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A lognormal central limit theorem for particle approximations of normalizing constants

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Abstract

Feynman-Kac models arise in a large variety of scientific disciplines including physics, chemistry and signal processing. Their mean field particle interpretations, termed commonly Sequential Monte Carlo or Particle Filters, have found numerous applications as they allow to sample approximately from sequences of complex probability distributions and estimate their associated normalizing constants. It is well-known that, under regularity assumptions, the relative variance of these normalizing constant estimates increases linearly with the time horizon n so that practitioners usually scale the number of particles N linearly w.r.t n to obtain estimates whose relative variance remains uniformly bounded w.r.t n. We establish here that, under this standard linear scaling strategy, the fluctuations of the normalizing constant estimates are lognormal as n, hence N, goes to infinity. For particle absorption models in a time-homogeneous environment and hidden Markov models in an ergodic random environment, we also provide more explicit descriptions of the limiting bias and variance.

Keywords: Feynman-Kac models; mean field interacting particle systems; nonlinear filtering; particle absorption models; quasi-invariant measures; central limit theorems.
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1 Introduction

1.1 Feynman–Kac models

Consider a Markov chain $(X_n)_{n\geq 0}$ on a measurable state space $(E,\mathcal{E})$ whose transitions are defined by a sequence of Markov kernels $(M_n)_{n\geq 1}$. We also introduce a collection of positive bounded and measurable functions $(G_n)_{n\geq 0}$ on E and associate to $(M_n)_{n\geq 1}$ and $(G_n)_{n\geq 0}$ the sequence of unnormalized Feynman–Kac measures $(\gamma_n)_{n\geq 0}$ on $(E,\mathcal{E})$, defined through their action on bounded real-valued measurable functions f by

$$\gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0\leq p<n} G_p(X_p)\right). \tag{1.1}$$

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The corresponding sequence of normalized Feynman–Kac probability measures $(\eta_n)_{n\geq 0}$ is thus defined by

$$\eta_n(f) := \gamma_n(f)/\gamma_n(1). \tag{1.2}$$

It is easily checked that the sequence of measures $(\gamma_n)_{n\geq 0}$ satisfies the evolution equation

$$\gamma_n(f) = \gamma_{n-1}(Q_n(f)) \tag{1.3}$$

where Q_n is a bounded positive integral operator given by

$$Q_n(f)(x) = \int G_{n-1}(x)\ M_n(x,dy)\ f(y). \tag{1.4}$$

Additionally, the normalizing constant $\gamma_n(1)$ satisfies the product formula

$$\gamma_n(1) = \mathbb{E}\left(\prod_{0\leq p<n} G_p(X_p)\right) = \prod_{0\leq p<n} \eta_p(G_p). \tag{1.5}$$

The detailed proofs of (1.3) and (1.5) can be found for instance in [4], section 2.1.1.

Throughout the paper, for any finite signed measure μ and bounded function f defined on the same space, we denote by $\mu(f)$ the Lebesgue integral of f with respect to μ, i.e. $\mu(f) := \int f(x)d\mu(x)$. Given a bounded integral operator $K(x,dx')$ from E into itself, we also denote by μK the measure resulting from the action of K on μ, i.e.

$$\mu K(dx') := \int \mu(dx)K(x,dx'),$$

while for a bounded measurable function f on E, we denote by $K(f)$ the bounded measurable function

$$K(f)(x) := \int K(x,dx')f(x').$$

We recall that the Dobrushin contraction or ergodic coefficient $\beta(K)$ of the Markov kernel K from E into itself is the quantity defined by

$$\beta(K) = \sup\left\{\|K(x,\bullet) - K(y,\bullet)\|_{\mathrm{tv}}\ ;\ (x,y)\in E^2\right\} \in [0,1], \tag{1.6}$$

where the total variation norm is given for any probability measures μ_1,μ_2 on $(E,\mathcal{E})$ by

$$\|\mu_1-\mu_2\|_{\mathrm{tv}} = \sup\left\{|\mu_1(f)-\mu_2(f)|\ ;\ f\in \mathrm{Osc}(E)\right\}.$$

Here $\mathrm{Osc}(E)$ stands for the set of $\mathcal{E}$-measurable functions f with oscillations $\mathrm{osc}(f) := \mathrm{Sup}_{x,y}|f(x)-f(y)| \leq 1$.

Feynman–Kac models appear in numerous scientific fields including, among others, signal processing, statistics, chemistry and statistical physics; see [5], [6] and [13]. Their interpretation is dependent on the application domain. We describe here briefly a few examples, two applications are discussed in more details in Sections 1.4.1 and 1.4.2.

Non-linear filtering. In a non-linear filtering framework, the measure η_n corresponds to the posterior distribution of the latent state X_n of a dynamic model at time n given the observations Y_n collected from time 0 to time $n-1$, and $\gamma_n(1)$ corresponds to the likelihood of these observations; that is

$$\eta_{n+1} = \mathrm{Law}\left(X_n \mid (Y_0,\ldots,Y_n)\right) \quad \text{and} \quad \gamma_{n+1}(1) = p_n(Y_0,\ldots,Y_n)$$

where p_n stands for the density of the observation sequence $(Y_p)_{0\leq p\leq n}$ w.r.t some reference measure.

Physics and Chemistry. In these contexts, Feynman–Kac models are widely used to describe molecular systems. The evolution equation (1.3) is interpreted as a discrete-time approximation of an imaginary time Schrödinger equation. The Markov kernel $M_n \simeq Id + L\ \Delta t$ corresponds to the discretization of continuous-time stochastic process X_t with infinitesimal generator L, $G_n = e^{-V\ \Delta t}$ where V is a potential energy and $\Delta t \ll 1$ a discretization time-step. With this notation, we observe that the integral operator (1.4) is such that $Q_n \simeq Id + L^V\ \Delta t$, where the Schrödinger operator L^V is defined by

$$L^V(f)(x) := L(f)(x) - V(x)f(x).$$

Using (1.3), this implies that

$$\gamma_n(f) - \gamma_{n-1}(f) \simeq \gamma_{n-1}\left(L^V(f)\right)\ \Delta t.$$

The resulting continuous-time model is given by the sequence of Feynman–Kac measures $(\gamma_t)_{t\geq 0}$ defined, in a weak sense, by the evolution equation

$$\frac{d}{dt}\gamma_t(f) = \gamma_t(L^V(f))$$

and its associated path space representation

$$\gamma_t(f) = \mathbb{E}\left(f(X_t)\ \exp\left\{-\int_0^t V(X_s)ds\right\}\right).$$

Here the normalizing constant $\gamma_t(1)$ corresponds to the free energy of the system, and its exponential decay is related to the top of the spectrum (whenever it exists) of the Schrödinger operator L^V. For a more thorough discussion on these continuous-time models and their applications in chemistry and physics, we refer the reader to [2], as well as to [17, 18, 20], the recent monograph [6] and the references therein.

1.2 Mean field particle models

A key issue with Feynman–Kac measures is that they are analytically intractable in most situations of interest. Over the past twenty years, particle methods, termed Diffusion or Quantum Monte Carlo methods in physics or Sequential Monte Carlo in statistics and applied probability, have emerged as the tools of choice to provide numerical approximations of these measures and of their associated normalizing constants. We give a brief overview of these methods here and refer the reader to [5, 6] for a more thorough and detailed treatment.

We first observe that the sequence $(\eta_n)_{n\geq 0}$ satisfies the following recursion for all $n \geq 1$

$$\eta_n = \Phi_n(\eta_{n-1}), \tag{1.7}$$

where Φ_n is a non-linear transformation on probability measures defined by

$$\Phi_n(\mu) := \Psi_{G_{n-1}}(\mu)M_n.$$

Here, given a bounded positive function G and a probability measure μ on E, Ψ_G denotes the Boltzmann-Gibbs transformation

$$\Psi_G(\mu)(dx) := \frac{1}{\mu(G)}G(x)\mu(dx). \tag{1.8}$$

It is possible to express Φ_n as

$$\Phi_n(\mu) = \mu K_{n,\mu}, \tag{1.9}$$

where $(K_{n,\mu})_{n\geq 1}$ is a collection of Markov kernels for any probability measure μ on E. There is not a unique $K_{n,\mu}$ satisfying (1.9). One can obviously use $K_{n,\mu}(x,dx') := \Phi_n(\mu)(dx')$ but there are alternatives. For example, if G_{n-1} takes its values in the interval $(0,1]$, $\Psi_{G_{n-1}}(\mu)$ can be expressed through a non-linear Markov transport equation

$$\Psi_{G_{n-1}}(\mu) = \mu S_{G_{n-1},\mu}, \tag{1.10}$$

with the non-linear Markov transition kernel

$$S_{G_{n-1},\mu}(x,dx') := G_{n-1}(x)\ \delta_x(dx') + (1-G_{n-1}(x))\ \ \Psi_{G_{n-1}}(\mu)(dx'),$$

so we can use

$$K_{n,\mu} := S_{G_{n-1},\mu} M_n. \tag{1.11}$$

The non-linear Markov representation (1.9) directly suggests a mean-field type particle approximation scheme for $(\eta_n)_{n\geq 0}$. For every $n \geq 0$, consider a $N-$tuple of elements of E denoted by $\xi_n^{(N)} = \left(\xi_n^{(N,i)}\right)_{1\leq i\leq N}$, whose empirical measure $\eta_n^N := \frac{1}{N}\sum_{j=1}^N \delta_{\xi_n^{(N,j)}}$ provides a particle approximation of η_n. The sequence $(\xi_n^{(N)})_{n\geq 0}$ evolves as an E^N-valued Markov chain whose initial distribution is given by $\mathbb{P}\left(\xi_0^{(N)} \in dx\right) = \prod_{i=1}^N \eta_0\left(dx_i\right)$, while the transition mechanism is specified for any $n \geq 1$ by

$$\mathbb{P}\left(\xi_n^{(N)} \in dx \ \middle|\ \mathcal{F}_{n-1}^N\right) = \prod_{i=1}^N\ K_{n,\eta_{n-1}^N}(\xi_{n-1}^{(N,i)}, dx^i). \tag{1.12}$$

Here $\mathcal{F}_{n-1}^N$ is the sigma-field generated by the random variables $(\xi_p^{(N)})_{0\leq p\leq n-1}$, and $dx := dx^1 \times \ldots \times dx^N$ stands for an infinitesimal neighborhood of a point $x = (x^1,\ldots,x^N) \in E^N$.

Using the identity (1.5), we can easily obtain a particle approximation $\gamma_n^N\left(1\right)$ of the normalizing constant $\gamma_n\left(1\right)$ by replacing the measures $(\eta_p)_{p=0}^{n-1}$ by their particle approximations $\left(\eta_p^N\right)_{p=0}^{n-1}$ to get

$$\gamma_n^N(1) := \prod_{0\leq p<n} \eta_p^N\left(G_p\right). \tag{1.13}$$

We define the normalized version of this estimate by

$$\overline{\gamma}_n^N(1) = \gamma_n^N(1)/\gamma_n(1) = \prod_{0\leq p<n} \eta_p^N(\overline{G}_p) \quad \text{with} \quad \overline{G}_n := G_n/\eta_n(G_n). \tag{1.14}$$

The main goal of this article is to establish a central limit theorem for $\log \overline{\gamma}_n^N(1)$ as $n \to \infty$ when the number of particles N is proportional to n.

1.3 Statement of the main result

To state our result, we need to introduce additional notations. We use the conventions $\Phi_0(\mu) := \eta_0$ for all μ, $K_{0,\mu}(x,\cdot) := \eta_0(\cdot)$ for all x, $\mathcal{F}_{-1}^N = \{\emptyset,\Omega\}$, $\eta_{-1} := \eta_0$ and $\eta_{-1}^N := \eta_0$. These conventions make (1.7)-(1.9)-(1.12) valid for $n = 0$.

