DOI: 10.1214/EJP.v3-30

Electronic Journal of Probability

Vol. 3 (1998) Paper no. 8, pages 1–15.

Journal URL
http://www.math.washington.edu/~ejpecp/
Paper URL
http://www.math.washington.edu/~ejpecp/EjpVol3/paper8.abs.html

MEASURE ATTRACTORS FOR STOCHASTIC NAVIER–STOKES EQUATIONS

Marek Capiński

Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland
E-mail: capinski@im.uj.edu.pl

Nigel J. Cutland

Department of Mathematics, University of Hull, Hull, HU6 7RX, England
E-mail: n.j.cutland@maths.hull.ac.uk

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AMS subject classification: Primary: 35Q30, 60H15, 60G60; Secondary: 35R60, 76D05, 60J25

The research of the first author was supported by an EPSRC Visiting Fellowship at the University of Hull and also partially by the KBN grant 2PO3A 064 08.

Submitted to EJP on 15 May, 1997. Final version accepted on May 20, 1998.

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MEASURE ATTRACTORS FOR STOCHASTIC NAVIER–STOKES EQUATIONS

MAREK CAPIŃSKI AND NIGEL J. CUTLAND

ABSTRACT. We show existence of measure attractors for 2-D stochastic Navier-Stokes equations with general multiplicative noise.

1. Introduction

This paper is concerned with existence of attractors in connection with *stochastic* Navier-Stokes equations in dimension 2. For *deterministic* Navier-Stokes equations, the existence of a global attractor in dimension 2 goes back to the work of Ladyzhenskaya [9] and Foias & Temam; for a full exposition see Chapter III (sec. 2) of Temam's book [15].

The new difficulties encountered when seeking attractors for the *stochastic* equations are twofold. First there is a problem with the very definition of attractors for stochastic equations - see the discussion below. Second, for the stochastic Navier–Stokes equations there is the issue of existence of solutions to the equations themselves. Whereas the deterministic equations were solved by Leray in 1933-4 (see [14] for a modern exposition), solutions for stochastic equations with a general form of noise were first constructed in 1991 [3] some eighteen years after the first results in this direction [2], which considered additive noise only.

The notion of attractor is concerned with the asymptotic behaviour of trajectories of semigroups of operators. Recall that a semigroup on a topological space X is a one parameter family $(S(t))_{t\geq 0}$ of operators with $S(t) : X \to X$, satisfying

1) $S(0) = \mathrm{id}_X$,
2) $S(t+s) = S(t) \circ S(s)$.

Date: August 20, 1998.

1991 *Mathematics Subject Classification.* Primary: 35Q30, 60H15, 60G60; Secondary: 35R60, 76D05, 60J25.

Key words and phrases. Stochastic Navier–Stokes equations, measure attractors.

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A set $A \subseteq X$ is an *attractor* for S if

1) it is invariant, that is, $S(t)A = A$, for all t.
2) there is an open neighbourhood U of A, called the *basin* of attraction, such that for all $x \in U$, $S(t)x \to A$ (in the sense that for each open neighbourhood V of A, $S(t)x \in V$ for t sufficiently large).

If 2) holds with $U = X$, then A is a *global attractor*. Condition 1) is trivially fulfilled for $A = \emptyset$, while 2) is for $A = X$. Of course, these conditions do not guarantee uniqueness.

Variations of this definition are possible and occur in the literature — for example 2) can be replaced by the stronger condition that A attracts sets from a certain class $\mathcal{B}$, i.e. for all $B \in \mathcal{B}$, $S(t)B \subseteq V$ for sufficiently large t. An example of such a class is where $\mathcal{B}$ is the family of all bounded sets (assuming now that X is metric).

For evolution equations the semigroup $S(t)$ is defined on the phase space by: $S(t)x$ is the solution to the equation in question with the initial value $x \in X$. To get the semigroup property it is necessary that the equation be homogeneous, that is, the coefficients be independent of time. This is not the case for stochastic equations. Moreover, the paths of solutions to stochastic differential equations are driven by a Wiener process so that there is little hope that they should get attracted by some set.

The above difficulties can be tackled in various ways. In [6], for example, the notion of semigroup is generalized to that of a *cocycle.* The attraction property is formulated in a special way, by introducing *stochastic attractors* as random sets that are stationary and attract at finite time the trajectories started at $-\infty$, where the equations are suitably extended to the whole real line. For this approach to make sense, however, one needs a stochastic flow. For Navier-Stokes equations stochastic flows are proven to exist only for some particular cases of noise: additive noise and special linear multiplicative noise. Thus this approach is not available for general stochastic Navier–Stokes equations. For further details, see [6] where the notion of stochastic attractor was introduced and existence proved for the special cases mentioned.

