e. must be identical. From this observation and Fubini's theorem it follows that given $(x, t) \in E \times]0, 1[$ there is an m^X-null set $N(x, t)$ such that (3.3) holds for all $(y, s) \in E \times [0, 1 - t[$ and all $z \notin N(x, t)$. Taking $s = 0$ we find that

$$\frac{\psi_0(y)}{\psi_t(y)} = \frac{\psi_0(z)}{\psi_t(z)} \tag{3.4}$$

for all $y \in E$, $0 < t < 1$, and $z \notin N(x, t)$. Thus, defining $\lambda_t := -\log[\psi_t(b)/\psi_0(b)]$ and $\psi := \psi_0$, we have, for each $x \in E$,

$$Z_t = e^{-\lambda_t} \frac{\psi(X_t)}{\psi(X_0)}, \qquad P^x\text{-a.s.}, \tag{3.5}$$

for all $t \in]0, 1[$, since $P^x(X_t \in N) = 0$ for any m^X-null set N. The multiplicativity of Z implies first that $\lambda_t = \lambda t$ for some real constant λ, and then that (3.5) holds for all $t > 0$. This yields (1.13), from which (1.11) follows immediately because Z is a P^x-martingale.

The dual assertions (1.12) and (1.14) are proved in the same way, and the fact that ψ and $\hat{\psi}$ correspond to the same "eigenvalue" λ follows easily from (1.5).

Turning to the final assertion, let ρ denote a strictly positive and finite version of the Radon-Nikodym derivative dm^Y/dm^X—the equivalence of m^X and m^Y follows immediately from (1.13). Using (1.5) (for X and for Y) one can check that $P_t(\rho/\psi\hat{\psi}) = \rho/\psi\hat{\psi}$ and $P_t(\psi\hat{\psi}/\rho) = \psi\hat{\psi}/\rho$, m^X-a.e. Consequently,

$$1 = P_t 1 = P_t\Big((\rho/\psi\hat{\psi})^{1/2}(\psi\hat{\psi}/\rho)^{1/2}\Big) \le \Big(P_t(\rho/\psi\hat{\psi})\, P_t(\psi\hat{\psi}/\rho)\Big)^{1/2} = 1,$$

which forces $\rho = \psi\hat{\psi}$, m^X-a.e, as claimed. $\square$

Proof of (1.10)(c). Formula (1.13) implies that for each $x \in E$ and $t > 0$,

$$q_t(x,y) = e^{-\lambda t}\frac{1}{\psi(x)\hat{\psi}(y)} p_t(x,y), \qquad m^X\text{-a.e. } y \in E, \tag{3.6}$$

because $\psi\hat{\psi} = dm^Y/dm^X$. For fixed x the two sides of (3.6) are finely continuous (as functions of $(y,t) \in E\times]0,\infty[$) with respect to the backward space-time process $(\hat{X}_t, r-t)_{t\ge 0}$. (As before, the equivalence of laws established in (1.10)(a) implies that $(\hat{X}_t, r-t)$ and $(\hat{Y}_t, r-t)$ have the same fine topologies.) Since $m^X \otimes \mathrm{Leb}$ is a reference measure for this space-time process, the equality in (3.6) holds for *all* $(y,t) \in E\times]0,\infty[$. The asserted equality of bridges now follows from (1.13) and (2.3). $\square$

4. Proof of (1.16)

We first show that $P^{x,y_0}_{t_1} = Q^{x,y_0}_{t_1}$ for all $x \in E$, where $t_1 := t_0/2$. To this end fix $x \in E$, let d be a metric on E compatible with its topology, and let $B(\delta)$ denote the d-ball of radius δ centered at x. Let F be a bounded $\mathcal{F}_{t_1-}$-measurable function of the form $G\circ\theta_s$, where $0 < s < t_1$ and $G \in \mathcal{F}_{(t_1-s)-}$. By Corollary (2.7),

$$\begin{aligned} &P^{x_0,y_0}_{t_0}(F\circ\theta_{t_1}\,|X_{t_1} \in B(\delta)) \\ &\qquad = \int_{B(\delta)} P^{z,y_0}_{t_1}(F)\, P^{x_0,y_0}_{t_0}(X_{t_1} \in dz)/P^{x_0,y_0}_{t_0}(X_{t_1} \in B(\delta)). \end{aligned} \tag{4.1}$$

(Notice that $P^{x_0,y_0}_{t_0}(X_{t_1} \in B(\delta)) > 0$ because of the strict positivity of the transition density function of X.) By Lemma (2.8), the mapping $z \mapsto P^{z,y_0}_{t_1}(F)$ is continuous. Since the probability measure

$$dz \mapsto 1_{B(\delta)}(z)\, P^{x_0,y_0}_{t_0}(X_{t_1} \in dz)/P^{x_0,y_0}_{t_0}(X_{t_1} \in B(\delta))$$

converges weakly to the unit mass at x as $\delta \to 0$, it follows from (3.7) that

$$P^{x,y_0}_{t_1}(F) = \lim_{\delta\to 0} P^{x_0,y_0}_{t_0}(F\circ\theta_{t_1}\,|\,X_{t_1} \in B(\delta)). \tag{4.2}$$

By hypothesis, the right side of (4.2) is unchanged if $P^{x_0,y_0}_{t_0}$ is replaced by $Q^{x_0,y_0}_{t_0}$; the same is therefore true of the left side, so $P^{x,y_0}_{t_1}(F) = Q^{x,y_0}_{t_1}(F)$. The monotone class theorem clinches the matter.

The arguments used in the proof of Theorem (1.10) (especially Step 4 of the proof of (1.10)(b)) can now be used to finish the proof. The dual assertion follows in the same way. $\square$

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