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On Wong-Zakai type approximations of reflected diffusions*

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Abstract

We study weak and strong convergence of Wong-Zakai type approximations of reflected stochastic differential equations on general domains satisfying the conditions (A) and (B) introduced by Lions and Sznitman. We assume that the diffusion coefficient is Lipschitz continuous but the drift coefficient need not be even continuous. In the case where the drift coefficient is also Lipschitz continuous we show that the rate of convergence is exactly the same as for usual Euler type approximation.

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1 Introduction

Let W be a standard d-dimensional Brownian motion and W^n denote the linear approximations of W, i.e. $W_0^n = W_0 = 0$ and

$$W_t^n = W_{\frac{k}{n}} + n(t-\frac{k}{n})(W_{\frac{k+1}{n}} - W_{\frac{k}{n}})\,), \quad t \in [\frac{k}{n}, \frac{k+1}{n}), \, n \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}.$$

In this paper we study convergence of d–dimensional Wong-Zakai type approximations on a domain D with reflecting boundary condition of the form

$$X_t^n = X_0 + \int_0^t \sigma^n(X_s^n)\,dW_s^n + \int_0^t b^n(X_s^n)\,ds + K_t^n, \quad t \in \mathbb{R}^+. \tag{1.1}$$

Here $X_0 \in \bar{D} = D \cup \partial D$, X^n is a reflecting process on $\bar{D}$, K^n is a bounded variation process with variation $|K^n|$ increasing only, when $X_t^n \in \partial D$ and $\sigma^n : \bar{D} \to \mathbb{R}^d \otimes \mathbb{R}^d$, $b^n : \bar{D} \to \mathbb{R}^d$ are uniformly bounded functions.

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Doss and Priouret [3] were first to observe that if ∂D is sufficiently smooth and $\sigma^n = \sigma \in \mathbb{C}_b^2(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$, $b^n = b \in \mathbb{C}_b^1(\bar{D}, \mathbb{R}^d)$ then (X^n, K^n) converges to the solution of the reflected Stratonovich stochastic differential equation (SDE) of the form

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dW_s + \int_0^t (\frac{1}{2}\sigma'\sigma(X_s) + b(X_s))\, ds + K_t, \quad t \in \mathbb{R}^+. \tag{1.2}$$

Then Doss and Priouret's results have been generalized in different directions. In particular, almost sure convergence of Wong–Zakai approximation has been proved by Pettersson [13] in the case of convex domain and constant diffusion coefficient. Ren and Xu [15, 16] subsequently refined this result in papers concerning multivalued SDEs, which correspond to SDEs with reflecting boundary conditions on convex domains. Evans and Stroock [4] have proved weak convergence of Wong–Zakai approximations in the case of not necessary smooth domains satisfying the conditions (A), (B) and (C) from the paper by Lions and Sznitman [10]. Recently, their result have been strengthened to L^p convergence by Aida and Sasaki [1] and Zhang [30]. Wong-Zakai type approximations of reflected Stratonovich SDEs with jumps have been considered in [8, 11, 24].

It is worth pointing out that in all the above mentioned papers at least Lipschitz continuity of the diffusion and drift coefficients is assumed. This assumption is sometimes too strong, because there are interesting situations in which (1.2) has a unique solution even if the drift coefficients $\sigma'\sigma$ or b are discontinuous. For instance, if σ is bounded and Lipschitz continuous then σ', or more precisely, the partial derivatives $\frac{\partial \sigma_{i,j}}{\partial x_l}$, $i,j,l = 1,\dots,d$, exist a.e. with respect to Lebesgue measure l^d on $\mathbb{R}^d$ and are a.e. bounded. Therefore the famous result by Stroock and Varadhan [25] gives weak uniqueness of (1.2) provided that $\sigma\sigma^\star$ is elliptic, b is bounded and ∂D is sufficiently smooth. In the present paper we assume that $D \subset \mathbb{R}^d$ satisfies conditions (A), (B) introduced in Lions and Sznitman [10]. Our main purpose is to find minimal conditions on the coefficients ensuring the convergence of (X^n, K^n) to (X, K). We will also address the problem of the rate of convergence.

The paper is organized as follows.

In Section 2 we study the convergence of Wong-Zakai type approximations of the form (1.1). We assume that there is $C > 0$ such that

$$\|a^n(x)\| + |b^n(x)|^2 \le C,\ x \in \bar{D}, \tag{1.3}$$

$$\|\sigma^n(x) - \sigma^n(y)\| \le C|x-y|,\ x, y \in \bar{D} \quad \text{and} \quad b^n \in \mathbb{C}(\bar{D}, \mathbb{R}^d), \tag{1.4}$$

where $a^n = \sigma^n\sigma^{n,\star}$, $n \in \mathbb{N}$. We show that if $\sigma^n \to_K \sigma$ (here "$\to_K$" denotes the uniform convergence on all compact subsets of D), then $\{(X^n, K^n)\}$ is tight in $\mathbb{C}\,(\mathbb{R}^+, \mathbb{R}^{2d})$ and its every limit point (X, K) is a solution of the reflected SDE with the same as in (1.2) diffusion part. If moreover $\sigma^n, \sigma \in \mathbb{C}^1(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$ and $\sigma^{n,\prime} \to_K \sigma'$, $b^n \to_K b$ or $\sigma^n \in \mathbb{C}^1(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$, $\sigma\sigma^*$ is elliptic and $l^d(G^c) = 0$, where G denotes the set of all $x \in \bar{D}$ such that $\sigma^{n,\prime}(x_n) \to \sigma'(x)$ and $b^n(x_n) \to b(x)$ for any $\{x_n\} \subset \bar{D}$ converging to x, then every limit point of $\{(X^n, K^n)\}$ is a weak solution of (1.2). As a result, under the assumption that (1.2) has the weak uniqueness (pathwise uniqueness) property we get conditions ensuring weak (strong) convergence of $\{(X^n, K^n)\}$ to (X, K). Thus we generalize convergence results for Wong-Zakai approximations given in the papers cited before.

Section 3 is devoted to the classical case, where $\sigma_n = \sigma$, $b_n = b$, $n \in \mathbb{N}$, $\sigma \in \mathbb{C}^1(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$ and $\sigma'\sigma, b$ are Lipschitz continuous functions. In this case we prove that

$$\{\sup_{t \le q} |X^n_t - X_t|^2\} = \mathcal{O}_P((\frac{\log n}{n})^{1/2}), \quad q \in \mathbb{R}^+,$$

where for a sequence of positive constants $h_n \searrow 0$ and a sequence of nonnegative random variables $\{Z_n\}$ the notation $\{Z_n\} = \mathcal{O}_P(h_n)$ means that $\{Z_n/h_n\}$ is bounded in probability. This rate is the same as for the usual Euler type approximations (see Remark 4.1). Note that for domains satisfying (A), (B), (C) and coefficients $\sigma \in \mathbb{C}^2_b(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$, $b \in \mathbb{C}^1_b(\bar{D}, \mathbb{R}^d)$ Aida and Sasaki [1] have recently obtained the following rate of convergence in L^2 norm: $E \sup_{t\le q} |X^n_t - X_t|^2 = \mathcal{O}((\frac{1}{n})^{\theta/6})$, $0 < \theta < 1$, $q \in \mathbb{R}^+$. It is much weaker than the corresponding rate of Euler type approximations, so it would be desirable to extend our results to convergence in L^2 norm. However, we have been unable to do this.

