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LAWS OF THE ITERATED LOGARITHM FOR TRIPLE INTERSECTIONS OF THREE DIMENSIONAL RANDOM WALKS

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Abstact: Let $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ and $X'' = \{X''_n, n \geq 1\}$ be three independent copies of a symmetric random walk in Z^3 with $E(|X_1|^2 \log_+ |X_1|) < \infty$. In this paper we study the asymptotics of I_n, the number of triple intersections up to step n of the paths of X, X' and X'' as $n \to \infty$. Our main result is

$$\limsup_{n\to\infty} \frac{I_n}{\log(n)\log_3(n)} = \frac{1}{\pi|Q|} \qquad \text{a.s.}$$

where Q denotes the covariance matrix of X_1. A similar result holds for J_n, the number of points in the triple intersection of the ranges of X, X' and X'' up to step n.

Keywords: random walks, intersections

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Laws of the Iterated Logarithm for Triple Intersections of Three Dimensional Random Walks

Jay Rosen*

Abstract

Let $X = \{X_n, n \geq 1\}$, $X' = \{X'_n, n \geq 1\}$ and $X'' = \{X''_n, n \geq 1\}$ be three independent copies of a symmetric random walk in Z^3 with $E(|X_1|^2 \log_+ |X_1|) < \infty$. In this paper we study the asymptotics of I_n, the number of triple intersections up to step n of the paths of X, X' and X'' as $n \to \infty$. Our main result is

$$\limsup_{n\to\infty} \frac{I_n}{\log(n)\log_3(n)} = \frac{1}{\pi|Q|} \qquad \text{a.s.}$$

where Q denotes the covariance matrix of X_1. A similar result holds for J_n, the number of points in the triple intersection of the ranges of X, X' and X'' up to step n.

1 Introduction

Let $X = \{X_n, n \geq 1\}, X' = \{X'_n, \geq 1\}$, and $X'' = \{X''_n, n \geq 1\}$ be three independent copies of a random walk in Z^3 with zero mean and finite variance. In this paper we study the asymptotics of the number of triple intersections up to step n of the paths of X, X' and X'' as $n \to \infty$, both the number of 'intersection times'

(1.1) $$I_n = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} 1_{\{X_i = X'_j = X''_k\}}$$

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and the number of 'intersection points'

$$J_n = |X(1,n) \cap X'(1,n) \cap X''(1,n)| \tag{1.2}$$

where $X(1,n)$ denotes the range of X up to time n and $|A|$ denotes the cardinality of the set A. For random walks with finite variance, dimension three is the 'critical case' for triple intersections, since $I_n, J_n \uparrow \infty$ almost surely but three independent Brownian motions in R^3 do not intersect. This implies that in some sense $I_n, J_n \uparrow \infty$ slowly. We also note that in dimension > 3 we have $I_\infty, J_\infty < \infty$ a.s.

We assume that X_n is adapted, which means that X_n does not live on any proper subgroup of Z^3. In the terminology of Spitzer [9] X_n is aperiodic.

We have the following two limit theorems.

Theorem 1 *Assume that* $E(|X_1|^2 \log_+ |X_1|) < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{I_n}{\log(n)\log_3(n)} = \frac{1}{\pi|Q|} \qquad \text{a.s.} \tag{1.3}$$

where Q *denotes the covariance matrix of* X_1.

As usual, $\log_j$ denotes the j-fold iterated logarithm.

In the particular case of the simple random walk on Z^3, where $Q = \frac{1}{3}I$, Theorem 1 states that

$$\limsup_{n\to\infty} \frac{I_n}{\log(n)\log_3(n)} = \frac{27}{\pi} \qquad \text{a.s.} \tag{1.4}$$

A similar result holds for J_n:

Theorem 2 *Assume that* $E(|X_1|^2 \log_+ |X_1|) < \infty$. *Then*

$$\limsup_{n\to\infty} \frac{J_n}{\log(n)\log_3(n)} = \frac{q^3}{\pi|Q|} \qquad \text{a.s.} \tag{1.5}$$

where q *denotes the probability that* X *will never return to its initial point.*

Le Gall [4] proved that $(\log n)^{-1} J_n$ converges in distribution to a gamma random variable. This paper is an outgrowth of my paper with Michael Marcus [6] in which we prove analogous laws of the iterated logarithm for intersections of two symmetric random walks in Z^4 with finite third moment. In this paper we also use some of the ideas of [4] along with techniques developed in [8, 7].

2 Proof of Theorem 1

We use $p_n(x)$ to denote the transition function for X_n. Recall

$$
\begin{aligned}
I_n &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} 1_{\{X_i=X'_j=X''_k\}} \\
&= \sum_{x\in\mathbf{Z}^3}\left\{\left(\sum_{i=1}^{n} 1_{\{X_i=x\}}\right)\left(\sum_{j=1}^{n} 1_{\{X'_j=x\}}\right)\left(\sum_{k=1}^{n} 1_{\{X''_k=x\}}\right)\right\} \qquad (2.1)
\end{aligned}
$$

We set

$$
h(n)=E(I_n)=\sum_{x\in\mathbf{Z}^3}\left\{\left(\sum_{i=1}^{n} p_i(x)\right)\left(\sum_{j=1}^{n} p_j(x)\right)\left(\sum_{k=1}^{n} p_k(x)\right)\right\}. \qquad (2.2)
$$

With

$$
u_t(x)=\sum_{r=1}^{t} p_r(x).
$$

we have

$$
h(t)=\sum_{x\in\mathbf{Z}^3}\left(u_t(x)\right)^3. \qquad (2.3)
$$

As shown in [9] the random walk X_n is adapted if and only if the origin is the unique element of T^3 satisfying $\phi(p)=1$ where $\phi(p)$ is the characteristic function of X_1 and $T^3=(-\pi,\pi]^3$ is the usual three dimensional torus. We use τ to denote the number of elements in the set $\{p\in T^3|\ |\phi(p)|=1\}$. We say that X is aperiodic if $\tau=1$. (In Spitzer[9] this is called strongly aperiodic). We will prove our theorems for X aperiodic, but using the ideas of section 2.4 in [5] it is then easy to show that they also hold if $\tau>1$. According to the local central limit theorem, P7.9 and P7.10 of [9],

$$
p_n(x)=\frac{q_n(Q^{-1/2}x)}{|Q|^{1/2}}+\inf\left(\frac{1}{n^{3/2}},\frac{1}{|x|^2n^{1/2}}\right)o(1_n) \qquad (2.4)
$$

where $q_t(x)$ denotes the transition density for Brownian motion in R^3 and Q denotes the covariance matrix of X_1. Then, arguing seperately in the regions $n\le n_0$, $n_0<n\le|x|^2$ and $n>|x|^2$ we have

$$
u_t(x)=\sum_{n=1}^{t}\frac{q_n(Q^{-1/2}x)}{|Q|^{1/2}}+\frac{o(1_{n_0})}{|x|}+\frac{O(n_0^{1/2})}{|x|^2} \qquad (2.5)
$$

Since

(2.6)
$$\int_0^\infty q_s(x)\,ds = \frac{1}{2\pi|x|}$$

we see that taking $n_0 = 1$ in (2.5) gives the bound

(2.7)
$$u_t(x) \le \frac{c}{1+|x|}$$

We also have

(2.8)
$$\begin{aligned}
&\sum_{x\in\mathbf{Z}^3}\left(\sum_{n=1}^{t}\frac{q_n(Q^{-1/2}x)}{|Q|^{1/2}}\right)^3\\
&\sim \int_{x\in\mathbf{R}^3}\left(\frac{\int_{s=1}^t q_s(Q^{-1/2}x)\,ds}{|Q|^{1/2}}\right)^3\,dx\\
&= \frac{1}{|Q|^{3/2}}\int_{x\in\mathbf{R}^3}\left(\int_{s=1}^t q_s(Q^{-1/2}x)\,ds\right)^3\,dx\\
&= \frac{1}{|Q|}\int_{x\in\mathbf{R}^3}\left(\int_{s=1}^t q_s(x)\,ds\right)^3\,dx\\
&= \frac{1}{(2\pi)^{9/2}|Q|}\int_{a=1}^t\int_{b=1}^t\int_{c=1}^t\\
&\qquad\left((abc)^{-3/2}\int_{x\in\mathbf{R}^3}\exp(-\frac{|x|^2}{2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}))\,dx\right)\,da\,db\,dc\\
&= \frac{1}{(2\pi)^3|Q|}\int_{a=1}^t\int_{b=1}^t\int_{c=1}^t\\
&\qquad(abc)^{-3/2}(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})^{-3/2}\,da\,db\,dc\\
&= \frac{1}{(2\pi)^3|Q|}\int_{a=1}^t\int_{b=1}^t\int_{c=1}^t(bc+ac+ab)^{-3/2}\,da\,db\,dc
\end{aligned}$$