We denote by V_n^N the centered local error random fields defined for $n \geq 0$ by

$$V_n^N := \sqrt{N}\ \left(\eta_n^N - \Phi_n(\eta_{n-1}^N)\right), \tag{1.15}$$

so that

$$\eta_n^N = \Phi_n(\eta_{n-1}^N) + \frac{1}{\sqrt{N}} V_n^N.$$

To describe the corresponding covariance structure, let us introduce for any $n \geq 0$, bounded measurable functions f_1, f_2 and probability measure μ the notation

$$\text{Cov}_{n,\mu}(f_1, f_2) := \mu\left[K_{n,\mu}(f_1 f_2) - K_{n,\mu}(f_1)K_{n,\mu}(f_2)\right].$$

We have the following explicit expression for conditional covariances

$$\mathbb{E}\left(V_n^N(f_1)V_n^N(f_2)\,\middle|\mathcal{F}_{n-1}^N\right) = \text{Cov}_{n,\eta_{n-1}^N}(f_1, f_2). \tag{1.16}$$

It is proved in [5, chapter 9] that $(V_n^N)_{n\geq 0}$ converges in law, as N tends to infinity, to a sequence of independent, Gaussian and centered random fields $(V_n)_{n\geq 0}$ with a covariance given for any bounded measurable functions f_1, f_2 on E by

$$C_{V_n}(f_1, f_2) := \mathbb{E}(V_n(f_1)V_n(f_2)) = \text{Cov}_{n,\eta_{n-1}}(f_1, f_2). \tag{1.17}$$

When $K_{n,\mu}(x,\cdot) := \Phi_n(\mu)$, (1.17) reduces to

$$C_{V_n}(f_1, f_2) = \eta_n(f_1 f_2) - \eta_n(f_1)\eta_n(f_2). \tag{1.18}$$

Let us now introduce the family of operators $(Q_{p,n})_{0\leq p\leq n}$ acting on the space of bounded measurable functions defined by

$$Q_{p,n}(f)(x) := \mathbb{E}\left(f(X_n)\prod_{p\leq q<n} G_q(X_q)\,\middle|\, X_p = x\right). \tag{1.19}$$

It is easily checked that $(Q_{p,n})_{0\leq p\leq n}$ forms a semigroup for which

$$\gamma_n = \gamma_p Q_{p,n}. \tag{1.20}$$

We also define

$$\overline{Q}_{p,n}(f) := \frac{Q_{p,n}(f)}{\eta_p Q_{p,n}(1)}. \tag{1.21}$$

Finally, we introduce the Markov kernel $P_{p,n}$ through its action on bounded measurable functions

$$P_{p,n}(f) := Q_{p,n}(f)/Q_{p,n}(1). \tag{1.22}$$

It is well-known in the literature, see for example [5, chapter 9], that, for any fixed n, the following convergence in distribution holds

$$\sqrt{N}\left(\overline{\gamma}_n^N(1) - 1\right) \xrightarrow[N\to+\infty]{d} \sum_{0\leq p<n} V_p(\overline{Q}_{p,n}(1)). \tag{1.23}$$

We are here interested in the fluctuations of $\overline{\gamma}_n^N(1)$ in an alternative scenario where both $n, N \to \infty$ with N proportional to n. It has been recently established in [3] that, under regularity conditions, the variance of $\overline{\gamma}_n^N(1)$ increases linearly with n so that one should increase N at least linearly w.r.t the time horizon n to obtain estimates whose variance is uniformly bounded w.r.t n. This linear variance growth had been empirically observed by practitioners for a long time and it is actually standard practice to set N proportional to n in applications; e.g. see [1] and [19] for recent work in computational statistics. However, the fluctuations of $\overline{\gamma}_n^N(1)$ under this linear scaling scheme have never been investigated. We establish in this paper that, in this regime, the observed behaviour is different from that described by (1.23). Indeed, the magnitude of the fluctuations of $\overline{\gamma}_n^N(1)$ around 1 do not vanish as n, N go to infinity, and they are described in the limit by a lognormal instead of a normal distribution.

Whereas no regularity assumption is required for (1.23) to hold, our result requires assumptions we now detail. First, the potential functions are assumed to satisfy

$$g_n := \sup G_n / \inf G_n < +\infty \quad \text{and} \quad g := \sup_{n \geq 0} g_n < +\infty. \tag{1.24}$$

Second, we assume that the Dobrushin coefficient of $P_{p,n}$, denoted $\beta(P_{p,n})$ and defined in (1.6), satisfies

$$\beta(P_{p,n}) \leq a \; e^{-\lambda(n-p)} \tag{1.25}$$

for some finite constant $a < +\infty$ and some positive $\lambda > 0$. Third, we assume that there exists a finite constant κ such that the kernels $K_{n,\mu}$ satisfy an inequality of the following form

$$\text{Sup}_{x \in E} \left| \left[K_{n,\mu_1} - K_{n,\mu_2} \right] (f)(x) \right| \leq \kappa \left| \left[\mu_1 - \mu_2 \right] (T_n(f, \mu_2)) \right|, \tag{1.26}$$

for any two probability measures μ_1, μ_2 on E and any measurable function f with oscillation $\text{osc}(f) \leq 1$, where $T_n(f, \mu_2)$ is a measurable map with oscillations inferior or equal to 1 that may depend on n, f, μ_2. This regularity condition can be extended to Lipschitz type control estimates (1.26) involving several functions $T_{n,u}(f, \mu_2)$ indexed by some parameter u on some probability space; see for instance [6, 10].

In the rest of the paper, unless otherwise stated, we assume that (1.24)-(1.25)-(1.26) hold.

Several sufficient conditions on the Markov kernels M_n under which (1.25) holds are discussed in [5, Section 4.3] and in Section 3.4 in [10]. Conditions under which (1.26) is satisfied are given in Section 2.

The main result of the paper is the following theorem.

Theorem 1.1. *Assume (1.24)-(1.25)-(1.26), and define v_n as*

$$v_n := \sum_{0 \leq p < n} \mathbb{E}\left(V_p(\overline{Q}_{p,n}(1))^2 \right) = \sum_{0 \leq q < n} \text{Cov}_{q,\eta_{q-1}}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1)).$$

Assume that N depends on n and satisfies

$$\lim_{n \to +\infty} \frac{n}{N} = \alpha \in (0, +\infty),$$

and that

$$\lim_{n \to +\infty} \frac{v_n}{n} = \sigma^2 \in (0, +\infty). \tag{1.27}$$

Then the following convergence in distribution holds

$$\log \overline{\gamma}_n^N(1) \xrightarrow[n \to +\infty]{d} \mathcal{N}\left(-\frac{1}{2}\alpha\sigma^2, \alpha\sigma^2 \right), \tag{1.28}$$

where $\mathcal{N}(u, v)$ denotes the normal distribution of mean u and variance v.

Remark 1.2. It follows from the continuous mapping theorem that $\overline{\gamma}_n^N(1)$ follows asymptotically a lognormal distribution. The relationship between the asymptotic bias and variance in (1.28) is unsurprising since $\mathbb{E}(\overline{\gamma}_n^N(1)) = 1$ for any n, N; see [5, Proposition 7.4.1.].

Remark 1.3. Under assumption (1.24), it can be easily checked that one always has $\sup_n \frac{v_n}{n} < +\infty$. If, in addition to (1.24)-(1.25)-(1.26), we assume that $\liminf_{n \to +\infty} \frac{v_n}{n} > 0$ instead of the stronger assumption (1.27), the proof of Theorem 1.1 still leads to a central limit theorem of the form

$$\frac{1}{\sqrt{\alpha \frac{v_n}{n}}} \left(\log \overline{\gamma}_n^N(1) + \frac{\alpha}{2} \frac{v_n}{n} \right) \xrightarrow[n \to +\infty]{d} \mathcal{N}(0, 1).$$

The result (1.28) was conjectured by Pitt et al. [19]. We believe that this theorem could be established under the weaker stability conditions of [11] or [23] at the price of a significantly more complex proof. Compelling empirical evidence can be found in [19] and [12].

The fact that $\overline{\gamma}_n^N(1)$ exhibits asymptotically lognormal fluctuations has been used in recent work to provide quantitative guidelines on how to select the number of particles in a particle filter when the normalizing constant estimate is used within a Metropolis–Hastings scheme; see [19], [12] and [21]. Another application of (1.28) is to the computation of Bayes factors for long time series data in scenarios where the likelihood of candidate models is computed using particle methods. Equation (1.28) suggests that it is possible to perform an approximate bias correction and obtain approximate confidence intervals for the particle estimate of the log-Bayes factor whenever estimates of the variances of normalizing constant estimates are available.

1.4 Some illustrations

In this section, we discuss two applications of Theorem 1.1 where the variance expression (1.27) can be made more explicit.

1.4.1 Particle absorption models

Consider a particle in an absorbing random medium, whose successive states $(X_n)_{n\geq 0}$ evolve according to a Markov kernel M. At time n, the particle is absorbed with probability $1-G(X_n)$ where G is a $[0,1)$-valued potential function. Letting $G_n := G$ for all $n \geq 0$ and $M_n := M$ for all $n \geq 1$, the connection with the Feynman–Kac formalism is the following: denoting by T the absorption time of the particle, we have $\gamma_n(1) = \mathbb{P}(T \geq n)$, and $\eta_n = \text{Law}(X_n \mid T \geq n)$. In this situation, the multiplicative formula (1.5) takes the form

$$\mathbb{P}(T \geq n) = \prod_{0 \leq m < n} \mathbb{P}(T \geq m+1 \mid T \geq m),$$

where

$$\mathbb{P}(T \geq m+1 \mid T \geq m) = \int G(x)\, \mathbb{P}(X_m \in dx \mid T \geq m) = \eta_m(G).$$

In this context, we have $\Phi_n = \Phi$ for all $n \geq 1$, and conditions (1.24)-(1.25) ensure that Φ admits a unique fixed point measure η_∞ such that

$$\text{Law}(X_n \,| T \geq n) \longrightarrow_{n\to\infty} \eta_\infty = \Phi(\eta_\infty).$$

Moreover, we have

$$\overline{Q}_{0,n}(1)(x) = \mathbb{P}(T \geq n \,| X_0 = x)/\mathbb{P}(T \geq n) \longrightarrow_{n\to\infty} h(x).$$

Setting $\overline{Q} = Q/\eta_\infty Q(1)$, we find that the function h satisfies the spectral equations

$$\overline{Q}(h) = h \Leftrightarrow Q(h) = \lambda h, \text{with } \lambda = \eta_\infty(G).$$

The measure η_∞ is the so-called quasi-invariant or Yaglom measure. Under some additional conditions, the parameter λ coincides with the largest eigenvalue of the integral operator Q and h is the corresponding eigenfunction. In statistical physics, Q comes from a discrete-time approximation of a Schrödinger operator and h is called the ground state function. For a more thorough discussion, we refer the reader to Chapters 2 and 3 in [5] and Chapter 7 in [6].

In this scenario, the limiting variance σ^2 appearing in (1.28) is given by

$$\sigma^2 = \text{Cov}_{1,\eta_\infty}(h,h). \tag{1.29}$$

In particular, if the Markov kernels used in the particle approximation scheme are given by $K_\eta(x, \bullet) = \Phi(\eta)$, then we obtain using (1.18) that $\sigma^2 = \eta_\infty\left([h-1]^2\right)$. The detailed statement and proof of these results are provided in Section 3.3.