In the sequel we will use the following notation. $\mathbb{R}^+ = [0, \infty)$, $\mathbb{C}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of continuous functions $x : \mathbb{R}^+ \to \mathbb{R}^d$ equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}^+$. $\rho^n_t = \max\{\frac{k}{n}; \frac{k}{n} \le t\}$, $n \in \mathbb{N}$, $t \in \mathbb{R}^+$. For any $x \in \mathbb{C}(\mathbb{R}^+, \mathbb{R}^d)$, $\{x^{\rho^n}\}$ is a sequence of discretizations of x of the form $x^{\rho^n}_t = x_{\rho^n_t} = x_{\frac{k}{n}}$, $t \in [\frac{k}{n}, \frac{k+1}{n})$, $k \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ is the space of all càdlàg mappings $x : \mathbb{R}^+ \to \mathbb{R}^d$, i.e. mappings which are right continuous and admit left-hand limits equipped with the Skorokhod J_1 topology. If $X = (X^1, \dots, X^d)$ is a semimartingale then $[[X]]_t = \{[X^i, X^j]_t\}_{i,j=1,\dots,d}$, $[X]_t = \sum_{i=1}^d [X^i]_t = \mathrm{tr}\,[[X]]_t$ and $Q^X = \frac{d[[X]]}{d[X]}$ is the Radon-Nikodym derivative. If $K = (K^1, \dots, K^d)$ is the process with locally finite variation, then $|K|_t = \sum_{i=1}^d |K^i|_t$, where $|K^i|_t = |K^i|_{[0,t]}$ is the total variation of K^i on $[0, t]$. $\mathbb{L}_p(D)$, $p \ge 1$, is the usual $\mathbb{L}_p$-space defined in terms of Lebesgue measure l^d on D. $\mathbb{R}^d \otimes \mathbb{R}^d$ is the set of all $(d \times d)$-matrices, σ^* is the matrix adjoint to σ, $\|\sigma\| = (\mathrm{tr}\,\sigma\sigma^*)^{1/2}$, I_d is the identity matrix of dimension d. As usual, we say that a sequence $\{X^n\}$ is C-tight if it is tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and its every limit point has continuous trajectories. "$\longrightarrow_{\mathcal{D}}$", "$\longrightarrow_{\mathcal{P}}$" denote convergence in law and in probability, respectively.

2 Convergence of Wong-Zakai type approximations

Let D be a connected domain in $\mathbb{R}^d$. Define the set $\mathcal{N}_x$ of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r},\ \mathcal{N}_{x,\infty} = \bigcap_{r>0} \mathcal{N}_{x,r},\ \mathcal{N}_{x,r} = \{\mathbf{n} \in \mathbb{R}^d : |\mathbf{n}| = 1,\ B(x - r\mathbf{n}, r) \cap D = \emptyset\},$$

where $B(z, r) = \{y \in \mathbb{R}^d : |y - z| \le r\}$, $z \in \mathbb{R}^d$, $r > 0$. Following Lions and Sznitman [10] and Saisho [19] we will consider domains satisfying the assumptions:

(A) There is $r_0 \in (0, \infty]$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$ for every $x \in \partial D$.

(B) There exist constants $\delta > 0$, $\beta \ge 1$ such that for every $x \in \partial D$ there is a unit vector $\mathbf{l}_x$ with the following property: $< \mathbf{l}_x, \mathbf{n} > \ge 1/\beta$ for every $\mathbf{n} \in \bigcup_{y \in B(x,\delta)\cap\partial D} \mathcal{N}_y$, where $< \cdot, \cdot >$ denotes the usual inner product in $\mathbb{R}^d$.

The remark below is to be found in [10, 19].

Remark 2.1. (i) $\mathbf{n} \in \mathcal{N}_{x,r}$ if and only if $< y - x, \mathbf{n} > + \frac{1}{2r}|y - x|^2 \ge 0$ for every $y \in \bar{D}$.
(ii) If $\mathrm{dist}(x, \bar{D}) < r_0$, $x \notin \bar{D}$ then there exists a unique $\Pi_{\bar{D}}(x) \in \partial D$ such that $|x - \Pi_{\bar{D}}(x)| = \mathrm{dist}(x, \bar{D})$. Moreover $(\Pi_{\bar{D}}(x) - x)/|\Pi_{\bar{D}}(x) - x| \in \mathcal{N}_{\Pi_{\bar{D}}(x)}$.
(iii) If D is a convex domain in $\mathbb{R}^d$ then $r_0 = \infty$.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space with filtration $(\mathcal{F}_t)$ satisfying the usual conditions and let W be an $(\mathcal{F}_t)$ adapted Wiener process. We will say that the SDE (1.2) has a (strong) solution if there exists a pair (X, K) of $(\mathcal{F}_t)$-adapted processes satisfying (1.2) and such that

$$X \text{ is } \bar{D}\text{-valued,} \tag{2.1}$$

K is a process with locally bounded variation such that $K_0 = 0$ and

$$K_t = \int_0^t \mathbf{n}_s \, d|K|_s, \qquad |K|_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}} \, d|K|_s, \quad t \in \mathbb{R}^+,$$

where $\mathbf{n}_s \in \mathcal{N}_{X_s}$ if $X_s \in \partial D$. (2.2)

Similarly we define strong solutions of (1.1). Recall also that the SDE (1.2) is said to have a weak solution if there exists a filtered probability space $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar{P})$, an $(\bar{\mathcal{F}}_t)$-adapted Wiener process $\bar{W}$ and a pair $(\bar{X}, \bar{K})$ of $(\bar{\mathcal{F}}_t)$-adapted processes such that the conditions (1.2), (2.1) and (2.2) hold for processes $(\bar{X}, \bar{K})$ instead of (X, K).

Note that $X = (X^1, \ldots, X^d)$, $K = (K^1, \ldots, K^d)$ are d-dimensional processes and (1.2) has the following equivalent form

$$X_t^i = X_0^i + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_s) \, dW_s^j + \frac{1}{2} \sum_{j=1}^d \sum_{l=1}^d \int_0^t \frac{\partial \sigma_{i,j}}{\partial x_l} \sigma_{l,j}(X_s) \, ds$$
$$+ \int_0^t b_i(X_s) \, ds + K_t^i, \quad t \in \mathbb{R}^+, \quad i = 1, \ldots, d.$$

In the sequel we say that σ, b, σ' have some property if the coefficients σ_{ij}, b_i, $\frac{\partial \sigma_{ij}}{\partial x_l}$ possesses this property for $i, j, l = 1, \ldots, d$.

Theorem 2.2. *Assume that D satisfies (A), (B). Let $\{(X^n, K^n)\}$ be a sequence of solutions of (1.1). If the coefficients σ^n, b^n satisfy (1.3), (1.4) and $\sigma^n \to_K \sigma$, then $\{(X^n, K^n)\}$ is tight in $C(\mathbb{R}^+, \mathbb{R}^{2d})$ and its each limit point $(\bar{X}, \bar{K})$ is a solution of the reflected SDE*

$$\bar{X}_t = \bar{X}_0 + \int_0^t \sigma(\bar{X}_s) \, d\bar{W}_s + \int_0^t \bar{H}_s \, ds + \bar{K}_t, \quad t \in \mathbb{R}^+, \tag{2.3}$$

where $\bar{W}$ is a Wiener process adapted to some filtration $(\bar{\mathcal{F}}_t)$, $\mathcal{L}(X_0, W) = \mathcal{L}(\bar{X}_0, \bar{W})$ and $\bar{H}$ is some d-dimensional bounded and progressively measurable process with respect $(\bar{\mathcal{F}}_t)$.

Proof. Let $Y^n = X_0 + \int_0^t \sigma^n(X_s^n) \, dW_s^n + \int_0^t b^n(X_s^n) \, ds$, $t \in \mathbb{R}^+$, $n \in \mathbb{N}$. We first prove that

$$\{Y^n\} \quad \text{is tight in } \mathbb{C}\,(\mathbb{R}^+, \mathbb{R}^d). \tag{2.4}$$

To check this we decompose Y^n as follows

$$Y_t^n = X_0 + \int_{\rho_t^n}^t \sigma^n(X_s^n) \, dW_s^n + \int_0^{\rho_t^n} \sigma^n(X_{s-}^{n,\rho^n}) \, dW_s^n + \int_0^{\rho_t^n} (\sigma^n(X_s^n) - \sigma^n(X_{s-}^{n,\rho^n})) \, dW_s^n$$
$$+ \int_0^t b^n(X_s^n) \, ds \equiv X_0 + I_t^{n,1} + I_t^{n,2} + I_t^{n,3} + I_t^{n,4}, \quad t \in \mathbb{R}^+.$$

Of course

$$\sup_{t \le q} |I_t^{n,1}| = \sup_{t \le q} \Big| \int_{\rho_t^n}^t \sigma^n(X_s^n) \, dW_s^n \Big| \le C \sup_{t \le q+1} |\Delta W_t^{\rho^n}| \longrightarrow 0, \quad P\text{-a.s.}, \ q \in \mathbb{R}^+.$$

