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Fluctuation exponents for directed polymers in the intermediate disorder regime*

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Abstract

We derive exact fluctuation exponents for a solvable model of one-dimensional directed polymers in random environment in the intermediate scaling regime. This regime corresponds to taking the inverse temperature to zero as the size of the system goes to infinity. The exponents satisfy the KPZ scaling relation and coincide with previous nonrigorous predictions. In the critical case, we recover the fluctuation exponent of the Hopf-Cole solution of the KPZ equation in equilibrium and close to equilibrium.

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1 Main results

1.1 Introduction

The *directed polymer in a random environment* is a statistical physics model that assigns Boltzmann-Gibbs weights to random walk paths as a function of the environment encountered by the walk. It was originally introduced in [31] as a model of an interface in two dimensions. Here is the standard lattice formulation in $d+1$ dimensions (d space dimensions, one time dimension).

The environment is a collection of i.i.d. random weights $\{\omega(i,x) : i \in \mathbb{N}, x \in \mathbb{Z}^d\}$ with probability distribution $\mathbb{P}$. Let P be the law of simple symmetric random walk $(S_t)_{t\in\mathbb{Z}_+}$ on $\mathbb{Z}^d$ with $S_0 = 0$. Denote expectation under P and $\mathbb{P}$ by E and $\mathbb{E}$, respectively. The *quenched partition function* of the directed polymer in environment ω and at inverse temperature $\beta > 0$ is

$$Z_{N,x}(\beta) = E\big[e^{\beta\sum_{i=1}^{N}\omega(i,S_i)},\, S_N = x\big], \tag{1.1}$$

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where $E[X, A] = E[X \cdot \mathbf{1}_A]$ is the expectation of X restricted to the event A. This is the *point-to-point* partition function because the endpoint S_N of the walk is constrained to be x. The version that allows S_N to fluctuate freely is the *point-to-line* partition function. In the point-to-point setting the *quenched polymer measure* on paths ending at x is

$$\begin{aligned} &Q^{\beta}_{N,x}\left(S_1 = x_1, \ldots, S_N = x_N\right) \\ &\quad = \frac{1}{Z_{N,x}(\beta)} e^{\beta \sum_{i=1}^N \omega(i,x_i)} P\left[S_1 = x_1, \ldots, S_N = x_N\right] \cdot \mathbf{1}\{x_N = x\}. \end{aligned} \tag{1.2}$$

These quenched quantities are functions of the environment ω and thereby random. The averaged distribution of the path is $P^{\beta}_{N,x}(\cdot) = \mathbb{E}Q^{\beta}_{N,x}(\cdot)$. We refer the reader to reviews [21, 27, 34] for a deeper discussion of the subject.

We restrict the discussion to the 1+1 dimensional case. Basic objects of study are the fluctuations of the free energy $\log Z_{N,Nx}(\beta)$ and the path $(S_t)_{0\le t\le N}$. On the crudest level the orders of magnitude of these fluctuations are described by two exponents χ and ζ:

- fluctuations of $\log Z_{N,Nx}(\beta)$ under $\mathbb{P}$ have order of magnitude N^{χ}
- fluctuations of the path S_t under $P^{\beta}_{N,Nx}$ have order of magnitude N^{ζ}

In the 1+1 dimensional case these exponents are expected to take the values $\chi = 1/3$ and $\zeta = 2/3$ independently of β, provided the i.i.d. weights $\omega(i,x)$ satisfy a moment bound. Furthermore, there are specific predictions for the limit distributions of the scaled quantities: for example, the GUE Tracy-Widom distribution for $\log Z_{N,Nx}(\beta)$. These properties are features of the *Kardar-Parisi-Zhang (KPZ) universality class* to which these models are expected to belong. See [23, 47] for recent surveys. The KPZ regime should be contrasted with the *diffusive regime* where $\chi = 0$, $\zeta = 1/2$, and the path satisfies a central limit theorem. Diffusive behavior is known to happen for $d \ge 3$ and small enough β [22].

There are four exactly solvable 1+1 dimensional positive temperature polymer models for which KPZ predictions have been partially proved:

(a) the semidiscrete polymer in a Brownian environment [41]
(b) the log-gamma polymer [45]
(c) the continuum directed random polymer, in other words, the solution of the Kardar-Parisi-Zhang (KPZ) equation [1, 4, 33]
(d) the strict-weak lattice polymer [26, 40]

In recent years a number of results have appeared, first for exponents and then for distributional properties. This is not a place for a thorough review, but let us cite some of the relevant papers: [4, 15, 16, 17, 10, 25, 39, 45, 46]. To do justice to history, we mention also that KPZ results appeared earlier for zero-temperature polymers (the $\beta \to \infty$ limit of (1.1)–(1.2), known as last-passage percolation), beginning with the seminal papers [7, 32].

Getting closer to the topic of the present paper, physics paper [3] introduced the study of the *intermediate disorder regime* in model (1.1)–(1.2). This means that β is scaled to zero as $N \to \infty$ by taking $\beta = \beta_0 N^{-\alpha}$. The window of interest is $0 \le \alpha \le 1/4$. At $\alpha = 0$ one sees the KPZ behavior with exponents $\chi = 1/3$ and $\zeta = 2/3$. At $\alpha = 1/4$ one has the critical case where exponents are diffusive ($\chi = 0$ and $\zeta = 1/2$) but fluctuations are different [2]. When $\alpha > 1/4$ the disorder is irrelevant and the polymer behaves like a simple random walk [1].

Article [3] conjectured the exponents for the entire range:

$$\chi(\alpha) = \tfrac{1}{3}(1 - 4\alpha) \quad \text{and} \quad \zeta(\alpha) = \tfrac{2}{3}(1 - \alpha) \quad \text{for } 0 \le \alpha \le 1/4. \tag{1.3}$$

In this paper we derive these intermediate disorder exponents for the semidiscrete polymer in the Brownian environment (introduced in [41], hence also called the O'Connell-Yor model). Along the way we offer some improvements to the earlier work [46] which treated the $\alpha = 0$ case. This model has two versions: a *stationary version* with particular boundary conditions that render the process of $\log Z$ increments shift-invariant, and the *point-to-point version* without boundary conditions represented by (1.1)–(1.2) above. In general we have better results for the stationary version. In case the reader is encountering polymer models with boundaries for the first time but can appreciate an analogy with the totally asymmetric simple exclusion process (TASEP), then the stationary polymer corresponds to stationary TASEP with Bernoulli occupations, while the point-to-point version of the polymer is the analogue of TASEP with step initial condition.

We list below the precise contributions of our paper:

(i) For the free energy we derive the exponent $\chi(\alpha) = \frac{1}{3}(1-4\alpha)$ for the entire range $0 \le \alpha \le 1/4$ for the stationary version and for $0 \le \alpha < 1/4$ for the point-to-point version. For the fixed temperature case ($\alpha = 0$) the lower bound $\chi \ge 1/3$ for the point-to-point version was not covered in [46], but is done here.

(ii) We have the path exponent $\zeta(\alpha) = \frac{2}{3}(1-\alpha)$ for the stationary version, and the upper bound $\zeta(\alpha) \le \frac{2}{3}(1-\alpha)$ for the point-to-point version.

(iii) We refine the prediction (1.3) in the following way. We introduce a macroscopic time parameter $\tau > 0$ and conclude that the fluctuations of $\log Z_{\tau N, \tau N x}(\beta_0 N^{-\alpha})$ are of magnitude $\tau^{1/3} N^{\chi(\alpha)}$ while the path fluctuations are of magnitude $\tau^{2/3} N^{\zeta(\alpha)}$. In other words, in the macroscopic variables we see again the exponents $\frac{1}{3}$ and $\frac{2}{3}$.

(iv) In the fixed temperature case ($\alpha = 0$) the lower bound $\chi \ge 1/3$ was already proved in [46] for the stationary version. Here we give a considerably simpler proof of the lower bound, including the case $\alpha = 0$.

(v) In the critical case $\alpha = 1/4$ we can connect with the KPZ equation. The macroscopic variable τ becomes the time parameter of the stochastic heat equation (SHE), and we obtain again the exponent of the stationary Hopf-Cole solution of the KPZ equation, first proved in [10]: $\mathbb{V}\mathrm{ar}[\log \mathcal{Z}(\tau, 0)] \asymp \tau^{\frac{2}{3}}$ where $\mathcal{Z}$ is the solution of SHE. Moreover, we prove similar bounds for solutions where the initial condition is a bounded perturbation of the stationary initial condition.

The structure of the present paper is similar to [46]. However, new arguments were needed to obtain estimates that hold uniformly for a broader range of parameters. In particular, for the upper bound proofs of the point-to-point case we found an approach based on the connection of Brownian last passage percolation with the Gaussian unitary ensemble.

Some further comments about the state of the field and the place of this work are in order. Presently one can identify the following three approaches to fluctuations of polymer models and of models in the KPZ class more broadly.

(a) The *resolvent method*. This is a fairly robust method used to establish superdiffusivity. It is quite general, for it can often be applied as long as a model has a tractable invariant distribution [13, 35, 42, 43, 44, 49]. A drawback of the method is that often it cannot determine the exact exponents but provides only bounds on them. However, here are two exceptions. In [49] the scaling exponent of a 2d TASEP model is identified exactly. In [42, 43] the method is used to give a comparison between the solvable 1d TASEP and more general 1d exclusion models to show that the scaling exponents are the same.

(b) The *coupling method*, represented by the present work and references [8, 9, 10, 11, 19, 45, 46]. This approach is able to identify exact exponents, but so far has depended on the presence of special structures such as a Burke-type property.

(c) *Exact solvability methods*. When it can be applied, this approach leads to the sharpest results, namely Tracy-Widom limit distributions. But it is the most specialized and technically very heavy. This approach became available for the semidiscrete polymer after determinantal expressions where found for the distribution of $\log Z$ [15, 16, 39]. For the related log-gamma polymer, see [17, 25]. For the ASEP the first scaling limits were proved using Fredholm determinant formulas based on the work of [48]. The recent work of [18] uses certain duality relations to get scaling limits for the same model. Their method can be thought of as a rigorous version of the physicists' replica trick.

The free energy exponent $\chi = 1/3$ in the fixed temperature case ($\alpha = 0$) is also a consequence of the distributional limits for $\log Z$ in [15, 16]. Presently these results cover the point-to-point case of the semidiscrete polymer for the entire fixed temperature range $0 < \beta < \infty$. It is expected that these methods should work also in the intermediate disorder regime (personal communication from the authors). However, these works do not yet give anything on the stationary versions of the models, or on the path fluctuations in either the point-to-point or stationary version.

The open problem that remains in the coupling approach used here is the lower bound for the path in the point-to-point case.

One more expected universal feature of polymer exponents worth highlighting here is the scaling relation $\chi = 2\zeta - 1$. This is expected to hold very generally across models and dimensions. The exponents we derive satisfy this identity. There is important recent work on this identity that goes beyond exactly solvable models: first [20], and then [5] with a simplified proof, derived this relation for first passage percolation under strong assumptions on the existence of the exponents. These results are extended to positive temperature directed polymers in [6].

Finally, we point out that the coupling method applied to directed polymers first appeared in the work [45] in the context of discrete polymers in a log-gamma environment. Most of the results of [46] have discrete analogues in [45]. The intermediate regime can also be investigated for the polymers in the log-gamma environment. Although this model is formulated for $\beta = 1$, the parameters of the environment can be tuned to emulate the situation $\beta \to 0$. We have obtained proofs for the fluctuation exponents of the log-gamma model in the intermediate scaling regime. The methods are very similar to the ones used here for the semidiscrete polymer model, but involve considerably heavier asymptotics so we decided not to include them in the present paper.

Organization of the paper. We introduce the directed polymer in a Brownian environment in its point-to-point and stationary versions and state our main theorems in Sections 1.2.1 and 1.2.2. Their proofs are in Section 2. In Section 1.3 we state our results for the KPZ equation. The proofs are given in Section 3. Some basic estimates on polygamma functions are provided in Section 4.

Notation and conventions. $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For $\theta > 0$, the usual gamma function is $\Gamma(\theta) = \int_0^\infty s^{\theta-1} e^{-s}\, ds$ and the Gamma(θ) distribution has density $f(x) = \Gamma(\theta)^{-1} x^{\theta-1} e^{-x}$ for $0 < x < \infty$. The digamma and trigamma functions are $\Psi_0 = \Gamma'/\Gamma$ and $\Psi_1 = \Psi_0'$. Ψ_1^{-1} is the inverse function of Ψ_1. See Section 4 for a few facts about polygamma functions.

The environment distribution $\mathbb{P}$ has expectation symbol $\mathbb{E}$. Generically expectation under a probability measure Q is denoted by E^Q. To simplify notation we drop integer parts. A real value s in a position that takes an integer should be interpreted as the integer part $\lfloor s \rfloor$.

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1.2 The semi-discrete polymer in a Brownian environment

We begin with the results for the semi-discrete polymer in a Brownian environment. This is a semi-discrete version of the generic polymer model described in (1.1). As already mentioned, the model has two versions: a point-to-point and a stationary version.