1.4.2 Non-linear filtering

Let $(X_n, Y_n)_{n\geq 0}$ be a Markov chain on some product state space $E_1 \times E_2$ whose transition mechanism takes the form

$$\mathbb{P}\left((X_n, Y_n) \in d(x,y) \mid (X_{n-1}, Y_{n-1})\right) = M_n(X_{n-1}, dx)\; g_n(y,x)\; \nu_n(dy),$$

where $(\nu_n)_{n\geq 0}$ is a sequence of positive measures on E_2, $(M_n)_{n\geq 0}$ is a sequence of Markov kernels from E_1 into itself, and $(g_n)_{n\geq 0}$ is a sequence of density functions on $E_2 \times E_1$. The aim of non-linear filtering is to infer the hidden Markov process $(X_n)_{n\geq 0}$ given a realization of the observation sequence $Y = y$. It is easily checked that

$$\eta_n = \mathsf{Law}\left(X_n \mid Y_m = y_m,\ 0 \leq m < n\right),$$

using $G_n := g_n(y_n, \bullet)$ in (1.1). Furthermore, the density denoted $p_n(y_0, \ldots, y_n)$ of the random sequence of observations $(Y_0, \ldots, Y_n)$ w.r.t. to the product measure $\otimes_{0\leq p\leq n}\nu_p$ evaluated at the observation sequence, that is the so-called marginal likelihood, is equal to the normalizing constant $\gamma_{n+1}(1)$. In this context, the multiplicative formula (1.5) takes the form

$$p_n(y_0, \ldots, y_n) = \prod_{0\leq m\leq n} q_m(y_m \mid y_l,\ 0 \leq l < m)$$

with

$$q_m(y_m \mid y_l,\ 0 \leq l < m) = \int g_m(y_m, x)\; \mathbb{P}\left(X_m \in dx \mid Y_l = y_l,\ 0 \leq l < m\right) = \eta_m(G_m).$$

For time-homogeneous models $(g_m, M_m) = (g, M)$ associated to an ergodic process Y satisfying a random environment version of Assumption (1.25) detailed in Section 3.4, the ergodic theorem implies that the normalized log-likelihood function converges to the entropy of the observation sequence

$$\begin{aligned}\frac{1}{n+1}\log p_n(Y_0, \ldots, Y_n) &= \frac{1}{n+1}\sum_{0\leq m\leq n}\log q_m(Y_m \mid Y_l,\ 0 \leq l < m)\\ &\longrightarrow_{n\to\infty}\quad \mathbb{E}\left(\log q(Y_0 \mid Y_m,\ m < 0)\right),\end{aligned}$$

where $q(Y_0 \mid Y_m,\ m < 0)$ is the conditional density of the random variable Y_0 w.r.t the infinite past. In Section 3.4, we shall prove the existence of a limiting measure η_∞^Y, and function h^Y such that

$$q(Y_0 \mid Y_m,\ m < 0) = \eta_\infty^Y(g(Y_0, \bullet))$$

and

$$\overline{Q}_{0,n+1}^Y(1)(x) := \frac{q_{0,n}((Y_0, \ldots, Y_n) \mid x)}{\int \eta_\infty^Y(dx)\; q_{0,n}((Y_0, \ldots, Y_n) \mid x)} \longrightarrow_{n\to\infty} h^Y(x)$$

where $q_{0,n}((Y_0, \ldots, Y_n)|x)$ stands for the conditional density of $(Y_0, \ldots, Y_n)$ given $X_0 = x$. Similar results have been recently established in [24] using slightly more restrictive assumptions. In this situation, the limiting variance σ^2 appearing in (1.28) satisfies

$$\sigma^2 = \mathbb{E}\left(\mathsf{Cov}_{1,\eta_\infty^{\theta^{-1}(Y)}}^{\theta^{-1}(Y)}(h^Y, h^Y)\right), \tag{1.30}$$

where θ denotes the shift operator, and, if the Markov kernels used by the particle approximation scheme are given by $K_{n,\eta}(x,\bullet) = \Phi_n(\eta)$ associated to the potential $G_n := g_n(Y_n,\bullet)$, then we obtain using (1.18)

$$\sigma^2 = \mathbb{E}\left(\eta_\infty^{\theta^{-1}(Y)}\left(\left[h^Y - 1\right]^2\right)\right).$$

The detailed statement and proof of these results are provided in Section 3.4.

1.5 Notations and conventions

We denote, respectively, by $\mathcal{M}(E)$, $\mathcal{P}(E)$ and $\mathcal{B}_b(E)$, the set of all finite signed measures on the measurable space $(E,\mathcal{E})$ equipped with the total variation norm $\|\bullet\|_{\rm tv}$, the subset of all probability measures, and the Banach space of all bounded and measurable functions f equipped with the uniform norm $\|f\| = \mathrm{Sup}_{x\in E}|f(x)|$. We also denote by $\|X\|_m = \mathbb{E}(|X|^m)^{1/m}$, the $\mathbb{L}_m$-norm of the random variable X, where $m \geq 1$.

In the sequel, the generic notation c is used to denote a constant that depends only on the model. To alleviate notations, we do not use distinct indices (e.g. $c_1, c_2, \ldots$) each time such a constant appears, and keep using the notation c even though the corresponding constant may vary from one statement to the other. However, to avoid confusion, we sometimes make a distinction between such constants by using c, c', c'' inside an argument. When the constant also depend on additional parameters $p_1,\ldots,p_\ell$, this is explicitly stated in the notation by writing $c(p_1,\ldots,p_\ell)$.

1.6 Organization of the paper

The rest of the paper is organized as follows. Section 2 establishes some basic regularity properties of the covariance operator. Section 3 analyzes the long-time behavior of Feynman–Kac semigroups and provides a precise description of the asymptotic behavior of the variance term v_n appearing in Theorem 1.1 in two special cases: time-homogeneous models and models in a stationary ergodic random environment.

The key result, Theorem 1.1, is established in Section 4. The main idea is to expand $\log\overline{\gamma}_n^N(1)$ in terms of local fluctuation terms of the form V_k^N. Broadly speaking, the contribution of quadratic terms in the expansion amounts to an asymptotically deterministic bias term whose fluctuations are controlled with variance bounds, while the contribution of linear terms is treated by invoking the martingale central limit theorem.

2 Regularity of the covariance function

We first note that, in the special case where $K_{n,\eta}(x,\bullet) = \Phi_n(\eta)$ for all x, Property (1.26) is in fact a consequence of (1.24). Indeed, we can then write

$$[\Phi_n(\mu_1) - \Phi_n(\mu_2)](f) = \frac{1}{\mu_1(G_{n-1})}\,[\mu_1-\mu_2]\left(G_{n-1}M_n(f - \Phi_n(\mu_2)(f))\right),$$

and check that for any $f \in \mathrm{Osc}(E)$

$$\|\left[K_{n,\mu_1} - K_{n,\mu_2}\right](f)\| \leq 2g\ \left|[\mu_1-\mu_2]\left(h_{n,\mu_2}\right)\right|,$$

where g is defined in (1.24) and

$$h_{n,\mu} = \frac{1}{2\|G_{n-1}\|}\ G_{n-1}M_n(f - \Phi_n(\mu)(f)) \in \mathrm{Osc}(E).$$

In the alternative case (1.11), we have

$$\left[K_{n,\mu_1} - K_{n,\mu_2}\right](f) = (1 - G_{n-1})\ \ \left[\Phi_n(\mu_1) - \Phi_n(\mu_2)\right](f)$$

so that (1.26) is also satisfied.

Observe that (1.26) immediately implies the following Lipschitz-type property

$$\sup_{x\in E}\|K_{n,\mu_1}(x,\cdot)-K_{n,\mu_2}(x,\cdot)\|_{\mathrm{tv}}\leq\kappa\|\mu_1-\mu_2\|_{\mathrm{tv}}. \tag{2.1}$$

Proposition 2.1. *There exists $c<\infty$ such that for any $\mu_1,\mu_2\in\mathcal{P}(E)$ and any functions $f_1,f_2\in\mathrm{Osc}(E)$*

$$|\mathrm{Cov}_{n,\mu_1}(f_1,f_2)-\mathrm{Cov}_{n,\mu_2}(f_1,f_2)|\leq c\|\mu_1-\mu_2\|_{\mathrm{tv}}. \tag{2.2}$$

Proof. We have

$$\begin{aligned}&\mathrm{Cov}_{n,\mu_1}(f_1,f_2)-\mathrm{Cov}_{n,\mu_2}(f_1,f_2)\\ &=[\Phi_n(\mu_1)-\Phi_n(\mu_2)](f_1f_2)+[\mu_2-\mu_1](K_{n,\mu_2}(f_1)K_{n,\mu_2}(f_2))\\ &\qquad\qquad+\mu_1(K_{n,\mu_2}(f_1)K_{n,\mu_2}(f_2)-K_{n,\mu_1}(f_1)K_{n,\mu_1}(f_2))\end{aligned}$$

and

$$[\Phi_n(\mu_1)-\Phi_n(\mu_2)]=\mu_1[K_{n,\mu_1}-K_{n,\mu_2}]+[\mu_1-\mu_2]K_{n,\mu_2}.$$

There is no loss of generality in assuming that $\mu_2(f_1)=\mu_2(f_2)=0$ so that $\|f_i\|\leq\mathrm{osc}(f_i)\leq 1$. Thus, using

$$\begin{aligned}&\|K_{n,\mu_2}(f_1)K_{n,\mu_2}(f_2)-K_{n,\mu_1}(f_1)K_{n,\mu_1}(f_2)\|\\ &\leq\|K_{n,\mu_1}(f_1)-K_{n,\mu_2}(f_1)\|+\ \|K_{n,\mu_1}(f_2)-K_{n,\mu_2}(f_2)\|,\end{aligned}$$

the desired conclusion follows from (2.1). ☐

Finally, we can also easily check that there exists $c<\infty$ such that for any $f_1,f_2,\phi_1,\phi_2\in\mathcal{B}_b(E)$

$$|\mathrm{Cov}_{n,\mu}(f_1,f_2)-\mathrm{Cov}_{n,\mu}(\phi_1,\phi_2)|\leq c\left(\|f_1\|\ \|f_2-\phi_2\|+\|\phi_2\|\ \|f_1-\phi_1\|\right). \tag{2.3}$$

3 Feynman–Kac semigroups

3.1 Contraction estimates

We denote by $(\Phi_{p,n})_{0\leq p\leq n}$ the semigroup of non-linear operators acting on probability measures defined by

$$\Phi_{p,n}:=\Phi_n\circ\cdots\circ\Phi_{p+1},$$

so that

$$\eta_n(f)=\Phi_{p,n}(\eta_p)(f)=\eta_pQ_{p,n}(f)/\eta_pQ_{p,n}(1)=\Psi_{Q_{p,n}(1)}(\eta_p)P_{p,n}(f). \tag{3.1}$$

We have

$$\sup_{\mu,\nu}\|\Phi_{p,n}(\mu)-\Phi_{p,n}(\nu)\|_{\mathrm{tv}}=\beta(P_{p,n}), \tag{3.2}$$

see for example [5, chapter 4]. We also define

$$g_{p,n}:=\sup_{x,y\in E}[Q_{p,n}(1)(x)/Q_{p,n}(1)(y)]\quad\text{and}\quad d_{p,n}(f)=\overline{Q}_{p,n}(f-\eta_n(f)).$$

We observe that $\overline{Q}_{n,n+1}(1)=G_n/\eta_n(G_n)=\overline{G}_n$ and

$$d_{p,n}(\overline{G}_n)=\overline{Q}_{p,n}(\overline{Q}_{n,n+1}(1)-1)=\overline{Q}_{p,n+1}(1)-\overline{Q}_{p,n}(1). \tag{3.3}$$

We will use the fact that the semigroup $Q_{p,n}$ satisfies a decomposition similar to (1.5); that is for any probability measure μ on E

$$\mu Q_{p,n}(1) = \prod_{p\leq q<n} \Phi_{p,q}(\mu)(G_q). \tag{3.4}$$

Combining (1.21) and (3.4), we can also write

$$\log \overline{Q}_{p,n}(1)(x) = \sum_{p\leq q<n} \left[\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)\right]. \tag{3.5}$$

Lemma 3.1. *For any* $0 \leq p \leq n$ *and any* $f \in \mathrm{Osc}(E)$*, we have*

$$g_{p,n} \leq b := \exp\left(a(g-1)/(1-e^{-\lambda})\right) \quad \textit{and} \quad \|d_{p,n}(f)\| \leq ab\, e^{-\lambda(n-p)}. \tag{3.6}$$

Additionally, for any $\mu, \nu \in \mathcal{P}(E)$*, we have*

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\mathrm{tv}} \leq ab\, e^{-\lambda(n-p)}\, \|\mu - \nu\|_{\mathrm{tv}}. \tag{3.7}$$

Proof. Using the decomposition (3.4), we have

$$\frac{Q_{p,n}(1)(x)}{Q_{p,n}(1)(y)} = \frac{\delta_x Q_{p,n}(1)}{\delta_y Q_{p,n}(1)} = \exp\left\{\sum_{p\leq q<n} \left(\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\delta_y)(G_q)\right)\right\}. \tag{3.8}$$

Using the inequality $|\log u - \log v| \leq \frac{|u-v|}{\min(u,v)}$, valid for any $u, v > 0$, we deduce the inequality

$$\frac{Q_{p,n}(1)(x)}{Q_{p,n}(1)(y)} \leq \exp\left\{\sum_{p\leq q<n} \widetilde{g}_q \times \left|\Phi_{p,q}(\delta_x)(\widetilde{G}_q) - \Phi_{p,q}(\delta_y)(\widetilde{G}_q)\right|\right\},$$

with $\widetilde{G}_q := G_q/\mathrm{osc}(G_q)$ (and the convention that $\widetilde{G}_q := 1$ if G_q is constant) and $\widetilde{g}_q := \mathrm{osc}(G_q)/\inf G_q \leq g_q - 1 \leq g - 1$.