An approach that is appropriate for general stochastic Navier-Stokes equations

$$\mathrm{d}u = (\nu\Delta u - \langle u, \nabla\rangle u + f(u))\mathrm{d}t + g(u)\mathrm{d}w_t \tag{1.1}$$

is to study time evolution of the initial measure, that is, the probability distribution of the initial value. Such a time evolving measure was called a *statistical solution* to the Navier–Stokes equations by Foias, who introduced this idea in [8] for the deterministic equations. In dimension 2 this evolution of measures is defined in a natural way by the solutions to the equation, and gives a semigroup on the space $\mathcal{M}(\mathbf{H})$ of probability measures defined on the underlying Hilbert space. To transport the initial measures along the solutions to (1.1) we have of course to restrict ourselves to the 2-dimensional case since only then do we have global uniqueness theorems.

Attractors for the Navier–Stokes equations in this setting were first studied by B.Schmalfuß [11], [12] (see also [10], which deals with a general class of equations that does not include the Navier–Stokes equations). The appropriate notion is that of a *measure attractor* for statistical solutions, which will be a subset of the space of measures $\mathcal{M}(\mathbf{H})$ (which we equip with the topology of weak convergence). Our phase space will be a naturally defined subset of $\mathcal{M}(\mathbf{H})$.

The paper [11] studies the case of periodic boundary conditions with additive noise. In the sequel [12], Schmalfuß considers Navier–Stokes equations under less general assumptions than considered here, and imposes strong and somewhat artificial restrictions on the class of measures that can be considered. See also the paper [13], which constructs a measure attractor from a pathwise stochastic attractor; necessarily this works only for the special kinds of noise in the Navier–Stokes equation for which a pathwise stochastic attractor exists.

In this paper we work initially with the most general assumptions for which there is currently known to exist a solution to the stochastic Navier–Stokes equation. Drawing on the uniqueness property we transport the initial measure along the trajectories of the solution, to give the semigroup $S(t)$.

To construct the measure attractor we need to impose some additional technical conditions on the growth of the feedback in the forces $f(u)$ and the noise $g(u)$. Techniques from nonstandard analysis are used to prove a crucial continuity property (Lemma 4.2) that is used to show that the attractor is compact. This allows us to relax the conditions on the coefficients f, g and consider a considerably broader space of measures as compared with [12], where it is assumed that f is constant and that $g(u)$ is essentially bounded.

2. Preliminaries

We introduce the following commonly used notation.

Denote by D a bounded domain in $\mathbb{R}^2$; write $\mathbf{H}$ for the closure of the set $\{u \in C_0^\infty(D, \mathbb{R}^2)\colon \operatorname{div} u = 0\}$ in the L^2 norm $|u| = (u,u)^{1/2}$, where

$$(u,v) = \sum_{j=1}^{2} \int_D u^j(x) v^j(x) \mathrm{d}x.$$

The letters u, v, w will be used for elements of $\mathbf{H}$. The space $\mathbf{V}$ is the closure of $\{u \in C_0^\infty(D, \mathbb{R}^2)\colon \operatorname{div} u = 0\}$ in the norm $|u| + \|u\|$ where $\|u\| = ((u,u))^{1/2}$ and

$$((u,v)) = \sum_{j=1}^{2} (\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}).$$

$\mathbf{H}$ and $\mathbf{V}$ are Hilbert spaces with scalar products $(\cdot,\cdot)$ and $((\cdot,\cdot))$ respectively, and $|\cdot| \le c\|\cdot\|$ for some constant c.

By A we denote the Stokes operator in $\mathbf{H}$ and by $\{e_k\}$ an orthonormal basis of its eigenfunctions with the corresponding eigenvalues λ_k, $\lambda_k \ge 0$, $\lambda_k \nearrow \infty$; for $u \in \mathbf{H}$ we write $u_k = (u, e_k)$. Let $\mathbf{H}_\alpha$ be the subspace of $\mathbf{H}$ of points where $|u|_\alpha^2 = \sum_{k=1}^\infty \lambda_k^\alpha u_k^2 < \infty$, $\alpha \ge 0$, and then $\mathbf{H}_{-\alpha}$ is the dual to $\mathbf{H}_\alpha$. Note that $\mathbf{V} = \mathbf{H}_1$, $\mathbf{V}' = \mathbf{H}_{-1}$.

We put

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_D u^j(x) \frac{\partial v^i}{\partial x_j}(x) w^i(x) dx = (\langle u, \nabla \rangle v, w)$$

whenever the integrals make sense. We recall some well-known properties of the trilinear form b, which we will need:

$$b(u,v,w) = -b(u,w,v) \tag{2.1}$$

$$|b(u,v,w)| \le c\,\|u\|^{1/2}|u|^{1/2}\|v\|^{1/2}|v|^{1/2}\|w\| \tag{2.2}$$

We will need the following stochastic Gronwall lemma (proved in fact in [12]).