Changing variables, first $x = ab, y = ac, z = bc$, and then $x = u^2, y = v^2, z = w^2$ we have

$$\begin{aligned}
&\int_{a=1}^t\int_{b=1}^t\int_{c=1}^t\frac{1}{(bc+ac+ab)^{3/2}}\,da\,db\,dc\\
&\qquad= \int\int\int_{1\le x,y,z\le t^2}\frac{1}{(x+y+z)^{3/2}}\frac{1}{2\sqrt{xyz}}\,dx\,dy\,dz
\end{aligned}$$

$$= 4 \int\int\int_{1 \le u,v,w \le t} \frac{1}{(u^2+v^2+w^2)^{3/2}} \, du \, dv \, dw$$

$$\sim 2\pi^2 \log t. \tag{2.9}$$

Hence, taking n_0 large in (2.5), we have

$$h(t) \sim \frac{1}{4\pi|Q|} \log t. \tag{2.10}$$

Thus the assertion of Theorem 1 can be written as

$$\limsup_{n\to\infty} \frac{I_n}{4h(n)\log_2 h(n)} = 1 \qquad \text{a.s.} \tag{2.11}$$

We begin our proof with some moment calculations.

$$\begin{aligned}
(2.12)\; E(I_t^n) &= \sum_{x_1,\dots,x_n} \left\{ E\left(\prod_{i=1}^n \sum_{r_i=1}^t 1_{\{X_{r_i}=x_i\}} \right) \right\}^3 \\
&= \sum_{x_1,\dots,x_n} \left\{ \sum_\pi \sum_{r_1 \le r_2 \le \dots \le r_n \le t} E\left(\prod_{i=1}^n 1_{\{X_{r_i}=x_{\pi(i)}\}} \right) \right\}^3 \\
&= \sum_{x_1,\dots,x_n} \left(\sum_\pi \sum_{r_1 \le r_2 \le \dots \le r_n \le t} \prod_{i=1}^n p_{r_i - r_{i-1}}(x_{\pi(i)} - x_{\pi(i-1)}) \right)^3 \\
&= n! \sum_{x_1,\dots,x_n} \left(\sum_{r_1 \le r_2 \le \dots \le r_n \le t} \prod_{i=1}^n p_{r_i - r_{i-1}}(x_i - x_{i-1}) \right) \\
&\qquad \left(\sum_\pi \sum_{s_1 \le s_2 \le \dots \le s_n \le t} \prod_{j=1}^n p_{s_j - s_{j-1}}(x_{\pi(j)} - x_{\pi(j-1)}) \right)^2
\end{aligned}$$

where $\sum_\pi$ runs over the set of permutations π of $\{1, 2, \dots, n\}$. We see from (2.12) that

$$\begin{aligned}
E(I_t^n) \;&\le\; n! \sum_{x_1,\dots,x_n} \left(\prod_{i=1}^n u_t(x_i - x_{i-1}) \right) \\
&\qquad \left(\sum_\pi \prod_{j=1}^n u_t(x_{\pi(j)} - x_{\pi(j-1)}) \right)^2,
\end{aligned} \tag{2.13}$$

while

$$E(I_t^n) \geq n! \sum_{x_1,\ldots,x_n} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1}) \right) \left(\sum_\pi \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right)^2 . \tag{2.14}$$

Note that

$$p_j(x) = P(X_j = x) \leq P(|X_j| \geq |x|) \leq \frac{Cj}{|x|^2} \tag{2.15}$$

so that

$$u_t(x) \leq C \frac{t^2}{|x|^2} \tag{2.16}$$

giving us the bound

$$u_t(x) \leq \frac{C}{1+|x|^{3/2}} \quad \text{for all } |x| > t^4. \tag{2.17}$$

Lemma 1 *For all integers $n, t \geq 0$ and for any $\epsilon > 0$*

$$E(I_t^n) \leq (1+\epsilon)\Psi(n)h^n(t) + R(n,t) \tag{2.18}$$

where

$$0 \leq R(n,t) \leq C(n!)^5 h^{n-1/3}(t) \tag{2.19}$$

and

$$\Psi(n) = \prod_{j=1}^{n-1}(4j+1).$$

Proof of Lemma 1: We will make use of several ideas of Le Gall, [4]. We begin by rewriting (2.13) as

$$E(I_t^n) \leq n! \sum_{y_1,\ldots,y_n} \left(\prod_{i=1}^n u_t(y_i) \right) \left(\sum_\pi \prod_{j=1}^n u_t(v_{\pi,j}) \right)^2 , \tag{2.20}$$

where $y_i = x_i - x_{i-1}$,

$$v_{\pi,j} = x_{\pi(j)} - x_{\pi(j-1)} = \sum_{k \in]\pi(j-1),\pi(j)]} y_k, \tag{2.21}$$

and (with a slight abuse of notation), $k \in]\pi(j-1), \pi(j)]$ means

$$k \in]\min(\pi(j-1), \pi(j)), \max(\pi(j-1), \pi(j))].$$

In view of (2.20), in order to prove our lemma it suffices to show that

$$\begin{aligned} & n! \sum_{y_1,\dots,y_n} \left(\prod_{i=1}^n u_t(y_i)\right) \left(\sum_\pi \prod_{j=1}^n u_t(v_{\pi,j})\right) \left(\sum_{\pi'} \prod_{j=1}^n u_t(v_{\pi',j})\right) \\ & = (1+\epsilon)\Psi(n) h^n(t) + R(n,t) \end{aligned} \tag{2.22}$$

with $R(n,t)$ as in (2.19). For each permutations σ of $\{1,2,\dots,n\}$ we define

$$\Delta_\sigma = \{(y_1,\cdots,y_n) |\ |y_{\sigma(1)}| \le |y_{\sigma(2)}| \le \cdots \le |y_{\sigma(n)}|\}$$

and rewrite the left hand side of (2.22) as

$$n! \sum_{\sigma,\pi,\pi'} \sum_{\Delta_\sigma} \left(\prod_{i=1}^n u_t(y_i)\right) \left(\prod_{j=1}^n u_t(v_{\pi,j})\right) \left(\prod_{k=1}^n u_t(v_{\pi',k})\right), \tag{2.23}$$

Note that by (2.7)

$$\begin{aligned} & \sum_{y \le |x| \le 9y} (u_t(x))^3 \\ & \le \sum_{y \le |x| \le 9y} C \frac{1}{1+|x|^3} \\ & \le C(\log 9y - \log y) = C\log(9). \end{aligned} \tag{2.24}$$

and that by (2.2)

$$\sum_x u_t^3(x) = h(t). \tag{2.25}$$

Let $A_{\sigma,k} = \{(y_1,\dots,y_n) \mid |y_{\sigma_{k-1}}| \le |y_{\sigma_k}| \le 9|y_{\sigma_{k-1}}|\}$. Using Hölder's inequality, (2.24) and (2.25) we have

$$\begin{aligned} & \sum_{(y_1,\dots,y_n)\in A_{\sigma,k}} \left(\prod_{i=1}^n u_t(y_i)\right) \left(\prod_{j=1}^n u_t(v_{\pi,j})\right) \left(\prod_{k=1}^n u_t(v_{\pi',k})\right) \\ & \le \left(\sum_{(y_1,\dots,y_n)\in A_{\sigma,k}} \prod_{i=1}^n (u_t(y_i))^3\right)^{1/3} h^{2n/3}(t) \\ & \le C h^{n-1/3}(t) \end{aligned} \tag{2.26}$$

Set

$$\hat{\Delta}_\sigma = \{(y_1, \cdots, y_n)|\ 9|y_{\sigma(k-1)}| < |y_{\sigma(k)}|,\ \forall k\}.$$

Using (2.26) we see that the sum in (2.23) differs from the sum over $\hat{\Delta}_\sigma$ by an error term which can be incorporated into $R(n,t)$. Up to the error terms described above, we can write the sum in (2.23) as

$$n! \sum_{\sigma,\pi,\pi'} \sum_{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i)\right)\left(\prod_{j=1}^n u_t(v_{\pi,j})\right)\left(\prod_{k=1}^n u_t(v_{\pi',k})\right). \tag{2.27}$$

Note that on $\hat{\Delta}_\sigma$

$$\sum_{j=1}^{k-1} |y_{\sigma(j)}| \le (\sum_{j=1}^{k-1}(1/9)^{k-j})|y_{\sigma(k)}| \le (1/8)|y_{\sigma(k)}| \tag{2.28}$$