It is also clear that uniform boundedness of $\{b^n\}$ implies tightness of $\{I^{n,4}\}$ and that

$$\sup_{t \le q} |\Delta I_t^{n,2}| \underset{\mathcal{P}}{\longrightarrow} 0 \quad \text{and} \quad \sup_{t \le q} |\Delta I_t^{n,3}| \underset{\mathcal{P}}{\longrightarrow} 0$$

for $q \in \mathbb{R}^+$. To check that $\{I^{n,2}\}$ and $\{I^{n,3}\}$ are C-tight we use Aldous criterion (see [2]). Since $I_0^{n,2} = I_0^{n,3} = 0$, it suffices to show that $I^{n,2}_{\gamma_n+\delta_n} - I^{n,2}_{\gamma_n} \longrightarrow_{\mathcal{P}} 0$ and $I^{n,3}_{\gamma_n+\delta_n} - I^{n,3}_{\gamma_n} \longrightarrow_{\mathcal{P}} 0$ for any $q \in \mathbb{R}^+$, any sequence $\{\gamma_n\}$ of $\mathcal{F}^{\rho^n}$ stopping times and any sequence of constants $\{\delta_n\}$ such that $\gamma_n + \delta_n \le q$ and $\delta_n \to 0$. Using uniform boundedness of $\{\sigma^n\}$ and the fact that $\int_0^{\rho^n_t} \sigma^n(X^{n,\rho^n}_{s-})\, dW^n_s = \int_0^t \sigma^n(X^{n,\rho^n}_{s-})\, dW^{\rho^n}_s$ we have

$$\begin{aligned} E|I^{n,2}_{\gamma_n+\delta_n} - I^{n,2}_{\gamma_n}|^2 &\le CE\int_{\gamma_n}^{\gamma_n+\delta_n} \|\sigma^n(X^{n,\rho^n}_{s-})\|^2\, d[W^{\rho^n}]_s \\ &\le CE\int_{\gamma_n}^{\gamma_n+\delta_n} \|\sigma^n(X^{n,\rho^n}_{s-})\|^2\, d\rho^n_s \\ &\le CE(\rho^n_{\gamma_n+\delta_n} - \rho^n_{\gamma_n}) \le C(\delta_n + \frac{1}{n}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Clearly,

$$\begin{aligned} E|I^{n,3}_{\gamma_n+\delta_n} - I^{n,3}_{\gamma_n}|^2 &\le CE(\int_{\gamma_n}^{\gamma_n+\delta_n} |X^n_s - X^{n,\rho^n}_{s-}|\, d|W^n|_s)^2 \\ &\le CE(\sum_{k=[n\gamma_n]+1}^{[\gamma_n+\delta_n]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} (\sigma^n(X^n_u)\, d|W^n|_u + b^n(X^n_u)du)\, d|W^n|_s)^2 \\ &\quad + CE(\sum_{k=[n\gamma_n]+1}^{[\gamma_n+\delta_n]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} (K^n_s - K^n_{\frac{k-1}{n}})\, d|W^n|_s)^2 \equiv A^{n,1} + A^{n,2} \end{aligned}$$

and

$$\begin{aligned} A^{n,1} &\le CE(\sum_{k=[n\gamma_n]+1}^{[\gamma_n+\delta_n]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} (d|W^n|_u + du)\, d|W^n|_s)^2 \\ &\le CE\big(([W^{\rho^n}]_{\gamma_n+\delta_n} - [W^{\rho^n}]_{\gamma_n}) + \max_{\{k;\frac{k}{n}\le q+1\}} |\Delta W^{\rho^n}_t|\delta_n\big)^2 \\ &\le C(E(\rho^n_{\gamma_n+\delta_n} - \rho^n_{\gamma_n})^2 + \delta_n^2) \le C(\delta_n^2 + \frac{1}{n^2}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $d|K^n|_u \le C(|\sigma^n(X^n_u)|\, d|W^n|_u + |b^n(X^n_u)|du)$ (see, e.g., [1, Lemma 2.4]), we also have

$$A^{n,2} \le C(\delta_n^2 + \frac{1}{n^2}) \to 0 \text{ as } n \to \infty.$$

Consequently, $\{I^{n,2}\}$ and $\{I^{n,3}\}$ are C-tight in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, which completes the proof of (2.4). Hence, by [21, Proposition 3,4],

$$\{|K^n|_q\} \quad \text{is bounded in probability} \tag{2.5}$$

and

$$\{(X^n, K^n)\} \quad \text{is tight in } \mathbb{C}(\mathbb{R}^+, \mathbb{R}^{2d}). \tag{2.6}$$

In particular, (2.6) implies that

$$\sup_{t\le q} |X^n_t - X^{n,\rho^n}_t| \underset{\mathcal{P}}{\longrightarrow} 0, \quad q \in \mathbb{R}^+. \tag{2.7}$$

By (2.6), (2.7) without loss of generality we may assume that

$$(X^n, X^{n,\rho^n}, I^{n,3} + I^{n,4}, K^n, Y^n, W^{\rho^n}) \underset{\mathcal{D}}{\longrightarrow} (\bar{X}, \bar{X}, \bar{I}, \bar{K}, \bar{Y}, \bar{W}) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{6d}), \tag{2.8}$$

where $\bar{W}$ is a Wiener process adapted to the natural filtration $(\bar{\mathcal{F}}_t)$ of the pair $(\bar{X}, \bar{W}, \bar{I})$. By [21, Proposition 4], to complete the proof, it suffices to show that (X^n, K^n, Y^n) converges in law to $(\bar{X}, \bar{K}, \bar{Y})$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$, where $\bar{Y}_t = \bar{X}_0 + \int_0^t \sigma(\bar{X}_s)\, d\bar{W}_s + \bar{I}_t$ and $\bar{I}_t = \int_0^t \bar{H}_s\, ds$, $t \in \mathbb{R}^+$, for some bounded and progressively measurable process $\bar{H}$. By the functional convergence theorem for stochastic integrals (see, e.g., [7, Theorem 2.6] or [9, Theorem 2.2]),

$$(I^{n,2}, I^{n,3} + I^{n,4}, X^n, K^n, W^{\rho^n}) \underset{\mathcal{D}}{\longrightarrow} \Big(\int_0^\cdot \sigma(\bar{X}_s)\, d\bar{W}_s, \bar{I}, \bar{X}, \bar{K}, \bar{W}\Big) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{5d}).$$

Hence $(X^n, K^n, Y^n) = (X_0 + I^{n,1} + I^{n,2} + I^{n,3} + I^{n,4}, K^n, Y^n) \longrightarrow_{\mathcal{D}} (\bar{X}, \bar{K}, \bar{Y})$ and what is left is to show that $\bar{I}$ has the desired form. By simple calculations for any $s < t \le q$

$$\begin{aligned}
|I^{n,3}_t - I^{n,3}_s| + |I^{n,4}_t - I^{n,4}_s| &\le C\Big(\int_s^t |X^n_u - X^{n,\rho^n}_{u-}|\, d|W^n|_u + (t-s)\Big) \\
&\le C\Big(\sum_{k=[ns]+1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} (d|W^n|_u + du)\, d|W^n|_s) + (t-s)\Big) \\
&\le C\big(([W^{\rho^n}]_t - [W^{\rho^n}]_s) + (\max_{k;\frac{k}{n}\le q} |\Delta W^{\rho^n}_t| + 1)(t-s)\big).
\end{aligned}$$

Since $\sup_{t\le q} |[W^{\rho^n}]_t - dt| \longrightarrow_{\mathcal{P}} 0$ and $\sup_{t\le q} |\Delta|W^{\rho^n}_t| \longrightarrow_{\mathcal{P}} 0$, it is clear that $|\bar{I}_t - \bar{I}_s| \le C(t-s)$, which completes the proof. □

Remark 2.3. To prove tightness of $\{(X^n, K^n)\}$ instead of Aldous criterion one can use Kolmogorov's tightness criterion and estimates from [1, Lemma 4.5].