Using (1.25) and (3.2), we deduce that

$$g_{p,n} \leq \exp\left\{a(g-1) \sum_{p\leq q<n} e^{-\lambda(q-p)}\right\} \leq b.$$

This ends the proof of the l.h.s. of (3.6). The proof of the r.h.s. of (3.6) comes from the following expression for $d_{p,n}(f)$

$$d_{p,n}(f) = \overline{Q}_{p,n}(1) \times P_{p,n}\left[f - \Psi_{Q_{p,n}(1)}(\eta_p) P_{p,n}(f)\right]$$

which implies, using the fact that $\|\overline{Q}_{p,n}(1)\| \leq g_{p,n}$, that

$$\|d_{p,n}(f)\| \leq g_{p,n}\, \beta(P_{p,n})\, \mathrm{osc}(f) \leq ab\, e^{-\lambda(n-p)}\, \mathrm{osc}(f). \tag{3.9}$$

From [5, Section 4.3], see also Proposition 3.1 in [10], we have

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\mathrm{tv}} \leq g_{p,n}\, \beta(P_{p,n})\, \|\mu - \nu\|_{\mathrm{tv}}.$$

Using (3.6), we conclude that

$$\|\Phi_{p,n}(\mu) - \Phi_{p,n}(\nu)\|_{\mathrm{tv}} \leq ab\, e^{-\lambda(n-p)}\, \|\mu - \nu\|_{\mathrm{tv}}.$$

This ends the proof of the lemma. □

3.2 Limiting semigroup

We now state a general theorem on the convergence of $\overline{Q}_{p,n}(1)$ as $n \to +\infty$.

Theorem 3.2. *There exists $c < \infty$ such that for all $0 \le p \le n$*

$$\left\|\overline{Q}_{p,n}(1) - \overline{Q}_{p,\infty}(1)\right\| \le c\, e^{-\lambda(n-p)}, \tag{3.10}$$

where the limiting function $\overline{Q}_{p,\infty}(1)$ is defined through the following series

$$\log \overline{Q}_{p,\infty}(1)(x) := \sum_{q \ge p} \left[\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)\right]. \tag{3.11}$$

Proof of Theorem 3.2. We first check that the function $\overline{Q}_{p,\infty}(1)$ is well defined, using the fact that, as in the proof of Lemma 3.1,

$$\left|\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)\right| \le a(g-1)e^{-\lambda(q-p)}.$$

One then obtains

$$\left|\log\ \overline{Q}_{p,n}(1)(x) - \log\ \overline{Q}_{p,\infty}(1)(x)\right| \le \sum_{q \ge n} \left|\log \Phi_{p,q}(\delta_x)(G_q) - \log \Phi_{p,q}(\eta_p)(G_q)\right|,$$

whence

$$\left|\log\ \overline{Q}_{p,n}(1)(x) - \log\ \overline{Q}_{p,\infty}(1)(x)\right| \le \sum_{q \ge n} a(g-1)\ e^{-\lambda(q-p)} \le c\ e^{-\lambda(n-p)}.$$

Using the inequality $|e^u - e^v| \le |u - v|\ \ \max\left(e^u, e^v\right)$, we finally check that

$$\left\|\overline{Q}_{p,n}(1) - \overline{Q}_{p,\infty}(1)\right\| \le c\ \left\|\log\ \overline{Q}_{p,n}(1) - \log\ \overline{Q}_{p,\infty}(1)\right\|,$$

as $\|\overline{Q}_{p,n}(1)\| \le g_{p,n} \le g$. This ends the proof of (3.10). □

3.3 The time-homogeneous case

Here we consider the special case of time-homogeneous models, where there exist G, M, K such that $G_n = G$ for all $n \ge 0$, $M_n = M$ and $K_n = K$ for all $n \ge 1$. Our assumptions imply the existence of a unique fixed point $\eta_\infty = \Phi(\eta_\infty)$ towards which η_n converges exponentially fast

$$\|\Phi^n(\eta_0) - \eta_\infty\|_{\mathrm{tv}} \le ab\ e^{-\lambda n} \text{ for any } n \ge 0. \tag{3.12}$$

This result and a more thorough discussion on invariant measures of Feynman–Kac semigroups and their connexions with particle absorption models, Yaglom limits and quasi-invariant measures can be found in [7, 8, 9], chapter 4 in [5], as well as chapters 12 and 13 in [6].

In this situation, Theorem 3.2 leads to a precise description of the asymptotic behavior of the variance term v_n appearing in Theorem 1.1. Define indeed the function h by

$$\log h(x) := \sum_{n \ge 0} \left[\log \Phi^n(\delta_x)(G_n) - \log \Phi^n(\eta_\infty)(G_n)\right].$$

In the stationary version of the model where $\eta_0 := \eta_\infty$, h corresponds to the limiting function $\overline{Q}_{0,\infty}(1)$ whose existence is asserted by Theorem 3.2. In this situation, it turns out that, by stationarity, $\overline{Q}_{n,\infty}(1) = h$ for all $n \ge 1$.

Proposition 3.3. *There exists $c < \infty$ such that for any $p \ge 0$*

$$||\overline{Q}_{p,\infty}(1) - h|| \le ce^{-\lambda p}. \tag{3.13}$$

Corollary 3.4. *The variance term v_n appearing in Theorem 1.1 satisfies*

$$\frac{v_n}{n} = \frac{1}{n} \sum_{0 \le q < n} \mathrm{Cov}_{q,\eta_{q-1}}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1)) = \mathrm{Cov}_{\eta_\infty}(h,h) + \mathrm{O}(1/n), \tag{3.14}$$

where we use the notation Cov_η to denote the common value of $\mathrm{Cov}_{q,\eta}$ for $q \ge 1$.

An alternative spectral characterization of h is given in the following corollary. In the homogeneous case, $Q_{p,p+1}$ does not depend on p, so we use the simpler notation Q.

Corollary 3.5. *In the homogeneous case, $(\eta_\infty Q(1), h)$ is characterized as the unique pair (ζ, f) such that $Q(f) = \zeta f$ and $\eta_\infty(f) = 1$.*

Proof of Proposition 3.3. Using the exponential convergence to η_∞ stated in (3.12), and the Lipschitz property (3.7), we have

$$\sum_{q \ge p} [\log \Phi_{p,q}(\eta_p)(G_q) - \log \Phi_{p,q}(\eta_\infty)(G_q)] \le c \sum_{q \ge p} e^{-\lambda((q-p)+p)} \le c' \ e^{-\lambda p}.$$

We conclude as in the proof of Theorem 3.2. □

Proof of Corollary 3.4. Using the Lipschitz property (2.2), and the fact that $\|\overline{Q}_{q,n}(1)\| \le g$ for all $q \le n$, we see that replacing each η_{q-1} in the l.h.s. of (3.14) by η_∞ leads to a $O(1/n)$ error term. Then, using Theorem 3.2 and (2.3), we see that we can replace each $\overline{Q}_{q,n}(1)$ term by $\overline{Q}_{q,\infty}(1)$ in the l.h.s. of (3.14), and commit no more than a $O(1/n)$ overall error. Finally, (3.13) allows us to replace each $\overline{Q}_{q,\infty}(1)$ by h, again with an overall $O(1/n)$ error term. □

Proof of Corollary 3.5. We consider the stationary version of the model where we start with $\eta_0 := \eta_\infty$.

We first check that one indeed has $\eta_\infty(h) = 1$ and $Q(h) = \eta_\infty(Q(1))h$. By Theorem 3.2, we have

$$\lim_{n \to +\infty} \left\|\overline{Q}_{0,n}(1) - h\right\| = 0. \tag{3.15}$$

Since by construction, $\eta_\infty \overline{Q}_{0,n}(1) = 1$, (3.15) yields $\eta_\infty(h) = 1$. Then, due to stationarity, one has $\overline{Q}_{p,n} = \overline{Q}^{n-p}$ where $\overline{Q}(f) := Q(f)/\eta_\infty Q(1)$, so that one can also deduce from (3.15) that $\overline{Q}(h) = h$, which yields $Q(h) = \eta_\infty(Q(1))h$.

Now consider a pair (ζ, f) such that $Q(f) = \zeta f$ and $\eta_\infty(f) = 1$, and let us show that $\zeta = \eta_\infty Q(1)$ and $f = h$.

By stationarity, one has

$$\overline{Q}_{0,n}(f) = Q^n(f)/\eta_\infty Q^n(1),$$

and we deduce from (3.4) and the stationarity of η_∞ that

$$\eta_\infty Q^n(1) = (\eta_\infty Q(1))^n .$$

Using the fact that $\Phi(\eta_\infty) = \eta_\infty$, we have the identity

$$\eta_\infty Q(f)/\eta_\infty(Q(1)) = \eta_\infty(f).$$

Since $Q(f) = \zeta f$ and $\eta_\infty(f) = 1$, it immediately follows that $\zeta = \eta_\infty(Q(1))$.

Consequently, the equality $Q(f) = \eta_\infty(Q(1))f$ implies that, for all $n \ge 1$, one has on the one hand

$$\overline{Q}_{0,n}(f) = f,$$

whereas on the other hand, we have for any bounded functions f_1, f_2

$$\overline{Q}_{0,n}(f_1 - f_2)(x) = \Phi_{0,n}(\delta_x)(f_1 - f_2) \times \overline{Q}_{0,n}(1)(x).$$

Letting $n \to \infty$, (3.12) and Theorem 3.2 yields

$$\lim_{n\to\infty} \overline{Q}_{0,n}(f_1 - f_2) = \eta_\infty(f_1 - f_2) \times h.$$

Using $f_1 := f$ and $f_2 := h$, we deduce that $f = h$. □

3.4 The random environment case

3.4.1 Description of the model

We consider a stationary and ergodic process $Y = (Y_n)_{n\in\mathbb{Z}}$ taking values in a measurable state space $(S, \mathcal{S})$. The process Y provides a random environment governing the successive transitions between step $n-1$ and step n in our model. In the sequel, we define and study the model for a given realization $y \in S^{\mathbb{Z}}$ of the environment. It is only in Corollary 3.7 that we exploit the ergodicity of Y to establish the almost sure limiting behavior of the variance v_n.

Specifically, we consider a family $(M_s)_{s\in S}$ of Markov kernels on E and a family $(G_s)_{s\in S}$ of positive bounded functions on E. For $n \in \mathbb{Z}$ and $y \in S^{\mathbb{Z}}$, we set $M_n^y := M_{y_n}$ and $G_n^y := G_{y_n}$. We then denote with a y superscript all the objects associated with the Feynman–Kac model using the sequence of kernels $(M_n^y)_{n\geq 1}$ and functions $(G_n^y)_{n\geq 0}$, i.e. the measures γ_n^y and η_n^y, the operators $\Phi_{p,n}^y$, $G_{p,n}^y$, $\mathrm{Cov}_{p,\eta}^y$, etc. To define the particle approximation scheme, we also consider a family of Markov kernels $(K_{(s,s'),\mu})_{s,s'\in S,\ \mu\in\mathcal{P}(E)}$ such that, for all s, s', μ, one has

$$\Psi_{G_s}(\mu) M_{s'} = \mu K_{(s,s'),\mu}.$$

We then use $K_{n,\mu}^y := K_{(y_{n-1},y_n),\mu}$ for all $n \geq 1$.

We define the shift operator on $S^{\mathbb{Z}}$ by setting $\theta(y) := (y_{n+1})_{n\in\mathbb{Z}}$ for any $y = (y_n)_{n\in\mathbb{Z}} \in S^{\mathbb{Z}}$. With our definitions, one has for all $0 \leq p \leq n$

$$Q_{p,n}^y = Q_{0,n-p}^{\theta^p(y)},\ \Phi_{p,n}^y = \Phi_{0,n-p}^{\theta^p(y)},$$

and, in particular,

$$\Phi_{0,n}^y = \Phi_{0,n-p}^{\theta^p(y)} \circ \Phi_{0,p}^y. \tag{3.16}$$

Our assumptions on the model are that E has a Polish space structure, and that the bounds listed in (1.24), (1.25) and (1.26) hold for M_n^y, G_n^y and $K_{n,\mu}^y$ uniformly over $y \in S^{\mathbb{Z}}$.