For given σ,π define the map $\phi = \phi_{\sigma,\pi} : \{1,2,\ldots,n\} \mapsto \{1,2,\ldots,n\}$ by

$$\phi(j) = \sigma(k_{\sigma,\pi,j}),$$

where

$$k_{\sigma,\pi,j} = \max\{k|\,\sigma(k) \in]\pi(j-1),\pi(j)]\}.$$

Note that on $\hat{\Delta}_\sigma$, $\phi(j)$ is the unique integer in $]\pi(j-1),\pi(j)]$ such that $|y_{\phi(j)}| = \sup_{k\in]\pi(j-1),\pi(j)]} |y_k|$. Futhermore, on $\hat{\Delta}_\sigma$, we see from (2.28) that $\frac{1}{2}|v_{\pi,j}| < |y_{\phi(j)}| < 2|v_{\pi,j}|$. Using Hölder's inequality, and the bounds (2.7), (2.17) we have

$$\begin{aligned}
&\sum_{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i)\right)\left(\prod_{j=1}^n u_t(v_{\pi,j})\right)\left(\prod_{k=1}^n u_t(v_{\pi',k})\right) \\
&\le \left(\sum_{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma} \prod_{j=1}^n (u_t(v_{\pi,j}))^3\right)^{1/3} h^{2n/3}(t) \\
&\le \left(\sum_{\substack{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma \\ |v_{\pi,j}|\le t^4,\forall j}} \prod_{j=1}^n (u_t(v_{\pi,j}))^3\right)^{1/3} h^{2n/3}(t) + Cnh^{n-1/3}(t) \\
&\le C\left(\sum_{\substack{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma \\ |y_j|\le 2t^4,\forall j}} \prod_{j=1}^n \frac{1}{1+|y_{\phi(j)}|^3}\right)^{1/3} h^{2n/3}(t) + Cnh^{n-1/3}(t).
\end{aligned} \tag{2.29}$$

We now show that

$$\sum_{\substack{(y_1,\dots,y_n)\in\hat{\Delta}_\sigma\\ |y_j|\le 2t^4,\forall j}} \prod_{j=1}^{n} \frac{1}{1+|y_{\phi(j)}|^3} \le Ch^{n-1}(t). \tag{2.30}$$

unless $\phi=\phi_{\sigma,\pi}:\{1,2,\dots,n\}\mapsto\{1,2,\dots,n\}$ is bijective.

To begin, we note that by (2.21) both $\{y_j,\ j=1,\dots,n\}$ and $\{v_{\pi,j},\ j=1,\dots,n\}$ generate $\{x_j,\ j=1,\dots,n\}$ in the sense of linear combinations, so that both sets consist of n linearly independent vectors. Furthermore, from (2.21) we see that each $v_{\pi,j}$ is a sum of vectors from $\{y_j,\ j=1,\dots,n\}$. However, from the definitions, we see that when we write out any vector in $\{v_{\pi,j}\ |\ k_{\sigma,\pi,j}\le m\}$ as such a sum, the sum will only involve vectors from $\{y_{\sigma(j)}\ |\ j\le m\}$. Hence $\{v_{\pi,j}\ |\ k_{\sigma,\pi,j}\le m\}$ will contain at most m linearly independent vectors. Therefore, for each $m=0,1,\dots,n-1$, the set $\{v_{\pi,j}\ |\ k_{\sigma,\pi,j}>m\}$ will contain at least $n-m$ elements. Equivalently, for each $m=0,1,\dots,n-1$, the set $\{j\ |\ \sigma^{-1}\phi(j)>m\}$ will contain at least $n-m$ elements. This shows that for each $m=0,1,\dots,n-1$, the product

$$\prod_{j=1}^{n}\frac{1}{1+|y_{\phi(j)}|^3}$$

will contain at least $n-m$ factors of the form

$$\frac{1}{1+|y_{\sigma(j)}|^3}$$

with $j>m$. We now return to (2.30) and sum in turn over the variables $y_{\sigma(n)},y_{\sigma(n-1)},\dots,y_{\sigma(1)}$ using the fact that

$$\sum_{\{y_{\sigma(j)}\in\mathbf{Z}^3|\ 9|y_{\sigma(j-1)}|\le|y_{\sigma(j)}|\le 2t^4\}} \frac{1}{1+|y_{\sigma(j)}|^3}\le Ch(t) \tag{2.31}$$

while for any $k>1$

$$\sum_{\{y_{\sigma(j)}\in\mathbf{Z}^3|\ 9|y_{\sigma(j-1)}|\le|y_{\sigma(j)}|\le 2t^4\}} \frac{1}{1+|y_{\sigma(j)}|^{3k}}\le C\frac{1}{1+|y_{\sigma(j-1)}|^{3(k-1)}}. \tag{2.32}$$

The above considerations show that as we sum successively over the variables $y_{\sigma(n)},y_{\sigma(n-1)},\dots,y_{\sigma(1)}$, at the stage when we sum over $y_{\sigma(j)}$, we will be summing a factor of the form $\frac{1}{1+|y_{\sigma(j)}|^{3k}}$ for some $k \geq 1$, while if $\phi = \phi_{\sigma,\pi} : \{1, 2, \ldots, n\} \mapsto \{1, 2, \ldots, n\}$ is not bijective we must have $k > 1$ at some stage. These considerations, together with (2.31) and (2.32) establish (2.30), and similarly for $\phi_{\sigma,\pi'}$.

Let Ω_n be the set of (σ, π, π') for which $\phi_{\sigma,\pi}$ and $\phi_{\sigma,\pi'}$ are both bijections. Up to the error terms described above, we can write the sum in (2.27) as

$$n! \sum_{(\sigma,\pi,\pi')\in\Omega_n} \sum_{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t(y_i)\right) \left(\prod_{j=1}^n u_t(v_{\pi,j})\right) \left(\prod_{k=1}^n u_t(v_{\pi',k})\right) \tag{2.33}$$

Since on $\hat{\Delta}_\sigma$ we have by (2.28) that $|y_{\phi(j)}| \geq 8|v_{\pi,j} - y_{\phi(j)}|$, we can then replace each occurence of $v_{\pi,j}$ in (2.33) by $y_{\phi(j)}$, bounding the error terms using

$$\begin{aligned} &\sum_{\{|x|\geq 8|a|\}} (u_t(x+a) - u_t(x))^3 \\ &\qquad \leq C \sum_{\{|x|\geq 8|a|\}} \left(\frac{|a|^3}{1+|x|^6} + \frac{1}{1+|x|^4}\right) \leq C \end{aligned} \tag{2.34}$$

which comes from (2.17) and Lemma 5 of the Appendix.

Thus, up to error terms described which can be incorporated into $R(n,t)$, we can write the sum in (2.33) as

$$n! \sum_{(\sigma,\pi,\pi')\in\Omega_n} \sum_{(y_1,\ldots,y_n)\in\hat{\Delta}_\sigma} \left(\prod_{i=1}^n u_t^3(y_i)\right). \tag{2.35}$$

Proceeding as above, up to the error terms described above, we can replace (2.35) by

$$n! \sum_{(\sigma,\pi,\pi')\in\Omega_n} \sum_{(y_1,\ldots,y_n)\in\Delta_\sigma} \left(\prod_{i=1}^n u_t^3(y_i)\right). \tag{2.36}$$

Since

$$n! \sum_{(y_1,\ldots,y_n)\in\Delta_\sigma} \left(\prod_{i=1}^n u_t^3(y_i)\right) \sim h^n(t),$$

and as by the remark following Lemma 2.5 in [4], we have $|\Omega_n| = \Psi(n)$, the lemma is proved. □

We will use $E^{v,w,z}$ to denote expectation with respect to the random walks X, X', X'' where $X_0 = v$, $X'_0 = w$ and $X''_0 = z$. We define

$$a(v,w,z,t) = \frac{h(v,w,z,t)}{h(t)} \tag{2.37}$$

where

$$\begin{aligned} h(v,w,z,t) &= E^{v,w,z}(I_t) \\ &= \sum_{x\in\mathbf{Z}^3} \left\{ \left(\sum_{i=1}^t p_i(x-v)\right) \left(\sum_{j=1}^t p_j(x-w)\right) \left(\sum_{k=1}^t p_k(x-z)\right) \right\} \end{aligned} \tag{2.38}$$

We will need the following lower bound.