1.2.1 Point-to-point semi-discrete polymer

The environment consists of a family of independent one-dimensional standard Brownian motions $\{B_i(\cdot) : i \geq 1\}$. These are two-sided Brownian motions with $B_i(0) = 0$. Polymer paths are nondecreasing càdlàg paths $x : [0, t] \to \mathbb{N}$ with nearest-neighbor jumps, $x(0) = 1$, and $x(t) = n$. A path can be coded in terms of its jump times $0 = s_0 < s_1 < \cdots < s_{n-1} < s_n = t$. At level k the path collects the increment $B_k(s_{k-1}, s_k) = B_k(s_k) - B_k(s_{k-1})$. The partition function in a fixed Brownian environment at inverse temperature $\beta > 0$ is, for $(n, t) \in \mathbb{N} \times [0, \infty)$,

$$Z_{n,t}(\beta) = \int\limits_{0<s_1<\cdots<s_{n-1}<t} \exp\bigl[\beta\bigl(B_1(0, s_1) + B_2(s_1, s_2) + \cdots + B_n(s_{n-1}, t)\bigr)\bigr]\, ds_{1,n-1}. \tag{1.4}$$

In the integral $ds_{1,n-1}$ is short for $ds_1 \cdots ds_{n-1}$. The limiting free energy density was computed for a fixed β in [38]:

$$\begin{aligned} \mathbf{F}(\beta) = \lim_{n\to+\infty} \frac{1}{n} \log Z_{n,n}(\beta) &= \inf_{t>0} \bigl\{ t\beta^2 - \Psi_0(t) \bigr\} - 2\log\beta \\ &= \Psi_1^{-1}(\beta^2)\beta^2 - \Psi_0(\Psi_1^{-1}(\beta^2)) - 2\log\beta \quad \text{for } \beta > 0. \end{aligned} \tag{1.5}$$

We consider this model in the intermediate disorder regime where $\beta = \beta_0 n^{-\alpha}$ for fixed $\beta_0 \in (0, \infty)$ and $\alpha \in [0, 1/4]$. If $0 < \alpha \leq 1/4$, $\log Z_{n,n}(\beta)$ concentrates asymptotically around the value $n\mathbf{F}(\beta_0 n^{-\alpha}) = n + O(n^{1-2\alpha})$. (See (4.2) and (4.3) for the asymptotics of the functions Ψ_0 and Ψ_1.)

Our first result identifies the free energy fluctuation exponent $\chi = \frac{1}{3}(1 - 4\alpha)$ for the point-to-point semi-discrete polymer in the intermediate disorder regime. In the fixed temperature case ($\alpha = 0$) the upper bound was proved in [46] but a lower bound proof with coupling methods is new even in this case. (To clarify, the correct exponent in the $\alpha = 0$ case has of course been identified in the weak convergence results [15, 16] with exact solvability methods.) Note that we see the intermediate regime exponent on the scaling parameter n, but for the macroscopic variable τ we see the exponent $\frac{1}{3}$ corresponding to the KPZ scaling.

Theorem 1.1. *Fix $\alpha \in [0, 1/4)$ and $0 < \beta_0 < \infty$. Let $\beta = \beta_0 n^{-\alpha}$. There exist finite positive constants C, n_0, b_0, τ_0 that depend on (α, β_0) such that the following bounds hold. For $\tau \geq \tau_0$, $n \geq n_0$ and $b \geq b_0$,*

$$\mathbb{P}\bigl\{ |\log Z_{\tau n,\tau n}(\beta) - \tau n\mathbf{F}(\beta)| \geq b\,\tau^{\frac{1}{3}} n^{\frac{1}{3}(1-4\alpha)} \bigr\} \leq C b^{-3/2} \tag{1.6}$$

and

$$C^{-1}\tau^{\frac{1}{3}} n^{\frac{1}{3}(1-4\alpha)} \;\leq\; \mathbb{E}|\log Z_{\tau n,\tau n}(\beta) - \tau n\mathbf{F}(\beta)| \;\leq\; C\tau^{\frac{1}{3}} n^{\frac{1}{3}(1-4\alpha)}. \tag{1.7}$$

We turn to the fluctuations of the polymer path. The quenched polymer measure $Q_{n,t,\beta}$ on paths is defined, in terms of the expectation of a bounded Borel function $f : \mathbb{R}^{n-1} \to \mathbb{R}$, by

$$E^{Q_{n,t,\beta}} f(\sigma_1, \dots, \sigma_{n-1}) = \frac{1}{Z_{n,t}(\beta)} \int\limits_{0<s_1<\cdots<s_{n-1}<t} f(s_1, \dots, s_{n-1}) \\ \times \exp\bigl[\beta\bigl(B_1(0,s_1) + \cdots + B_n(s_{n-1},t)\bigr)\bigr]\, ds_{1,n-1}.$$

The jump times as functions of the path are denoted by σ_i. Averaged (or *annealed*) probability and expectation are denoted by $P_{n,t,\beta}(\cdot) = \mathbb{E}Q_{n,t,\beta}(\cdot)$ and $E_{n,t,\beta}(\cdot) = \mathbb{E}E^{Q_{n,t,\beta}}(\cdot)$.

In the point-to-point setting the path exponent ζ describes the order of magnitude of the deviations of the path from the diagonal. A path close to the diagonal in the rectangle $\{1, \dots, n\} \times [0, t]$ would have $\sigma_i \approx it/n$. The next theorem shows that the path exponent ζ is bounded above by its conjectured value $\frac{2}{3}(1-\alpha)$.

Theorem 1.2. *Fix $\alpha \in [0, 1/4)$ and $0 < \beta_0 < \infty$. Let $\beta = \beta_0 n^{-\alpha}$. There exist finite positive constants C, n_0, b_0, τ_0 that depend on (α, β_0) such that the following bound holds. For all $0 < \gamma < 1$, $\tau \geq \tau_0$, $b \geq b_0$, and $n \geq n_0$,*

$$P_{n,t,\beta}\left\{ |\sigma_{\gamma\tau n} - \gamma\tau n| \geq b\, \tau^{\frac{2}{3}} n^{\frac{2}{3}(1-\alpha)} \right\} \leq Cb^{-3}. \tag{1.8}$$

1.2.2 Stationary semi-discrete polymer

The proofs of the above theorems rely on comparison with a stationary version of the model. Enlarge the environment by adding another Brownian motion B independent of $\{B_i\}_{i\geq 1}$. Introduce a parameter $\theta \in (0, \infty)$ and restrict to $\beta = 1$ for a moment. The stationary partition function is, for $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$Z^{\theta}_{n,t} = \int\limits_{-\infty<s_0<s_1<\cdots<s_{n-1}<t} \exp\bigl[-B(s_0) + \theta s_0 + B_1(s_0,s_1) + + \cdots + B_n(s_{n-1},t)\bigr]\, ds_{0,n-1}. \tag{1.9}$$

This model has a useful stationary structure described by [41]. Let $Y_0(t) = B(t)$ and, for $k \geq 1$, define inductively

$$r_k(t) = \log \int_{-\infty}^{t} e^{Y_{k-1}(s,t) - \theta(t-s) + B_k(s,t)} ds \tag{1.10}$$

$$Y_k(t) = Y_{k-1}(t) + r_k(0) - r_k(t). \tag{1.11}$$

Induction shows that

$$Z^{\theta}_{n,t}\, e^{B(t)-\theta t} = \exp\Bigl(\sum_{k=1}^{n} r_k(t)\Bigr). \tag{1.12}$$

For each fixed $t \geq 0$, $\{r_k(t)\}_{k\geq 1}$ are i.i.d. and $e^{-r_k(t)}$ has Gamma(θ) distribution [41]. Thus the law of $Z^{\theta}_{n,t}\, e^{B(t)-\theta t}$ is independent of t. This stationarity is part of a broader Burke-type property (see [46, Section 3.1] for more details).

Extend definition (1.4) to $1 \leq k \leq n \in \mathbb{N}$ and $s < t \in \mathbb{R}$ by $Z_{(k,k),(s,t)}(\beta) = e^{B_k(s,t)}$ and

$$Z_{(k,n),(s,t)}(\beta) = \int\limits_{s<s_k<\cdots<s_{n-1}<t} \exp\bigl[\beta\bigl(B_k(s,s_k) + B_{k+1}(s_k,s_{k+1}) \\ + \cdots + B_n(s_{n-1},t)\bigr)\bigr]\, ds_{k,n-1}, \tag{1.13}$$

and abbreviate the $\beta = 1$ case as $Z_{(k,n),(s,t)} = Z_{(k,n),(s,t)}(1)$. The stationary partition function can be recovered by integrating these point-to-point partition functions against the boundary Brownian motion:

$$Z^{\theta}_{n,t} = \int_{-\infty}^{t} ds_0 \, e^{-B(s_0)+\theta s_0} \, Z_{(1,n),(s_0,t)}.$$

We include the inverse temperature in the stationary partition function by defining

$$\begin{aligned} Z^{\theta,\beta}_{n,t} = \int\limits_{-\infty<s_0<s_1<\cdots<s_{n-1}<t} & \exp\bigl[-\beta B(s_0) + \beta\theta s_0 + \beta\bigl(B_1(s_0,s_1) \\ & + \cdots + B_n(s_{n-1},t)\bigr)\bigr] \, ds_{0,n-1}. \end{aligned} \tag{1.14}$$

The following theorem identifies the fluctuation exponent χ for the stationary model. A key difference between the point-to-point and stationary versions is that KPZ fluctuations appear in the stationary version only in a particular characteristic direction (n,t) determined by the parameters. In other directions the diffusive fluctuations of the boundaries dominate (see [45], Corollary 2.2, in the context of discrete polymers in a log-gamma environment). Once we choose $\beta = \beta_0 n^{-\alpha}$, to make the diagonal a characteristic direction we are forced to pick $\theta = \beta\Psi_1^{-1}(\beta^2) \sim \beta^{-1}$. To simplify notation we suppress the n-dependence of the parameters β and θ.

Theorem 1.3. *Let $\alpha \in [0, 1/4]$, $0 < \beta_0 < \infty$, $\beta = \beta_0 n^{-\alpha}$, and $\theta = \beta\Psi_1^{-1}(\beta^2)$. Then there exist positive constants C, n_0, τ_0 depending only on α and β_0 such that*

$$C^{-1}\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-4\alpha)} \le \mathbb{V}\mathrm{ar}(\log Z^{\theta,\beta}_{\tau n,\tau n}) \le C\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-4\alpha)} \tag{1.15}$$

for all $\tau \ge \tau_0$ and $n \ge n_0$.

The stationary quenched polymer measure $Q^{\theta,\beta}_{n,t}$ lives on nondecreasing cádlág paths $x : (-\infty, t] \to \{0, 1, \ldots, n\}$ with boundary conditions $x(-\infty) = 0$, $x(t) = n$. We represent paths again in terms of jump times $-\infty < \sigma_0 < \sigma_1 < \cdots < \sigma_{n-1} \le t$ where $x(\sigma_i-) = i < i+1 = x(\sigma_i)$. The path measure is defined by

$$\begin{aligned} E^{Q^{\theta,\beta}_{n,t}} f(\sigma_0, \sigma_1, \ldots, \sigma_{n-1}) = \frac{1}{Z^{\theta,\beta}_{n,t}} & \int\limits_{-\infty<s_0<\cdots<s_{n-1}<t} f(s_0, s_1, \ldots, s_{n-1}) \\ & \times \exp\bigl[-\beta B(s_0) + \beta\theta s_0 + \beta\left(B_1(s_0,s_1) + \cdots + B_n(s_{n-1},t)\right)\bigr] \, ds_{0,n-1}. \end{aligned} \tag{1.16}$$

Averaged probability and expectation are denoted by $P^{\theta,\beta}_{n,t}(\cdot) = \mathbb{E}Q^{\theta,\beta}_{n,t}(\cdot)$ and $E^{\theta,\beta}_{n,t}(\cdot) = \mathbb{E}E^{Q^{\theta,\beta}_{n,t}}(\cdot)$. When $\beta = 1$, we simply remove it from the notation.

In the stationary case we can identify the exact path exponent $\zeta = \frac{2}{3}(1-\alpha)$.

Theorem 1.4. *Let $\alpha \in [0, \frac{1}{4}]$, $0 < \beta_0 < \infty$, $\beta = \beta_0 n^{-\alpha}$, $\gamma \in (0,1)$, and $\theta = \beta\Psi_1^{-1}(\beta^2)$. Then there exist positive constants C, n_0, τ_0 depending only on α and β_0 such that these bounds hold. For $\tau \ge \tau_0$, $n \ge n_0$ and $b \ge 1$*

$$P^{\theta,\beta}_{\tau n,\tau n}\left\{ |\sigma_{\gamma\tau n} - \gamma\tau n| > b\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-\alpha)} \right\} \le Cb^{-3} \tag{1.17}$$

and

$$C^{-1}\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-\alpha)} \le E^{\theta,\beta}_{\tau n,\tau n}|\sigma_{\gamma\tau n} - \gamma\tau n| \le C\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-\alpha)} \tag{1.18}$$

Note that Theorems 1.1 and 1.2 (on the point-to-point polymer) are restricted to $\alpha < 1/4$ while Theorems 1.3 and 1.4 (on the stationary polymer) allow $\alpha = 1/4$ as well. We will prove the theorems about the stationary polymer first. The results for the point-to-point case follow via various comparisons with the stationary case. During these comparisons one picks up certain error terms (one instance of this can be seen by comparing equations (2.16) and (2.31) with (2.38)). In the critical $\alpha = 1/4$ case these error terms become too large, hence the need the for the extra $\alpha < 1/4$ restriction in the point-to-point case. It appears to us that this restriction is purely technical.

1.3 The KPZ equation close to equilibrium

The Kardar-Parisi-Zhang (KPZ) equation was introduced in [33] as a model of a randomly growing interface in $1+1$ dimension: if we let $h(t,x)$ denote the height of the interface at site $x \in \mathbb{R}$ and time $t \geq 0$, then the evolution of the interface is represented by the (ill-posed) stochastic partial differential equation

$$\partial_t h \quad = \quad \tfrac{1}{2}\Delta h + \tfrac{1}{2}\left(\nabla h\right)^2 + \dot{\mathscr{W}}, \tag{1.19}$$

where $\dot{\mathscr{W}}$ is a space-time white noise.