3.4.2 Contraction properties

Rewriting (1.25), (3.2) and (3.7) in the present context, we have for any $y \in S^{\mathbb{Z}}$

$$\beta\left(P_{0,n}^y\right) = \sup_{\mu,\nu} \|\Phi_{0,n}^y(\mu) - \Phi_{0,n}^y(\nu)\|_{\mathrm{tv}} \leq a\, e^{-\lambda n} \tag{3.17}$$

and

$$\|\Phi_{0,n}^y(\mu) - \Phi_{0,n}^y(\nu)\|_{\mathrm{tv}} \leq ab\, e^{-\lambda n}\, \|\mu - \nu\|_{\mathrm{tv}}, \tag{3.18}$$

with the constant b defined in (3.6). Using (3.16), we have

$$\Phi_{0,n+m}^{\theta^{-(n+m)}(y)} = \Phi_{0,n}^{\theta^{-n}(y)} \circ \Phi_{0,m}^{\theta^{-(n+m)}(y)},$$

so that, using (3.17), one obtains

$$\sup_{\mu,\nu}\|\Phi_{0,n}^{\theta^{-n}(y)}(\mu)-\Phi_{0,n+m}^{\theta^{-(n+m)}(y)}(\nu)\|_{\rm tv}\leq a\;e^{-\lambda n}.$$

Arguing as in [15, 24], we conclude that, for any $f\in\mathcal{B}_b(E)$ and any $\mu\in\mathcal{P}(E)$, $\Phi_{0,n}^{\theta^{-n}(y)}(\mu)(f)$ is a Cauchy sequence so that $\Phi_{0,n}^{\theta^{-n}(y)}(\mu)$ weakly converges to a measure η^y_∞, as $n\to\infty$. In addition, for any $n\geq 0$, we have

$$\Phi^y_{0,n}(\eta^y_\infty)=\eta_\infty^{\theta^n(y)} \tag{3.19}$$

and exponential convergence to equilibrium

$$\sup_{\mu}\|\Phi_{0,n}^{\theta^{-n}(y)}(\mu)-\eta^y_\infty\|_{\rm tv}\leq a\;e^{-\lambda n}. \tag{3.20}$$

We now restate the conclusion of Theorem 3.2 in the present context : there exists $c<\infty$ such that for all $0\leq p\leq n$

$$\left\|\overline{Q}^y_{p,n}(1)-\overline{Q}^y_{p,\infty}(1)\right\|\leq c\;e^{-\lambda(n-p)}, \tag{3.21}$$

where the limiting function $\overline{Q}^y_{p,\infty}(1)$ is defined through the series

$$\log\overline{Q}^y_{p,\infty}(1)(x):=\sum_{q\geq p}\left[\log\Phi^y_{p,q}(\delta_x)(G^y_q)-\log\Phi^y_{p,q}(\eta^y_p)(G^y_q)\right]. \tag{3.22}$$

We now define the map h^y by

$$\log h^y(x):=\sum_{q\geq 0}\left[\log\Phi^y_{0,q}(\delta_x)(G^y_q)-\log\Phi^y_{0,q}(\eta^y_\infty)(G^y_q)\right].$$

Proposition 3.6. *There exists $c<\infty$ such that for any $y\in S^{\mathbb{Z}}$ and any $p\geq 0$*

$$||\overline{Q}^y_{p,\infty}(1)-h^{\theta^p(y)}||\leq ce^{-\lambda p}. \tag{3.23}$$

Proof of Proposition 3.6. Setting $q:=q-p$ in (3.22), we find that

$$\log\overline{Q}^y_{p,\infty}(1)(x)=\sum_{q\geq 0}\left[\log\Phi_{0,q}^{\theta^p(y)}(\delta_x)(G_q^{\theta^p(y)})-\log\Phi_{0,q}^{\theta^p(y)}(\eta^y_p)(G_q^{\theta^p(y)})\right]$$

for any $p\geq 0$, while

$$\log h^{\theta^p(y)}(x)=\sum_{q\geq 0}\left[\log\Phi_{0,q}^{\theta^p(y)}(\delta_x)(G_q^{\theta^p(y)})-\log\Phi_{0,q}^{\theta^p(y)}(\eta_\infty^{\theta^p(y)})(G_q^{\theta^p(y)})\right].$$

Using (3.20), we obtain

$$\|\eta^y_p-\eta_\infty^{\theta^p(y)}\|_{\rm tv}\leq a\;e^{-\lambda p}.$$

Combining this bound with (3.18), we deduce that

$$\left|\Phi_{0,q}^{\theta^p(y)}(\eta^y_p)(G_q^{\theta^p(y)})-\Phi_{0,q}^{\theta^p(y)}(\eta_\infty^{\theta^p(y)})(G_q^{\theta^p(y)})\right|\leq ce^{-\lambda(p+q)}.$$

We can now conclude as in the proof of Theorem 3.2. □

Introduce the map $\mathcal{C}$ defined on $S^{\mathbb{Z}}$ by

$$\mathcal{C}(y):=\mathrm{Cov}^{\theta^{-1}(y)}_{1,\eta_\infty^{\theta^{-1}(y)}}(h^y,h^y).$$

We add to (1.24)-(1.25)-(1.26) the assumption that $\mathcal{C}$ is measurable with respect to the product $\sigma-$algebra on $S^{\mathbb{Z}}$.

Arguing as in the proof of (3.14), then applying Birkhoff's ergodic theorem (on canonical space), we deduce the following asymptotic behavior for the variance v_n.

Corollary 3.7. *We have*

$$\frac{v_n}{n} = \frac{1}{n} \sum_{0 \le p < n} \text{Cov}^y_{p,\eta^y_{p-1}}(\overline{Q}^y_{p,n}(1), \overline{Q}^y_{p,n}(1)) = \frac{1}{n} \sum_{1 \le p < n} \mathcal{C}(\theta^p(y)) + O(1/n)$$

so the ergodic theorem yields

$$\lim_{n\to\infty} \frac{v_n}{n} = \mathbb{E}\left(\text{Cov}^{\theta^{-1}(Y)}_{1,\eta^{\theta^{-1}(Y)}_\infty}(h^Y, h^Y) \right) \text{ a.s.}$$

4 Fluctuation analysis

4.1 Moment bounds

In addition to the local error fields V_n^N defined in (1.15), we consider the global error fields W_n^N defined by

$$W_n^N = \sqrt{N}\ \left(\eta_n^N - \eta_n\right) \ \Leftrightarrow\ \eta_n^N = \eta_n + \frac{1}{\sqrt{N}}\, W_n^N. \tag{4.1}$$

We now recall some key moment estimates on V_n^N and W_n^N, see [5, chapter 4] or [6, chapter 9]. Under our assumptions, one has for any $n \ge 0$, $N \ge 1$, $f \in \text{Osc}(E)$ and $m \ge 1$

$$\left\| V_n^N(f) \right\|_m \le c(m), \tag{4.2}$$

and

$$\left\| W_n^N(f) \right\|_m \le c(m). \tag{4.3}$$

4.2 Expansion of the particle estimate of log-normalizing constants

Starting from the product-form expression (1.14), we apply a second-order expansion to the logarithm of each factor. Using (4.3), we have that for any $n \ge 0$ and $N \ge 1$

$$\log \overline{\gamma}_n^N(1) = \frac{1}{\sqrt{N}} \sum_{0\le p<n} W_p^N(\overline{G}_p) - \frac{1}{2N} \sum_{0\le p<n} \left(W_p^N(\overline{G}_p)\right)^2 + \frac{1}{\sqrt{N}} \left(\frac{n}{N}\right) C(n,N), \tag{4.4}$$

where the remainder term satisfies $||C(n,N)||_m \le c(m)$ for all $m \ge 1$.

4.3 Second order perturbation formulae

We derive an expansion of $W_n^N(f)$ in terms of the local error terms V_p^N introduced in (1.15), up to an error term of order $1/N$. The key result we prove is the following.

Theorem 4.1. *For all $n \ge 0$, $N \ge 1$ and any function $f \in \text{Osc}(E)$,*

$$W_n^N(f) = \mathcal{W}_n^N(f) + \frac{1}{N}\, \mathcal{R}_n^N(f) \tag{4.5}$$

with

$$\mathcal{W}_n^N(f) = \sum_{p=0}^{n} V_p^N\left[d_{p,n}(f)\right] - \frac{1}{\sqrt{N}} \sum_{0\le p<n} \left[\sum_{q=0}^{p} V_q^N\left[d_{q,p}(\overline{G}_p)\right]\right] \left[\sum_{q=0}^{p} V_q^N\left[d_{q,n}(f)\right]\right],$$

where the remainder measure $\mathcal{R}_n^N$ is such that $||\mathcal{R}_n^N(f)||_m \le c(m)$ for all $m \ge 1$.

To prove Theorem 4.1, we start with the following exact decomposition of $W_n^N(f)$ into a first term of order 1 involving the V_p^N for $p = 0, ..., n$ plus a remainder term of order $1/\sqrt{N}$.

Theorem 4.2 ([6, chapter 14, p. 432]). *For all* $n \geq 0$, $N \geq 1$ *and any function* $f \in \mathrm{Osc}(E)$, *we have the decomposition*

$$W_n^N(f) = \sum_{p=0}^{n} V_p^N \left[d_{p,n}(f)\right] + \frac{1}{\sqrt{N}} \, S_n^N(f), \tag{4.6}$$

with the second order remainder

$$S_n^N(f) := -\sum_{0 \leq p < n} \frac{1}{\eta_p^N(\overline{G}_p)} \, W_p^N(\overline{G}_p) \, W_p^N \left[d_{p,n}(f)\right].$$

Under our assumptions, the remainder term satisfies for all $m \geq 1$

$$||S_n^N(f)||_m \leq c(m). \tag{4.7}$$

Decomposing $1/\eta_p^N(\overline{G}_p)$ into a term of order 1 plus a term of order $1/\sqrt{N}$ as follows

$$\frac{1}{\eta_p^N(\overline{G}_p)} = 1 - \frac{1}{\eta_p^N(\overline{G}_p)} \frac{1}{\sqrt{N}} \, W_p^N(\overline{G}_p), \tag{4.8}$$

we refine Theorem 4.2 into the following decomposition, which now has an error term of order $1/N$.

Corollary 4.3. *For all* $n \geq 0$, $N \geq 1$ *and any function* $f \in \mathrm{Osc}(E)$, *the following decomposition holds*

$$W_n^N(f) = \sum_{p=0}^{n} V_p^N \left[d_{p,n}(f)\right] - \frac{1}{\sqrt{N}} \sum_{0 \leq p < n} W_p^N(\overline{G}_p) \, W_p^N \left[d_{p,n}(f)\right] + \frac{1}{N} \, \mathcal{R}_n^N(f), \tag{4.9}$$

where the remainder term is such that $||\mathcal{R}_n^N(f)||_m \leq c(m)$ *for all* $m \geq 1$.

Proof. Using (4.8), we obtain (4.9) with the remainder term

$$\mathcal{R}_n^N(f) := \sum_{0 \leq p < n} \frac{1}{\eta_p^N(\overline{G}_p)} \, W_p^N(\overline{G}_p)^2 \, W_p^N \left[d_{p,n}(f)\right],$$

and, for any $m \geq 1$, we have

$$\mathbb{E}\left(\left|\mathcal{R}_n^N(f)\right|^m\right)^{\frac{1}{m}} \leq g \sum_{0 \leq p < n} \mathbb{E}\left(\left|W_p^N(\overline{G}_p)\right|^{4m}\right)^{\frac{1}{2m}} \, \mathbb{E}\left(\left|W_p^N \left[d_{p,n}(f)\right]\right|^{2m}\right)^{\frac{1}{2m}}.$$

Combining (4.3) and (3.6), we obtain

$$\mathbb{E}\left(\left|\mathcal{R}_n^N(f)\right|^m\right)^{\frac{1}{m}} \leq c(m) \sum_{0 \leq p < n} e^{\lambda(n-p)},$$

for some finite constant $c(m) < \infty$. This ends the proof of the corollary. □

We are now ready to derive Theorem 4.1, by replacing the W_p^N terms appearing in the previous corollary by their expansions in terms of the V_p^N provided by Theorem 4.2. Here is the proof of Theorem 4.1.