Lemma 2 *For all integers $n, t \geq 0$ and for any $\epsilon > 0$*

$$E^{v,w,z}(I_t^n) \geq (1-\epsilon)\Psi(n) a(v,w,z,t/n) h^n(t/n) - R'(n,t) \tag{2.39}$$

where

$$0 \leq R'(n,t) \leq C(n!)^5 h^{n-1/3}(t) \tag{2.40}$$

Proof of Lemma 2: We first note that as in (2.14)

$$\begin{aligned} &E^{v,w,z}(I_t^n) \\ &\geq n! \sum_{x_1,\ldots,x_n} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1})\right) \left(\sum_\pi \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)})\right) \\ &\qquad \left(\sum_{\pi'} \prod_{j=1}^n u_{t/n}(x_{\pi'(j)} - x_{\pi'(j-1)})\right) \end{aligned} \tag{2.41}$$

where now we use the convention $x_0 = v, x_{\pi(0)} = w, x_{\pi'(0)} = z$. We then use (2.22), observing that if $\phi_{\sigma,\pi}$ is bijective we must have $\phi_{\sigma,\pi}(j) = 1$ for some j and this must be $j = 1$ since $1 \in]\pi(j-1), \pi(j)]$ is possible only for $j = 1$. Thus, $v_{\pi,1}$ is replaced in (2.27) by y_1, and a similar analysis applies to $v_{\pi',1}$. $\Box$.

Lemma 3 *For all $t \geq 0$ and $x = O(\log\log h(t))$ we have*

$$P\left(\frac{I_t}{4h(t)} \geq x\right) \leq C\sqrt{x}e^{-x}. \tag{2.42}$$

Proof of Lemma 3: We first note that if $n = O(\log\log h(t))$ then

$$\frac{(n!)^5}{h^{1/3}(t)} \to 0 \tag{2.43}$$

as $t \to \infty$, so that by Lemma 1 we have

$$E(I_t^n) \le C\Psi(n)h^n(t). \tag{2.44}$$

Then Chebyshev's inequality gives us

$$P\left(\frac{I_t}{4h(t)} \ge x\right) \le C\frac{\Psi(n)}{(4x)^n} = C\frac{\sqrt{n}n^n e^{-n}}{x^n}(1 + O(1/n)) \tag{2.45}$$

for any $n = O(\log\log h(t))$. Taking $n = [x]$ then yields (2.42). □.

Lemma 4 *For all $\epsilon > 0$ there exists an x_0 and a $t' = t'(\epsilon, x_0)$ such that for all $t \ge t'$ and $x_0 \le x = O(\log\log h(t))$ we have*

$$P\left(\frac{I_t}{4h(t)} \ge (1-\epsilon)x\right) \ge C_\epsilon e^{-x} \tag{2.46}$$

and

$$P^{v,w,z}\left(\frac{I_t}{4h(t)} \ge (1-\epsilon)x\right) \ge C_\epsilon\left(a(v,w,z,2t/(3x))e^{-x} - e^{-(1+\epsilon')x}\right) \tag{2.47}$$

for some $\epsilon' > 0$.

Proof of Lemma 4: This follows from Lemmas 2, 3 and (2.43) by the methods used in the proof of Lemma 2.7 in [7]. □

Proof of Theorem 1: For $\theta > 1$ we define the sequence $\{t_n\}$ by

$$h(t_n) = \theta^n. \tag{2.48}$$

By Lemma 3 we have that for all integers $n \ge 2$ and all $\epsilon > 0$

$$P\left(\frac{I_{t_n}}{4h(t_n)\log\log h(t_n)} \ge (1+\epsilon)\right) \le Ce^{-(1+\epsilon)\log n}. \tag{2.49}$$

Therefore, by the Borel-Cantelli lemma

$$(2.50)\qquad \limsup_{n\to\infty} \frac{I_{t_n}}{4h(t_n)\log\log h(t_n)} \le 1+\epsilon \qquad a.s.$$

By taking θ arbitrarily close to 1 it is simple to interpolate in (2.50) to obtain

$$(2.51)\qquad \limsup_{n\to\infty} \frac{I_n}{4h(n)\log\log h(n)} \le 1+\epsilon \qquad a.s.$$

We now show that for any $\epsilon > 0$

$$(2.52)\qquad \limsup_{n\to\infty} \frac{I_{t_n}}{4h(t_n)\log\log h(t_n)} \ge 1-\epsilon \qquad a.s.$$

for all θ sufficiently large. It is sufficient to show that

$$(2.53)\qquad \limsup_{n\to\infty} \frac{I_{t_n}-I_{t_{n-1}}}{4h(t_n)\log\log h(t_n)} \ge 1-\epsilon \qquad a.s.$$

Let $s_n = t_n - t_{n-1}$ and note that, as in (2.60) of [7], we have $h(s_n) \sim h(t_n)$. We also note that

$$(2.54)\qquad |I_{t_n} - I_{t_{n-1}} - I_{s_n}\circ\Theta_{t_{n-1}}| \le I_{t_n,t_n,t_{n-1}} + I_{t_n,t_{n-1},t_n} + I_{t_{n-1},t_n,t_n}$$

where Θ_n denotes the shift on paths defined by

$$(X_i, X'_j, X"_k)(\Theta_n\omega) = (X_{n+i}, X'_{n+j}, X"_{n+k})(\omega)$$

and

$$(2.55)\qquad I_{n,m,p} = \sum_{x\in\mathbf{Z}^3}\left\{\left(\sum_{i=1}^{n} 1_{\{X_i=x\}}\right)\left(\sum_{j=1}^{m} 1_{\{X'_j=x\}}\right)\left(\sum_{k=1}^{p} 1_{\{X''_k=x\}}\right)\right\}.$$

As in Lemma 1, we can show that for $t \ge t'$, and for all integers $n \ge 0$ and any $\epsilon > 0$

$$(2.56)\qquad \begin{aligned} E(I^n_{t,t,t'}) \;\le\; & (1+\epsilon)\Psi(n)h^{2n/3}(t)h^{n/3}(t') \\ & +O\left((n!)^5 h^{2n/3}(t)h^{n/3-1/3}(t')\right) \end{aligned}$$

which, as before, leads to

$$\limsup_{n\to\infty} \frac{I_{t_n,t_n,t_{n-1}}}{4h(t_n)\log\log h(t_n)} \tag{2.57}$$
$$= \limsup_{n\to\infty} \frac{I_{t_n,t_n,t_{n-1}}}{4\sqrt[3]{\theta h^2(t_n)h(t_{n-1})}\log\log h(t_n)}$$
$$\le \frac{1+\epsilon}{\sqrt[3]{\theta}} \qquad a.s.$$

Using this for θ large, (2.54), Levy's Borel-Cantelli lemma (see Corollary 5.29 in [1]) and the Markov property, we see that (2.53) will follow from

$$\sum_{n=1}^{\infty} P^{X_{t_{n-1}},X'_{t_{n-1}},X''_{t_{n-1}}}\left(\frac{I_{s_n}}{4h(s_n)\log\log h(s_n)} \ge 1-\epsilon\right) = \infty \qquad a.s. \tag{2.58}$$

If we apply Lemma 4 with $t = s_n$ and $x = \log\log s_n$ we see that (2.58) will follow from

$$\sum_{n=1}^{\infty} a(X_{t_{n-1}}, X'_{t_{n-1}}, X''_{t_{n-1}}, s_n/\log n)\frac{1}{n^{1-\epsilon'}} = \infty \qquad a.s. \tag{2.59}$$

We begin by showing

$$\sum_{n=1}^{\infty} E(a(X_{t_{n-1}}, X'_{t_{n-1}}, X''_{t_{n-1}}, s_n/\log n))\frac{1}{n^{1-\epsilon'}} = \infty \tag{2.60}$$

To see this we note that

$$E(a(X_t, X'_t, X''_t, m)) \tag{2.61}$$
$$= \frac{\sum_{x\in\mathbf{Z}^3}\left(\sum_{i=1}^m p_{i+t}(x)\right)^3}{h(m)}$$

so that

$$E(a(X_{t_{n-1}}, X'_{t_{n-1}}, X''_{t_{n-1}}, s_n/\log n)) \tag{2.62}$$
$$= \frac{\sum_{x\in\mathbf{Z}^d}\left(\sum_{i=1}^{s_n/\log n} p_{i+t_{n-1}}(x)\right)^3}{h(s_n/\log n)}$$

$$= \frac{h(t_{n-1}+s_n/\log n)-h(t_{n-1})}{h(s_n/\log n)}$$
$$-\frac{3\sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{t_{n-1}} p_i(x)\right)^2\left(\sum_{j=1}^{s_n/\log n)} p_{j+t_{n-1}}(x)\right)\right\}}{h(s_n/\log n)}$$
$$-\frac{3\sum_{x\in\mathbf{Z}^d}\left\{\left(\sum_{i=1}^{t_{n-1}} p_i(x)\right)\left(\sum_{j=1}^{s_n/\log n)} p_{j+t_{n-1}}(x)\right)^2\right\}}{h(s_n/\log n)}.$$