Lemma 2.1. *If h, y, g, f are real-valued stochastic processes satisfying*

$$\mathrm{d}y(t) = h(t)y(t)\mathrm{d}t + g(t)\mathrm{d}w_t - f(t)\mathrm{d}t \tag{2.3}$$

and $f \ge 0$, then

$$\mathbb{E}\Big(\exp\{-\int_0^t h(s)ds\} \cdot y(t)\Big) \le \mathbb{E}\, y(0).$$

Proof. Write (2.3) in the form

$$y(t) = y(0) + \int_0^t h(s)y(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}w_s - \int_0^t f(s)\mathrm{d}s.$$

Put $z(t) = \exp\{-\int_0^t h(s)ds\}$. By Ito's formula we have

$$\mathrm{d}(yz) = -zf\mathrm{d}t + zg\mathrm{d}w$$

and after taking the mathematical expectation, the integral form of the resulting equality gives the result. □

Note that if h is deterministic, then we have

$$\mathbb{E}\, y(t) \leq \exp\{\int_0^t h(s)\mathrm{d}s\}\mathbb{E}\, y(0)$$

which easily follows from

$$\mathbb{E}\, y(t) \leq \mathbb{E}\, y(0) + \int_0^t h(s)\mathbb{E}\, y(s)\mathrm{d}s$$

using the classical Gronwall lemma.

3. Solutions of stochastic Navier-Stokes equations

Let w_t be Wiener process in $\mathbf{H}$ with trace class covariance operator Q.

Definition 3.1. A weak solution of the stochastic Navier–Stokes equations with the initial function u is a stochastic process $v(t, \omega; u)$ satisfying

(i) $v(t) \in L^\infty(0, T; \mathbf{H}) \cap L^2(0, T; \mathbf{V}) \cap C(0, T; \mathbf{H}_{\mathrm{weak}})$ a.s. for all T.
(ii) for all t

$$v(t) = u + \int_0^t [\nu A v(s) - B(v(s)) + f(v(s))]\mathrm{d}s + \int_0^t g(v(s))\mathrm{d}w_s$$

holds as an identity in $\mathbf{V}'$.

We formulate some conditions on the coefficients of the equation. They are needed to obtain existence, uniqueness and regularity of solutions.

C1. $|f(u) - f(v)|_{-1}^2 + |g(u) - g(v)|_{\mathbf{H},\mathbf{H}}^2 \leq c_1|u - v|^2$.
C2. For some $\alpha \in (0, 1]$, $|f(u)|_{\alpha-1} + |g(u)|_{\mathbf{H},\mathbf{H}_\alpha} \leq c_2(1 + |u|_\alpha)$.

Theorem 3.2. **(i)** *If* C1 *holds, then there exists a unique weak solution satisfying*

$$\mathbb{E}\left(\sup_{s\in[0,t]} |v(s)|^2 + \nu \int_0^t \|v(s)\|^2 \mathrm{d}s\right) \leq c(t)(1 + |v(0)|^2) \tag{3.1}$$

(ii) *If, additionally,* C2 *holds, then for* α *given by* C2 *the solution satisfies*

$$\mathbb{E}\left(\sup_{t_0\leq t\leq T} \xi(t)|v(t)|_\alpha^2\right) \leq c(T)(1 + \mathbb{E}|v(t_0)|_\alpha^2) \tag{3.2}$$

for all $t_0 > 0$*, where* $\xi(t) = \exp\{-c\int_0^t \|v(s)\|^2\mathrm{d}s\}$ *(which is positive a.e., by* (3.1)*). Hence then* $|v(t)|_\alpha < \infty$ *a.e. for all* $t > 0$*.*

For the proof see [4], where (i) is proved (in fact *existence* is shown for dimension $n \leq 4$, but uniqueness is only known for dimension 2); (ii) requires a slight modification of the proof of Theorem 6.5.4 of [4]. For the final part, use (3.1) to find $t_0 \leq t$ with $\mathbb{E}\|v(t_0)\|^2 < \infty$ and then apply (3.2). In [12] existence is proved for $n = 2$ under equivalent conditions.