Proof. Using (4.9), we have

$$W_n^N(f) = \mathcal{V}_n^{(N,1)}(f) + \frac{1}{\sqrt{N}} \, \mathcal{V}_n^{(N,2)}(f) + \frac{1}{N} \, \mathcal{R}_n^N(f),$$

with

$$\mathcal{V}_n^{(N,1)}(f) := \sum_{p=0}^{n} V_p^N \left[d_{p,n}(f)\right],$$
$$\mathcal{V}_n^{(N,2)}(f) := -\sum_{0\le p<n} W_p^N(\overline{G}_p)\ W_p^N\left[d_{p,n}(f)\right].$$

This implies that

$$\sum_{0\le p<n} W_p^N(\overline{G}_p)\ W_p^N\left[d_{p,n}(f)\right] = \mathcal{I}_n^{(0)} + \frac{1}{\sqrt{N}}\ \mathcal{I}_n^{(1)}(f) + \frac{1}{N}\ \mathcal{I}_n^{(2)}(f) + \frac{1}{N^2}\ \mathcal{I}_n^{(3)}(f),$$

with

$$\mathcal{I}_n^{(0)}(f) = \sum_{0\le p<n} \mathcal{V}_p^{(N,1)}(\overline{G}_p)\ \mathcal{V}_p^{(N,1)}(d_{p,n}(f)),$$
$$\mathcal{I}_n^{(1)}(f) = \sum_{0\le p<n} \left[\mathcal{V}_p^{(N,1)}(\overline{G}_p)\ \mathcal{V}_p^{(N,2)}(d_{p,n}(f)) + \mathcal{V}_p^{(N,2)}(\overline{G}_p)\ \mathcal{V}_p^{(N,1)}(d_{p,n}(f))\right],$$
$$\mathcal{I}_n^{(2)}(f) = \sum_{0\le p<n} \left\{\mathcal{R}_p^N(\overline{G}_p)\ \left[\mathcal{V}_p^{(N,1)}(d_{p,n}(f)) + \frac{1}{\sqrt{N}}\ \mathcal{V}_p^{(N,2)}(d_{p,n}(f))\right]\right.$$
$$\left. + \mathcal{R}_p^N(d_{p,n}(f))\ \left[\mathcal{V}_p^{(N,1)}(\overline{G}_p) + \frac{1}{\sqrt{N}}\ \mathcal{V}_p^{(N,2)}(\overline{G}_p)\right]\right\},$$
$$\mathcal{I}_n^{(3)}(f) = \sum_{0\le p<n} \mathcal{R}_p^N(\overline{G}_p)\ \mathcal{R}_p^N(d_{p,n}(f)).$$

Arguing as in the previous proof, we see that $\sup_{1\le i\le 3} \mathbb{E}\left(\left|\mathcal{I}_n^{(i)}(f)\right|^m\right)^{\frac{1}{m}} \le c(m)$, which yields the conclusion. □

4.4 Fluctuations of local random fields

As mentioned in Section 1.3, when N goes to infinity, the fields $(V_n^N)_{n\ge 0}$ converge in distribution to a sequence of independent centered Gaussian random fields $(V_n)_{n\ge 0}$ whose covariances are characterized by

$$C_{V_n}(f,\phi) := \mathbb{E}(V_n(f)V_n(\phi)) = \mathrm{Cov}_{n,\eta_{n-1}}(f,\phi),$$

for any $f,\phi \in \mathcal{B}_b(E)$.

We recall that for any $n\ge 1$, $q\ge 1$ and any $q-$tensor product function

$$f = \otimes_{1\le i\le q} f_i \in \mathrm{Osc}(E)^{\otimes q},$$

the q-moments of a centered Gaussian random field V are given by the Isserlis-Wick formula [16, 26]

$$\mathbb{E}\left(V^{\otimes q}(f)\right) = \sum_{\boldsymbol{i}\in\pi(q)} \prod_{1\le \ell\le q/2} \mathbb{E}(V(f_{i_{2\ell-1}})V(f_{i_{2\ell}})), \tag{4.10}$$

where $\pi(q)$ denotes the set of pairings of $\{1,\ldots,q\}$, i.e. the set of partitions $\boldsymbol{i}$ of $\{1,\ldots,q\}$ into pairs $\boldsymbol{i}_1 = \{i_1,i_2\},\ldots,\boldsymbol{i}_{q/2} = \{i_{q-1},i_q\}$. Note that when q is odd, $\mathbb{E}\left(V^{\otimes q}(f)\right) = 0$.

In the following proposition, we give quantitative bounds on the convergence speed of product-form functionals of the fields $(V_n^N)_{n\ge 0}$.

Proposition 4.4. *For any* $p\ge 1$*, there exists* $c(p)<\infty$ *such that for any* $f = (f_i)_{1\le i\le p} \in \mathrm{Osc}(E)^p$*, any integers* $a = (a_i)_{1\le i\le p}$*,* $n\ge 0$ *and any* $N\ge 1$

$$\left|\mathbb{E}(V_{a_1}^N(f_1)\cdots V_{a_p}^N(f_p)) - \mathbb{E}(V_{a_1}(f_1)\cdots V_{a_p}(f_p))\right| \le \frac{c(p)}{\sqrt{N}}.$$

To prove the proposition, we use the following lemma.

Lemma 4.5. *Consider a sequence of N independent random variables $(Z_i)_{1\leq i\leq N}$ with distributions $(\mu_i)_{1\leq i\leq N}$ on E, and define the empirical random fields V^N for $f\in \mathrm{Osc}(E)$ by*

$$V^N(f) := N^{-1/2}\sum_{j=1}^{N}(f(Z_j)-\mu_j(f)).$$

Let $\overline{V}^N$ denote a centered Gaussian random field with covariance function defined for any $f,\phi\in\mathrm{Osc}(E)$ by

$$C_{\overline{V}^N}(f,\phi)=\mathbb{E}\left(\overline{V}^N(f)\overline{V}^N(\phi)\right)=\frac{1}{N}\sum_{i=1}^{N}\mathrm{cov}_{\mu_i}(f,\phi),$$

where

$$\mathrm{cov}_{\mu_i}(f,\phi) := \mu_i\left([f-\mu_i(f)]\left[\phi-\mu_i(\phi)\right]\right).$$

For any $1\leq q\leq N$, there exists $c(q)<\infty$ such that for any $q-$tensor product function

$$f=\otimes_{1\leq i\leq q} f_i\in \mathrm{Osc}(E)^{\otimes q},$$

the following inequality holds

$$\left|\mathbb{E}\left(\left[V^N\right]^{\otimes q}(f)\right)-\mathbb{E}\left(\left[\overline{V}^N\right]^{\otimes q}(f)\right)\right|\leq c(q)\;N^{-\rho(q)}, \tag{4.11}$$

where $\rho(q):=1$ for even q, and $\rho(q):=1/2$ for odd q.

Proof. We write

$$V^N(f_i)=\frac{1}{\sqrt{N}}\sum_{1\leq j\leq N} f_i^{(j)}(Z_j)\quad\text{with}\quad f_i^{(j)}=f_i-\mu_j(f_i).$$

Expanding the product, we get

$$N^{q/2}\;\mathbb{E}\left(\left[V^N\right]^{\otimes q}(f)\right)=\sum_{1\leq j_1,\ldots,j_q\leq N}\mathbb{E}(f_1^{(j_1)}(Z_{j_1})\cdots f_q^{(j_q)}(Z_{j_q})).$$

Each term in the above r.h.s. such that an index j_i appears exactly once in the list $(j_1,\ldots,j_q)$ must be zero, so the only terms that may contribute to the sum are those for which every index appears at least twice. When q is odd, the number of such combinations of indices is bounded above by $c(q)N^{(q-1)/2}$, for some finite constant $c(q)<\infty$ depending only on q. Since each expectation is bounded in absolute value by 1, we can conclude.

Now assume that q is even. Consider a pairing $\boldsymbol{i}$ of $\{1,\ldots,q\}$ given by $\boldsymbol{i}_1=\{i_1,i_2\},\ldots,\boldsymbol{i}_{q/2}=\{i_{q-1},i_q\}$, and a combination of indices $j_1,\ldots,j_q$ such that $j_a=j_b$ whenever a,b belong to the same pair, while $j_a\neq j_b$ otherwise. Denoting by k_r the value of j_a when $a\in\boldsymbol{i}_r$, and using independence, we see that the contribution of this combination to the sum is

$$\mathbb{E}(f_1^{(j_1)}(Z_{j_1})\cdots f_q^{(j_q)}(Z_{j_q}))=\mathrm{cov}_{\mu_{k_1}}(f_{i_1},f_{i_2})\cdots\mathrm{cov}_{\mu_{k_{q/2}}}(f_{i_{q-1}},f_{i_q}).$$

Every combination of indices in which every index appears exactly twice is of the form just described. Then, the number of combinations in which every index appears at least twice, but which are not of the previous form, is $O(N^{q/2-1})$. Consequently, we have

$$\begin{aligned}&N^{q/2}\;\mathbb{E}(\left(V^N\right)^{\otimes q}(f))\\&=\sum_{\boldsymbol{i}\in\pi(q)}\;\sum_{k\in\langle q/2,N\rangle}\mathrm{cov}_{\mu_{k_1}}(f_{i_1},f_{i_2})\cdots\mathrm{cov}_{\mu_{k_{q/2}}}(f_{i_{q-1}},f_{i_q})+\mathrm{O}\left(N^{q/2-1}\right),\end{aligned}$$

where $\langle p, N\rangle$ stands for the set of all $(N)_p = N!/(N-p)!$ one-to-one mappings from $[p] := \{1,\ldots,p\}$ into $[N]$. Moreover, for any function $\varphi \in \mathbb{R}^{[N]^{[p]}}$ such that $|\varphi| \leq 1$, we have

$$\left|\frac{1}{(N)_p}\sum_{k\in\langle p,N\rangle}\varphi(k) - \frac{1}{N^p}\sum_{k\in[N]^{[p]}}\varphi(k)\right| \leq \frac{(p-1)^2}{N}.$$

A detailed proof of the above inequality is provided in Proposition 8.6.1 in [5] (cf. for instance the derivation of formula (8.14) p. 269). Now note that

$$\begin{aligned}
&\textstyle\sum_{\boldsymbol{i}\in\pi(q)} \ \frac{1}{N^{q/2}}\sum_{k\in[N]^{[q/2]}} \mathrm{cov}_{\mu_{k_1}}(f_{i_1}, f_{i_2})\cdots\mathrm{cov}_{\mu_{k_{q/2}}}(f_{i_{q-1}}, f_{i_q})\\
&= \textstyle\sum_{\boldsymbol{i}\in\pi(q)} \ \prod_{1\leq\ell\leq q/2}\frac{1}{N}\sum_{1\leq j\leq N}\mathrm{cov}_{\mu_j}(f_{i_{2\ell-1}}, f_{i_{2\ell}})\\
&= \textstyle\sum_{\boldsymbol{i}\in\pi(q)} \ \prod_{1\leq\ell\leq q/2} C_{\overline{V}^N}(f_{i_{2\ell-1}}, f_{i_{2\ell}}) = \mathbb{E}\left(\left(\overline{V}^N\right)^{\otimes q}(f)\right),
\end{aligned}$$

where the last identity uses Wick's formula (4.10).