Also note that

$$\frac{h(t_{n-1}+s_n/\log n)-h(t_{n-1})}{h(s_n/\log n)} \geq \frac{h(s_n/\log n)-h(t_{n-1})}{h(s_n/\log n)} \sim 1-\frac{1}{\theta}. \tag{2.63}$$

This follows fairly easily since $h(t) \sim c\log(t)$. (For the details, in a more general setting, see the proof of Theorem 1.1 of [7], especially that part of the proof surrounding (2.82)). Furthermore, we have by Hölder's inequality

$$\begin{aligned} &\frac{\sum_{x\in\mathbf{Z}^3}\left\{\left(\sum_{i=1}^{t_{n-1}} p_i(x)\right)\left(\sum_{j=1}^{s_n/\log n)} p_{j+t_{n-1}}(x)\right)^2\right\}}{h(s_n/\log n)} \\ &\leq \frac{h^{1/3}(t_{n-1})h^{2/3}(t_n)}{h(s_n/\log n)} \\ &\sim \frac{1}{\theta^{1/3}}. \end{aligned} \tag{2.64}$$

Taking θ large establishes (2.60).

Furthermore, since $a(v,w,z,t) \leq 1$ (use (2.38) and Hölder's inequality), we see that for any $\epsilon' < 1/2$

$$\sum_{n=1}^{\infty} E\left(a(X_{t_{n-1}}, X'_{t_{n-1}}, X''_{t_{n-1}}, s_n/\log n)\frac{1}{n^{1-\epsilon'}}\right)^2 < \infty. \tag{2.65}$$

(2.59) will now follow from the Paley-Zygmund lemma, (see e.g. Inequality II on page 8 of [2]), once we show that

(2.66)

$$\frac{E(a(X_{t_{n-1}},X'_{t_{n-1}},X''_{t_{n-1}},s_n/\log n)a(X_{t_{m-1}},X'_{t_{m-1}},X''_{t_{m-1}},s_m/\log m))}{E(a(X_{t_{n-1}},X'_{t_{n-1}},X''_{t_{n-1}},s_n/\log n))E(a(X_{t_{m-1}},X'_{t_{m-1}},X''_{t_{m-1}},s_m/\log m))} \le 1+2\epsilon$$

for all $\epsilon > 0$, when $n > m \ge N(\epsilon)$ for some $N(\epsilon)$ sufficiently large. To prove (2.66) we begin by noting that as in (2.61)

$$E(h(X_t,X'_t,X''_t,s)) = \sum_{x\in\mathbf{Z}^3}\left(\sum_{i=1}^{s} p_{i+t}(x)\right)^3 \tag{2.67}$$

and for $t' < t$
(2.68)

$$\begin{aligned}
&E(h(X_{t'},X'_{t'},X''_{t'},s')h(X_t,X'_t,X''_t,s))\\
&\quad= \sum_{x,y,x',y',x'',y''} h(x,x',x'',s')p_{t'}(x)p_{t'}(x')p_{t'}(x'')h(y,y',y'',s)\\
&\qquad\cdot p_{t-t'}(y-x)p_{t-t'}(y'-x')p_{t-t'}(y''-x'')\\
&\quad= \sum_{x,x',x''} h(x,x',x'',s')p_{t'}(x)p_{t'}(x')p_{t'}(x'')\\
&\qquad\cdot\sum_{u\in\mathbf{Z}^3}\left\{\left(\sum_{i=1}^{s}p_{i+t-t'}(u-x)\right)\left(\sum_{j=1}^{s}p_{j+t-t'}(u-x')\right)\right.\\
&\qquad\qquad\left.\cdot\left(\sum_{k=1}^{s}p_{k+t-t'}(u-x'')\right)\right\}\\
&\quad\le \sum_{x,x',x''} h(x,x',x'',s')p_{t'}(x)p_{t'}(x')p_{t'}(x'')\sum_{u\in\mathbf{Z}^3}\left(\sum_{j=1}^{s}p_{j+t-t'}(u)\right)^3\\
&\quad= \sum_{x\in\mathbf{Z}^3}\left(\sum_{i=1}^{s'}p_{i+t'}(x)\right)^3\sum_{u\in\mathbf{Z}^3}\left(\sum_{j=1}^{s}p_{j+t-t'}(u)\right)^3.
\end{aligned}$$

From (2.67), (2.68) we see that

$$\begin{aligned}
&\frac{E(h(X_{t_{m-1}},X'_{t_{m-1}},X''_{t_{m-1}},s_m/\log m)h(X_{t_{n-1}},X'_{t_{n-1}},X''_{t_{n-1}},s_n/\log n))}{E(h(X_{t_{m-1}},X'_{t_{m-1}},X''_{t_{m-1}},s_m/\log m))E(h(X_{t_{n-1}},X'_{t_{n-1}},X''_{t_{n-1}},s_n/\log n))}\\
&\le \frac{\sum_{u\in\mathbf{Z}^3}\left(\sum_{j=1}^{s_n/\log n}p_{j+t_{n-1}-t_{m-1}}(u)\right)^3}{\sum_{u\in\mathbf{Z}^3}\left(\sum_{j=1}^{s_n/\log n}p_{j+t_{n-1}}(u)\right)^3}. \qquad (2.69)
\end{aligned}$$

Arguing as in (2.62)-(2.64) we see that (2.66) follows. This completes the proof of Theorem 1. □

3 Proof of Theorem 3

We begin with some moment calculations. Recall

$$
\begin{aligned}
J_n &= |X(1,n)\cap X'(1,n)\cap X''(1,n)| \\
&= \sum_{x\in\mathbf{Z}^4} 1_{\{x\in X(1,n)\}}1_{\{x\in X'(1,n)\}}1_{\{x\in X''(1,n)\}}
\end{aligned} \tag{3.1}
$$

As usual set

$$T_x = \inf\{k|\, X_k = x\},$$

and note that

$$
\begin{aligned}
E(J_t^n) &= E\left[(\sum_x 1_{\{x\in X(1,t)\}}1_{\{x\in X'(1,t)\}}1_{\{x\in X''(1,t)\}})^n \right] \\
&= \sum_{x_1,\ldots,x_n} E\left(\prod_{i=1}^n 1_{\{x\in X(1,t)\}}1_{\{x\in X'(1,t)\}}1_{\{x\in X''(1,t)\}}\right) \\
&= \sum_{x_1,\ldots,x_n} \left\{ E\left(\prod_{i=1}^n 1_{\{x_i\in X(1,t)\}}\right)\right\}^3 \\
&\le \sum_{x_1,\ldots,x_n} \left\{ \sum_\pi P(T_{x_{\pi(1)}}\le T_{x_{\pi(2)}}\le\cdots\le T_{x_{\pi(n)}}\le t)\right\}^3 \\
&= n! \sum_{x_1,\ldots,x_n} (P(T_{x_1}\le T_{x_2}\le\cdots\le T_{x_n}\le t)) \\
&\qquad \cdot\left(\sum_\pi P(T_{x_{\pi(1)}}\le T_{x_{\pi(2)}}\le\cdots\le T_{x_{\pi(n)}}\le t)\right)^2
\end{aligned} \tag{3.2}
$$

where $\sum_\pi$ runs over the set of permutations π of $\{1,2,\ldots,n\}$. Set

$$v_t(x) = P(T_x\le t).$$

Then we see from (3.2) that

$$
\begin{aligned}
E(J_t^n) &\le n! \sum_{x_1,\ldots,x_n} \left(\prod_{i=1}^n v_t(x_i - x_{i-1})\right) \\
&\qquad \left(\sum_\pi \prod_{j=1}^n v_t(x_{\pi(j)} - x_{\pi(j-1)})\right)^2,
\end{aligned} \tag{3.3}
$$

while

$$E(J_t^n) \geq n! \sum_{x_1,\dots,x_n} \left(\prod_{i=1}^{n} v_{t/n}(x_i - x_{i-1}) \right) \left(\sum_{\pi} \prod_{j=1}^{n} v_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right)^2 . \tag{3.4}$$