We take initial conditions of the form $\mathcal{B}+\varphi$ where $\mathcal{B}$ is a double-sided one-dimensional Brownian motion and φ is a bounded function. We will always consider the so called Hopf-Cole solution of (1.19). Let $\mathcal{Z}$ be the (well-defined) solution of the stochastic heat equation

$$\partial_t \mathcal{Z}^\varphi \quad = \quad \tfrac{1}{2}\Delta \mathcal{Z}^\varphi + \mathcal{Z}^\varphi \dot{\mathscr{W}}, \qquad \mathcal{Z}^\varphi(0,x) = e^{\varphi(x)+\mathcal{B}(x)}. \tag{1.20}$$

Then $h = \log \mathcal{Z}$ formally solves (1.19). [14] showed that the Hopf-Cole solution is the correct scaling limit of a weakly asymmetric microscopic growth model in $1+1$ dimension. A rigorous solution theory for (1.19) on the circle has been developed in [30]. For a more detailed overview of the KPZ equation and KPZ universality class, we refer the reader to the review [23] and its references.

It is expected that, for a wide family of initial conditions, the fluctuations of $\log \mathcal{Z}(t,x)$ are of order $t^{1/3}$. This was first proved in [10] in the stationary case, that is, when $\varphi = 0$. The proof was based on the convergence of the rescaled height function of the weakly asymmetric exclusion process to the Hopf-Cole solution of the KPZ equation [14], together with non-asymptotic fluctuation bounds on the current of the asymmetric simple exclusion process [11]. It is not clear that this approach can be extended to the case of non-zero φ.

When the initial condition is $\mathcal{Z}(0,x) = \delta_0(x)$, the asymptotic distribution of the fluctuations of $\log \mathcal{Z}$ is identified in [4] as the Tracy-Widom distribution. The proof is based on heavy asymptotic analysis of exact formulas for the weakly asymmetric simple exclusion process.

We will extend the result of [10] to the case of a bounded perturbation φ. Our approach is different as we use an approximation of $\mathcal{Z}$ by partition functions of the Brownian semidiscrete directed polymer in the critical case $\alpha = \frac{1}{4}$ rather than by particle systems.

Building on the techniques of [3], the unpublished preprint [37] shows that a suitable renormalization of the partition function of the semi-discrete model with $\alpha = \frac{1}{4}$ converges to $\mathcal{Z}^\varphi$. More precisely, let $\varphi_n(x) = \varphi\big(-\frac{x}{\sqrt{n}}\big)$ and let

$$Z^{\theta,\beta,\varphi}_{n,t} = \int_{-\infty}^{t} \exp\big[\varphi_n(s_0) - \beta B(s_0) + \beta\theta s_0\big]\, Z_{(1,n)(s_0,t)}(\beta)\, ds_0. \tag{1.21}$$

The renormalized partition function is

$$\mathcal{Z}_n^{\varphi}(\tau) = e^{-\tau n - \frac{1}{2}\tau\sqrt{n}}\, Z_{\tau n, \tau n}^{\theta,\beta,\varphi}. \tag{1.22}$$

When $\varphi = 0$, we simply denote this by $\mathcal{Z}_n(\tau)$.

Theorem 1.5. *[37] Let $\beta = \beta_n = n^{-1/4}$ and $\theta = \beta_n \Psi_1^{-1}(\beta_n^2)$. Then as $n \to +\infty$, the process $(\mathcal{Z}_n^{\varphi}(\tau), \tau \geq 0)$ converges in law to $(\mathcal{Z}^{\varphi}(\tau, 0), \tau \geq 0)$, where $\mathcal{Z}^{\varphi}$ solves the stochastic heat equation* (1.20).

Combined with Theorem 1.3 this gives

Theorem 1.6. *Let φ be a bounded function and let $\mathcal{Z}^{\varphi}$ be the solution of the stochastic heat equation* (1.20). *Assuming the conclusion of Theorem 1.5, there exist constants C_1, C_2, $\tau_0 > 0$ such that*

$$C_1 \tau^{\frac{2}{3}} \leq \mathbb{V}\mathrm{ar}[\log \mathcal{Z}^{\varphi}(\tau, 0)] \leq C_2 \tau^{\frac{2}{3}},$$

for all $\tau > \tau_0$.

We note that our results for the path of the stationary polymer could in principle have a meaning in the context of the SHE. In [1], $\mathcal{Z}$ is identified as the partition function of a continuum directed polymer. Theorem 1.4 strongly suggests that the fluctuations of the path of the continuum polymer are of order $t^{2/3}$, in agreement with the KPZ scaling.

An alternative proof of Theorem 1.6 appeared in [24, Remark 1.9] posted after the present paper.

2 Proofs for the semi-discrete polymer model

The proofs of our Theorems 1.1, 1.2, 1.3 and 1.4 are given in this section. We first prove the results for the stationary model. The results for the point-to-point model are then done by comparison.

2.1 Preliminaries

We recall some facts from [46]. Throughout this section we take $\beta = 1$ as we can reduce the situation to this by Brownian scaling (see Section 2.2). The stationary model can be written as

$$Z_{n,t}^{\theta} = \int_0^t e^{-B(s)+\theta s} Z_{(1,n),(s,t)}\, ds + \sum_{j=1}^{n} \left(\prod_{k=1}^{j} e^{r_k(0)} \right) Z_{(j,n),(0,t)} \tag{2.1}$$

where the r_k processes are defined recursively in (1.10). Recall that the random variables $r_k(0)$ are i.i.d. and $e^{-r_k(0)}$ has Gamma(θ) distribution.

It is convenient to define $Z_{0,t}^{\theta} = \exp(-B(t) + \theta t)$. The processes r_k and Y_k give space and time increments of the partition function:

$$\begin{aligned} r_k(t) &= \log Z_{k,t}^{\theta} - \log Z_{k-1,t}^{\theta} \\ Y_k(s,t) &= Y_k(t) - Y_k(s) = \theta(t-s) - \log Z_{k,t} + \log Z_{k,s}. \end{aligned} \tag{2.2}$$

The appearance of polygamma functions in our results is natural because of the identities

$$\mathbb{E}[r_k(t)] = -\Psi_0(\theta) \quad \text{and} \quad \mathbb{V}\mathrm{ar}[r_k(t)] = \Psi_1(\theta). \tag{2.3}$$

From (2.2) and (2.3) one immediately gets

$$\mathbb{E}(\log Z_{n,t}^{\theta}) = -n\Psi_0(\theta) + \theta t. \tag{2.4}$$

A formula for the variance was derived in Theorem 3.6 in [46]:

$$\mathbb{V}\mathrm{ar}(\log Z_{n,t}^{\theta}) = n\Psi_1(\theta) - t + 2E_{n,t}^{\theta}(\sigma_0^+) = t - n\Psi_1(\theta) + 2E_{n,t}^{\theta}(\sigma_0^-) = E_{n,t}^{\theta}\,|\sigma_0|\,. \tag{2.5}$$

We will also need the following lemma from [46]:

Lemma 2.1. *[46, Lemma 4.3] For $\theta, \lambda > 0$,*

$$\left|\mathbb{V}\mathrm{ar}(\log Z_{n,t}^{\lambda}) - \mathbb{V}\mathrm{ar}(\log Z_{n,t}^{\theta})\right| \le n|\Psi_1(\lambda) - \Psi_1(\theta)|.$$

Finally, we note a shift-invariance of the stationary model (Remark 3.1 of [46]):

$$E^{Q_{n,t}^{\theta}} f(\sigma_0, \sigma_1, \dots, \sigma_{n-1}) \stackrel{d}{=} E^{Q_{n,0}^{\theta}} f(t+\sigma_0, t+\sigma_1, \dots, t+\sigma_{n-1}). \tag{2.6}$$

This follows from the stationarity of $Z_n^{\theta}(t)\exp(B(t)-\theta t)$ by observing that the density of $(\sigma_0, \dots, \sigma_{n-1})$ under $Q_{n,t}^{\theta}$ can also be written as

$$\frac{1}{\widehat{Z}_n^{\theta}(t)} \exp\bigl[\widehat{B}(s_0,t) + \widehat{B}_1(s_0,s_1) + \cdots + \widehat{B}_n(s_{n-1},t)\bigr] \times \mathbf{1}\{s_0 < \cdots < s_{n-1} < t\} \tag{2.7}$$

where $\widehat{B}(u) = B(u) - \theta u/2$ (and similarly for $\widehat{B}_k$) and $\widehat{Z}_n^{\theta}(t) = Z_n^{\theta}(t)\exp(B(t)-\theta t)$.

Using the same ideas one can also show a shift-invariance property in n (see the proof of Theorem 6.1 in [46]):

$$E^{Q_{n,t}^{\theta}} f(\sigma_k, \sigma_{k+1}, \dots, \sigma_{n-1}) \stackrel{d}{=} E^{Q_{n-k,t}^{\theta}} f(\sigma_0, \sigma_1, \dots, \sigma_{n-k-1}). \tag{2.8}$$

2.2 Rescaled models and characteristic direction

For the proofs we scale β away via the following identity in law which is obtained by Brownian scaling:

$$Z_{(1,n),(0,t)}(\beta) \stackrel{d}{=} \beta^{-2(n-1)} Z_{(1,n),(0,\beta^2 t)}(1). \tag{2.9}$$

We drop $\beta = 1$ from the notation and write $Z_{(1,n),(0,t)} = Z_{(1,n),(0,t)}(1)$. The regime $\beta = \beta_0 n^{-\alpha}$ corresponds to studying $Z_{(1,n),(0,\beta_0^2 n^{1-2\alpha})}$. Similarly we scale β away from the stationary partition function (1.14):

$$Z_{n,t}^{\theta,\beta} \stackrel{d}{=} \beta^{-2n} Z_{n,\,\beta^2 t}^{\beta^{-1}\theta,1}. \tag{2.10}$$

As we take (n,t) to infinity in the stationary model, we have to follow approximately a characteristic direction determined by θ. The characteristic direction is found by minimizing the right-hand side of (2.4) with respect to θ, or equivalently, by arranging the cancellation of the first two terms on the right of (2.5). The following condition on the triples (n,t,θ) expresses the fact that (n,t) is close to the characteristic direction:

$$|n\Psi_1(\theta) - t\,| \le \kappa n^{2/3}\theta^{-4/3} \qquad \text{with a fixed constant } \kappa \ge 0. \tag{2.11}$$

By the scaling relation (2.10), we can see that the choice of parameters in Theorem 1.3 corresponds to the characteristic direction.

2.3 Upper bounds for the stationary model

The main tool for our upper bounds is the following lemma. The proof of the upper bound in Theorem 1.3 will follow by a particular choice of the parameters and can be found at the end of this section. When we write $\sigma_0^{\pm}$ we mean that the statement is true for both σ_0^+ and σ_0^-.

Lemma 2.2. *Fix $\theta_0 > 0$ and $\kappa \geq 0$. Assume that $\theta > 0$, $n \in \mathbb{N}$ and $t > 0$ satisfy* (2.11) *and $\theta_0 \leq \theta \leq \theta_0^{-1}\sqrt{n}$. Then there are constants $\delta > 0$ and $c < \infty$ that depend only on θ_0 such that, for all $n\theta^{-1} \geq u \geq 2\kappa n^{2/3}\theta^{-4/3}$, we have*

$$\mathbb{P}\big\{Q^{\theta}_{n,t}(\sigma_0^{\pm} \geq u) \geq e^{-\delta\theta^2u^2n^{-1}}\big\} \leq c(1+\kappa)\frac{n^{8/3}}{\theta^{16/3}u^4} + c\frac{n^2}{\theta^4u^3}. \tag{2.12}$$

When $u \geq n\theta^{-1} \vee 2\kappa n^{2/3}\theta^{-4/3}$, bound (2.12) *continues to hold for σ_0^+, but for σ_0^- we have this bound:*

$$\mathbb{P}\big\{Q^{\theta}_{n,t}(\sigma_0^{-} \geq u) \geq e^{-\delta\theta u}\big\} \leq 2e^{-c\theta u}. \tag{2.13}$$

Furthermore, we have these bounds:

$$E^{\theta}_{n,t}(\sigma_0^{\pm}) \leq c(1+\kappa)\frac{n^{2/3}}{\theta^{4/3}}, \tag{2.14}$$

$$P^{\theta}_{n,t}\big\{\sigma_0^{\pm} \geq bn^{\frac{2}{3}}\theta^{-\frac{4}{3}}\big\} \leq c(1+\kappa)b^{-3} \qquad \text{for } b \geq (2\kappa)\vee 1, \tag{2.15}$$

$$\text{and} \qquad \mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t}) \leq c(1+\kappa)n^{2/3}\theta^{-4/3}. \tag{2.16}$$

Proof. We introduce $a = \delta\theta^2u^2n^{-1}$. We fix the positive parameter r (its value will be determined later), and set $\lambda = \theta + ru\theta^2n^{-1}$. From the definition of the path measure, we have

$$\begin{aligned}
&Q^{\theta}_{n,t}(\sigma_0^+ \geq u)\\
&= \frac{1}{Z^{\theta}_{n,t}} \int\limits_{u<s_0<\cdots<s_{n-1}<t} \exp\big[-B(s_0) + \theta s_0 + B_1(s_0,s_1) + \cdots + B_n(s_{n-1},t)\big]\, ds_{0,n-1}\\
&\leq \frac{1}{Z^{\theta}_{n,t}} \int\limits_{u<s_0<\cdots<s_{n-1}<t} e^{(\theta-\lambda)u} \exp\big[-B(s_0) + \lambda s_0 + B_1(s_0,s_1) + \cdots + B_n(s_{n-1},t)\big]\, ds_{0,n-1}\\
&\leq \frac{Z^{\lambda}_{n,t}}{Z^{\theta}_{n,t}} e^{(\theta-\lambda)u}.
\end{aligned} \tag{2.17}$$

Consequently,

$$\begin{aligned}
\mathbb{P}\big\{Q^{\theta}_{n,t}(\sigma_0^+ \geq u) \geq e^{-a}\big\} &\leq \mathbb{P}\big\{\log Z^{\lambda}_{n,t} - \log Z^{\theta}_{n,t} \geq (\lambda-\theta)u - a\big\}\\
&= \mathbb{P}\big\{\overline{\log Z^{\lambda}_{n,t}} - \overline{\log Z^{\theta}_{n,t}} \geq n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda-\theta) + (\lambda-\theta)u - a\big\},
\end{aligned} \tag{2.18}$$

where $\overline{X} = X - \mathbb{E}X$ denotes the centering of the random variable X. Because of (2.4) we have $\overline{\log Z^{\lambda}_{n,t}} = \log Z^{\lambda}_{n,t} + n\Psi_0(\theta) - \theta t$.