This yields

$$N^{q/2}\,\mathbb{E}(\left(V^N\right)^{\otimes q}(f)) = (N)_{q/2}\,\mathbb{E}\left(\left(\overline{V}^N\right)^{\otimes q}(f)\right) + \mathrm{O}\left(N^{q/2-1}\right).$$

We end the proof of (4.11) using the fact that $0 \leq (1-(N)_p/N^p) \leq (p-1)^2/N$ for any $p \leq N$. ☐

Lemma 4.6. *Let $q > 0$ be an even number and $m \geq 1$ an integer. There exists $c(q,m) < \infty$ such that for any functions $(f_i)_{1\leq i\leq q} \in \mathrm{Osc}(E)^q$, any $n \geq 0$ and any $N \geq 1$, we have*

$$\left\|\prod_{1\leq\ell\leq q/2}\mathrm{Cov}_{n,\eta^N_{n-1}}(f_{2\ell-1}, f_{2\ell}) - \prod_{1\leq\ell\leq q/2}\mathrm{Cov}_{n,\eta_{n-1}}(f_{2\ell-1}, f_{2\ell})\right\|_m \leq c(q,m)/\sqrt{N}. \quad (4.12)$$

Proof. Arguing as in the proof of Proposition 2.1, combining (4.2) and (1.26), we obtain that for any $f_1, f_2 \in \mathrm{Osc}(E)$

$$\sqrt{N}\ \left\|\mathrm{Cov}_{n,\eta^N_{n-1}}(f_1,f_2) - \mathrm{Cov}_{n,\eta_{n-1}}(f_1,f_2)\right\|_m \leq c^{'}(m). \quad (4.13)$$

We end the proof of (4.12) using the bound

$$\left|\prod_{1\leq i\leq m}u_i - \prod_{1\leq i\leq m}v_i\right| \leq \sup(|u_i|,|v_i|;\ 1\leq i\leq m)^{m-1}\sum_{1\leq i\leq m}|u_i - v_i|,$$

valid for any $u = (u_i)_{1\leq i\leq m} \in \mathbb{R}^m$ and $v = (v_i)_{1\leq i\leq m} \in \mathbb{R}^m$. ☐

We now come to the proof of Proposition 4.4.

Proof of Proposition 4.4. Assume that the a_i are ordered so that $a_1 \leq \ldots \leq a_\ell < a_{\ell+1} = \cdots = a_{\ell+q}$, where $\ell + q = p$. Set

$$A^N := V^N_{a_1}(f_1)\cdots V^N_{a_\ell}(f_\ell) \quad\text{and}\quad B^N := V^N_a(f_{\ell+1})\cdots V^N_a(f_{\ell+q})$$

where $a := a_p$. Given $\mathcal{F}^N_{a-1}$, we let $\overline{V}^N_a$ be a sequence of Gaussian random fields with covariance function defined for any $f, \phi \in \mathrm{Osc}(E)$ by

$$C_{\overline{V}^N_a}(f,\phi) = \mathrm{Cov}_{a,\eta^N_{a-1}}(f,\phi)$$

and we set

$$\overline{B}^N := \overline{V}_a^N(f_{\ell+1})\cdots\overline{V}_a^N(f_{\ell+q}) \quad \text{and} \quad B := V_a(f_{\ell+1})\cdots V_a(f_{\ell+q})$$

Now $\mathbb{E}(A^N B^N) = \mathbb{E}(A^N \times \mathbb{E}(B^N|\mathcal{F}_{a-1}^N))$, and, by Lemma 4.5 applied to conditionally independent samples, one has the deterministic bound

$$\left|\mathbb{E}(B^N|\mathcal{F}_{a-1}^N) - \mathbb{E}\left(\overline{B}^N\,|\mathcal{F}_{a-1}^N\right)\right| \le c(q)/\sqrt{N}.$$

Moreover, combining (4.12) with Wick's formula (4.10), we obtain

$$\mathbb{E}\left(\overline{B}^N\,|\mathcal{F}_{a-1}^N\right) = \sum_{\boldsymbol{i}\in\pi(q)}\prod_{1\le r\le q/2}\mathrm{Cov}_{a,\eta_{a-1}^N}(f^{(\ell+2r-1)}, f^{(\ell+2r)}),$$

hence it follows that

$$\sqrt{N}\ \left\|\mathbb{E}\left(\overline{B}^N\,|\mathcal{F}_{a-1}^N\right) - \mathbb{E}\left(B\right)\right\|_m \le c(m).$$

Using the decomposition

$$\mathbb{E}(A^N B^N) - \mathbb{E}(A^N\mathbb{E}(B)) = \mathbb{E}\left(A^N\ \left[\mathbb{E}(B^N|\mathcal{F}_{a-1}^N) - \mathbb{E}(B)\right]\right),$$

we conclude that

$$\left|\mathbb{E}(A^N B^N) - \mathbb{E}(A^N)\ \mathbb{E}(B)\right| \le c'(q)/\sqrt{N}.$$

One then concludes by iterating the argument. □

4.5 Expansion of the particle estimates continued

By inserting the expansions obtained in Section 4.3 in the development obtained in (4.4), we obtain, after some rearrangement, the following proposition.

Proposition 4.7. *For any $n \ge 0$, $N \ge 1$, we have the second order decomposition*

$$\begin{aligned}
&\frac{1}{\sqrt{N}}\sum_{0\le q<n} W_q^N(\overline{G}_q) - \frac{1}{2N}\sum_{0\le q<n} W_q^N(\overline{G}_q)^2 \\
&= \frac{1}{\sqrt{N}}\sum_{0\le q<n} V_q^N(\overline{Q}_{q,n}(1)) \\
&\quad -\frac{1}{2N}\sum_{0\le k\le p<n}\left[V_k^N(\overline{Q}_{k,p+1}(1) - \overline{Q}_{k,p}(1))\ V_k^N(\overline{Q}_{k,p+1}(1) + \overline{Q}_{k,p}(1))\ \right] \\
&\qquad -\frac{1}{N}U_n^N - \frac{1}{2N}Y_n^N + \frac{1}{\sqrt{N}}\left(\frac{n}{N}\right)C_2(n,N),
\end{aligned} \tag{4.14}$$

where U_n^N and Y_n^N are centered random variables given by

$$U_n^N := \sum_{0\le k\ne l\le q<p<n} V_k^N\left(d_{k,q}(\overline{G}_q)\right) V_l^N\left(d_{l,p}(\overline{G}_p)\right), \tag{4.15}$$

$$Y_n^N := \sum_{0\le k<l\le q<n} V_k^N\left[d_{k,q}(\overline{G}_q)\right] V_l^N\left[d_{l,q}(\overline{G}_q)\right],$$

and the remainder term satisfies $||C_2(n,N)||_m \le c(m)$ for all $m \ge 1$.

Proof. By Theorem 4.1, we may replace W_q^N by the random fields $\mathcal{W}_q^N$ given in (4.5) in the linear terms of the expression we want to expand, i.e. the l.h.s. of (4.14), while committing at most an error of the form

$$\frac{1}{\sqrt{N}}\left(\frac{n}{N}\right) C_3(n,N),$$

where for all $m \geq 1$

$$||C_3(n,N)||_m \leq c(m).$$

On the other hand, using the cruder expansion provided by Theorem 4.2, we may replace W_q^N by just $\sum_{p=0}^{q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]$ in the quadratic terms appearing in the l.h.s. of (4.14), and commit an overall error of the form

$$\frac{1}{\sqrt{N}}\left(\frac{n}{N}\right) C_4(n,N),$$

where for all $m \geq 1$

$$||C_4(n,N)||_m \leq c'(m).$$

By the definition of $\mathcal{W}_q^N$ given in (4.5), we have

$$\begin{aligned}\mathcal{W}_q^N(\overline{G}_q) &= \sum_{p=0}^{q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]\\ &\quad - \frac{1}{\sqrt{N}}\sum_{0\leq p<q}\left[\sum_{k=0}^{p} V_k^N\left[d_{k,p}(\overline{G}_p)\right]\right]\left[\sum_{k=0}^{p} V_k^N\left[d_{k,q}(\overline{G}_q)\right]\right]\end{aligned}$$

so that

$$\begin{aligned}&\frac{1}{\sqrt{N}}\sum_{0\leq q<n}\mathcal{W}_q^N(\overline{G}_q)\\ &= \frac{1}{\sqrt{N}}\sum_{0\leq p<n} V_p^N\left[\sum_{p\leq q<n} d_{p,q}(\overline{G}_q)\right]\\ &\qquad -\frac{1}{N}\sum_{0\leq q<n}\sum_{0\leq p<q}\left[\sum_{k=0}^{p} V_k^N\left[d_{k,p}(\overline{G}_p)\right]\right]\left[\sum_{k=0}^{p} V_k^N\left[d_{k,q}(\overline{G}_q)\right]\right].\end{aligned}$$

By (3.3), we recall that

$$\sum_{p\leq q<n} d_{p,q}(\overline{G}_q) = \sum_{p\leq q<n}\left[\overline{Q}_{p,q+1}(1) - \overline{Q}_{p,q}(1)\right] = \overline{Q}_{p,n}(1) - 1,$$

so on the one hand we have

$$\sum_{0\leq p<n} V_p^N\left[\sum_{p\leq q<n} d_{p,q}(\overline{G}_q)\right] = \sum_{0\leq p<n} V_p^N\left[\overline{Q}_{p,n}(1)\right],$$

whereas, on the other hand, we have

$$
\begin{aligned}
&\textstyle\sum_{0\le p<q<n}\left[\sum_{k=0}^{p} V_k^N\left[d_{k,p}(\overline{G}_p)\right]\right]\left[\sum_{k=0}^{p} V_k^N\left[d_{k,q}(\overline{G}_q)\right]\right]\\
&=\sum_{0\le k\le p<q<n} V_k^N\left[d_{k,p}(\overline{G}_p)\right] V_k^N\left[d_{k,q}(\overline{G}_q)\right]+U_n^N\\
&=\sum_{0\le k<q<n} V_k^N\left[\sum_{k\le p<q} d_{k,p}(\overline{G}_p)\right] V_k^N\left[d_{k,q}(\overline{G}_q)\right]+U_n^N\\
&=\sum_{0\le k<q<n} V_k^N\left[\overline{Q}_{k,q}(1)\right] V_k^N\left[d_{k,q}(\overline{G}_q)\right]+U_n^N.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\frac{1}{\sqrt{N}}\sum_{0\le q<n}\mathcal{W}_q^N(\overline{G}_q)\\
&=\frac{1}{\sqrt{N}}\sum_{0\le p<n} V_p^N\left[\overline{Q}_{p,n}(1)\right]\\
&\qquad\qquad -\frac{1}{N}\sum_{0\le q<n}\sum_{0\le p<q} V_p^N\left[\overline{Q}_{p,q}(1)\right] V_p^N\left[d_{p,q}(\overline{G}_q)\right]-\frac{1}{N}\,U_n^N.
\end{aligned}
$$

It remains to analyze the quadratic part, which we write as

$$
\sum_{0\le q<n}\left(\sum_{0\le p\le q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]\right)^2=\sum_{0\le q<n}\sum_{0\le p\le q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]^2+Y_n^N.
$$

Now notice that

$$
\begin{aligned}
&-\sum_{0\le p<q} V_p^N\left[\overline{Q}_{p,q}(1)\right] V_p^N\left[d_{p,q}(\overline{G}_q)\right]-\frac{1}{2}\sum_{0\le p\le q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]^2\\
&=-\frac{1}{2}V_q^N\left[d_{q,q}(\overline{G}_q)\right]^2-\sum_{0\le p<q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]\;V_p^N\left[\frac{1}{2}\;d_{p,q}(\overline{G}_q)+\overline{Q}_{p,q}(1)\right]\\
&=-\frac{1}{2}V_q^N\left[d_{q,q}(\overline{G}_q)\right]^2\\
&\qquad -\textstyle\sum_{0\le p<q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]\;V_p^N\left[\frac{1}{2}\left[\overline{Q}_{p,q+1}(1)-\overline{Q}_{p,q}(1)\right]+\overline{Q}_{p,q}(1)\right]\\
&=-\frac{1}{2}V_q^N\left[\overline{Q}_{q,q+1}(1)-\overline{Q}_{q,q}(1)\right]^2\\
&\qquad -\textstyle\frac{1}{2}\sum_{0\le p<q} V_p^N\left[\overline{Q}_{p,q+1}(1)-\overline{Q}_{p,q}(1)\right]\;V_p^N\left[\overline{Q}_{p,q+1}(1)+\overline{Q}_{p,q}(1)\right].
\end{aligned}
$$

Recalling that $\overline{Q}_{q,q}(1)=1$, we conclude that

$$
\begin{aligned}
&-\sum_{0\le p<q} V_p^N\left[\overline{Q}_{p,q}(1)\right] V_p^N\left[d_{p,q}(\overline{G}_q)\right]-\frac{1}{2}\sum_{0\le p\le q} V_p^N\left[d_{p,q}(\overline{G}_q)\right]^2\\
&=-\textstyle\frac{1}{2}\sum_{0\le k\le q} V_k^N\left[\overline{Q}_{k,q+1}(1)-\overline{Q}_{k,q}(1)\right]\;V_k^N\left[\overline{Q}_{k,q+1}(1)+\overline{Q}_{k,q}(1)\right].
\end{aligned}
$$

□

The next step is to show that both centered terms U_n^N and Y_n^N yield negligible contributions in (4.14).