Let

$$f_r(x) = P(T_x = r)$$

so that

$$v_t(x) = \sum_{r=1}^{t} f_r(x).$$

We have

$$p_j(x) = \sum_{i=1}^{j} f_i(x) p_{j-i}(0) \tag{3.5}$$

where as usual we set $p_0(x) = 1_{\{x=0\}}$. From this we see that

$$\begin{aligned} u_t(x) &= \sum_{j=1}^{t} p_j(x) \\ &= \sum_{j=1}^{t} \sum_{i=1}^{j} f_i(x) p_{j-i}(0) \\ &= \sum_{i=1}^{t} \sum_{j=i}^{t} f_i(x) p_{j-i}(0) \\ &= \sum_{i=1}^{t} f_i(x)(1 + u_{t-i}(0)). \end{aligned} \tag{3.6}$$

Consequently we have

$$u_t(x) \leq v_t(x)(1 + u_t(0)) \tag{3.7}$$

and

$$u_{2t}(x) \geq v_t(x)(1 + u_t(0)). \tag{3.8}$$

Now it is well known that

$$\frac{1}{1 + u_t(0)} \downarrow q \tag{3.9}$$

so that for any $\epsilon > 0$ we can find $t_0 < \infty$ such that

$$qu_t(x) \le v_t(x) \le (q+\epsilon)u_{2t}(x) \tag{3.10}$$

for all $t \ge t_0$ and x. Hence (3.3) and (3.4) give us

$$\begin{aligned} E(J_t^n) \;\; &\le \;\; (q+\epsilon)^{2n} n! \sum_{x_1,\dots,x_n} \left(\prod_{i=1}^n u_{2t}(x_i - x_{i-1}) \right) \\ &\qquad \left(\sum_\pi \prod_{j=1}^n u_{2t}(x_{\pi(j)} - x_{\pi(j-1)}) \right)^2, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} E(J_t^n) \;\; &\ge \;\; q^{2n} n! \sum_{x_1,\dots,x_n} \left(\prod_{i=1}^n u_{t/n}(x_i - x_{i-1}) \right) \\ &\qquad \left(\sum_\pi \prod_{j=1}^n u_{t/n}(x_{\pi(j)} - x_{\pi(j-1)}) \right)^2. \end{aligned} \tag{3.12}$$

The proof of Theorem 3 now follows exactly along the lines of the proof of Theorem 1. □

4 Appendix

Lemma 5 *Let X_n be a mean-zero adapted random walk in Z^3. Assume that $E(|X_1|^2 \log_+ |X_1|) < \infty$. Then for some $C < \infty$*

$$|u(x+a) - u(x)| \le \frac{C|a|}{1+|x|^2}, \tag{4.1}$$

for all a, x satisfying $|a| \le |x|/8$.

Furthermore, for some $C < \infty$

$$|u_t(x+a) - u_t(x)| \le \frac{C|a|}{1+|x|^2}, \tag{4.2}$$

for all a, x, t satisfying $|a| \le |x|/8$ and $|x|^{1/4} < t$.

In [3], Lawler shows that the usual Green's function asymptotics do not necessarily hold for all mean zero finite variance random walks on Z^4. We expect that a similar analysis would show that finite variance is not enough to guarantee (4.1). Our Lemma says that $E(|X_1|^2 \log_+ |X_1|) < \infty$ is sufficient.

Proof of Lemma 5: Let

$$\phi(p) = E(e^{ipX_1})$$

denote the characteristic function of X_1. We have

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int_{[-\pi,\pi]^3} \frac{e^{ipx}}{1-\phi(p)}\, dp. \tag{4.3}$$

Let $Q = \{Q_{i,j}\}$ denote the covariance matrix of $X_1 = (X_1^{(1)}, X_1^{(2)}, X_1^{(3)})$, i.e. $Q_{i,j} = E(X_1^{(i)} X_1^{(j)})$. We write

$$Q(p) = \frac{1}{2} \sum_{i,j=1}^{3} Q_{i,j} p_i p_j.$$

Using our assumption that $E(X_1) = 0$ and $E(|X_1|^2 \log_+ |X_1|) < \infty$, we observe that for $|p| \le 1$

$$\begin{aligned} &|1 - \phi(p) - Q(p)| \\ &= |E(1 - e^{ip\cdot X_1} + ip \cdot X_1 + (1/2)(ip \cdot X)^2)| \\ &\le c|p|^3 E(1_{\{|X_1| \le 1/|p|\}} |X_1|^3) + c|p|^2 E(1_{\{|X_1| > 1/|p|\}} |X_1|^2) \\ &\le c|p|^2 / \log_+(1/|p|) \\ &= o(|p|^2) \end{aligned} \tag{4.4}$$

Similarly, we have

$$\begin{aligned} |\phi_1(p) + Q_1| &= |E(iX_1^{(1)} e^{ip\cdot X_1} + X_1^{(1)} p \cdot X_1)| \\ &\le |E(iX_1^{(1)}(e^{ip\cdot X_1} - 1 - ip \cdot X_1))| \\ &\le c|p|^2 E(1_{\{|X_1| \le 1/|p|\}} |X_1|^3) + c|p| E(1_{\{|X_1| > 1/|p|\}} |X_1|^2) \\ &= o(|p|) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} |\phi_{1,1}(p) + Q_{1,1}| &= |E(-(X_1^{(1)})^2 (e^{ip\cdot X_1} - 1))| \\ &\le c|p| E(1_{\{|X_1| \le 1/|p|\}} |X_1|^3) + cE(1_{\{|X_1| > 1/|p|\}} |X_1|^2) \\ &= o(1_{|p|}) \end{aligned} \tag{4.6}$$

Using the third line of (4.4) we see that

$$\begin{aligned}(4.7)\quad \int_{|p|\le 1} &\frac{|1-\phi(p)-Q(p)|}{|p|^5}\,dp \\ &\le cE((\int_{\{|p|\le 1/|X_1|\}} \frac{1}{|p|^2}\,dp)|X_1|^3) + cE((\int_{\{|p|>1/|X_1|\}} \frac{1}{|p|^3}\,dp)|X_1|^2) \\ &\le cE(|X_1|^2 \log_+ |X_1|) < \infty.\end{aligned}$$

Similarly, using (4.5) and (4.6) we see that

$$\int_{|p|\le 1} \frac{|\phi_1(p)+Q_1|}{|p|^4}\,dp < \infty \tag{4.8}$$

and

$$\int_{|p|\le 1} \frac{|\phi_{1,1}(p)+Q_{1,1}|}{|p|^3}\,dp < \infty \tag{4.9}$$

Let $q_t(x)$ denote the transition density for Brownian motion in R^3 and set

$$v_\delta(x) = \int_\delta^\infty q_t(x)\,dt = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{ipx} \frac{e^{-\delta|p|^2/2}}{|p|^2/2}\,dp. \tag{4.10}$$

We have

$$\frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)}\,dp. \tag{4.11}$$

Note that

$$v_\delta(x) \uparrow v_0(x) = \int_0^\infty q_t(x)\,dt = \frac{1}{2\pi|x|} \tag{4.12}$$

as $\delta \to 0$.

If $x = (x_1, x_2, x_3)$, we can assume, without loss of generality, that $|x| \ne 0$ and that $|x_1| = \max_j |x_j|$. We have

$$\begin{aligned}(4.13)\quad \frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}} &= \frac{1}{(2\pi)^{3/2}} \int_A e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)}\,dp \\ &+ \frac{1}{(2\pi)^{3/2}} \int_B e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)}\,dp + \frac{1}{(2\pi)^{3/2}} \int_C e^{ipx} \frac{e^{-\delta Q(p)}}{Q(p)}\,dp\end{aligned}$$

where $A = [-\pi,\pi]^3$, $B = [-\pi,\pi]^c \times [-\pi,\pi]^2$, and $C = R \times ([-\pi,\pi]^2)^c$. Note that

$$C = \{|p_2| > \pi\} \bigcup \{|p_3| > \pi\}. \tag{4.14}$$

We have

$$u(x+a)-u(x)-(\frac{v_\delta(Q^{-1/2}(x+a))}{|Q|^{1/2}}-\frac{v_\delta(Q^{-1/2}x)}{|Q|^{1/2}}) \tag{4.15}$$
$$=\frac{1}{(2\pi)^{3/2}}\int_A(e^{ip(x+a)}-e^{ipx})\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$-\frac{1}{(2\pi)^{3/2}}\int_B(e^{ip(x+a)}-e^{ipx})\frac{e^{-\delta Q(p)}}{Q(p)}\,dp$$
$$-\frac{1}{(2\pi)^{3/2}}\int_C(e^{ip(x+a)}-e^{ipx})\frac{e^{-\delta Q(p)}}{Q(p)}\,dp$$

To prove (4.1) it suffices to show that in the limit as $\delta\to 0$ the right hand side is $O(|a|/|x|^2)$.