By the monotonicity of $\Psi_2(z) = \Psi_0''(z)$ (see (4.1)) for any $\lambda > \theta > 0$

$$0 \geq \Psi_0(\lambda) - \Psi_0(\theta) - \Psi_1(\theta)(\lambda-\theta) = \int_\theta^\lambda \int_\theta^y \Psi_2(z)dzdy \geq -\tfrac{1}{2}|\Psi_2(\theta)|(\lambda-\theta)^2.$$

Assumptions (2.11) and $u \geq 2\kappa n^{2/3}\theta^{-4/3}$ imply

$$|n\Psi_1(\theta) - t\,| \leq u/2 \tag{2.19}$$

and so the right-hand side inside the probability (2.18) develops as follows:

$$\begin{aligned}
&n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda-\theta) + (\lambda-\theta)u - a\\
&\qquad = n\big(\Psi_0(\lambda) - \Psi_0(\theta) - \Psi_1(\theta)(\lambda-\theta)\big) + (n\Psi_1(\theta) - t)(\lambda-\theta) + u(\lambda-\theta) - a\\
&\qquad \geq -\frac{n}{2}|\Psi_2(\theta)|(\lambda-\theta)^2 + \frac{u}{2}(\lambda-\theta) - a \geq \left(-\frac{r^2c_0}{2} + \frac{r}{2} - \delta\right)\theta^2u^2n^{-1}\\
&\qquad \geq \delta\theta^2u^2n^{-1}.
\end{aligned}$$

Above we introduced

$$c_0 = \sup_{x \ge \theta_0} |\Psi_2(x)| x^2 < \infty \tag{2.20}$$

(which is finite by (4.1) and (4.2)) and then chose $r = (2c_0)^{-1}$ and $\delta = c_0^{-1}/16$.

In the following c denotes a constant that depends only on θ_0, but may change from line to line. From line (2.18), using Lemma 2.1 we get

$$\begin{aligned}
\mathbb{P}\big\{Q^{\theta}_{n,t}(\sigma_0^+ \ge u) \ge e^{-a}\big\} &\le \mathbb{P}\big\{\overline{\log Z^{\lambda}_{n,t}} - \overline{\log Z^{\theta}_{n,t}} \ge \delta\theta^2 u^2 n^{-1}\big\} \\
&\le c\frac{n^2}{\theta^4 u^4}\mathbb{V}\mathrm{ar}\big[\log Z^{\lambda}_{n,t} - \log Z^{\theta}_{n,t}\big] \\
&\le c\frac{n^2}{\theta^4 u^4}\big(\,\mathbb{V}\mathrm{ar}\big[\log Z^{\theta}_{n,t}\big] + n|\Psi_1(\lambda) - \Psi_1(\theta)|\,\big) \\
&\le c\frac{n^2}{\theta^4 u^4}\big(E^{\theta}_{n,t}(\sigma_0^+) + u\big).
\end{aligned} \tag{2.21}$$

Above we used (2.5), (2.19), and the following estimate:

$$|\Psi_1(\lambda) - \Psi_1(\theta)| \le |\Psi_2(\theta)|(\lambda - \theta) \le c_0\theta^{-2}(\lambda - \theta) = c_0 r u/n = u/(2n).$$

Let $u_0 \ge 2\kappa n^{2/3}\theta^{-4/3}$.

$$\begin{aligned}
E^{\theta}_{n,t}(\sigma_0^+) &\le u_0 + \int_{u_0}^{t} du\, P^{\theta}_{n,t}\left[\sigma_0^+ \ge u\right] \\
&\le u_0 + \int_{u_0}^{t} du\left\{\int_{e^{-a}}^{1} dr\, \mathbb{P}\big[Q^{\theta}_{n,t}(\sigma_0^+ \ge u) \ge r\big] + e^{-a}\right\} \\
&\le u_0 + \frac{cn^2}{\theta^4}\int_{u_0}^{t} du\left(\frac{E^{\theta}_{n,t}(\sigma_0^+)}{u^4} + \frac{1}{u^3}\right) + \int_{u_0}^{t} e^{-\delta\theta^2 u^2/n}\, du \\
&\le u_0 + \frac{cn^2}{\theta^4 u_0^3}E^{\theta}_{n,t}(\sigma_0^+) + \frac{cn^2}{\theta^4 u_0^2} + \frac{\delta^{-1}n}{2\theta^2 u_0}e^{-\delta\theta^2 u_0^2/n}.
\end{aligned}$$

The last term comes from $\int_m^\infty e^{-x^2}\,dx \le (2m)^{-1}e^{-m^2}$ for $m > 0$. Now choose $u_0 = 2(1 + c + \kappa)n^{2/3}\theta^{-4/3}$. The inequality above can be rearranged to give

$$\begin{aligned}
E^{\theta}_{n,t}(\sigma_0^+) &\le (c + 4\kappa)\frac{n^{2/3}}{\theta^{4/3}} + \frac{cn^{1/3}}{\theta^{2/3}}\exp(-\delta n^{1/3}\theta^{-2/3}) \\
&\le c(1+\kappa)\frac{n^{2/3}}{\theta^{4/3}}.
\end{aligned} \tag{2.22}$$

Above, c has been redefined but still depends only on θ_0. This proves (2.14) for σ_0^+. Substitute this back up in (2.21) to get

$$\mathbb{P}\big\{Q^{\theta}_{n,t}(\sigma_0^+ \ge u) \ge e^{-\delta\theta^2 u^2 n^{-1}}\big\} \le c(1+\kappa)\frac{n^{8/3}}{\theta^{16/3}u^4} + c(1+u)\frac{n^2}{\theta^4 u^4} \tag{2.23}$$

which proves (2.12) as $\theta \le \theta_0\sqrt{n}$. To prove (2.15) apply (2.12) with $u = bn^{2/3}\theta^{-4/3}$, and use $b \ge (2\kappa) \vee 1$:

$$\begin{aligned}
P^{\theta}_{n,t}\big\{\sigma_0^+ \ge bn^{\frac{2}{3}}\theta^{-\frac{4}{3}}\big\} &\le e^{-\delta\theta^2 u^2 n^{-1}} + \mathbb{P}\big\{Q^{\theta}_{n,t}(\sigma_0^+ \ge bn^{\frac{2}{3}}\theta^{-\frac{4}{3}}) \ge e^{-\delta\theta^2 u^2 n^{-1}}\big\} \\
&\le e^{-\delta\theta^{-2/3}n^{1/3}b^2} + c(1+\kappa)b^{-4} + cb^{-3} \\
&\le c(1+\kappa)b^{-3}.
\end{aligned} \tag{2.24}$$

The proof for σ_0^- is similar, but we need some modifications. We take $\lambda = \theta - ru\theta^2 n^{-1}$. Note that by choosing $r < 1/2$ and using $n\theta^{-1} \geq u$ we have $\lambda > \theta/2$. Now we use the inequality

$$Q^\theta_{n,t}(\sigma_0^- \geq u) \leq \frac{Z^\lambda_{n,t}}{Z^\theta_{n,t}} e^{-(\theta-\lambda)u},$$

instead of (2.17), its proof being similar. Bound (2.12) now follows exactly the same way as for σ_0^+.

In order to get the bound (2.13) for $u \geq n\theta^{-1}$ we set $\lambda = (1-r)\theta$ with $0 < r < 1/2$ to be specified later and proceed with the proof exactly the same way as in the previous case. We have

$$\begin{aligned}
&\mathbb{P}\big\{Q^\theta_{n,t}(\sigma_0^- \geq u) \geq e^{-\delta\theta u}\big\} \\
&\qquad \leq \mathbb{P}\big\{\overline{\log Z^\lambda_{n,t}} - \overline{\log Z^\theta_{n,t}} \geq n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda-\theta) + (\theta-\lambda)u - \delta\theta u\big\}.
\end{aligned}$$

Below we use the Taylor expansion of Ψ_0 and (2.19).

$$\begin{aligned}
&n(\Psi_0(\lambda) - \Psi_0(\theta)) - t(\lambda-\theta) + (\theta-\lambda)u - \delta\theta u \\
&\qquad = n\big(\Psi_0(\lambda) - \Psi_0(\theta) - \Psi_1(\theta)(\lambda-\theta)\big) + (n\Psi_1(\theta) - t)(\lambda-\theta) + u(\theta-\lambda) - \delta\theta u \\
&\qquad \geq -\frac{n}{2}|\Psi_2(\lambda)|(\lambda-\theta)^2 + \frac{u}{2}(\theta-\lambda) - \delta\theta u \geq -nC(\theta_0)r^2 + \frac{1}{2}ru\theta - \delta\theta u \geq c_0\theta u
\end{aligned}$$

with a fixed positive c_0. In order to get the last bound we need to choose r and δ small enough in terms of $c(\theta_0)$. This gives

$$\begin{aligned}
\mathbb{P}\big\{Q^\theta_{n,t}(\sigma_0^+ \geq u) \geq e^{-\delta\theta u}\big\} &\leq \mathbb{P}\big\{\overline{\log Z^\lambda_{n,t}} - \overline{\log Z^\theta_{n,t}} \geq c_0\theta u\big\} \\
&\leq \mathbb{P}\big\{\overline{\log Z^\lambda_{n,t}} + B(t) \geq c_0\theta u/2\big\} + \mathbb{P}\big\{\overline{\log Z^\theta_{n,t}} + B(t) \leq -c_0\theta u/2\big\} \qquad (2.25)
\end{aligned}$$

By (1.12) $\log Z^\theta_{n,t} + B(t) - \theta t = \sum_{k=1}^n r_k(t)$ where $e^{-r_k(t)}$ are i.i.d. Gamma(θ) variables. (Recall that $\overline{X} = X - EX$.) We will use the following large deviation estimate: for $y \geq c > 0$ there exists a constant $c_2 > 0$ depending only on c such that

$$\mathbb{P}\Big(\Big|\sum_{k=1}^n \overline{r_k(t)}\Big| \geq ny\Big) \leq e^{-c_2\theta ny}. \qquad (2.26)$$

This will follow from a standard exponential Markov inequality. For the right tail for $0 < q < \theta$ we get

$$\mathbb{P}\Big(\sum_{k=1}^n \overline{r_k(t)} \geq ny\Big) \leq e^{-qny}\big(\mathbb{E}e^{q\,\overline{r_k(t)}}\big)^n \leq e^{-n\big(\log\Gamma(\theta) - \log\Gamma(\theta-q) - q\Psi_0(\theta) + qy\big)}, \qquad (2.27)$$

using the fact that if ξ is Gamma(θ) then $\mathbb{E}\xi^{-q} = \frac{\Gamma(\theta-q)}{\Gamma(\theta)}$. If $q < \theta/2$ then we can bound the multiplier of n in the exponent as

$$\begin{aligned}
\log\Gamma(\theta) - \log\Gamma(\theta-q) - q\Psi_0(\theta) + qy &= -\int_{\theta-q}^{\theta}\int_s^\theta \Psi_1(v)dv\,ds + qs \\
&\leq -c_3q^2\theta^{-1} + qy \qquad (2.28)
\end{aligned}$$

with a constant $c_3 > 0$ depending only on θ_0. Using $y \geq c$ we may choose a small enough $0 < c' < 1/2$ so that with $q = c'\theta$ so that the right side of (2.28) is bounded by $e^{-c_2\theta ny}$ with some constant c_2. The left tail bound of (2.26) follows similarly.

Returning to (2.25) we can bound the second term on the right as

$$\mathbb{P}\big\{\overline{\log Z^\theta_{n,t}} + B(t) \leq -c_0\theta u/2\big\} \leq e^{-\tilde{c}\theta^2 u}$$

where $\tilde{c}$ depends on θ_0. The first term on the right of (2.25) can be bounded similarly by using $\lambda = (1-r)\theta$ and the fact that r can be chosen to be small. This completes the proof of (2.13) for σ_0^-.

To prove (2.14) for σ_0^- we use $E^\theta_{n,t}(\sigma_0^+ - \sigma_0^-) = t - n\Psi_1(\theta)$ and the fact that we already have (2.14) for σ_0^+. To prove (2.15) for σ_0^- we can follow (2.24) in the case $bn^{\frac{2}{3}}\theta^{-\frac{4}{3}} \le n\theta^{-1}$ and use the bound (2.13) with a similar argument if $bn^{\frac{2}{3}}\theta^{-\frac{4}{3}} \ge n\theta^{-1}$.

Finally, bound (2.16) follows from (2.5), (2.11) and (2.14). $\square$

Proof of the upper bound in Theorem 1.3. Introduce variables

$$\tilde{n} = \tau n,\ t = \tau\beta_0^2 n^{1-2\alpha} \text{ and } \tilde{\theta} = \Psi_1^{-1}(\beta_0^2 n^{-2\alpha}). \tag{2.29}$$

By the scaling identity (2.10), $\mathbb{V}\mathrm{ar}(\log Z^{\theta,\beta}_{\tau n,\tau n}) = \mathbb{V}\mathrm{ar}(\log Z^{\tilde{\theta}}_{\tilde{n},t})$. Condition (2.11) is satisfied by $(\tilde{n}, t, \tilde{\theta})$ with $\kappa = 0$, $\tilde{\theta} \ge \Psi_1^{-1}(\beta_0^2) > 0$ and $\tilde{\theta} \le C\beta_0^{-2}n^{2\alpha} \le C'\sqrt{\tilde{n}}$, as long as $\tau \ge \tau_0$ for a constant $\tau_0 = \tau_0(\beta_0)$. This means that we may apply Lemma 2.2 with $(\tilde{n}, t, \tilde{\theta})$. The bound (2.16) gives

$$\mathbb{V}\mathrm{ar}\big[\log Z^{\tilde{\theta}}_{\tilde{n},t}\big] \le c\frac{\tilde{n}^{2/3}}{\tilde{\theta}^{4/3}} \le C\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-4\alpha)}$$

where C depends only on β_0. $\square$

2.4 Lower bound for the stationary model

In this section we prove the lower bound in Theorem 1.3. Again, the proof will follow by a particular choice of the parameters in the next proposition and can be found at the end of the section.