Theorem 2.4. *Let the assumptions of Theorem 2.2 hold. If one of the two following conditions is satisfied*

(i) $\sigma^n, \sigma \in \mathbb{C}^1(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$ *and* $\sigma^{n,\prime} \underset{K}{\to} \sigma'$, $b^n \underset{K}{\to} b$,

(ii) $\sigma^n \in \mathbb{C}^1(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$, $\sigma\sigma^*$ *is elliptic and* $l^d(\mathbb{R}^d \setminus G) = 0$, *where*

$$G = \{x \in \bar{D};\ for\ any\ \{x_n\} \subset \bar{D},\ if\ x_n \to x\ then\ \sigma^{n,\prime}(x_n) \to \sigma'(x),\ b^n(x_n) \to b(x)\},$$

then each limit point of $\{(X^n, K^n)\}$ *is a weak solution of (1.2).*

Proof. First let us assume (i). Note that

$$\begin{aligned}
I^{n,3}_t &= \sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} (\sigma^n(X^n_s) - \sigma^n(X^n_{\frac{k-1}{n}}))\, dW^n_s = \sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} \sigma^{n,\prime}(X^n_u)\, dX^n_u\, dW^n_s \\
&= \sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} \sigma^{n,\prime}\sigma^n(X^n_u)\, dW^n_u\, dW^n_s + \sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} \sigma^{n,\prime}(X^n_u)\, dK^n_u\, dW^n_s \\
&= \sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} \sigma^{n,\prime}\sigma^n(X^{n,\rho^n}_{u-})\, dW^n_u\, dW^n_s + R^n_t = C^n_t + R^n_t, \quad t \in \mathbb{R}^+,
\end{aligned}$$

where

$$\begin{aligned}
R^n_t \equiv &\sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} (\sigma^{n,\prime}\sigma^n(X^n_u) - \sigma^{n,\prime}\sigma^n(X^{n,\rho^n}_{u-}))\, dW^n_u\, dW^n_s \\
&+ \sum_{k=1}^{[nt]} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \int_{\frac{k-1}{n}}^{s} \sigma^{n,\prime}(X^n_u)\, dK^n_u\, dW^n_s\big) \equiv R^{n,1}_t + R^{n,2}_t, \quad t \in \mathbb{R}^+.
\end{aligned}$$

More precisely, i-th coordinate of R^n_t has the form

$$\begin{aligned}(R^n_t)_i &= (R^{n,1}_t)_i + (R^{n,2}_t)_i \\ &= \sum_{j=1}^d\sum_{l=1}^d\sum_{m=1}^d\sum_{k=1}^{[nt]}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}(\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_u) - \frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}(X^{n,\rho^n}_{u-}))\,dW^{n,m}_u\,dW^{n,j}_s \\ &\quad + \sum_{j=1}^d\sum_{l=1}^d\sum_{k=1}^{[nt]}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}\frac{\partial\sigma^n_{i,j}}{\partial x_l}(X^n_u)\,dK^{n,l}_u\,dW^{n,j}_s.\end{aligned}$$

Clearly,

$$|R^{n,1}_t| \le C\max_{j,l,m}\sup_{s\le t}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_u) - \frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^{n,\rho^n}_{u-})|[W^{\rho^n}]_t, \quad t\in\mathbb{R}^+,$$

and

$$|R^{n,2}_t| \le C\max_{j,l}\sup_{s\le t}|\Delta W^{\rho^n}||\frac{\partial\sigma^n_{i,j}}{\partial x_l}(X^n_u)||K^n|_t, \quad t\in\mathbb{R}^+.$$

Since $\sigma^{n,\prime}\sigma^n \to_K \sigma'\sigma$, it follows by (2.7) that $\sup_{t\le q}|R^n_t| \longrightarrow_{\mathcal{P}} 0$, $q\in\mathbb{R}^+$. Observe now that i-th coordinate of C^n_t has the form

$$\begin{aligned}(C^n_t)_i &= \sum_{j=1}^d\sum_{l=1}^d\sum_{m=1}^d\sum_{k=1}^{[nt]}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}(X^{n,\rho^n}_{u-})\,dW^{m,n}_u\,dW^{j,n}_s \\ &= \frac{1}{2}\sum_{j=1}^d\sum_{l=1}^d\sum_{m=1}^d\int_0^{\rho^n_t}\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}(X^{n,\rho^n}_{u-})\,d[W^{m,\rho^n}, W^{j,\rho^n}]_s, \quad t\in\mathbb{R}^+.\end{aligned}$$

Since $\sup_{t\le q}\|[[W^{\rho^n}]]_t - tI_d\| \longrightarrow_{\mathcal{P}} 0$, $q\in\mathbb{R}^+$, it follows from [7, Theorem 2.6] or [9, Theorem 2.2] that

$$(I^{n,2}, C^n, I^{n,4}, X^n, K^n, W^{\rho^n}) \underset{\mathcal{D}}{\longrightarrow} (\int_0^{\cdot}\sigma(\bar X_s)\,d\bar W_s, \bar C, \int_0^{\cdot} b(\bar X_s)ds, \bar X, \bar K, \bar W) \tag{2.9}$$

in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{6d})$, where i-th coordinate of $\bar C$ equals $\bar C^i_t = \frac{1}{2}\sum_{j=1}^d\sum_{l=1}^d\int_0^t\frac{\partial\sigma_{i,j}}{\partial x_l}\sigma_{l,j}(\bar X_s)\,ds$. Therefore $(\bar X, \bar K)$ satisfies the reflected SDE

$$\bar X_t = \bar X_0 + \int_0^t\sigma(\bar X_s)\,d\bar W_s + \int_0^t(\frac{1}{2}\sigma'\sigma(\bar X_s) + b(\bar X_s))\,ds + \bar K_t, \quad t\in\mathbb{R}^+, \tag{2.10}$$

on $(\bar\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t), \bar P)$, i.e. $(\bar X, \bar K)$ is a weak solution of (1.2).

Now we assume (ii). We are going to show that as before each limit point $(\bar X, \bar K)$ of $\{(X^n, K^n)\}$ satisfies (2.10) (From Theorem 2.2 we only know that $(\bar X, \bar K)$ satisfies (2.3)). To this end, we first show Krylov's estimates for solutions of (2.3). Let $\tau^R = \inf\{t\ge 0 : |\bar X_t| > R\}$. We will show that for any $R>0$ there exists $C>0$ depending only on a, d, R, q such that

$$E\int_0^{q\wedge\tau^R} f(\bar X_s)\,ds \le C\|f\|_{\mathbb{L}_d(B(0,R)\cap\bar D)} \tag{2.11}$$

for all non-negative measurable $f : \bar D \to \mathbb{R}^+$. As in [17, Lemma 1] (see also [18]) we will apply Krylov's estimates for continuous semimartingales. Let $\bar X^c$ denote the martingale part of $\bar X$. Then $Q^{\bar X^c}_s = a(\bar X_s)/\mathrm{tr}\,a(\bar X_s)$, which implies that $(\det Q^{\bar X^c}_s)^{1/d} = (\det a(\bar X_s))^{1/d}/\mathrm{tr}\,a(\bar X_s)$. Since a is elliptic, it follows that $(\mathrm{tr}\,a(\bar X_s)^{-1} \le c(\det Q^{\bar X^c}_s)^{1/d}$. Consequently,

$$ds = (d\,\mathrm{tr}\,a(\bar X_s))^{-1}d < \bar X^c >_s \le \frac{c}{d}(\det Q^{\bar X^c}_s)^{1/d}\,d < \bar X^c >_s$$

and

$$E\int_0^{q\wedge\tau^R} f(\bar X_s)\,ds \le \frac{c}{d}E\int_0^{q\wedge\tau^R}(\det Q_s^{\bar X^c})^{1/d} f(\bar X_s)\,d<\bar X^c>_s\ .$$

By [1, Lemma 2.4] and boundedness of $\bar H$, $E|\bar K|_q<\infty$ and $E|\int_0^\cdot \bar H_s\,ds|_q<\infty$. Therefore applying [12, Theorem 6] to f we get (2.11). From (2.11) and the fact that $\tau^R\nearrow\infty$ it follows that

$$\int_0^q \mathbf{1}_{\{\bar X_s\in G^c\}}\,ds=0,\quad q\in\mathbb{R}^+. \tag{2.12}$$

We will now use (2.12) to show that $\sup_{t\le q}|R^n_t|\longrightarrow_{\mathcal P}0$, $q\in\mathbb{R}^+$, and $(\bar X,\bar K)$ satisfies (2.10). To this end, let us set $A^n_k\equiv|\sqrt{n}(W_{\frac{k}{n}}-W_{\frac{k-1}{n}})|^2$, $n,k\in\mathbb{N}$ and observe that

$$\begin{aligned}
\sup_{t\le q}|R^{n,1}_t| &\le C\max_{j,l,m}\sum_{\{k;\frac{k}{n}\le q\}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_u)-\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^{n,\rho^n}_{u-})|ds\,A^n_k\\
&\le C\max_{j,l,m}\sum_{\{k;\frac{k}{n}\le t\}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_u)-\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^{n,\rho^n}_{u-})|ds\,A^n_k\mathbf{1}_{\{A^n_k\le M\}}\\
&\quad+C\max_{j,l,m}\sum_{\{k;\frac{k}{n}\le t\}}\int_{\frac{k-1}{n}}^{\frac{k}{n}}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_u)-\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^{n,\rho^n}_{u-})|ds\,A^n_k\mathbf{1}_{\{A^n_k>M\}}\\
&\equiv R^{n,3}_q(M)+R^{n,4}_q(M).
\end{aligned}$$