We first show that

$$\lim_{\delta\to 0}|\int_C e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp|\le\frac{c}{|x|^2}. \tag{4.16}$$

To see this we integrate by parts twice in the p_1 direction to get

$$\int_C e^{ipx}\frac{e^{-\delta Q(p)}}{Q(p)}\,dp=\frac{i^2}{x_1^2}\int_C e^{ipx}D_1^2(\frac{e^{-\delta Q(p)}}{Q(p)})\,dp \tag{4.17}$$

and

$$D_1^2(\frac{e^{-\delta Q(p)}}{Q(p)})=D_1^2(e^{-\delta Q(p)})\frac{1}{Q(p)}+2D_1(e^{-\delta Q(p)})D_1(\frac{1}{Q(p)}) \tag{4.18}$$
$$+e^{-\delta Q(p)}D_1^2(\frac{1}{Q(p)})$$

Note that $\inf_{p\in B\bigcup C}Q(p)\ge d>0$. Also, $D_1^j(\frac{1}{Q(p)})$ is homogeneous in p of degree $-(2+j)$, so that the last term in (4.18) is integrable on C even when we take $\delta=0$. Since

$$D_1(e^{-\delta Q(p)})=-\delta Q_1(p)e^{-\delta Q(p)} \tag{4.19}$$

and $Q_1(p)D_1(\frac{1}{Q(p)})$ is homogeneous in p of degree -2, scaling out δ shows that the integral of the absolute value of the second term in (4.18) is bounded by

$$\delta^{1/2}\int\frac{e^{-Q(p)}}{|p|^2}\,dp\le c\delta^{1/2}. \tag{4.20}$$

The first term in (4.18) is handled similarly, proving (4.16).

For the first two integrals on the right hand side of (4.15) we integrate by parts in the p_1 direction to obtain

$$
\begin{aligned}
(4.21)\qquad & \frac{i}{(x_1+a_1)}\frac{1}{(2\pi)^{3/2}}\int_A e^{ip(x+a)}D_1\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\\
& -\frac{i}{x_1}\frac{1}{(2\pi)^{3/2}}\int_A e^{ipx}D_1\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\\
& -\frac{i}{(x_1+a_1)}\frac{1}{(2\pi)^{3/2}}\int_B e^{ip(x+a)}D_1(\frac{e^{-\delta Q(p)}}{Q(p)})\,dp\\
& +\frac{i}{x_1}\frac{1}{(2\pi)^{3/2}}\int_B e^{ipx}D_1(\frac{e^{-\delta Q(p)}}{Q(p)})\,dp
\end{aligned}
$$

where we have used the fact that the boundary terms coming from the integrals over A and B cancel. (These boundary terms are easily seen to be finite).

We claim that (4.21) is equal to

$$
\begin{aligned}
(4.22)\qquad & \frac{i}{x_1}\frac{1}{(2\pi)^{3/2}}\int_A e^{ipx}(e^{ipa}-1)D_1\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp\\
& -\frac{i}{x_1}\frac{1}{(2\pi)^{3/2}}\int_B e^{ipx}(e^{ipa}-1)D_1(\frac{e^{-\delta Q(p)}}{Q(p)})\,dp\\
& +O(|a|/|x|^2).
\end{aligned}
$$

To establish our last claim, using the fact that

$$|\frac{1}{(x_1+a_1)}-\frac{1}{x_1}|\le c|a|/|x|^2,$$

it suffices to show that

$$(4.23)\qquad \int_A |D_1\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)|\,dp<\infty$$

and

$$(4.24)\qquad \int_B |D_1(\frac{e^{-\delta Q(p)}}{Q(p)})|\,dp<\infty$$

with bounds uniform in $0 < \delta \leq 1$. (4.24) is easily seen to be bounded independently of $\delta \leq 1$, using ideas similar to those we used in bounding the integral in (4.17). As for (4.23), we first observe that

$$D_1 \left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) = \left(\frac{\phi_1(p)}{(1-\phi(p))^2} + \frac{Q_1(p)}{(Q(p))^2} \right) + D_1 \left(\frac{1-e^{-\delta Q(p)}}{Q(p)} \right) \tag{4.25}$$

The integral of the last term is bounded easily as before. As to the first term in (4.25), we write

$$\begin{aligned} &\frac{\phi_1(p)}{(1-\phi(p))^2} + \frac{Q_1(p)}{(Q(p))^2} \\ &= \frac{\phi_1(p)+Q_1(p)}{(1-\phi(p))^2} + Q_1(p) \left(\frac{1}{(Q(p))^2} - \frac{1}{(1-\phi(p))^2} \right) \\ &= \frac{\phi_1(p)+Q_1(p)}{(1-\phi(p))^2} + Q_1(p) \left(\frac{(1-\phi(p))^2-(Q(p))^2}{(Q(p))^2(1-\phi(p))^2} \right) \\ &= \frac{\phi_1(p)+Q_1(p)}{(1-\phi(p))^2} \\ &\qquad + (1-\phi(p)-Q(p)) \left(\frac{Q_1(p)(1-\phi(p)+Q(p))}{(Q(p))^2(1-\phi(p))^2} \right) \end{aligned} \tag{4.26}$$

The integrals of the two terms in the last equality are bounded by (4.8) and (4.7) respectively. This establishes our claim that (4.21) is equal to (4.22).

To bound the integrals in (4.22) we now integrate by parts once more in the p_1 direction to obtain

$$\begin{aligned} &= \frac{i^2}{x_1^2} \frac{1}{(2\pi)^{3/2}} \int_A e^{ipx} D_1 \left\{ (e^{ipa}-1) D_1 \left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) \right\} dp \\ &\quad - \frac{i^2}{x_1^2} \frac{1}{(2\pi)^{3/2}} \int_B e^{ipx} D_1 \left\{ (e^{ipa}-1) D_1 (\frac{e^{-\delta Q(p)}}{Q(p)}) \right\} dp. \end{aligned} \tag{4.27}$$

Once again, the (finite) boundary terms cancel. (Actually, each boundary term is $O(1/|x|^2)$.) Using the bounds (4.23) and (4.24), we find that (4.27) equals

$$= \frac{i^2}{x_1^2}\frac{1}{(2\pi)^{3/2}}\int_A e^{ipx}(e^{ipa}-1)D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right)dp$$
$$-\frac{i^2}{x_1^2}\frac{1}{(2\pi)^{3/2}}\int_B e^{ipx}(e^{ipa}-1)D_1^2(\frac{e^{-\delta Q(p)}}{Q(p)})\,dp$$
$$+O(|a|/|x|^2) \tag{4.28}$$

As in the proof of (4.16), we see that

$$\lim_{\delta\to 0}|\int_B e^{ipx}(e^{ipa}-1)D_1^2(\frac{e^{-\delta Q(p)}}{Q(p)})\,dp|\leq c. \tag{4.29}$$

To handle the first integral in (4.28) we note that

$$(e^{ipa}-1)D_1^2\left(\frac{1}{1-\phi(p)}-\frac{e^{-\delta Q(p)}}{Q(p)}\right) \tag{4.30}$$
$$=(e^{ipa}-1)D_1^2\left(\frac{1}{1-\phi(p)}-\frac{1}{Q(p)}\right)$$
$$+(e^{ipa}-1)(1-e^{-\delta Q(p)})D_1^2(\frac{1}{Q(p)})$$
$$-2(e^{ipa}-1)D_1(e^{-\delta Q(p)})D_1(\frac{1}{Q(p)})$$
$$-(e^{ipa}-1)D_1^2(e^{-\delta Q(p)})\frac{1}{Q(p)}$$

Once again it is easy to control the last three terms on the right hand side of (4.30), while for the first term we use

$$D_1^2\left(\frac{1}{1-\phi(p)}-\frac{1}{Q(p)}\right)=\left(\frac{\phi_{1,1}(p)}{(1-\phi(p))^2}+\frac{Q_{1,1}}{(Q(p))^2}\right)$$
$$+2\left(\frac{(\phi_1(p))^2}{(1-\phi(p))^3}-\frac{(Q_1(p))^2}{(Q(p))^3}\right). \tag{4.31}$$

As in (4.26), we write out the first term on the right hand side of (4.31) as

$$\frac{\phi_{1,1}(p)}{(1-\phi(p))^2}+\frac{Q_{1,1}}{(Q(p))^2} \tag{4.32}$$

$$= \frac{\phi_{1,1}(p) + Q_{1,1}}{(1-\phi(p))^2}$$

$$+(1-\phi(p)-Q(p))\left(\frac{Q_{1,1}(1-\phi(p)+Q(p))}{(Q(p))^2(1-\phi(p))^2}\right)$$

Hence, we can bound

$$\begin{aligned} &|e^{ipa}-1|\left|\frac{\phi_{1,1}(p)}{(1-\phi(p))^2}+\frac{Q_{1,1}}{(Q(p))^2}\right| \\ &\leq \frac{c|a||\phi_{1,1}(p)+Q_{1,1}|}{|p|^3}+\frac{c|a||1-\phi(p)-Q(p)|}{|p|^5}. \end{aligned} \tag{4.33}$$

The integrals are now bounded using (4.6) and (4.4) respectively.