Proposition 2.3. *Let $\theta_0 > 0$ and $\kappa \ge 0$. There are positive constants δ_1, δ_2, n_0 that depend on (κ, θ_0) such that*

$$\mathbb{P}\big\{\log Z^\theta_{n,t} - \mathbb{E}(\log Z^\theta_{n,t}) \ge \delta_1 n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\big\} \ge \delta_2 \tag{2.30}$$

whenever $n \ge n_0$, (n,t,θ) satisfies (2.11), and $\theta_0 \le \theta \le \theta_0^{-1}\sqrt{n}$.

Moreover, under the previous assumptions we also have

$$\mathbb{V}\mathrm{ar}\big[\log Z^\theta_{n,t}\big] \ge cn^{\frac{2}{3}}\theta^{-\frac{4}{3}} \tag{2.31}$$

Proof. It is sufficient to prove estimate (2.30) since the lower bound (2.31) follows from this easily. Fix a constant $0 < b < \theta^{1/3}n^{1/3}$ and set $\lambda = \theta + b\theta^{2/3}n^{-1/3} < 2\theta$ and $\bar{t} = t + n\Psi_1(\lambda) - n\Psi_1(\theta)$. Then we have

$$v = t - \bar{t} = n(\Psi_1(\theta) - \Psi_1(\lambda)) \ge n|\Psi_2(\lambda)|(\theta - \lambda) \ge 4^{-1}bn^{\frac{2}{3}}\theta^{-\frac{4}{3}} \tag{2.32}$$

where we used $|\Psi_2(\lambda)| \ge \lambda^{-2} \ge \theta^{-2}/4$ from (4.2). We shall take $b \in (0,\infty)$ large enough in the course of the argument, which is not problematic as $\theta^{1/3}n^{1/3}$ will be large for large enough n by our assumption $\theta \ge \theta_0$.

Fix a $c_1 \in (0, 1/2)$. By the shift-invariance (2.6) we have

$$\begin{aligned} Q^\lambda_{n,t}(\sigma_0^+ \le c_1 v) &= Q^\lambda_{n,t}(\sigma_0 \le c_1 v) \\ &\overset{d}{=} Q^\lambda_{n,\bar{t}}(t - \bar{t} + \sigma_0 \le c_1 v) = Q^\lambda_{n,\bar{t}}(\sigma_0 \le -(1-c_1)v) \\ &= Q^\lambda_{n,\bar{t}}(\sigma_0^- \ge (1-c_1)v). \end{aligned}$$

Since $\lambda \le 2\theta$, $(n, \bar{t}, \lambda)$ satisfies (2.11) with κ replaced by $2^{4/3}\kappa$. We can apply the upper bound (2.12) to σ^- with $(n, \bar{t}, \lambda)$ and $u = (1-c_1)v$ because

$$(1-c_1)v \ge \frac{1}{8}bn^{\frac{2}{3}}\theta^{-\frac{4}{3}} \ge 2(2^{4/3}\kappa)\lambda^{-4/3}n^{\frac{2}{3}}$$

if we choose $b > 2^{4+4/3}\kappa$. After collecting all the terms on the right of (2.12) we get the upper bound

$$P(Q^{\lambda}_{n,t}(\sigma_0^+ \le c_1 v) > \varepsilon_0) \le Cb^{-3}$$

for any fixed $\varepsilon_0 > 0$ if b is large enough (and hence also n) relative to ε_0 and θ_0. This choice of b is needed to ensure that $\varepsilon_0 > e^{-\delta\lambda^2 u^2 n^{-1}}$.

A similar shifting argument gives

$$Q^{\lambda}_{n,t}(\sigma_0^+ \ge (1+c)(t-\bar{t})) \overset{d}{=} Q^{\lambda}_{n,\bar{t}}(\sigma_0^+ \ge c(t-\bar{t})),$$

and the upper bound (2.12) can be applied with λ and $\bar{t}$. Hence, we can fix constants $0 < c_1 < c_2 < \infty$ such that, for a given $\varepsilon_0 > 0$ and large n,

$$\mathbb{P}\big[\, Q^{\lambda}_{n,t}\{c_1 v \le \sigma_0 \le c_2 v\} \ge 1-\varepsilon_0 \big] \;\ge\; 1 - Cb^{-3}. \tag{2.33}$$

Observe that c_1 can be taken as close to 0 as we wish.

Introduce temporary notations $s_1 = c_1 v$ and $s_2 = c_2 v$. Assumptions (2.11) and $\theta \ge \theta_0 > 0$ guarantee that $s_2 < t$ for large enough n. Hence

$$Q^{\lambda}_{n,t}\{s_1 \le \sigma_0 \le s_2\} = \frac{1}{Z^{\lambda}_{n,t}} \int_{s_1}^{s_2} e^{-B(s)+\lambda s} Z_{1,n}(s,t)ds.$$

Using (2.11),

$$\begin{aligned}
\mathbb{E}(\log Z^{\lambda}_{n,t}) - \mathbb{E}(\log Z^{\theta}_{n,t}) &= n(\Psi_0(\theta) - \Psi_0(\lambda)) + (\lambda-\theta)t \\
&\ge n\int_{\theta}^{\lambda} (\Psi_1(\theta) - \Psi_1(\xi))\, d\xi - \kappa n^{2/3}\theta^{-4/3}(\lambda-\theta) \\
&\ge \tfrac{1}{2} n|\Psi_2(\lambda)|(\lambda-\theta)^2 - \kappa b\theta^{-2/3} n^{1/3} \\
&\ge \tfrac{1}{8} n^{1/3} b^2 \theta^{-2/3} - \kappa b \theta^{-2/3} n^{1/3} \ge 2c_* b^2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}},
\end{aligned}$$

with the constant $c_* = \frac{1}{32}$ if we choose $b \ge 16\kappa$. Then from (2.33)

$$\begin{aligned}
1 - Cb^{-3} &\le \mathbb{P}\Big[\, Q^{\lambda}_{n,t}\{s_1 \le \sigma_0 \le s_2\} \ge 1-\varepsilon_0 \Big] \\
&\le \mathbb{P}\Big[\int_{s_1}^{s_2} e^{-B(s)+\lambda s} Z_{1,n}(s,t)ds \ge (1-\varepsilon_0) e^{\mathbb{E}(\log Z^{\theta}_{n,t}) + c_* b^2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}}} \Big] \qquad (2.34) \\
&\qquad + \mathbb{P}\big(Z^{\lambda}_{n,t} \le e^{\mathbb{E}(\log Z^{\lambda}_{n,t}) - c_* b^2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}}} \big). \qquad (2.35)
\end{aligned}$$

Bound probability (2.35) with Chebyshev:

$$\mathbb{P}\big(Z^{\lambda}_{n,t} \le e^{\mathbb{E}(\log Z^{\lambda}_{n,t}) - c_* b^2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}}} \big) \le c_*^{-2} b^{-4} \theta^{\frac{4}{3}} n^{-\frac{2}{3}} \mathbb{V}\mathrm{ar}(\log Z^{\lambda}_{n,t}).$$

The variance is estimated by (2.1) and (2.32):

$$\mathbb{V}\mathrm{ar}(\log Z^{\lambda}_{n,t}) \le \mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t}) + n|\Psi_1(\lambda) - \Psi_1(\theta)| \le C(1+b) n^{\frac{2}{3}} \theta^{-\frac{4}{3}}.$$

This implies that (2.35) $\le Cb^{-3}$.

Let A denote the event in probability (2.34):

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}\Big[e^{-B(s_1)+\lambda s_1} \int_{s_1}^{s_2} e^{-B(s_1,s)+\lambda(s-s_1)} Z_{1,n}(s,t)ds \ge (1-\varepsilon_0) e^{\mathbb{E}(\log Z^{\theta}_{n,t}) + c_* b^2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}}} \Big] \\
&= \mathbb{P}\Big[e^{-B(s_1)+\theta s_1} \int_{s_1}^{s_2} e^{-B(s_1,s)+\lambda(s-s_1)} Z_{1,n}(s,t)ds \ge (1-\varepsilon_0) e^{\mathbb{E}(\log Z^{\theta}_{n,t}) + c_* b^2 n^{\frac{1}{3}} \theta^{-\frac{2}{3}} - (\lambda-\theta)s_1} \Big].
\end{aligned}$$

We wish to replace λ by θ inside the integral to match it with the parameter in $\mathbb{E}(\log Z^{\theta}_{n,t})$ on the right-hand side. For this we use the Cameron-Martin-Girsanov formula to add a drift $\lambda-\theta$ to the Brownian motion $\{B(s_1,s): s_1 \le s \le s_2\}$. Note that the other random objects in the event A, namely $\{B(s_1); B_i(\cdot): 1 \le i \le n\}$, are independent of $B(s_1,\cdot)$. Let

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = e^{(\lambda-\theta)B(s_1,s_2)-\frac{1}{2}(\theta-\lambda)^2(s_2-s_1)}$$

so that, under $\widetilde{\mathbb{P}}$, $B(s_1,s) \overset{d}{=} \tilde{B}(s_1,s) + (\lambda-\theta)(s-s_1)$ where $\tilde{B}$ is a standard Brownian motion. By Cauchy-Schwarz

$$\mathbb{P}(A) = \widetilde{\mathbb{E}}\Big[\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\mathbf{1}(A)\Big] \le \sqrt{\widetilde{\mathbb{E}}\Big[\Big(\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\Big)^2\Big]}\sqrt{\widetilde{\mathbb{P}}(A)}.$$

The first expectation is finite:

$$\begin{aligned}\widetilde{\mathbb{E}}\Big[\Big(\frac{d\mathbb{P}}{d\widetilde{\mathbb{P}}}\Big)^2\Big] &= \widetilde{\mathbb{E}}e^{2(\theta-\lambda)B(s_1,s_2)+(\theta-\lambda)^2(s_2-s_1)} = \widetilde{\mathbb{E}}e^{2(\theta-\lambda)(\tilde{B}(s_1,s_2)+(\lambda-\theta)(s_2-s_1))+(\theta-\lambda)^2(s_2-s_1)}\\ &= \widetilde{\mathbb{E}}e^{2(\theta-\lambda)\tilde{B}(s_1,s_2)-(\theta-\lambda)^2(s_2-s_1)} = e^{(\theta-\lambda)^2(s_2-s_1)} \le e^{Cb^3}\end{aligned}$$

where C depends only on θ_0. We bound the probability $\widetilde{\mathbb{P}}(A)$ as follows: recall c_0 from (2.20),

$$\begin{aligned}&\widetilde{\mathbb{P}}(A)\\ &= \widetilde{\mathbb{P}}\Big[e^{-B(s_1)+\theta s_1}\int_{s_1}^{s_2} e^{-B(s_1,s)+\lambda(s-s_1)}Z_{1,n}(s,t)ds \ge (1-\varepsilon_0)e^{E\log Z^{\theta}_{n,t}+c_*b^2n^{\frac{1}{3}}\theta^{-\frac{2}{3}}-(\lambda-\theta)s_1}\Big]\\ &= \widetilde{\mathbb{P}}\Big[e^{-B(s_1)+\theta s_1}\int_{s_1}^{s_2} e^{-\tilde{B}(s_1,s)+\theta(s-s_1)}Z_{1,n}(s,t)ds \ge (1-\varepsilon_0)e^{E\log Z^{\theta}_{n,t}+c_*b^2n^{\frac{1}{3}}\theta^{-\frac{2}{3}}-(\lambda-\theta)s_1}\Big]\\ &\quad\le \mathbb{P}\big(Z^{\theta}_{n,t} \ge (1-\varepsilon_0)e^{E\log Z^{\theta}_{n,t}+c_*b^2n^{\frac{1}{3}}\theta^{-\frac{2}{3}}-(\lambda-\theta)s_1}\big)\\ &\quad\le \mathbb{P}\big(\overline{\log Z^{\theta}_{n,t}} \ge \log(1-\varepsilon_0)+c_*b^2n^{\frac{1}{3}}\theta^{-\frac{2}{3}}-c_0c_1b^2n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\big)\\ &\quad\le \mathbb{P}\big(\overline{\log Z^{\theta}_{n,t}} \ge \tfrac{1}{2}c_*b^2n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\big)\end{aligned}$$

where the last line follows after choosing c_1 small enough.