4. Construction of the semigroup

Let $\mathcal{M}(\mathbf{H})$ be the set of all probability Borel measures on $\mathbf{H}$. We equip the set $\mathcal{M}(\mathbf{H})$ with two topologies $\mathcal{T}_1, \mathcal{T}_2$ denoting the resulting spaces by

$$\mathcal{M}_1 = (\mathcal{M}(\mathbf{H}), \mathcal{T}_1)$$

and

$$\mathcal{M}_2 = (\mathcal{M}(\mathbf{H}), \mathcal{T}_2).$$

We say that

$$\mu_n \to \mu \text{ in } \mathcal{M}_1 \text{ if } \int \vartheta(u) \mathrm{d}\mu_n \to \int \vartheta(u) \mathrm{d}\mu \text{ for } \vartheta \in C_{\mathrm{wb}}(\mathbf{H}, \mathbb{R}),$$

$$\mu_n \to \mu \text{ in } \mathcal{M}_2 \text{ if } \int \vartheta(u) \mathrm{d}\mu_n \to \int \vartheta(u) \mathrm{d}\mu \text{ for } \vartheta \in C_{\mathrm{b}}(\mathbf{H}, \mathbb{R}),$$

where $C_{\mathrm{wb}}(\mathbf{H}, \mathbb{R})$ denotes the set of all bounded weakly continuous functions on $\mathbf{H}$, and $C_{\mathrm{b}}(\mathbf{H}, \mathbb{R})$ the set of bounded continuous functions. Thus $\mathcal{T}_i$ is the conventional weak topology on $\mathcal{M}(\mathbf{H})$ regarded as Borel measures on the space $\mathbf{H}$ with the topology τ_i, where τ_1 is the weak topology on $\mathbf{H}$ and τ_2 is the strong (norm) topology on $\mathbf{H}$. Each $(\mathbf{H}, \tau_i)$ is completely regular, and the Borel sets for these two topologies on $\mathbf{H}$ coincide. Since τ_2 is a metric topology, the measures in $\mathcal{M}(\mathbf{H})$ are all Radon with respect to τ_2, and hence also with respect to τ_1. Note that $\mathcal{M}_2$ is metrizable, whereas $\mathcal{M}_1$ is not; and the topology of $\mathcal{M}_2$ is stronger than that of $\mathcal{M}_1$. Both spaces are completely regular.

Our phase space consists of measures from $\mathcal{M}(\mathbf{H})$ with finite second moment:

$$X = \{\mu \in \mathcal{M}(\mathbf{H}) : \int |u|^2 \mathrm{d}\mu(u) < \infty\}.$$

Write

$$\mathcal{B}^r = \{\mu \in X : \int |u|^2 \mathrm{d}\mu(u) \leq r\}.$$

The following is noted for future use.

Lemma 4.1. *$\mathcal{B}^r$ is compact in $\mathcal{M}_1$.*

Proof. For $\mu \in \mathcal{B}^r$, by Chebyshev, we have $\mu(B(d)) \leq 1 - r/d^2$, where $B(d)$ is the closed ball in $\mathbf{H}$ of radius d, which is weakly compact. This shows that $\mathcal{B}^r$ is tight (as a family of measures on the space $(\mathbf{H}, \tau_1)$) and is thus compact in $\mathcal{M}_1$ by the Prohorov-Varadarajan theorem (see [16]).

Alternatively, this can be seen directly by the following Loeb space argument. Let $M \in {}^*\mathcal{B}^r$ and let M_L denote the corresponding Loeb measure on ${}^*\mathbf{H}$. Then M_L-a.a. $U \in {}^*\mathbf{H}$ has $|U|$ finite and so U is weakly nearstandard. Thus a Borel probability measure μ is defined on $\mathbf{H}$ by

$$\mu(B) = M_L(\text{w-st}^{-1}(B))$$

where w-st denotes the weak standard part mapping.

Clearly $\mu \in \mathcal{B}^r$.

For $\vartheta \in C_{\mathrm{wb}}(\mathbf{H}, \mathbb{R})$ we have

$$\begin{aligned}
\int_{{}^*\mathbf{H}} {}^*\vartheta(U)\mathrm{d}M(U) &\approx \int_{{}^*\mathbf{H}} {}^\circ{}^*\vartheta(U)\mathrm{d}M_L(U) \quad \text{by Loeb theory} \\
&= \int_{{}^*\mathbf{H}} \vartheta({}^\circ U)\mathrm{d}M_L(U) \quad \text{as } \vartheta \text{ is weakly continuous} \\
&= \int_{\mathbf{H}} \vartheta(u)\mathrm{d}\mu(u) \quad \text{by definition of } \mu.
\end{aligned}$$

Thus $M \approx \mu \in \mathcal{B}^r$ in $\mathcal{M}_1$ and so by the nonstandard criterion for compactness $\mathcal{B}^r$ is compact. □

The measure $\mu_t = S(t)\mu$ is defined by

$$\int_{\mathbf{H}} \vartheta(u)\mathrm{d}\mu_t(u) = \int_{\mathbf{H}} \mathbb{E}\,\vartheta(v(t,u))\mathrm{d}\mu(u)$$

for continuous and bounded ϑ, where $v(t,u)$ is the solution to the stochastic Navier–Stokes equation with the initial condition u. Thus in general $S(t) : \mathcal{M}(\mathbf{H}) \to \mathcal{M}(\mathbf{H})$. However due to the energy inequality, $S(t) : X \to X$. To see this take $\vartheta_n = |u|^2 \wedge n$, note that by (3.1) we have $\int \vartheta_n(u)\mathrm{d}\mu_t(u) \leq c(t)(1 + \int |u|^2 \mathrm{d}\mu_0)$, and let $n \to \infty$.