By the Skorokhod representation theorem we may assume that $X^n\to\bar X$ in $\mathbb{C}(\mathbb{R}^+,\mathbb{R}^d)$ P-a.s. Therefore by (2.12) for any $M>0$ we have

$$\begin{aligned}
R^{n,3}_q(M) &\le C\max_{j,l,m}\int_0^{\rho^n_q}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_s)-\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^{n,\rho^n}_{s-})|ds\\
&=C\max_{j,l,m}\int_0^{\rho^n_q}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^n_s)-\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,m}((X^{n,\rho^n}_{s-})|\mathbf{1}_{\{\bar X_s\in G\}}ds\\
&\longrightarrow 0,\ \ P\text{-a.s.}
\end{aligned}$$

It is clear that all A^n_k have the same χ^2 distribution. Using this and boundedness of $\sigma'\sigma$ gives

$$ER^{n,4}_q(M)\le CE\big(\sum_{\{k;\frac{k}{n}\le q\}}\frac{1}{n}A^n_k\mathbf{1}_{\{A^n_k>M\}}\big)\le Cq\,E\chi^2{}_{\{\chi^2>M\}}\longrightarrow 0\quad\text{as } M\nearrow\infty.$$

Therefore $\sup_{t\le q}|R^{n,1}_t|\longrightarrow_{\mathcal P}0$, $q\in\mathbb{R}^+$. Since, as in case (i),

$$\sup_{t\le q}|R^{n,2}_t|\le C\max_{j,l}\sup_{t\le q}|\Delta W^{\rho^n}_t||\frac{\partial\sigma^n_{i,j}}{\partial x_l}(X^n_t)||K^n|_q\underset{\mathcal P}{\longrightarrow}0,$$

we get $\sup_{t\le q}|R^n_t|\longrightarrow_{\mathcal P}0$, $q\in\mathbb{R}^+$. To check (2.10), we first observe that uniform boundedness of $\sigma^{n,\prime}\sigma^n$ implies that

$$E|C^n_q-\tilde C^n_q|^2\to 0,\quad q\in\mathbb{R}^+,$$

where

$$(\tilde C^n_t)_i=\frac12\sum_{j=1}^d\sum_{l=1}^d\int_0^{\rho^n_t}\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,j}(X^{n,\rho^n}_{s-})\,d\rho^n_s=\frac12\sum_{j=1}^d\sum_{l=1}^d\int_0^{\rho^n_t}\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,j}(X^{n,\rho^n}_{s-})\,ds,$$

$i=1,\dots,d$. Since $C^n-\tilde{C}^n$ is a martingale, this also implies that $\sup_{t\le q}|C^n_t-\tilde{C}^n_t|\longrightarrow_{\mathcal{P}}0$, $q\in\mathbb{R}^+$. Moreover, assuming without loss of generality that $X^n\to\bar{X}$ P-a.s. and using (2.12) we see that for any $i=1,\dots,d$,

$$\begin{aligned}\sup_{t\le q}|(\tilde{C}^n_t)_i-(\bar{C}_t)_i| &\le \frac{C}{n}+\sum_{j=1}^d\sum_{l=1}^d\int_0^{\rho^n_t}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,j}(X^{n,\rho^n}_{s-})-\frac{\partial\sigma_{i,j}}{\partial x_l}\sigma_{l,j}(\bar{X}_s)|\,ds\\ &=\frac{C}{n}+\sum_{j=1}^d\sum_{l=1}^d\int_0^{\rho^n_t}|\frac{\partial\sigma^n_{i,j}}{\partial x_l}\sigma^n_{l,j}(X^{n,\rho^n}_{s-})-\frac{\partial\sigma_{i,j}}{\partial x_l}\sigma_{l,j}(\bar{X}_s)|\mathbf{1}_{\{\bar{X}_s\in G\}}\,ds\\ &\longrightarrow 0,\quad P\text{-a.s.}\end{aligned}$$

Similarly we prove that

$$\begin{aligned}\sup_{t\le q}|I^{n,4}_t-\int_0^t b(\bar{X}_s)\,ds| &\le \int_0^q|b^n(X^n_s)-b(\bar{X}_s)|\,ds\\ &=\int_0^q|b^n(X^n_s)-b(\bar{X}_s)|\mathbf{1}_{\{\bar{X}_s\in G\}}\,ds\longrightarrow 0,\quad P\text{-a.s.}\end{aligned}$$

Therefore (2.10) holds true and the proof is complete. □

We recall that we say that the weak uniqueness holds for (1.2) if the laws $\mathcal{L}(\bar{X},\bar{K})$, $\mathcal{L}(\bar{X}',\bar{K}')$ of any weak solutions $(\bar{X},\bar{K})$, $(\bar{X}',\bar{K}')$ of (1.2), possibly defined on different probability spaces, are the same. SDE (1.2) is weakly unique for instance if b is bounded, σ is bounded and Lipschitz continuous, $\sigma\sigma^*$ is uniformly elliptic and ∂D is regular (see Stroock and Varadhan [25] for more details).

Corollary 2.5. *Under assumptions of Theorem 2.2, if weak uniqueness holds for (1.2), then*

$$(X^n,K^n)\underset{\mathcal{D}}{\longrightarrow}(X,K)\quad in\ \mathbb{C}\,(\mathbb{R}^+,\mathbb{R}^{2d}),$$

where (X,K) is a unique weak solution of (1.2).

Proof. Follows immediately from Theorem 2.4. □

Example 2.6. Assume that $d=2$, $D=B(0,2)$, $\sigma_{1,1}(x)=|x_1|+1$, $\sigma_{2,2}(x)=|x_2|+1$, $\sigma_{1,2}(x)=\sigma_{2,1}(x)=0$, $X_0=b=0$. By [25] there exists a unique weak solution (X,K) of the reflected SDE

$$X^i_t=\int_0^t(|X^i_s|+1)dW^i_s+\frac{1}{2}\int_0^t(\mathbf{1}_{\{X^1_s>0\}}-\mathbf{1}_{\{X^i_s<0\}})(|X^i_s|+1)ds+K^i_t,\quad i=1,2,\ t\in\mathbb{R}^+$$

(These equations are not independent because K^i depends essentially on both coordinates X^1 and X^2). Set $\sigma^n_{i,i}(x)=\sigma_{i,i}(x)$ if $|x_1|>1/n$ and $\sigma^n_{i,i}(x)=nx_i^2/2+1+1/(2n)$ if $|x_i|\le 1/n$, $i=1,2$, $\sigma^n_{1,2}=\sigma^n_{2,1}=0$, $b^n=0$, $n\in\mathbb{N}$. By simple calculations, $\sigma^n\in\mathbb{C}^1(B(0,2),\mathbb{R}^2\otimes\mathbb{R}^2)$ and $\sigma^{n,\prime}(x_n)\to\sigma'(x)$ for any $x\notin G^c=\{x\in B(0,2);x=(0,x_2)\,\text{or}\,(x_1,0)\}$ and any $\{x_n\}$ such that $x_n\to x$. Therefore the assumptions of Corollary 2.5 are satisfied and

$$(X^n,K^n)\underset{\mathcal{D}}{\longrightarrow}(X,K)\quad\text{in }\mathbb{C}\,(\mathbb{R}^+,\mathbb{R}^{2d}),$$

where $\{(X^n,K^n)\}$ is a sequence of Wong-Zakai type approximations of the form (1.1).

We say that the pathwise uniqueness holds for (1.2) if for any probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ with filtration $(\bar{\mathcal{F}}_t)$, any $\bar{X}_0\in\bar{D}$ and any $(\bar{\mathcal{F}}_t)$ Wiener process $\bar{W}$ such that $\mathcal{L}(\bar{X}_0,\bar{W})=\mathcal{L}(X_0,W)$ we have $P[(\bar{X}_t,\bar{K}_t)=(\bar{X}'_t,\bar{K}'_t),t\in\mathbb{R}^+]=1$ for any two $(\bar{\mathcal{F}}_t)$ adapted strong solutions $(\bar{X},\bar{K})$, $(\bar{X}',\bar{K}')$ on $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathcal{P}})$ of the SDE (1.2).