Put the estimates back on lines (2.34)–(2.35) to conclude that

$$(1-Cb^{-3})^2\exp(-Cb^3) \le \mathbb{P}\big\{\overline{\log Z^{\theta}_{n,t}} \ge \tfrac{1}{2}c_*b^2n^{1/3}\theta^{-\frac{2}{3}}\big\}$$

for a constant C that depends only on (κ,θ_0). This completes the proof of the proposition. □

Proof of the lower bound in Theorem 1.3. Proposition 2.3 applied to variables $(\tilde{n},t,\tilde{\theta})$ from (2.29) gives

$$\mathbb{V}\mathrm{ar}(\log Z^{\theta,\beta}_{\tau n,\tau n}) = \mathbb{V}\mathrm{ar}(\log Z^{\tilde{\theta}}_{\tilde{n},t}) \ge c\tilde{n}^{2/3}\tilde{\theta}^{-4/3} \ge c\tau^{2/3}n^{\frac{2}{3}(1-4\alpha)}.$$ □

2.5 Bounds on the path for the stationary model

Proof of Theorem 1.4. We start with the proof of the upper bound (1.17). We introduce the familiar rescaling $\tilde{\theta} = \Psi_1^{-1}(\beta_0^2 n^{-2\alpha})$, $\tilde{n} = \tau n$ and $t = \tau\beta_0^2 n^{1-2\alpha}$. Then

$$\begin{aligned}P^{\theta,\beta}_{\tau n,\tau n}\Big\{|\sigma_{\gamma\tau n}-\gamma\tau n| > b\tau^{\frac{2}{3}}n^{\frac{2}{3}(1-\alpha)}\Big\} &\le P^{\tilde{\theta}}_{\tilde{n},t}\Big\{|\sigma_{\gamma\tilde{n}}-\gamma t| > b\beta_0^{-\frac{2}{3}}\tilde{n}^{\frac{2}{3}}\tilde{\theta}^{-\frac{4}{3}}\Big\}\\ &= P^{\tilde{\theta}}_{(1-\gamma)\tilde{n},(1-\gamma)t}\Big\{|\sigma_0| > b\beta_0^{-\frac{2}{3}}\tilde{n}^{\frac{2}{3}}\tilde{\theta}^{-\frac{4}{3}}\Big\} \le Cb^{-3}.\end{aligned} \tag{2.36}$$

The first inequality is from Brownian scaling (2.10). After the change of variable from the Brownian scaling step, the quantity on the right of the inequality inside the braces develops as follows:

$$\beta^2 b\tau^{\frac{2}{3}} n^{\frac{2}{3}(1-\alpha)} = \beta_0^2 b\tau^{\frac{2}{3}} n^{\frac{2}{3}} n^{-\frac{8}{3}\alpha} \geq b\beta_0^{-\frac{2}{3}} \tilde{n}^{\frac{2}{3}} \tilde{\theta}^{-\frac{4}{3}}$$

using $\Psi_1^{-1}(x) \geq x^{-1}$ (4.2). The second step (equality) in (2.36) comes from shift invariance in t and n: (2.6) and (2.8). The last inequality in (2.36) is the upper bound (2.15) for $\sigma_0^{\pm}$. Note that $((1-\gamma)\tilde{n}, (1-\gamma)t, \tilde{\theta})$ satisfies (2.11) with $\kappa = 0$. The constant c from (2.15) depends only on the lower bound $\tilde{\theta}_0 = \Psi_1^{-1}(\beta_0^2)$ and so C above depends only on β_0. This completes the proof of (1.17).

We now prove the bound (1.18): using Brownian scaling (2.10) and shift invariance (2.6), (2.8) again,

$$E^{\theta,\beta}_{\tau n,\tau n}|\sigma_{\gamma\tau n} - \gamma\tau n| = \beta^{-2} E^{\tilde{\theta}}_{\tilde{n},t}|\sigma_{\gamma\tilde{n}} - \gamma t| = \beta^{-2} E^{\tilde{\theta}}_{(1-\gamma)\tilde{n},(1-\gamma)t}|\sigma_0|.$$

By (2.5)

$$\beta^{-2} E^{\tilde{\theta}}_{(1-\gamma)\tilde{n},(1-\gamma)t}|\sigma_0| = \beta_0^{-2} n^{4\alpha} \mathbb{V}\mathrm{ar}[\log Z^{\tilde{\theta}}_{(1-\gamma)\tilde{n}(1-\gamma)t}].$$

Now using (2.16) and (2.31) we have

$$C_1^{-1}(1-\gamma)^{2/3}\tilde{n}^{2/3}\tilde{\theta}^{-4/3} \leq \mathbb{V}\mathrm{ar}\log[Z^{\tilde{\theta}}_{(1-\gamma)\tilde{n}(1-\gamma)t}] \leq C_1((1-\gamma)^{2/3}\tilde{n}^{2/3}\tilde{\theta}^{-4/3} + 1)$$

Using the asymptotics for $\tilde{\theta}$ we get

$$C_1^{-1}\tau^{2/3} n^{\frac{2}{3}(1-\alpha)} \leq E^{\theta,\beta}_{\tau n,\tau n}|\sigma_{\gamma\tau n} - \gamma\tau n| \leq C_1 \tau^{2/3} n^{\frac{2}{3}(1-\alpha)}. \qquad \square$$

2.6 Bounds for the point-to-point model

This section derives bounds on the path and free energy fluctuations in the point-to-point case without boundaries, with $\beta = 1$, uniformly in (n, t, θ). Theorems 1.1 and 1.2 follow after a Brownian scaling step. For $n \in \mathbb{N}$, $t > 0$ and events D on the paths write $Z^{\theta}_{n,t}(D) = Z^{\theta}_{n,t} Q^{\theta}_{n,t}(D)$ for the unnormalized quenched measure.

Theorem 2.4. *Fix $0 < \theta_0 < \infty$. Let $\theta = \Psi_1^{-1}(t/n)$ satisfy $\theta_0 \leq \theta \leq \theta_0^{-1}\sqrt{n}$. Then there exist constants n_0, b_0, C that depend only on θ_0 so that for $n \geq n_0$, $b \geq b_0$ we have*

$$\mathbb{P}\{|\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta))| \geq bn^{\frac{1}{3}}\theta^{-\frac{2}{3}}\} \leq Cb^{-3/2} + \theta e^{-b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}} \tag{2.37}$$

and

$$C^{-1}(n^{\frac{1}{3}}\theta^{-\frac{2}{3}} - \log n) \;\leq\; \mathbb{E}|\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta))| \;\leq\; Cn^{\frac{1}{3}}\theta^{-\frac{2}{3}} + \theta e^{-b_0\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}}. \tag{2.38}$$

Note that the assumptions of the theorem force $c_1^{-1}\sqrt{n} \leq t \leq c_1 n$ for a positive constant c_1. The strategy of our proofs is the following. If we consider $Z^{\theta}_{n,t}$ with θ defined according to the theorem then $E^{\theta}_{n,t}(\sigma_0) = 0$ by (2.13). Since we expect σ_0 to be fairly close to its mean, this would suggest that $Z^{\theta}_{n,t}$ is fairly close to $Z_{(1,n),(0,t)}$. The main components of the proofs will rely on comparisons between the partition functions of the two models and on the results proved about the stationary model.

The centering $\theta t - n\Psi_0(\theta)$ inside probability (2.37) is the right choice, as can be seen from the exact expression for the free energy (1.5) and Brownian scaling.

Proof of the lower bound in (2.38). Note that (n, t, θ) satisfy (2.11) with $\kappa = 0$. Let

$$f_n = \theta t - n\Psi_0(\theta) = \mathbb{E}(\log Z^{\theta}_{n,t}). \tag{2.39}$$

Define the event $A = \{\log Z^{\theta}_{n,t} \ge f_n + \delta_1 n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\}$. By Proposition 2.3 there exist $\delta_1, \delta_2 > 0$ such that

$$\mathbb{P}\left(A\right) \ge \delta_2. \tag{2.40}$$

From (2.1) we get the simple bound

$$Z^{\theta}_{n,t} \ge e^{r_1(0)} Z_{(1,n),(0,t)}. \tag{2.41}$$

Utilizing this,

$$\begin{aligned}
f_n &= \mathbb{E}[(\log Z^{\theta}_{n,t})\mathbf{1}_A] + \mathbb{E}[(\log Z^{\theta}_{n,t})\mathbf{1}_{A^c}] \\
&\ge \mathbb{P}(A)(f_n + \delta_1 n^{\frac{1}{3}}\theta^{-\frac{2}{3}}) + \mathbb{E}[(\log Z^{\theta}_{n,t} - f_n)\mathbf{1}_{A^c}] + f_n \mathbb{P}(A^c) \\
&\ge f_n + \delta_1 \mathbb{P}(A) n^{\frac{1}{3}}\theta^{-\frac{2}{3}} + \mathbb{E}\left[\left(\log Z_{(1,n),(0,t)} - f_n\right)\mathbf{1}_{A^c}\right] + \mathbb{E}[r_1(0)\mathbf{1}_{A^c}].
\end{aligned}$$

Rearranging and using (2.40),

$$\begin{aligned}
\delta_1\delta_2 n^{\frac{1}{3}}\theta^{-\frac{2}{3}} &\le \delta_1 \mathbb{P}(A) n^{\frac{1}{3}}\theta^{-\frac{2}{3}} \le \mathbb{E}\left[\left(f_n - \log Z_{(1,n),(0,t)}\right)\mathbf{1}_{A^c}\right] - \mathbb{E}r_1(0)\mathbf{1}_{A^c} \\
&\le \mathbb{E}\left|f_n - \log Z_{(1,n),(0,t)}\right| + \mathbb{E}|r_1(0)| \\
&\le \mathbb{E}\left|f_n - \log Z_{(1,n),(0,t)}\right| + C(\theta_0)\log n.
\end{aligned}$$

We used $\mathbb{E}|r_1(0)| \le \sqrt{\mathbb{E}r_1(0)^2} = \sqrt{\Psi_1(\theta) + \Psi_0(\theta)^2}$ from (2.3). This proves the lower bound in (2.38). □

Proof of the upper bounds in (2.37) *and* (2.38). Inequality (2.41) gives

$$\mathbb{P}\left(\log Z^{\theta}_{n,t} - \log Z_{(1,n),(0,t)} \le -b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\right) \le \mathbb{P}\big(e^{-r_1(0)} \ge e^{b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}}\big) \le \theta e^{-b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}}$$

where the last step comes from Markov's inequality and $e^{-r_1(0)} \sim \text{Gamma}(\theta)$. Estimate (2.16) gives the Chebyshev bound

$$\mathbb{P}\left(|\log Z^{\theta}_{n,t} - f_n| \ge b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\right) \le b^{-2} n^{-\frac{2}{3}}\theta^{\frac{4}{3}}\,\mathbb{V}\mathrm{ar}(\log Z^{\theta}_{n,t}) \le Cb^{-2} \le Cb^{-3/2}$$

which gives

$$\mathbb{P}\left(\log Z_{(1,n),(0,t)} - f_n \ge b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\right) \le Cb^{-3/2} + \theta e^{-b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}}. \tag{2.42}$$

The bound on the other tail will be proved in two steps. In Lemma 2.8 below we show that there exists $c_0 > 0$ depending on θ_0 so that, for all $n \in \mathbb{N}$, if $b > c_0 n^{2/3}\theta^{2/3}$ then

$$\mathbb{P}\left(\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) \le -b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\right) \le Ce^{-C^{-1}bn^{1/3}\theta^{1/3}}. \tag{2.43}$$

In Lemma 2.7 below we will show that if $b \le c_0 n^{2/3}\theta^{2/3}$ then there are constants C and n_0 depending on c_0 and θ_0 so that

$$\mathbb{P}\left(\frac{Z^{\theta}_{n,t}}{Z_{(1,n),(0,t)}} \ge e^{b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}}\right) \le Cb^{-3/2} \tag{2.44}$$

for all $n > 1$. Using the Chebyshev bound again with (2.44) and then combining it with (2.43) we get

$$\mathbb{P}\left(\log Z_{(1,n),(0,t)} - f_n \le -b\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}\right) \le Cb^{-3/2} + Ce^{-C^{-1}bn^{1/3}\theta^{1/3}}. \tag{2.45}$$

The estimates (2.42) and (2.45) together establish (2.37).

Integrating out b in (2.37) gives

$$\mathbb{E}|\log Z^{\theta}_{n,t} - \log Z_{(1,n),(0,t)}| \le Cn^{\frac{1}{3}}\theta^{-\frac{2}{3}} + C\theta e^{-b_0\, n^{\frac{1}{3}}\theta^{-\frac{2}{3}}}.$$

Combining the above with

$$\mathbb{E}|\log Z^{\theta}_{n,t} - f_n| \le \sqrt{\mathbb{V}\mathrm{ar}\log Z^{\theta}_{n,t}} \le Cn^{\frac{1}{3}}\theta^{-\frac{2}{3}} \tag{2.46}$$

verifies the upper bound of (2.38). □

Except for the technical estimates postponed to Section 2.7, this completes the proof of Theorem 2.4.

Proof of Theorem 1.1. Recall that $\beta = \beta_0 n^{-\alpha}$. Introduce $\tilde{n} = \tau n, t = \tau\beta_0^2 n^{1-2\alpha}$ and $\theta = \Psi_1^{-1}(t/\tilde{n}) = \Psi_1^{-1}(\beta_0^2 n^{-2\alpha})$. From Brownian scaling (2.9) and the explicit free energy density $\mathbf{F}(\beta)$ in (1.5),

$$\begin{aligned}
&\log Z_{\tau n,\tau n}(\beta_0 n^{-\alpha}) - \tau n\mathbf{F}(\beta)\\
&\quad\overset{d}{=} -2(\tau n-1)\log(\beta_0 n^{-\alpha}) + \log Z_{(1,\tau n),(0,\beta_0^2\tau n^{1-2\alpha})} - \tau n\mathbf{F}(\beta_0 n^{-\alpha})\\
&\quad= \log Z_{(1,\tilde{n}),(0,t)} - 2(\tilde{n}-1)\log\beta - \tilde{n}(\theta\beta_0^2 n^{-2\alpha} - \Psi_0(\theta) - 2\log\beta)\\
&\quad= \log Z_{(1,\tilde{n}),(0,t)} - (\theta t - \tilde{n}\Psi_0(\theta)) + 2\log\beta.
\end{aligned}$$

The bounds claimed in Theorem 1.1 follow by quoting Theorem 2.4 for $(\tilde{n}, t)$. Note that because $\alpha \in [0, 1/4)$, the terms $\log\beta$ and $\log n$ are lower order than $n^{\frac{1}{3}(1-4\alpha)}$, and $\theta e^{-bn^{1/3}\theta^{-2/3}}$ is lower order than $b^{-3/2}$. □

We turn to the path fluctuations for the model without boundaries.