We shall need the following lemma

Lemma 4.2. *Assume* C1*, and fix* $t \geq 0$*. Then*

- **(i)** *for any continuous and bounded* $\vartheta : \mathbf{H} \to \mathbb{R}$*, the mapping* $u \mapsto \mathbb{E}\,\vartheta(v(t,u))$ *is continuous in* $\mathbf{H}$*;*
- **(ii)** *for any bounded weakly continuous* $\theta : \mathbf{H} \to \mathbb{R}$*, the mapping* $u \mapsto \mathbb{E}\,\theta(v(t,u))$ *is weakly continuous on bounded sets in* $\mathbf{H}$*;*
- **(iii)** *if* C2 *holds, then for any bounded continuous* $\vartheta : \mathbf{H} \to \mathbb{R}$*, the mapping* $u \mapsto \mathbb{E}\,\vartheta(v(t,u))$ *is weakly continuous on bounded sets in* $\mathbf{H}$.

Proof. **(i)** Suppose that $u_n \to u$ in $\mathbf{H}$. Let $y_n(t) = v(t, u_n) - v(t, u)$. Arguing as in the proof of the energy inequality (3.1) we get

$$|y_n(t)|^2 \leq |u_n - u|^2 + \int_0^t (c + \|v(s,u)\|^2)|y_n(s)|^2 \mathrm{d}s + \int_0^t \psi(s)\mathrm{d}w_s$$

with suitable ψ. The stochastic Gronwall lemma gives

$$\mathbb{E}\left(\exp\{-\int_0^t \|v(s,u)\|^2 \mathrm{d}s\}|y_n(t)|^2 \right) \leq |u_n - u|^2.$$

Since $\int_0^t \|v(s,u)\|^2 \mathrm{d}s$ is finite almost surely, $|y_n(t)|$ converges to zero a.s, all t. Hence $\vartheta(v(t,u_n)) \to \vartheta(v(t,u))$ a.s. and since this convergence is dominated we have the result.

(ii) For this it is convenient to give a simple nonstandard proof utilising the machinery developed in [4] for constructing solutions.

Write $\varphi(u) = \mathbb{E}(\theta(v(t,u))$. Working on the bounded (in $\mathbf{H}$) set $B(r) = \{u \in \mathbf{H} : |u| \leq r\}$, let $U \in {}^*\mathbf{H}$ with $|U| \leq r$ and let $u = {}^\circ U$ (this symbol denotes the weak standard part in $\mathbf{H}$), so that $U \approx_{\text{weak}} u$. We have to show that ${}^*\varphi(U) \approx \varphi(u)$.

Writing $V(\tau) = {}^*v(\tau, U)$ we have an internal solution to stochastic Navier–Stokes equations, living on an internal (i.e. nonstandard) probability space $\mathbf{\Omega}$. The proof of Theorem 6.4.1 of [4], with ${}^*\mathbf{H}$ in place of $\mathbf{H}_N$, shows that the process v defined by $v({}^\circ\tau) = {}^\circ V(\tau)$ a.s. is a weak solution to the stochastic Navier–Stokes equation, living on the Loeb space constructed from $\mathbf{\Omega}$, with $v(0) = u$. Note that the standard part on $V(\tau)$ is the weak standard part, so that in particular $v(t) \approx_{\text{weak}} V(t)$ a.s. The uniqueness of solutions means that $\varphi(u) = \mathbb{E}\theta(v(t))$, where the expectation is with respect to the Loeb measure on $\mathbf{\Omega}$. Then we have

$$\begin{aligned} \varphi(u) &= \mathbb{E}\,\theta(v(t)) \\ &= \mathbb{E}\,{}^\circ\left({}^*\theta(V(t))\right) \quad \text{since } \theta \text{ is weakly continuous} \\ &\approx \mathbb{E}\,{}^*\theta(V(t)) \quad \text{by Loeb theory} \\ &= \mathbb{E}\,{}^*\theta({}^*v(t,U) \\ &= {}^*\varphi(U) \end{aligned}$$

as required.