In the case of discontinuous coefficients results on pathwise uniqueness of (1.2) are known only for $d=1$, $D=\mathbb{R}^+$. For instance, Semrau [20] have proved that if b is bounded and σ is Lipschitz continuous and uniformly positive, then (1.2) is pathwise unique. Some results on pathwise uniqueness can also be found in the earlier paper by Zhang [29].

Corollary 2.7. *Under the assumptions of Theorem 2.2, if pathwise uniqueness holds for (1.2) then*

$$\sup_{t\le q}(|X^n_t - X_t| + |K^n_t - K_t|) \underset{\mathcal{P}}{\longrightarrow} 0, \quad q\in\mathbb{R}^+,$$

where (X,K) is the unique strong solution of (1.2).

Proof. We have to prove that $\{(X^n,K^n)\}$ converges in probability. For this purpose, as Gyöngy and Krylov [6], it suffices to show that from any subsequences $(l)\subset(n),(m)\subset(n)$ it is possible to choose further subsequences $(l_k)\subset(l),(m_k)\subset(m)$ such that

$$(X^{l_k},K^{l_k},X^{m_k},K^{m_k}) \underset{\mathcal{D}}{\longrightarrow} (\bar X,\bar K,\bar X,\bar K) \quad \text{in } \mathbb{C}\,(\mathbb{R}^+,\mathbb{R}^{4d}).$$

From the proof of Theorem 2.2 we deduce that $\{(X^l,K^l,X^m,K^m,W)\}$ is tight in $\mathbb{C}\,(\mathbb{R}^+,\mathbb{R}^{5d})$. Therefore we can choose subsequences $(l_k)\subset(l),(m_k)\subset(m)$ such that

$$(X^{l_k},K^{l_k},X^{m_k},K^{m_k},W) \underset{\mathcal{D}}{\longrightarrow} (\bar X,\bar K,\bar X',\bar K',\bar W), \quad \text{in } \mathbb{C}\,(\mathbb{R}^+,\mathbb{R}^{5d}),$$

where $\bar W$ is a Wiener process adapted to the natural filtration $\mathcal{F}^{\bar X,\bar X',\bar K,\bar K',\bar W}$ and $\mathcal{L}(\bar X_0,\bar W)=\mathcal{L}(X_0,W)$. By arguments from the proof of Theorem 2.4, $(\bar X,\bar K)$ and $(\bar X',\bar K')$ are two solutions to (1.2) with $\bar W$ instead of W and $\bar X_0$ instead of X_0. Since (1.2) is pathwise unique, $(\bar X,\bar K)=(\bar X',\bar K')$, and consequently, $\{(X^n,K^n)\}$ converges in probability in $\mathbb{C}\,(\mathbb{R}^+,\mathbb{R}^{2d})$ to some processes (X,K). It follows that (2.9) holds with the convergence in probability, which implies that (X,K) is a strong solution of (1.2). Using once again pathwise uniqueness property of (1.2) completes the proof. ☐

Example 2.8. Assume that $d=1$, $D=\mathbb{R}^+$, $\sigma(x)=\min(|x-2|+1,3)$, $X_0=b=0$. By [20] there exists a unique strong solution $(X.K)$ of the reflected SDE

$$\begin{aligned} X_t = &\int_0^t \min(|X_s-2|+1,3)\,dW_s \\ &+\frac{1}{2}\int_0^t (\mathbf{1}_{\{2<X_s<4\}} - \mathbf{1}_{\{X_s<2\}})\min(|X_s-2|+1,3)\,ds + K_t. \end{aligned}$$

Set

$$\sigma^n(x) = \begin{cases} \sigma(x), & x\in[0,2-\frac{1}{n}]\cup[2+\frac{1}{n},4-\frac{1}{n}]\cup[4+\frac{1}{n},\infty), \\ \frac{n}{2}(x-2)^2+\frac{1}{2n}+1, & x\in(2-\frac{1}{n},2+\frac{1}{n}), \\ -\frac{n}{4}(x-4)^2+\frac{1}{2}(x-4)-\frac{1}{4n}+1, & x\in(4-\frac{1}{n},4+\frac{1}{n}), \end{cases}$$

and $b^n=0$, $n\in\mathbb{N}$. Clearly, $\sigma^n\in\mathbb{C}^1(\mathbb{R}^+,\mathbb{R})$ and $\sigma^{n,\prime}(x_n)\to\sigma'(x)$ for any $x\notin\{2,4\}$ and any $\{x_n\}$ such that $x_n\to x$. Let (X^n,K^n) denote the Wong-Zakai type approximation of the form (1.1). Then by Corollary 2.7,

$$\sup_{t\le q}(|X^n_t - X_t| + |K^n_t - K_t|) \underset{\mathcal{P}}{\longrightarrow} 0, \quad q\in\mathbb{R}^+.$$

3 Rate of convergence of Wong-Zakai type approximations

In this section we consider Wong-Zakai type approximations of (1.2) with $\sigma^n = \sigma$, $b^n = b$, i.e.,

$$X^n_t = X_0 + \int_0^t \sigma(X^n_s)\,dW^n_s + \int_0^t b(X^n_s)\,ds + K^n_t, \quad t \in \mathbb{R}^+.$$

We assume that the coefficients satisfy the following condition

$$\sigma \in \mathbb{C}^1(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d), \quad \sigma,\ \sigma'\sigma,\ b \text{ are bounded and Lipschitz continuous.} \tag{3.1}$$

We will compare the Wong-Zakai type approximation with the classical Euler approximation $(\bar{X}^n, \bar{K}^n)$ of the form $\bar{X}^n_0 = \bar{Y}^n_0 = X_0$, $\bar{K}^n_0 = 0$,

$$\begin{cases} \bar{Y}^n_{\frac{k}{n}} & = \bar{Y}^n_{\frac{k-1}{n}} + \sigma(\bar{X}^n_{\frac{k-1}{n}})(W_{\frac{k}{n}} - W_{\frac{k-1}{n}}) + (\frac{1}{2}\sigma'\sigma + b)(\bar{X}^n_{\frac{k-1}{n}})\frac{1}{n} \\ \bar{X}^n_{\frac{k}{n}} & = \Pi_{\bar{D}}\big(\bar{X}^n_{\frac{k-1}{n}} + (\bar{Y}^n_{\frac{k}{n}} - \bar{Y}^n_{\frac{k-1}{n}})\big) \\ \bar{K}^n_{\frac{k}{n}} & = \bar{K}^n_{\frac{k-1}{n}} + (\bar{X}^n_{\frac{k}{n}} - \bar{X}^n_{\frac{k-1}{n}}) - (\bar{Y}^n_{\frac{k}{n}} - \bar{Y}^n_{\frac{k-1}{n}}) \end{cases}$$

and $\bar{X}^n_t = \bar{X}^n_{\frac{k-1}{n}}$, $\bar{K}^n_t = \bar{K}^n_{\frac{k-1}{n}}$, $\bar{Y}^n_t = \bar{Y}^n_{\frac{k-1}{n}}$, $t \in [\frac{k-1}{n}, \frac{k}{n})$, $k \in \mathbb{N}$. One can observe that $(\bar{X}^n, \bar{K}^n)$ is a solution of discrete reflected SDE

$$\bar{X}^n_t = X_0 + \int_0^t \sigma(\bar{X}^n_{s-})\,dW^{\rho^n}_s + \int_0^t (\frac{1}{2}\sigma'\sigma + b)(\bar{X}^n_{s-})\,d\rho^n_s + \bar{K}^n_t, \quad t \in \mathbb{R}^+, \tag{3.2}$$

which means that the pair $(\bar{X}^n, \bar{K}^n)$ is a solution of the Skorokhod problem associated with $\bar{Y}^n = X_0 + \int_0^\cdot \sigma(\bar{X}^n_{s-})dW^{\rho^n}_s + \int_0^\cdot (\frac{1}{2}\sigma'\sigma + b)(\bar{X}^n_{s-})\,d\rho^n_s$.