Theorem 2.5. *Fix $0 < \theta_0 < \infty$ and $0 < \varepsilon_0, \varepsilon_1 < 1/2$. Then there exist positive constants b_0, C, C_1 that depend only on θ_0, ε_0 such that the following holds. If $\theta = \Psi_1^{-1}(t/n)$ satisfies*

$$\theta_0 \le \theta \le \theta_0^{-1} n^{1/2-\varepsilon_0} \tag{2.47}$$

then for $n \ge n_0$, $b \ge b_0$, and $\varepsilon_1 \le \gamma \le 1-\varepsilon_1$ we have

$$P_{n,t}\left(|\sigma_{\lfloor n\gamma\rfloor} - \gamma t| > bn^{\frac{2}{3}}\theta^{-\frac{4}{3}}\right) \le Cb^{-3}. \tag{2.48}$$

The measure $P_{n,t}$ is the averaged measure $P_{n,t,\beta}$ with $\beta = 1$ introduced above Theorem 1.2.

Proof. Since $0 \le \sigma_k \le t = n\Psi_1(\theta) \le cn\theta^{-1}$ we may assume that

$$b \le cn^{1/3}\theta^{1/3} \tag{2.49}$$

with a constant c depending only on θ_0.

Let $\ell = \lfloor n\gamma\rfloor, t' = \gamma t$ and $u = b\,n^{\frac{2}{3}}\theta^{-\frac{4}{3}}$. By the definitions and (2.41)

$$\begin{aligned}
Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) &= \frac{1}{Z_{(1,n)(0,t)}}\int\limits_{|s-t'|>u} Z_{(1,\ell)(0,s)}Z_{(\ell+1,n)(s,t)}\,ds\\
&\le \frac{e^{-r_1(0)}}{Z_{(1,n)(0,t)}}\int\limits_{|s-t'|>u} Z^{\theta}_{\ell,s}\,Z_{(\ell+1,n)(s,t)}\,ds \;=\; \frac{e^{-r_1(0)}Z^{\theta}_{n,t}}{Z_{(1,n)(0,t)}}\,Q^{\theta}_{n,t}\left(|\sigma_\ell - t'| > u\right).
\end{aligned}$$

Let $h \in (b^{-3}, 1)$ (note that if b_0 is large enough then the interval is non-empty) and set $r = \delta b^2/(3(1-\gamma))$ with δ from Lemma 2.2.

$$\begin{aligned}\mathbb{P}\left(Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) > h\right) \quad &\le \quad \mathbb{P}(e^{-r_1(0)} \ge \theta b^3) + \mathbb{P}\left[\frac{Z^\theta_{n,t}}{Z_{(1,n)(0,t)}} \ge e^{rn^{\frac{1}{3}}\theta^{-\frac{2}{3}}}\right] \\ &\quad + \mathbb{P}\left[Q^\theta_{n,t}\left(|\sigma_\ell - t'| > u\right) > e^{-rn^{\frac{1}{3}}\theta^{-\frac{2}{3}}} h\theta^{-1}b^{-3}\right]\end{aligned}$$

On the right hand side, the first term is bounded by Cb^{-3} by Markov's inequality, since $e^{-r_1(0)} \sim \text{Gamma}(\theta)$. The second term is bounded by $Cr^{-3/2} \le Cb^{-3}$ by inequality (2.44) above (inequality (2.44) is proved in Lemma 2.7 below). To see that we can actually apply (2.44) note that by (2.49) we have

$$r = \frac{\delta b^2}{3(1-\gamma)} \le Cn^{2/3}\theta^{2/3}$$

with a constant depending on θ_0 and ε_1 which was the condition needed for (2.44).

Finally, the shift invariance (2.8) and Lemma 2.2 give, for large enough n and b and uniformly for $h \in (b^{-3}, 1)$,

$$\begin{aligned}\mathbb{P}&\left[Q^\theta_{n,t}\left(|\sigma_\ell - t'| > u\right) > e^{-rn^{\frac{1}{3}}\theta^{-\frac{2}{3}}} h\theta^{-1}b^{-3}\right] \le \mathbb{P}\left[Q^\theta_{n-\ell,t-t'}(|\sigma_0| > u) > e^{-rn^{\frac{1}{3}}\theta^{-\frac{2}{3}}}\theta^{-1}b^{-6}\right] \\ &\le \mathbb{P}\left[Q^\theta_{n-\ell,t-t'}(|\sigma_0| > u) > e^{-\delta\theta^2u^2/(n-\ell)}\right] \le Cb^{-3}.\end{aligned}$$

It is above that we need $\theta \le \theta_0^{-1} n^{1/2-\varepsilon_0}$ for $\varepsilon_0 > 0$, for otherwise the right-hand side $e^{-rn^{\frac{1}{3}}\theta^{-\frac{2}{3}}}\theta^{-1}b^{-6}$ cannot be bounded below by $e^{-\delta\theta^2u^2/(n-\ell)}$. Collecting the estimates gives

$$\mathbb{P}\left[Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) > h\right] \le Cb^{-3}$$

and from this

$$\begin{aligned}&P_{(1,n),(0,t)}\left(|\sigma_{\lfloor n\gamma\rfloor} - \gamma t| > bn^{\frac{2}{3}}\theta^{-\frac{4}{3}}\right) \\ &\qquad \le b^{-3} + \int_{b^{-3}}^1 \mathbb{P}\left[Q_{(1,n),(0,t)}\left(|\sigma_\ell - t'| > u\right) > h\right] dh \;\le\; Cb^{-3}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.2. We again introduce $\tilde{n} = \tau n, t = \tau\beta_0^2 n^{1-2\alpha}$ and $\theta = \Psi_1^{-1}(t/\tilde{n}) = \Psi_1^{-1}(\beta_0^2 n^{-2\alpha})$. Assumption (2.47) is satisfied because $\alpha < 1/4$. Using (2.9) and Theorem 2.5 the theorem follows. □

2.7 The tail estimates

In this section we prove the missing components of the proofs of Theorem 1.4 and Theorem 2.4. We begin with some definitions.

Augment the family $Z_{(j,k),(s,t)} = Z_{(j,k),(s,t)}(1)$ defined for $j \ge 1$ in (1.13) by introducing, for $k \in \mathbb{N}$ and $t \in \mathbb{R}_+$,

$$Z_{(0,0),(0,t)} = e^{-B(t)}, \qquad Z_{(0,k),(0,t)} = \int\limits_{0<s_0<\cdots<s_{k-1}<t} \exp\left[-B(s_0) + B_1(s_0,s_1) + \cdots + B_k(s_{k-1},t)\right] ds_{0,k-1}. \tag{2.50}$$

It is also convenient to set, for $A \subseteq \mathbb{R}$,

$$Z^\theta_{0,t}(\sigma_0 \in A) = \mathbf{1}_{A\cap\mathbb{R}_+}(t)\exp[-B(t) + \theta t]. \tag{2.51}$$

The following bounds are proved in Lemma 3.8 of [46] .

Lemma 2.6. *[46] Let $\theta > 0$. For $0 < s < t$ and $n \in \mathbb{Z}_+$*

$$\frac{Z^{\theta}_{n+1,t}(\sigma_0 > 0)}{Z^{\theta}_{n,t}(\sigma_0 > 0)} \le \frac{Z_{(0,n+1),(0,t)}}{Z_{(0,n),(0,t)}} \le \frac{Z^{\theta}_{n+1,t}(\sigma_0 < 0)}{Z^{\theta}_{n,t}(\sigma_0 < 0)} \tag{2.52}$$

and

$$\frac{Z^{\theta}_{n,t}(\sigma_0 > 0)}{Z^{\theta}_{n,s}(\sigma_0 > 0)} \ge \frac{Z_{(0,n),(0,t)}}{Z_{(0,n),(0,s)}} \ge \frac{Z^{\theta}_{n,t}(\sigma_0 < 0)}{Z^{\theta}_{n,s}(\sigma_0 < 0)}. \tag{2.53}$$

The second inequality of (2.53) *makes sense only for $n \ge 1$.*

For $A \subset \mathbb{R}$ note the identity

$$\frac{Z^{\theta}_{n,t}(\sigma_0 \in A)}{Z_{1,n}(0,t)} = \int_A \exp(-B(s) + \theta s)\frac{Z_{1,n}(s,t)}{Z_{1,n}(0,t)}ds.$$

We will also define a reversed system: construct a new environment $\tilde\omega$ with

$$\tilde B(s) = -(B_n(t) - B_n(t-s)), \quad \tilde B_i(s) = B_{n-i}(t) - B_{n-i}(t-s), \quad 1 \le i \le n-1.$$

Quantities that use environment $\tilde\omega$ are marked with a tilde. From the definitions one checks that

$$Z_{(1,n),(s,t)} = \tilde Z_{(0,n-1),(0,t-s)} \quad \text{for any } t > 0 \text{ and } s \in (-\infty, t). \tag{2.54}$$

Lemma 2.7. *Let $\theta = \Psi_1^{-1}(t/n)$ and assume that $\theta_0 \le \theta \le \theta_0^{-1}\sqrt{n}$ with a fixed $\theta_0 > 0$. Fix a $c_0 > 0$. Then there exist a finite, positive constant C depending on θ_0, c_0 such that if $n > 1$ and $b \le c_0 n^{2/3}\theta^{2/3}$, then*

$$\mathbb{P}\left(\frac{Z^{\theta}_{n,t}}{Z_{(1,n),(0,t)}} \ge e^{b\, n^{\frac13}\theta^{-\frac23}}\right) \le Cb^{-3/2}. \tag{2.55}$$

Proof. Note that once we prove (2.55) for $b > b_0$ with a constant b_0 depending on θ_0, c_0 then we can get it for all b by adjusting the constant C. Thus we may assume that b is big enough compared to θ_0 and c_0.

Let $u = \sqrt{b}n^{2/3}\theta^{-4/3}$ and $\nu = \varepsilon\sqrt{b}n^{-1/3}\theta^{2/3}$ where $\varepsilon > 0$ will be specified later. Then

$$\begin{aligned}\mathbb{P}\left(\frac{Z^{\theta}_{n,t}}{Z_{(1,n),(0,t)}} \ge e^{b\, n^{\frac13}\theta^{-\frac23}}\right) &= \mathbb{P}\left(\frac{Z^{\theta}_{n,t}(|\sigma_0| \le u)}{Z_{(1,n),(0,t)}\, Q^{\theta}_{n,t}(|\sigma_0| \le u)} \ge e^{b\, n^{\frac13}\theta^{-\frac23}}\right) \\ &\le \mathbb{P}\left(\frac{Z^{\theta}_{n,t}(|\sigma_0| \le u)}{Z_{(1,n),(0,t)}} \ge \frac12 e^{b\, n^{\frac13}\theta^{-\frac23}}\right) + \mathbb{P}\left(Q^{\theta}_{n,t}(|\sigma_0| \le u) \le 1/2\right)\end{aligned} \tag{2.56}$$

The second probability can be bounded as

$$\mathbb{P}\left(Q^{\theta}_{n,t}(|\sigma_0| \le u) \le 1/2\right) = \mathbb{P}\left(Q^{\theta}_{n,t}(|\sigma_0| > u) \ge 1/2\right) \le Cb^{-3/2} \tag{2.57}$$

by (2.12) of Lemma 2.2 The first probability can be bounded by

$$\mathbb{P}\left(\frac{Z^{\theta}_{n,t}(0 \le \sigma_0 \le u)}{Z_{(1,n),(0,t)}} \ge \frac14 e^{b\, n^{\frac13}\theta^{-\frac23}}\right) + \mathbb{P}\left(\frac{Z^{\theta}_{n,t}(-u \le \sigma_0 < 0)}{Z_{(1,n),(0,t)}} \ge \frac14 e^{b\, n^{\frac13}\theta^{-\frac23}}\right).$$

We will bound the first term, the second will follow similarly.

Introduce the new parameter $\lambda = \theta - \nu$. Note that by choosing $\varepsilon^2 \le (4c_0)^{-1}$ we can assume $\lambda > \theta/2$.

Begin with (2.54) and then apply comparison (2.53):

$$\begin{aligned}\frac{Z_{(1,n),(s,t)}}{Z_{(1,n),(0,t)}} &= \frac{\tilde Z_{(0,n-1),(0,t-s)}}{\tilde Z_{(0,n-1),(0,t)}} \le \frac{\tilde Z^{\lambda}_{n-1,t-s}(\sigma_0<0)}{\tilde Z^{\lambda}_{n-1,t}(\sigma_0<0)} = \frac{\tilde Z^{\lambda}_{n-1,t-s}}{\tilde Z^{\lambda}_{n-1,t}}\cdot\frac{\tilde Q^{\lambda}_{n-1,t-s}(\sigma_0<0)}{\tilde Q^{\lambda}_{n-1,t}(\sigma_0<0)}\\ &= \exp\bigl(\tilde Y_{n-1}(t-s,t)-\lambda s\bigr)\cdot\frac{\tilde Q^{\lambda}_{n-1,t-s}(\sigma_0<0)}{\tilde Q^{\lambda}_{n-1,t}(\sigma_0<0)}\\ &\le \exp\bigl(\tilde Y_{n-1}(t-s,t)-\lambda s\bigr)\cdot\frac{1}{\tilde Q^{\lambda}_{n-1,t}(\sigma_0<0)}.\end{aligned}$$

where we used (2.2) for the reversed system. Specializing the above to our context and substituting it in the probability that is to be bounded:

$$\begin{aligned}&\mathbb{P}\left(\frac{Z^{\theta}_{n,t}(0<\sigma_0\le u)}{Z_{(1,n),(0,t)}}\ge \tfrac14 e^{n^{\frac13}\theta^{-\frac23}b}\right)\\ &\quad= \mathbb{P}\left(\int_0^u \exp(-B(s)+\theta s)\frac{Z_{(1,n),(s,t)}}{Z_{(1,n),(0,t)}}\,ds \ge \tfrac14 e^{n^{\frac13}\theta^{-\frac23}b}\right)\\ &\quad\le \mathbb{P}\left(\int_0^u \frac{\exp(-B(s)+\tilde Y_{n-1}(t-s,t)+(\theta-\lambda)s)}{\tilde Q^{\lambda}_{n-1,t}(\sigma_0<0)}\,ds \ge \tfrac14 e^{n^{\frac13}\theta^{-\frac23}b}\right)\\ &\quad\le \mathbb{P}\left(\tilde Q^{\lambda}_{n-1,t}(\sigma_0<0)\le 1/2\right) &(2.58)\\ &\qquad+\ \mathbb{P}\left(\int_0^u \exp(-B(s)+\tilde Y_{n-1}(t-s,t)+\nu s)\,ds\ge \tfrac18 e^{n^{\frac13}\theta^{-\frac23}b}\right). &(2.59)\end{aligned}$$

To treat probability (2.58) set

$$\begin{aligned}\bar u = (n-1)\Psi_1(\lambda)-n\Psi_1(\theta) &\ge -\Psi_1(\theta)+(n-1)\Psi_2(\theta)(\lambda-\theta)\\ &\ge -C_1\theta^{-1}+C_2\varepsilon\sqrt{b}n^{\frac23}\theta^{-\frac43}\ge C'\sqrt{b}n^{\frac23}\theta^{-\frac43},\end{aligned}$$

where we used our assumptions on θ, $n>1$, the bounds (4.2), and took b large enough in relation to θ_0. Use invariance (2.6) of Q and upper bound (2.12):

$$\begin{aligned}\mathbb{P}(\tilde Q^{\lambda}_{n-1,t}(\sigma_0<0)<1/2) &= \mathbb{P}(\tilde Q^{\lambda}_{n-1,t}(\sigma_0>0)\ge 1/2)\\ &= \mathbb{P}(\tilde Q^{\lambda}_{n-1,t+\bar u}(\sigma_0>\bar u)\ge 1/2)\le C(\theta_0)b^{-3/2}.\end{aligned}\tag{2.60}$$

To justify our use of the upper bound, note that

$$(n-1)\Psi_1(\lambda)-t-\bar u\ = n\Psi_1(\theta)-t\ =0$$

so the upper bound (2.12) is valid for $\bar u$.