(iii) This is similar to (ii), but now we have only that θ is continuous (which is weaker than weakly continuous). However, the additional condition C2 gives that $|V(t)|_\alpha$ is finite a.s. – see Theorem 3.2. Thus $V(t)$ is *strongly* nearstandard a.s., and the continuity of ϑ is now sufficient to give ${}^*\vartheta(V(t)) \approx \vartheta(v(t))$. The rest of the argument is as set out in (ii). □

Proposition 4.3. *Assume* C1. *Then $S(t)$ defines a semigroup on X. Moreover*

(i) *$S(t)$ is continuous from $\mathcal{M}_2$ to $\mathcal{M}_2$.*
(ii) *$S(t)$ is continuous from $\mathcal{M}_1$ to $\mathcal{M}_1$ on $\mathcal{B}^r$ for any $r > 0$.*
(iii) *if* C2 *holds then for $t > 0$, $S(t)$ is continuous from $\mathcal{M}_1$ to $\mathcal{M}_2$ on $\mathcal{B}^r$ for any $r > 0$.*

Proof. The semigroup property follows from the Markov property of the process $v(t)$, which can be proved in the same way as was done in [7], Theorem 9.8, for Lipschitz nonlinearities. The proof only requires uniqueness of solutions and continuous dependence on the initial condition, which hold in our case.

(i) To prove continuity consider a sequence μ_n convergent to μ in $\mathcal{M}_2$. Let $\vartheta : \mathbf{H} \to \mathbb{R}$ be continuous and bounded. Then by Lemma 4.2

$$\begin{aligned} \int \vartheta(u)\mathrm{d}[S(t)\mu_n] &= \int \mathbb{E}\,\vartheta(v(t,u))\mathrm{d}\mu_n \\ &\to \int \mathbb{E}\,\vartheta(v(t,u))\mathrm{d}\mu \\ &= \int \vartheta(u)\mathrm{d}[S(t)\mu] \end{aligned}$$

as required.

(ii) and **(iii)** are proved similarly using Lemma 4.2 **(ii)** and **(iii)** respectively. ☐

We conclude this section with some information about the compactness properties of the sets $S(t)\mathcal{B}^r$.

Proposition 4.4. **(i)** *Assume* C1. *For any $r > 0$, $t \geq 0$ the set $S(t)\mathcal{B}^r$ is compact in $\mathcal{M}_1$.*
(ii) *Assume* C1–2. *For any $r > 0$, $t > 0$, the set $S(t)\mathcal{B}^r$ is compact in $\mathcal{M}_2$.*

Proof. **(i)** The set of measures $\mathcal{B}^r$ is compact in $\mathcal{M}_1$ by Lemma 4.1, so this follows immediately from the continuity of $S(t)$ on $\mathcal{B}^r$ in $\mathcal{M}_1$. (Proposition 4.3 (ii))

(ii) This is similar to (i) using the fact that $S(t)$ is continuous on $\mathcal{B}^r$ from $\mathcal{M}_1$ to $\mathcal{M}_2$. ☐

5. Construction of the attractor

To establish the existence of a measure attractor we need some extra conditions on the coefficients f, g, as follows.

C3. $\dfrac{|f(u)|_{-1}^2}{\gamma} + \mathrm{tr}Q.|g(u)|^2_{\mathbf{H},\mathbf{H}} \le c + \delta\|u\|^2$ for some $\gamma, \delta > 0$ with $\gamma + \delta < 2\nu$.

Note that this condition is weaker than the following alternative linked conditions on f and g.

D1. $|f(u)|_{-1} \le c + \delta_1\|u\|$,
D2. $|g(u)|_{\mathbf{H},\mathbf{H}} \le c + \delta_2\|u\|$
for $\delta_1, \delta_2 > 0$ with $2\delta_1 + \delta_2^2.\mathrm{tr}Q < 2\nu$.

Proof. Square D1 and D2, and use Young's inequality on $2c\delta_1\|u\|$ and $2c\delta_2\|u\|$, to find δ_3, δ_4 with $\delta_1 < \delta_3$ and $\delta_2 < \delta_4$ and $2\delta_3 + \delta_4^2.\mathrm{tr}Q < 2\nu$ such that

$$|f(u)|_{-1}^2 \le c^2 + \delta_3^2\|u\|^2$$

and

$$|g(u)|^2_{\mathbf{H},\mathbf{H}} \le c^2 + \delta_4^2\|u\|^2$$

Now this gives C3 with $\gamma = \delta_3$ and $\delta = \delta_3 + \delta_4^2$. □

Proposition 5.1. *Assume conditions* C1 *and* C3. *Then there exists a ρ such that for each r there exists t_r such that for $t > t_r$, $S(t)\mathcal{B}^r \subset \mathcal{B}^\rho$.*

Proof. Fix $u \in \mathbf{H}$ and let $v(t) = v(t, u)$. Then taking expectations after applying Itô's formula gives

$$\mathbb{E}|v(t)|^2 + 2\nu\int_0^t \mathbb{E}\|v(s)\|^2 ds = |u|^2 + 2\int_0^t \mathbb{E}(f(v(s)), v(s))ds + \int_0^t \mathbb{E}\,\mathrm{tr}[g(v(s))Qg(v(s))^{\mathrm{T}}]ds. \tag{5.1}$$

Now

$$\mathrm{tr}[g(v(s))Qg(v(s))^{\mathrm{T}}] \le \mathrm{tr}Q.|g(v(s))|^2_{\mathbf{H},\mathbf{H}} \tag{5.2}$$

and

$$2(f(v(s)), v(s)) \le 2|f(v(s))|_{-1}\|v(s)\| \le \frac{|f(v(s))|_{-1}^2}{\gamma} + \gamma\|v(s)\|^2$$

using Young's inequality.