Proposition 4.8. *There exists* $c<\infty$ *such that for any* $n\geq 0,\ N\geq 1$

$$\mathbb{E}((U_n^N)^2)\leq c\ \left(n+\frac{n^2}{\sqrt{N}}\right).$$

Proof. From (4.15), we obtain

$$\mathbb{E}((U_n^N)^2)=\sum\mathbb{E}\left(V_k^N\left(d_{k,q}(\overline{G}_q)\right)V_l^N\left(d_{l,p}(\overline{G}_p)\right)V_{k'}^N\left(d_{k',q'}(\overline{G}_{q'})\right)V_{l'}^N\left(d_{l',p'}(\overline{G}_{p'})\right)\right)$$

with

$$\sum=\sum_{0\leq k\neq l\leq q<p<n}\ \sum_{0\leq k'\neq l'\leq q'<p'<n}.$$

First consider replacing each V_k^N by the corresponding V_k in the above expectations. By Proposition 4.4 together with the exponential estimates (3.6), the overall error due to this replacement is bounded by

$$\frac{c}{\sqrt{N}}\sum\exp(-\lambda(q-k+p-l+q'-k'+p'-l'))\leq c'\,\frac{n^2}{\sqrt{N}}.$$

Now consider the corresponding sum associated with the Gaussian fields V_k which is given by

$$\sum\mathbb{E}\left(V_k\left(d_{k,q}(\overline{G}_q)\right)V_l\left(d_{l,p}(\overline{G}_p)\right)V_{k'}\left(d_{k',q'}(\overline{G}_{q'})\right)V_{l'}\left(d_{l',p'}(\overline{G}_{p'})\right)\right).$$

We only have a non-zero term when either $k=k'$ and $l=l'$ or $k=l'$ and $k'=l$.

Restricting summation to this subset of indices, we claim that

$$\sum\exp(-\lambda(q-k+p-l+q'-k'+p'-l'))\leq c''\times n. \tag{4.16}$$

To see this, consider first the subset of indices such that $0\leq k<l<n$, $k'=k$ and $l'=l$ and write $(q-k+p-l+q'-k'+p'-l')=2(l-k)+(q-l+p-l+q'-l+p'-l)$. The corresponding sum is thus bounded above by

$$\sum_{0\leq k<l<n}\exp(-2\lambda(l-k))\times\sum_{p,p',q,q'\geq l}\exp(-\lambda(q-l+p-l+q'-l+p'-l)). \tag{4.17}$$

Clearly, the r.h.s. of the product in (4.17) is bounded above by a constant since $\lambda>0$, and, for the same reason, the expression $\sum_{l=k+1}^{+\infty}\exp(-2\lambda(l-k))$ is bounded above by a constant which does not depend on k. Since (4.17) is a sum over at most n values of k, we deduce that the sum in (4.17) is bounded above by a constant times n. The other three subsets of indices contributing to (4.16) (namely $\{0\leq l<k<n, k'=k, l'=l\}$; $\{0\leq k<l<n, k'=l, l'=k\}$; $\{0\leq l<k<n, k'=l, l'=k\}$) can be dealt with in exactly the same way, leading to the bound in (4.16).

□

Using a similar argument, we obtain the following result.

Proposition 4.9. *There exists* $c<\infty$ *such that for any* $n\geq 0, N\geq 1$

$$\mathbb{E}((Y_n^N)^2)\leq c\ \left(n+\frac{n^2}{\sqrt{N}}\right).$$

Now we consider the remaining term in (4.14), i.e.

$$H_n^N := \sum_{0\le k\le p<n} \left(V_k^N\left[\overline{Q}_{k,p+1}(1)-\overline{Q}_{k,p}(1)\right]\ V_k^N\left[\overline{Q}_{k,p+1}(1)+\overline{Q}_{k,p}(1)\right]\ \right),$$

and show that it can be replaced by its expectation up to a negligible random term.

Proposition 4.10. *There exists $c<\infty$ such that for any $n\ge 0, N\ge 1$ the following variance bound holds*

$$\mathbb{V}(H_n^N)\le c\, n.$$

Proof. If we set

$$J_{k,p} := \overline{Q}_{k,p+1}(1)-\overline{Q}_{k,p}(1) \quad \text{and} \quad K_{k,p} := \overline{Q}_{k,p+1}(1)+\overline{Q}_{k,p}(1),$$

then we find that

$$\mathbb{E}\left(H_n^N\right) = \sum_{0\le k\le p<n} \mathbb{E}\left(V_k^N\left[J_{k,p}\right]\ V_k^N\left[K_{k,p}\right]\ \right),$$

whence

$$\begin{aligned}&(\mathbb{E}\left(H_n^N\right))^2\\ &= \sum_{0\le k\le p<n}\sum_{0\le k'\le p'<n} \mathbb{E}\left(V_k^N\left[J_{k,p}\right]\ V_k^N\left[K_{k,p}\right]\ \right)\mathbb{E}\left(V_{k'}^N\left[J_{k',p'}\right]\ V_{k'}^N\left[K_{k',p'}\right]\ \right),\end{aligned}$$

while

$$\begin{aligned}&\mathbb{E}\left(\left(H_n^N\right)^2\right)\\ &= \sum_{0\le k\le p<n}\sum_{0\le k'\le p'<n} \mathbb{E}\left(V_k^N\left[J_{k,p}\right]\ V_k^N\left[K_{k,p}\right]V_{k'}^N\left[J_{k',p'}\right]\ V_{k'}^N\left[K_{k',p'}\right]\ \right).\end{aligned}$$

Observe that the terms in the above two sums coincide whenever $k\neq k'$. Therefore, it remains to bound the contribution of the terms such that $k=k'$ in both sums. In both expressions, the corresponding sum is bounded above in absolute value by

$$\sum_{0\le k\le p,p'<n} c'\ e^{-\lambda(p'-k+p-k)}\le c''\ n.$$

This ends the proof of the proposition. ∎

Proposition 4.11. *There exists $c<\infty$ such that for any $n\ge 0, N\ge 1$*

$$\mathbb{E}(H_n^N) = v_n+\epsilon_n^N \quad \textit{with} \quad |\epsilon_n^N|\le c\ n/\sqrt{N}.$$

Proof. Recalling that $\overline{Q}_{p,n}(1)-1=\sum_{p\le k<n}\left(\overline{Q}_{p,k+1}-\overline{Q}_{p,k}\right)$, we prove that

$$\begin{aligned}&V_p(\overline{Q}_{p,n}(1))^2\\ &= \left(\textstyle\sum_{p\le k<n} V_p\left(\overline{Q}_{p,k+1}-\overline{Q}_{p,k}\right)\right)^2\\ &= \textstyle\sum_{p\le k<n} V_p\left(\overline{Q}_{p,k+1}-\overline{Q}_{p,k}\right)^2\\ &\qquad +2\textstyle\sum_{p\le l<n} V_p\left(\sum_{p\le k<l}\left[\overline{Q}_{p,k+1}-\overline{Q}_{p,k}\right]\right)V_p\left(\overline{Q}_{p,l+1}-\overline{Q}_{p,l}\right)\\ &= \textstyle\sum_{p\le l<n} V_p\left(\overline{Q}_{p,l+1}-\overline{Q}_{p,l}\right)^2\\ &\qquad\qquad +2\textstyle\sum_{p\le l<n} V_p\left(\overline{Q}_{p,l}\right)V_p\left(\overline{Q}_{p,l+1}-\overline{Q}_{p,l}\right).\end{aligned}$$

This yields the formula

$$V_p(\overline{Q}_{p,n}(1))^2 = \sum_{p\leq l<n} V_p\left(\overline{Q}_{p,l+1} - \overline{Q}_{p,l}\right)\ V_p\left(\overline{Q}_{p,l+1} + \overline{Q}_{p,l}\right).$$

Replacing each V_k^N by V_k in the expectation of H_n^N, we obtain

$$\begin{aligned}&\textstyle\sum_{0\leq p\leq l<n} \mathbb{E}\left(V_p\left[\overline{Q}_{p,l+1}(1) - \overline{Q}_{p,l}(1)\right] V_p\left[\overline{Q}_{p,l+1}(1) + \overline{Q}_{p,l}(1)\right]\right)\\ &= \textstyle\sum_{0\leq p<n} \mathbb{E}\left(V_p(\overline{Q}_{p,n}(1))^2\right) = v_n.\end{aligned}$$

To control the error introduced by the replacement, we use Proposition 4.4, (3.3) and (3.6). It follows that the overall error can be bounded above by

$$c \sum_{0\leq k\leq p<n} \frac{e^{-\lambda(p-k)}}{\sqrt{N}} \leq c'\ \frac{n}{\sqrt{N}}.$$

This ends the proof of the proposition. □

4.6 Central limit theorem

This section establishes the proof of Theorem 1.1.

Proof. Using the decomposition (4.4) and Propositions 4.8 to 4.11, we obtain

$$\log \overline{\gamma}_n^N(1) = \frac{1}{\sqrt{N}} \sum_{0\leq q<n} V_q^N(\overline{Q}_{q,n}(1)) - \frac{1}{2N}\ v_n + \varepsilon_n^N,$$

with ε_n^N going to zero in probability as n goes to infinity. Thus, to prove the theorem, it remains to show that

$$\frac{1}{\sqrt{v_n}} \sum_{0\leq q<n} V_q^N(\overline{Q}_{q,n}(1))$$

converges in distribution to a standard normal. We do so using the central limit theorem for martingale difference arrays (see e.g. [14, 22]). The martingale property just comes from the fact that we have for any $q \geq 0$ and any bounded function f_q

$$\mathbb{E}\left(V_q^N(f_q)|\mathcal{F}_{q-1}^N\right) = 0 \text{ a.s.}$$

We now have to show that

$$\frac{1}{v_n} \sum_{0\leq q<n} \mathbb{E}\left(\left[V_q^N(\overline{Q}_{q,n}(1))\right]^2 |\mathcal{F}_{q-1}^N\right)$$

converges to 1 in probability. One easily checks from the definition that

$$\mathbb{E}\left(\left[V_q^N(\overline{Q}_{q,n}(1))\right]^2 |\mathcal{F}_{q-1}^N\right) = \text{Cov}_{q,\eta_{q-1}^N}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1)).$$

We observe that

$$v_n = \sum_{0\leq q<n} \text{Cov}_{q,\eta_{q-1}}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1))$$

and

$$\begin{aligned}d_n^N &:= \left|\tfrac{1}{v_n} \textstyle\sum_{0\leq q<n} \mathbb{E}\left(\left[V_q^N(\overline{Q}_{q,n}(1))\right]^2 |\mathcal{F}_{q-1}^N\right) - 1\right|\\ &\leq \tfrac{1}{v_n} \textstyle\sum_{0\leq q<n} \left|\text{Cov}_{q,\eta_{q-1}^N}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1)) - \text{Cov}_{q,\eta_{q-1}}(\overline{Q}_{q,n}(1), \overline{Q}_{q,n}(1))\right|.\end{aligned}$$

Using (4.13), we see that

$$\mathbb{E}(d_n^N) \leq c \, \left(\frac{n}{v_n}\right) \frac{1}{\sqrt{N}},$$

so we can conclude using (1.27).

The last point to be checked is the asymptotic negligibility condition, that is, for all $\epsilon > 0$, we have to prove that

$$\frac{1}{v_n} \sum_{0 \leq q < n} \mathbb{E}\left(\left[V_q^N(\overline{Q}_{q,n}(1))\right]^2 1\!\!1\left(\left[V_q^N(\overline{Q}_{q,n}(1))\right]^2 \geq \epsilon \, v_n\right) | \mathcal{F}_{q-1}^N\right)$$

goes to zero in probability. By Schwarz's inequality and (4.2), the expectation of this expression is bounded above by

$$c' \, \left(\frac{n}{v_n}\right) \frac{1}{(\epsilon v_n)^{1/2}},$$

This ends the proof of the theorem. □

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