Remark 3.1. The rate of convergence of the Euler scheme for (1.2) is studied in [22]. In particular, in [22, Theorem 4] it is proved that for any $q \in \mathbb{R}^+$ and $\epsilon > 0$,

$$\{\sup_{t\le q} |\bar{X}^n_t - X_t|^2\} = \mathcal{O}_P((\frac{1}{n})^{1/2-\epsilon}).$$

In fact, the calculations from [22, Theorem 4] when combined with the following L^p estimates of modulus of continuity for Itô process Y with bounded coefficients

$$E \sup_{s,t\le q,\, |s-t|\le 1/n} |Y_t - Y_s|^{2p} = \mathcal{O}\big((\frac{\log n}{n})^p\big), \quad q \in \mathbb{R}^+, p \in \mathbb{N} \tag{3.3}$$

(see [5, 23]) give better rate. Namely, one can show that

$$\{\sup_{t\le q} |\bar{X}^n_t - X_t|^2\} = \mathcal{O}_P((\frac{\log n}{n})^{1/2}), \quad q \in \mathbb{R}^+.$$

Theorem 3.2. *Assume (A), (B) and (3.1). Then*

$$\{\sup_{t\le q} |X^n_t - X_t|^2\} = \mathcal{O}_P((\frac{\log n}{n})^{1/2}), \quad q \in \mathbb{R}^+.$$

Proof. We follow the proof of [1, Theorem 2.9] and [24, Theorem 4.2]. By [1, Lemma 2.4] there is $C > 0$ such that for $t \in (\frac{k}{n}, \frac{k+1}{n}]$,

$$|X^n_t - X^{n,\rho^n}_{t-}| \le C(\int_{\frac{k}{n}}^t ||\sigma(X^n_s||\,d|W^n|_s + \int_{\frac{k}{n}}^t |b(X^n_s|\,ds, \quad t \in \mathbb{R}^+.$$

Hence

$$\sup_{t\le q} |X^n_t - X^{n,\rho^n}_{t-}| \le C(\sup_{t\le q+1} |\Delta W^{\rho^n}_t| + \frac{1}{n}), \quad q \in \mathbb{R}^+. \tag{3.4}$$

Moreover,

$$\sup_{t\le q}|\Delta\bar X^n_t|\le 2\sup_{t\le q}|\Delta\bar Y^n_t|\le C(\sup_{t\le q}|\Delta W^{\rho^n}_t|+\frac{1}{n}),\quad q\in\mathbb{R}^+. \tag{3.5}$$

Set $\tau^n_b=\inf\{t>0;\max(|\bar X^n_t|,|\bar K^n|_t|,X^{n,\rho^n}_t|,|K^{n,\rho^n}|_t)>b\}$, $n\in\mathbb{N}$, $b\in\mathbb{N}$. Of course the processes $\bar X^n,|\bar K^n|,X^{n,\rho^n},|K^{n,\rho^n}|$ stopped at τ^n_b- are bounded by b. Since by arguments from the proof of Theorem 2.2 and [22, Theorem 4] for any $q\in\mathbb{R}^+$ the sequences $\{\sup_{t\le q}|X^n_t|\}$, $\{|K^n|_q\}$, $\{\sup_{t\le q}|\bar X^n_t|\}$ and $\{|\bar K^n|_q\}$ are bounded in probability,

$$\lim_{b\to\infty}\limsup_{n\to\infty}P(\tau^n_b\le q)=0,\quad q\in\mathbb{R}^+. \tag{3.6}$$

By [19, Lemma 2.3(i)] and the integration by parts formula,

$$\begin{aligned}|\bar X^n_t-X^{n,\rho^n}_t|^2&\le[\bar Y^n-Y^{n,\rho^n}]_t+\frac{1}{r_0}\int_0^t|\bar X^n_{s-}-X^{n,\rho^n}_{s-}|\,d(|\bar K^n|+|K^{n,\rho^n}|)_s\\&\quad+2\int_0^t(\bar X^n_{s-}-X^{n,\rho^n}_{s-})\,d(\bar Y^n-Y^{n,\rho^n})_s+R^{n,1}_t\\&\equiv I^{n,1}_t+I^{n,2}_t+I^{n,3}_t+R^{n,1}_t,\end{aligned} \tag{3.7}$$

where $R^{n,1}_t\equiv\frac{1}{r_0}\int_0^{\rho^n_t}|(\bar X^n_s-\bar X^n_{s-})-(X^n_s-X^{n,\rho^n}_{s-})|^2d(|\bar K^n|+|K^n|)_s-2\int_0^{\rho^n_t}(Y^{n,\rho^n}_s-Y^n_s)d(\bar K^n-K^n)_s$. By (3.3) and (3.4), $E\sup_{t\le q\wedge\tau^n_b-}|R^{n,1}_t|\le C(\frac{\log n}{n})^{1/2}$. Set $\Delta W_k\equiv W_{\frac{k}{n}}-W_{\frac{k-1}{n}}$, $k\in\mathbb{N}$. Since

$$\begin{aligned}\bar Y^n_t-Y^{n,\rho^n}_t&=\int_0^t\sigma(\bar X^n_{s-})-\sigma(X^{n,\rho^n}_{s-})\,dW^{\rho^n}_s-\int_0^{\rho^n_t}\sigma(X^n_s)-\sigma(X^{n,\rho^n}_{s-})\,dW^n_s\\&\quad+\int_0^t(\frac12\sigma'\sigma+b)(\bar X^n_{s-})\,d\rho^n_s-\int_0^{\rho^n_t}b(X^n_s)\,ds,\end{aligned}$$

it follows by (3.1) that

$$[\bar Y^n-Y^{n,\rho^n}]_t\le C\big(\sum_{k=1}^{[nt]}|\bar X^n_{\frac{k-1}{n}}-X^n_{\frac{k-1}{n}}|^2|\Delta W_k|^2+\sup_{s\le t}|X^n_t-X^{n,\rho^n}_{t-}|^2[W^{\rho^n}]_t+\frac{t}{n}\big).$$

By the above and (3.4),

$$I^{n,1}_t\le C\int_0^t|\bar X^n_{s-}-X^{n,\rho^n}_{s-}|^2\,d[W^{\rho^n}]_s+R^{n,2}_t,$$

where $E\sup_{t\le q\wedge\tau^n_b-}|R^{n,2}_t|\le C(\frac{\log n}{n})^{1/2}$. To estimate $I^{n,3}_t$ we first observe that for any $n,k\in\mathbb{N}$,

$$\begin{aligned}\Delta(\bar Y^n-Y^{n,\rho^n})_{\frac{k}{n}}&=(\sigma(\bar X^n_{\frac{k-1}{n}})-\sigma(X^{n,\rho^n}_{\frac{k-1}{n}}))\Delta W_k-n\int_{\frac{k-1}{n}}^{\frac{k}{n}}(\sigma(X^n_s)-\sigma(X^{n,\rho^n}_{\frac{k-1}{n}}))\,ds\,\Delta W_k\\&\quad+\frac12\sigma'\sigma(\bar X^n_{\frac{k-1}{n}})\Delta W_k\Delta W_k+\frac12\sigma'\sigma(\bar X^n_{\frac{k-1}{n}})(\frac1n I_d-\Delta W_k\Delta W_k)\\&\quad+(b(\bar X^n_{\frac{k-1}{n}})-b(X^{n,\rho^n}_{\frac{k-1}{n}}))\frac1n-\int_{\frac{k-1}{n}}^{\frac{k}{n}}(b(X^n_s)-b(X^{n,\rho^n}_{\frac{k-1}{n}}))\,ds\\&\equiv\Delta^{n,1}_k+\Delta^{n,2}_k+\Delta^{n,3}_k+\Delta^{n,4}_k+\Delta^{n,5}_k+\Delta^{n,6}_k.\end{aligned}$$

By Itô's formula applied to the bounded variation process X^n,

$$\begin{aligned}\Delta_k^{n,2} &= -n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}\sigma'(X_u^n)dX_u^n\,ds\,\Delta W_k\\ &= -n^2\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}\sigma'\sigma(X_u^n)\,du\,ds\,\Delta W_k\Delta W_k\\ &\quad - n\int_{\frac{k-1}{n}}^{\frac{k}{n}}\int_{\frac{k-1}{n}}^{s}(\sigma' b(X_u^n)du + dK_u^n)ds\Delta W_k\\ &\equiv -\frac{1}{2}\sigma'\sigma(X_{\frac{k-1}{n}}^n)\Delta W_k\Delta W_k + \Delta_k^{n,7} = -\Delta_k^{n,3} + \Delta_k^{n,7}.\end{aligned}$$