Similarly we write out the second term on the right hand side of (4.31) as

(4.34)

$$\begin{aligned} &\frac{(\phi_1(p))^2}{(1-\phi(p))^3}-\frac{(Q_1(p))^2}{(Q(p))^3} \\ &= \frac{(\phi_1(p))^2-(Q_1(p))^2}{(1-\phi(p))^3}+(Q_1(p))^2\left(\frac{1}{(Q(p))^3}-\frac{1}{(1-\phi(p))^3}\right) \\ &= \frac{(\phi_1(p))^2-(Q_1(p))^2}{(1-\phi(p))^3}+(Q_1(p))^2\left(\frac{(1-\phi(p))^3-(Q(p))^3}{(Q(p))^3(1-\phi(p))^3}\right) \\ &= (\phi_1(p)+Q_1(p))\left(\frac{\phi_1(p)-Q_1(p)}{(1-\phi(p))^3}\right)+(1-\phi(p)-Q(p))\times \\ &\quad\times\left(\frac{(Q_1(p))^2((1-\phi(p))^2+(1-\phi(p))Q(p)+(Q(p))^2)}{(Q(p))^3(1-\phi(p))^3}\right) \end{aligned}$$

and we can bound

$$\begin{aligned} &|e^{ipa}-1|\left|\frac{(\phi_1(p))^2}{(1-\phi(p))^3}-\frac{(Q_1(p))^2}{(Q(p))^3}\right| \\ &\leq \frac{c|a||\phi_1(p)+Q_1(p)|}{|p|^4}+\frac{c|a||1-\phi(p)-Q(p)|}{|p|^5}. \end{aligned} \tag{4.35}$$

The integrals are now bounded using (4.5) and (4.4) respectively, completing the proof of (4.1).

To prove (4.2) we first note that

$$u_{n-1}(x) = \frac{1}{(2\pi)^{3/2}} \int_{[-\pi,\pi]^3} \frac{e^{ipx}(1-\phi^n(p))}{1-\phi(p)}\, dp. \tag{4.36}$$

Set

$$v_\delta^n(x) = \int_\delta^n q_t(x)\, dt = \frac{1}{(2\pi)^{3/2}} \int_{R^3} e^{ipx} \frac{e^{-\delta|p|^2/2} - e^{-n|p|^2/2}}{|p|^2/2}\, dp. \tag{4.37}$$

We note that by the mean-value theorem

$$\begin{aligned} |q_t(x+a) - q_t(x)| &\leq C|a| \sup_{0\leq\theta\leq 1} \frac{|x+\theta a|}{t} q_t(x+\theta a) \\ &\leq C|a| \frac{|x|}{t} q_{2t}(x) \end{aligned} \tag{4.38}$$

where we have used the fact that under our assumptions

$$\frac{1}{2}|x| \leq |x+\theta a| \leq \frac{3}{2}|x|.$$

Since $t^{-1}q_t(x)$ is, up to a constant multiple, the transition density for Brownian motion in R^5, which has Green's function $C|x|^{-3}$, we have

$$|v_\delta^n(x+a) - v_\delta^n(x)| \leq C|a| \int_0^\infty \frac{|x|}{t} q_t(x)\, dt \leq C\frac{|a|}{|x|^2}. \tag{4.39}$$

Therefore, it suffices to bound as before an expression of the form (4.15) where u is replaced by u_{n-1} and v_δ is replaced by v_δ^n. All bounds involving v_δ^n on B and C are handled as before. One must verify that in each case no (divergent) factors involving n will remain. For example, whereas in the bound for the second term on the right hand side of (4.18) we were satisfied with a bound $c\delta^{1/2}$, see (4.20), when δ is replaced with n we now argue that

$$|Q_1(p)D_1(\frac{1}{Q(p)})| \leq \frac{cQ_1^2(p)}{Q^2(p)} \leq c|p|^2$$

on C, (where $Q(p) \geq d > 0$) and scaling out n now gives us a bound of $n^{-3/2}$.

The analogue of the first term on the right hand side of (4.15) is

$$\int_A (e^{ip(x+a)} - e^{ipx}) \left(\frac{1-\phi^n(p)}{1-\phi(p)} - \frac{e^{-\delta Q(p)} - e^{-nQ(p)}}{Q(p)} \right) dp \tag{4.40}$$
$$= \int_A (e^{ip(x+a)} - e^{ipx}) \left(\frac{1}{1-\phi(p)} - \frac{e^{-\delta Q(p)}}{Q(p)} \right) dp$$
$$- \int_A (e^{ip(x+a)} - e^{ipx}) \left(\frac{\phi^n(p)}{1-\phi(p)} - \frac{e^{-nQ(p)}}{Q(p)} \right) dp.$$

The first term is precisely the first term on the right hand side of (4.15), so that it only remains to bound the second term. It will be necessary to modify the preceeding proof at various stages. For example, in analogy with (4.23), let us show

$$\int_A |D_1 \left(\frac{\phi^n(p)}{1-\phi(p)} - \frac{e^{-nQ(p)}}{Q(p)} \right)| \, dp < \infty. \tag{4.41}$$

We write out

$$D_1 \left(\frac{\phi^n(p)}{1-\phi(p)} - \frac{e^{-nQ(p)}}{Q(p)} \right) \tag{4.42}$$
$$= n \left(\frac{\phi_1(p)\phi^{n-1}(p)}{1-\phi(p)} + \frac{Q_1(p)e^{-nQ(p)}}{Q(p)} \right)$$
$$+ \frac{\phi_1(p)\phi^n(p)}{(1-\phi(p))^2} + \frac{Q_1(p)e^{-nQ(p)}}{Q^2(p)}$$

By aperiodicity, for any $\epsilon > 0$ we have that $\sup_{|p|\geq\epsilon} |\phi(p)| \leq \gamma$ for some $\gamma < 1$, so that, using our assumption that $n - 1 > |x|^{1/4}$, we find that the factor $\phi^n(p)$ gives us rapid falloff in $|x|$, and clearly this also holds for $e^{-nQ(p)}$. In the region $|p| \leq \epsilon$ we will use (4.4)-(4.6). It is then easily seen that the integral (over $|p| \leq \epsilon$) of the second line in (4.42) is bounded after scaling in n. The third line is more delicate since both $\phi_1(p)/(1-\phi(p))^2$ and $Q_1(p)/Q^2(p)$ look like $|p|^{-3}$. To handle this we write

$$\frac{\phi_1(p)\phi^n(p)}{(1-\phi(p))^2} + \frac{Q_1(p)e^{-nQ(p)}}{Q^2(p)} \tag{4.43}$$
$$= (\phi_1(p) + Q_1(p)) \frac{\phi^n(p)}{(1-\phi(p))^2} - Q_1(p) \left(\frac{\phi^n(p)}{(1-\phi(p))^2} - \frac{e^{-nQ(p)}}{Q^2(p)} \right)$$

$$= (\phi_1(p) + Q_1(p))\frac{\phi^n(p)}{(1-\phi(p))^2} - Q_1(p)\phi^n(p)\left(\frac{1}{(1-\phi(p))^2} - \frac{1}{Q^2(p)}\right) - \frac{Q_1(p)}{Q^2(p)}\left(\phi^n(p) - e^{-nQ(p)}\right).$$

Using $|\phi^n(p)| \leq 1$, the first two terms on the right hand side are handled exactly the way we handled the second line of (4.26). As for the last term, we use (recall that $|p| \leq \epsilon$)

$$\frac{|Q_1(p)|}{Q^2(p)}|\phi^n(p) - e^{-nQ(p)}| \leq \frac{|Q_1(p)|}{Q^2(p)}|\phi(p) - e^{-Q(p)}|ne^{-nQ(p)/2}. \tag{4.44}$$

Since (4.4) shows that $|\phi(p) - e^{-Q(p)}| = O(|p|^2)$, the integral of this last term can also be bounded by scaling in n. The rest of the proof can be handled similarly.

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