Substituting these two estimates in (5.1) and using condition C3 we have

$$\frac{d}{dt}\mathbb{E}|v(t)|^2 + \varepsilon\,\mathbb{E}\|v(t)\|^2 \le c \tag{5.3}$$

where $\varepsilon = 2\nu - \gamma - \delta$. Now $|v(t)|^2 \leq \lambda_1^{-1}\|v(t)\|^2$ and so

$$\frac{d}{dt}\mathbb{E}|v(t)|^2 + c_1\mathbb{E}|v(t)|^2 \leq c$$

where $c_1 = \lambda_1\varepsilon$. Then, by Gronwall's lemma we deduce

$$\mathbb{E}\,|v(t)|^2 \leq |u|^2 \exp\{-c_1 t\} + c_2(1 - \exp\{-c_1 t\})$$

where $c_2 = c/c_1$. So if the initial measure is in $\mathcal{B}^r$ then μ_t is in $\mathcal{B}^\beta$ with

$$\beta = \beta(t) = r\exp\{-c_1 t\} + c_2(1 - \exp\{-c_1 t\}).$$

So let $\rho > c_2$ and solve $\beta(t_r) = \rho$, to give

$$t_r = -\frac{1}{c_1}\log[(\rho - c_2)/(r - c_2)].$$

Clearly, if $t > t_r$, then $\beta(t) < \beta(t_r) = \rho$. □

The construction of an attractor goes along the same lines as in the deterministic case, see [15], p. 17. This is because the problem of the time evolution of the probability distribution of the process v has been cast in the general abstract framework of semigroup (deterministic) acting on a metric space.

For $i = 1, 2$ we write

$$\mathcal{A}_i = \bigcap_{\tau > 0} \mathrm{cl}_{\mathcal{M}_i}\left(\bigcup_{\sigma \geq \tau} S(\sigma)\mathcal{B}^\rho\right).$$

In other words,

$$\mathcal{A}_i = \bigcap_{\tau > 0} \mathrm{cl}_{\mathcal{M}_i}\left(\mathcal{C}_\tau\right), \qquad \mathcal{C}_\tau = \bigcup_{\sigma \geq \tau} S(\sigma)\mathcal{B}^\rho.$$

Theorem 5.2. *Assume conditions* C1,C3

- **(i)** *The set $\mathcal{A}_1$ is invariant, compact in $\mathcal{M}_1$, and attracts the bounded sets in X (i.e. the sets $\mathcal{B}^r$ for $r > 0$).*
- **(ii)** *If, in addition,* C2 *holds, the set $\mathcal{A}_2$ is invariant, compact in $\mathcal{M}_2$, and attracts the bounded sets in X.*

Proof. The proof is a modification of the classical proof given in [15], but we give it for completeness of exposition. The proofs for $i = 1, 2$ are identical, so let us fix i, and assume that the topology $\mathcal{M}_i$ is used throughout.

The sets $\mathcal{A}_i$ are compact because $\mathcal{C}_\tau$ is relatively compact in each $\mathcal{M}_i$ for $\tau > t_\rho + 1$: if $\sigma \geq \tau$, then $S(\sigma)\mathcal{B}^\rho \subset S(1)\mathcal{B}^\rho$ which is compact by Proposition 4.4.

Next we show that for each $r > 0$, for each open set U containing $\mathcal{A}$

$$S(t)\mathcal{B}^r \subseteq U$$

for t sufficiently large. If not, there are sequences $t_n \nearrow \infty$, $\mu_n \in S(t_n)\mathcal{B}^r$ such that $\mu_n \notin U$. For $t_n > t_\rho + 1$, we have $S(t_n)\mathcal{B}^r \subset S(1)\mathcal{B}^\rho$ which is compact by Proposition 4.4. Hence we can extract a subsequence, denote it by μ_n for simplicity, convergent to a measure μ, so $\mu \notin U$.

To get a contradiction it is sufficient to show that $\mu \in \mathcal{A}$. We will show that for any τ, we have $\mu_n \in \mathcal{C}_\tau = \bigcup_{\sigma \geq \tau} S(\sigma)\mathcal{B}^\rho$ for sufficiently large n. In fact we will show that $\mu_n \in S(\tau)\mathcal{B}^\rho$ for sufficiently large n. Let n be such that $t_n > \tau + t_r$, so $t_n - \tau - t_r > 0$. Then

$$\begin{aligned} \mu_n \in S(t_n)\mathcal{B}^r &= S(\tau + t_r + (t_n - \tau - t_r))\mathcal{B}^r \\ &= S(\tau)S(t_r + (t_n - \tau - t_r))\mathcal{B}^r \\ &\subset S(\tau)\mathcal{B}^\rho. \end{aligned}$$

Hence $\mu \in \overline{\mathcal{C}_\tau}$ for all τ so $\mu \in \mathcal{A}$.