Clearly,

$$\begin{aligned}|\Delta_k^{n,7}| &\le C\big((\sup_{u\in[\frac{k-1}{n},\frac{k}{n})}|X_u^n - X_{\frac{k-1}{n}}^n||\Delta W_k|^2 + |\Delta W_k|(\frac{1}{n} + |K^n|_{[\frac{k-1}{n},\frac{k}{n}]})\big)\\ &\le C\big((\sup_{t\le q+1}|\Delta W_t^{\rho^n}| + \frac{1}{n})|\Delta W_k|^2 + \sup_{t\le q}|\Delta W_t^{\rho^n}|(\frac{1}{n} + |K^n|_{[\frac{k-1}{n},\frac{k}{n}]})\big).\end{aligned}$$

Similarly, $|\Delta_k^{n,6}| \le C(\sup_{t\le q+1}|\Delta W_t^{\rho^n}| + \frac{1}{n})\frac{1}{n}$. Consequently,

$$\begin{aligned}|I_t^{n,3}| &\le \sum_{k=1}^{[nt]}(\bar X_{\frac{k-1}{n}}^n - X_{\frac{k-1}{n}}^{n,\rho^n})(\sigma(\bar X_{\frac{k-1}{n}}^n) - \sigma(X_{\frac{k-1}{n}}^{n,\rho^n}))\Delta W_k + \sum_{k=1}^{[nt]}(\bar X_{\frac{k-1}{n}}^n - X_{\frac{k-1}{n}}^{n,\rho^n})\Delta_k^{n,4}\\ &\quad + \sum_{k=1}^{[nt]}(\bar X_{\frac{k-1}{n}}^n - X_{\frac{k-1}{n}}^{n,\rho^n})(b(\bar X_{\frac{k-1}{n}}^n) - b(X_{\frac{k-1}{n}}^{n,\rho^n}))\frac{1}{n} + R_t^{n,3}\\ &\le |\int_0^t(\bar X_{s-}^n - X_{s-}^{n,\rho^n})(\sigma(\bar X_{s-}^n) - \sigma(X_{s-}^{n,\rho^n}))\,dW_s^{\rho^n}| + \sum_{k=1}^{[nt]}(\bar X_{\frac{k-1}{n}}^n - X_{\frac{k-1}{n}}^{n,\rho^n})\Delta_k^{n,4}\\ &\quad + |\int_0^t(\bar X_{s-}^n - X_{s-}^{n,\rho^n})(b(\bar X_{s-}^n) - b(X_{s-}^{n,\rho^n}))\,d\rho_s^n| + R_t^{n,3},\end{aligned}$$

where $E\sup_{t\le q\wedge\tau_b^n-}|R_t^{n,3}| \le C(\frac{\log n}{n})^{1/2}$. Since by Burkholder's inequality,

$$E\sup_{t\le q\wedge\tau_b^n-}|\sum_{k=1}^{[nt]}(\bar X_{\frac{k-1}{n}}^n - X_{\frac{k-1}{n}}^{n,\rho^n})\Delta_k^{n,4}|^2 \le \frac{C}{n},$$

substituting previous estimates into (3.7) we conclude that

$$\begin{aligned}|\bar X_t^n - X_t^{n,\rho^n}|^2 &\le C\int_0^t|\bar X_{s-}^n - X_{s-}^{n,\rho^n}|^2\,d\rho_s^n\\ &\quad + \frac{1}{r_0}\int_0^t|\bar X_{s-}^n - X_{s-}^{n,\rho^n}|^2\,d(|\bar K^n| + |K^{n,\rho^n}|)_s\\ &\quad + |\int_0^t(\bar X_{s-}^n - X_{s-}^{n,\rho^n})(\sigma(\bar X_{s-}^n) - \sigma(X_{s-}^{n,\rho^n}))\,dW_s^{\rho^n}| + R_t^n, \qquad (3.8)\end{aligned}$$

where $\epsilon_n = E\sup_{t\le q\wedge\tau_b^n-}|R_t^n| \le C(\frac{\log n}{n})^{1/2}$. Fix $q\in\mathbb{R}^+$ and $b\in\mathbb{N}$. By (3.8) there is $C>0$ such that for every $(\mathcal{F}_t^{\rho^n})$ stopping time τ^n,

$$\begin{aligned}E\sup_{t<\gamma_n}|\bar X_t^n - X_t^{n,\rho^n}|^2 &\le CE\int_0^{\gamma^n-}\sup_{u\le s}|\bar X_{u-}^n - X_{u-}^{n,\rho^n}|^2\,d(\rho^n + |\bar K^n| + |K^{n,\rho^n}|)_s\\ &\quad + E(\sup_{t<\gamma^n}|\int_0^t(\bar X_{s-}^n - X_{s-}^{n,\rho^n})(\sigma(\bar X_{s-}^n) - \sigma(X_{s-}^{n,\rho^n}))\,dW_s^{\rho^n}|) + \epsilon_n,\end{aligned}$$

where $\gamma^n = \tau^n \wedge q \wedge \tau^n_b$. Since σ is Lipschitz continuous, it follows by the version of Metivier-Pellaumail inequality proved in Pratelli [14] and Schwartz inequalities that

$$\begin{aligned} I^{n,8} &\equiv E(\sup_{t<\gamma^n} |\int_0^t (\bar{X}^n_{s-} - X^{n,\rho^n}_{s-})(\sigma(\bar{X}^n_{s-}) - \sigma(X^{n,\rho^n}_{s-}))\, dW^{\rho^n}_s|) \\ &\leq cE(\int_0^{\gamma_n-} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|^4\, d([W^{\rho^n}] + \rho^n)_s)^{1/2} \\ &\leq cE \sup_{t<\gamma^n} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|(\int_0^{\gamma_n-} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|^2\, d([W^{\rho^n}] + \rho^n)_s)^{1/2} \\ &\leq c(E \sup_{s<\gamma^n} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|^2)^{1/2}(E\int_0^{\gamma_n-} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|^2\, d\rho^n_s)^{1/2}. \end{aligned}$$

Therefore there is $c' > 0$ such that

$$I^{n,8} \leq \frac{1}{2} E \sup_{s<\gamma^n} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|^2 + c'E\int_0^{\gamma_n-} |\bar{X}^n_{s-} - X^{n,\rho^n}_{s-}|^2\, d\rho^n_s.$$

By the above estimates,

$$E \sup_{t<\gamma_n} |\bar{X}^n_t - X^{n,\rho^n}_t|^2 \leq CE\int_0^{\gamma^n-} \sup_{u\leq s} |\bar{X}^n_{u-} - X^{n,\rho^n}_{u-}|^2\, d(\rho^n + |\bar{K}^n| + |K^{n,\rho^n}|)_s + 2\epsilon_n. \quad (3.9)$$

Set $D^n = \bar{X}^n - X^{n,\rho^n}$, $A^n = \rho^n + |\bar{K}^n| + |K^{n,\rho^n}|$ and observe that by (3.9), for every $(\mathcal{F}^{\rho^n}_t)$ stopping time τ^n,

$$\begin{aligned} E \sup_{t<\tau^n} |D^{n,(q\wedge\tau^n_b)-}_t|^2 = E \sup_{t<\gamma^n} |D^n_t|^2 &\leq CE\int_0^{\gamma^n-} \sup_{u\leq s} |D^n_{u-}|^2\, dA^n_s + 2\epsilon_n \\ &= CE\int_0^{\tau^n-} \sup_{u\leq s} |D^{n,(q\wedge\tau^n_b)-}_{u-}|^2\, dA^{n,(q\wedge\tau^n_b)-}_s + 2\epsilon_n. \end{aligned}$$

Since $A^{n,(q\wedge\tau^n_b)-}_\infty \leq q + 2b$, it follows by a stochastic version of Gronwall's lemma (see, e.g., [11, Lemma 2] or [21, Lemma 3 (ii)]) that

$$E \sup_{t<q\wedge\tau^n_b} |\bar{X}^n_t - X^{n,\rho^n}_t|^2 = E\sup_t |D^{n,(q\wedge\tau^n_b)-}_t|^2 \leq 2\epsilon_n \exp\{C(q+2b))\} \leq C(\frac{\log n}{n})^{1/2}.$$

Using (3.4), (3.6) and Remark 3.1 completes the proof. □

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