It remains to show invariance. We have the following characterization: $\mu \in \mathcal{A}_i$ if and only if there is a sequence (generalized) $\mu_\alpha \in \mathcal{B}^\rho$ and $t_\alpha \to \infty$ such that $S(t_\alpha)\mu_\alpha \to \mu$ in the corresponding topology.

If $\nu \in S(t)\mathcal{A}_i$ then $\nu = S(t)\mu$, $\mu \in \mathcal{A}_i$. Using the continuity of $S(t)$ (Proposition 4.3) we then have $S(t)S(t_\alpha)\mu_\alpha = S(t+t_\alpha)\mu_\alpha \to S(t)\mu = \nu$ and so $\nu \in \mathcal{A}_i$.

Conversely, fix t and let $\mu \in \mathcal{A}_i$ and take sequences μ_α and t_α with $S(t_\alpha)\mu_\alpha \to \mu$ as above. We have to show that $\mu = S(t)\nu$ for some $\nu \in \mathcal{A}_i$. Then consider $t_\alpha > t$ and the sequence $S(t_\alpha - t)\mu_\alpha$. This is relatively compact, so, taking a subsequence if necessary, we have $S(t_\alpha - t)\mu_\alpha \to \nu$ for some ν, which is in $\mathcal{A}_i$ by the above characterisation. To complete the proof we have $S(t_\alpha)\mu_\alpha = S(t)S(t_\alpha - t)\mu_\alpha \to S(t)\nu$, again using the continuity of $S(t)$. Since $S(t_\alpha)\mu_\alpha \to \mu$ this means that $\mu = S(t)\nu$. □

Since we have made use of nonstandard techniques earlier, it is of interest to see how the attractors $\mathcal{A}_i$ may be defined and the above theorem proved taking advantage of the extra structure provided.

First define the set

$$\begin{aligned} \mathcal{D} &= \bigcup_{\tau-\text{infinite}} {}^*S(\tau)\, {}^*\mathcal{B}^\rho \\ &= \bigcup_{\tau-\text{infinite}} {}^*\mathcal{C}_\tau \\ &= \bigcap_{n \in \mathbb{N}} {}^*\mathcal{C}_n \end{aligned}$$

where the last equality is proved by a simple overflow argument. From the definition it is obvious that

$$^*S(t)\mathcal{D} = \mathcal{D} \tag{5.4}$$

for finite t.

It is clear from Proposition 5.1 that $\mathcal{D} \subseteq {}^*S(1)^*\mathcal{B}^\rho$ and so $\mathcal{D}$ consists of nearstandard points in each $\mathcal{M}_i$. Then we have

$$\begin{aligned} \mathcal{A}_i &= \mathrm{st}_{\mathcal{M}_i}(\mathcal{D} \\ &= \mathrm{st}_{\mathcal{M}_i}\Big(\bigcap_{n\in\mathbb{N}} {}^*\mathcal{C}_n\Big) \\ &= \bigcap_{n\in\mathbb{N}} \mathrm{st}_{\mathcal{M}_i}({}^*\mathcal{C}_n). \end{aligned}$$

The invariance of $\mathcal{A}_i$ now follows from (5.4) and the fact that for any finite $t > 0$

$$S(t)\mathcal{A}_i = \mathrm{st}_{\mathcal{M}_i}({}^*S(t)\mathcal{D})$$

using the continuity of $S(t)$ (Proposition 4.3).

Finally, to establish the absorbtion property, let $r > 0$, and take an open set U containing $\mathcal{A}_i$. Then for each infinite τ we have $\mathrm{st}_{\mathcal{M}_i}{}^*\mathcal{C}_\tau \subseteq U$ and so ${}^*\mathcal{C}_\tau \subseteq {}^*U$. By underflow this means that $\mathcal{C}_n \subseteq U$ for some finite n. Now take $t \geq t_r + n$ (where t_r is given by Proposition 5.1) and then we have

$$S(t)\mathcal{B}^r \subseteq S(t - t_r)\mathcal{B}^\rho \subseteq \mathcal{C}_n \subseteq U$$

as required.

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Marek Capiński: Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

E-mail address: capinski@im.uj.edu.pl

Nigel J. Cutland: Department of Pure Mathematics and Statistics, University of Hull, Hull, HU6 7RX, England

E-mail address: n.j.cutland@maths.hull.ac.uk