For probability (2.59), observe first that $s\mapsto \tilde Y_{n-1}(t-s,t)$ is a standard Brownian motion which is independent of B by construction. By introducing $B^{\dagger}(s)=\frac{1}{\sqrt2}\bigl(-B(s)+\tilde Y_{n-1}(t-s,t)\bigr)$ we need to bound

$$\mathbb{P}\Bigl(\int_0^u \exp(\sqrt2 B^{\dagger}(s)+\nu s)ds\ge \tfrac18 e^{\varepsilon^{-1}\nu u}\Bigr)\le \mathbb{P}\Bigl(\int_0^u \exp(\sqrt2 B^{\dagger}(s)+\nu s)ds\ge e^{3\nu u}\Bigr).$$

where the upper bound follows by choosing ε small enough (for fixed b_0,n_0,θ_0). We will show that

$$\mathbb{P}\Bigl(\int_0^u \exp(\sqrt2 B^{\dagger}(s)+\nu s)ds\ge e^{3\nu u}\Bigr)\le Ce^{-\frac14\nu^2 u}\tag{2.61}$$

if $\nu > 0$, $u > 0$. Note that

$$\begin{aligned}\mathbb{P}\Big(\int_0^u \exp(\sqrt{2}B^\dagger(s)+\nu s)ds \ge e^{3\nu u}\Big) &\le \mathbb{P}\Big(\int_{-\infty}^u \exp(\sqrt{2}B^\dagger(s)+\nu s)ds \ge e^{3\nu u}\Big)\\ &\le \mathbb{P}(\exp(\sqrt{2}B^\dagger(u)+\nu u)\ge e^{2\nu u})\\ &+\mathbb{P}\Big(\int_{-\infty}^u \exp\big(\sqrt{2}(B^\dagger(s)-B^\dagger(u))+\nu(s-u)\big)ds \ge e^{\nu u}\Big).\end{aligned}$$

The first probability is $\mathbb{P}(\sqrt{2}B^\dagger(u)\ge \nu u) \le C\exp(-\nu^2 u/4)$. For the second probability we note that by Dufresne's identity [28] the integral has the same distribution as the reciprocal of a Gamma(ν) random variable. Thus the second term is

$$\mathbb{P}(\text{Gamma}(\nu)\le e^{-\nu u}) \le \frac{1}{\nu\Gamma(\nu)}e^{-\nu^2 u} \le Ce^{-\nu^2 u}$$

which proves the estimate (2.61).

Collecting everything we get that

$$P\left(\frac{Z^\theta_{n,t}(0<\sigma_0\le u)}{Z_{(1,n),(0,t)}} \ge \tfrac{1}{4}e^{n^{\frac{1}{3}}\theta^{-\frac{2}{3}}b}\right) \le Cb^{-3/2} + Ce^{-\frac{1}{4}\varepsilon^2 b^{3/2}} \le C'b^{-3/2}.$$

The case of $-u<\sigma<0$ goes similarly, with small alterations. Now $\lambda = \theta+\nu \le 3\theta/2$. Utilizing (2.54) and comparison (2.53) the ratio is developed as follows:

$$\begin{aligned}\frac{Z^\theta_{n,t}(-u\le\sigma_0<0)}{Z_{(1,n),(0,t)}} &= \int_{-u}^0 \exp(-B(s)+\theta s)\frac{Z_{(1,n),(s,t)}}{Z_{(1,n),(0,t)}}\,ds\\ &\le \int_{-u}^0 \frac{\exp(-B(s)-\tilde{Y}_{n-1}(t,t-s)-(\theta-\lambda)s)}{\tilde{Q}^\lambda_{n-1,t}(\sigma_0>0)}\,ds.\end{aligned}$$

The rest follows along the same lines as above. This proves (2.55). □

Lemma 2.8. *Fix $\theta_0>0$, suppose that $\theta_0<\theta<\theta_0^{-1}\sqrt{n}$ and let $t=n\Psi_1(\theta)$. Then there exist constants c_0, C depending on θ_0 so that for all $n\in\mathbb{N}$*

$$P\left(\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) < -x\right) \le c_1 e^{-c_1^{-1}x\theta}, \qquad \text{for } x\ge c_0 n. \tag{2.62}$$

The same bound holds for the upper tail.

Proof. We first note that

$$\begin{aligned}Z_{(1,n)(0,t)} &= \int_{0<s_1<\cdots<s_{n-1}<t} \exp\{B_1(0,s_1)+\cdots+B_n(s_{n-1},1)\}\,ds_{1,n-1}\\ &> \frac{t^{n-1}}{(n-1)!}\exp\left(\min_{0<s_1<\cdots<s_{n-1}<t}(B_1(0,s_1)+\cdots+B_n(s_{n-1},1))\right)\\ &\overset{d}{=} \frac{t^{n-1}}{(n-1)!}\exp(-\sqrt{nt}\,\lambda^n_{\max})\end{aligned}$$

where $\lambda^n_{\max}$ is the largest eigenvalue of an $n\times n$ GUE random matrix where the non-diagonal entries have variance $1/n$. (This is the normalization where the support of the spectrum converges to $[-2,2]$.) The fact that

$$\begin{aligned}\min_{0<s_1<\cdots<s_{n-1}<1}(B_1(0,s_1)+\cdots+B(s_{n-1},1)) &\overset{d}{=} -\max_{0<s_1<\cdots<s_{n-1}<1}(B_1(0,s_1)+\cdots+B(s_{n-1},1))\\ &\overset{d}{=} -\sqrt{n}\,\lambda^n_{\max}\end{aligned}$$

was proved independently in [29] and [12].

It is known that the random variable $\lambda^n_{\max}$ converges to 2, and the following uniform tail bound holds in n for $K > K_0 > 2$ (see e.g. [36] and the references within):

$$P(\lambda^n_{\max} > K) \le Ce^{-C^{-1}nK^2} \tag{2.63}$$

with a constant C depending only on K_0. We will use this bound with $K_0 = 3$.

We have

$$\begin{aligned} &P\left(\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) < -x\right) \\ &\quad \le P\left(e^{-(\theta t - n\Psi_0(\theta))}\frac{t^{n-1}}{(n-1)!}\exp(-\sqrt{nt}\,\lambda^n_{\max}) < e^{-x}\right) \\ &\quad = P\left(-\theta t + n\Psi_0(\theta) + x + (n-1)\log t - \log(n-1)! < \sqrt{nt}\lambda^n_{\max}\right). \end{aligned} \tag{2.64}$$

Using Stirling's formula, $t = n\Psi_1(\theta)$, the bounds $\theta_0 \le \theta \le \theta_0^{-1}n^{1/2}$ and the bounds (4.2), (4.3) on Ψ_0, Ψ_1 we get that

$$|-\theta t + n\Psi_0(\theta) + (n-1)\log t - \log(n-1)!| \le Cn$$

where C depends on θ_0. This gives

$$P\left(\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) < -x\right) \le P\left(-Cn + x < \sqrt{nt}\lambda^n_{\max}\right)$$

Choosing $c_0 > 2C$ we get

$$P\left(-Cn + x < \sqrt{nt}\lambda^n_{\max}\right) \le P\left(\tfrac{1}{2}x < \sqrt{nt}\lambda^n_{\max}\right) \le P(\tfrac{1}{2}xn^{-1}\Psi_1(\theta)^{-1/2} < \lambda^n_{\max}).$$

Now choose c_0 large enough so that $\frac{1}{2}c_0\Psi_1(\theta)^{-1/2} > 3$ for $\theta \ge \theta_0$ (possible by the monotonicity of Ψ_1). Now use (2.63):

$$\begin{aligned} P\left(\log Z_{(1,n),(0,t)} - (\theta t - n\Psi_0(\theta)) < -x\right) &\le P(\tfrac{1}{2}xn^{-1}\Psi_1(\theta)^{-1/2} < \lambda^n_{\max}) \\ &\le Ce^{-C^{-1}n(\frac{1}{2}xn^{-1}\Psi_1(\theta)^{-1/2})^2} \\ &\le c_1 e^{-c_1^{-1}\theta x} \end{aligned}$$

where in the last step we used the bounds on $\Psi_1(\theta)$ and $x \ge c_0 n$. □

3 Proofs for the KPZ equation

Proof of Theorem 1.6. We first start with the case $\varphi = 0$. From the definition (1.22), the scaling identity (2.10) and the identity (2.4) we get

$$\frac{1}{\tau}\mathbb{E}\log\mathcal{Z}_n(\tau) = -n - \frac{1}{2}\sqrt{n} - 2n\log\beta_n - n\Psi_0(\Psi_1^{-1}(\beta_n^2)) + \beta_n^2 n\Psi_1^{-1}(\beta_n^2). \tag{3.1}$$

Using the asymptotics in (4.3) with $\beta_n = n^{-1/4}$ we get that the right side of (3.1) is uniformly bounded by a constant which means that

$$|\mathbb{E}\log\mathcal{Z}_n(\tau)| \le A\tau \tag{3.2}$$

for some constant $A > 0$. The upper bound in Theorem 1.3 gives

$$\mathbb{V}\mathrm{ar}\log\mathcal{Z}_n(\tau) \le C\tau^{2/3},$$

with a constant C. This implies that $\{\log\mathcal{Z}_n(\tau) : n \ge 1\}$ is uniformly integrable, and hence from Theorem 1.5 we get $\mathbb{E}\log\mathcal{Z}_n(\tau) \to \mathbb{E}\log\mathcal{Z}(\tau,0)$. Note that this could have been obtained by exact computations for the stochastic heat equation as well. Together with Theorem 1.5, this gives the convergence in law

$$\log \mathcal{Z}_n(\tau) - \mathbb{E}\log \mathcal{Z}_n(\tau) \Rightarrow \log \mathcal{Z}(\tau) - \mathbb{E}\log \mathcal{Z}(\tau).$$

Fatou's lemma and Theorem 1.3 then give

$$\mathbb{V}\mathrm{ar}\log \mathcal{Z}(\tau) \leq \liminf_n \mathbb{E}\left[(\log \mathcal{Z}_n(\tau) - \mathbb{E}\log \mathcal{Z}_n(\tau))^2\right] \leq C\tau^{\frac{2}{3}},$$

for some $C > 0$. As for the lower bound,

$$\mathbb{P}\left\{\log \mathcal{Z}(\tau) - \mathbb{E}\log \mathcal{Z}(\tau) \geq c\tau^{1/3}\right\} \geq \overline{\lim_n}\, \mathbb{P}\left\{\log \mathcal{Z}_n(\tau) - \mathbb{E}\log \mathcal{Z}_n(\tau) \geq c\tau^{1/3}\right\} \geq \delta,$$

for some constants c, $\delta > 0$, and τ large enough, thanks to the lower bound in Proposition 2.3. This proves that there exist some constant $C > 0$ such that

$$C^{-1}\tau^{\frac{2}{3}} \leq \mathbb{V}\mathrm{ar}\log \mathcal{Z}(\tau) \quad \leq \quad C\tau^{\frac{2}{3}},$$

for τ large enough.

We now turn to the case $|\varphi| \leq K$ for some $0 < K < +\infty$. From (1.21), we can verify that

$$e^{-K}\mathcal{Z}_n^{\varphi}(\tau) \leq \mathcal{Z}_n(\tau) \leq e^{K}\mathcal{Z}_n^{\varphi}(\tau).$$

This implies that

$$\mathbb{E}|\log \mathcal{Z}_n^{\varphi}(\tau) - \log \mathcal{Z}_n(\tau)| \leq K$$

and

$$\mathbb{V}\mathrm{ar}\log \mathcal{Z}_n^{\varphi}(\tau) \leq 8K^2 + 2\mathbb{V}\mathrm{ar}\log \mathcal{Z}_n(\tau),$$

which in turn implies the uniform integrability of $\{\log \mathcal{Z}_n^{\varphi}(\tau) \,:\, n \geq 1\}$. Fatou's lemma and the upper bound on $\mathbb{V}\mathrm{ar}\log \mathcal{Z}_n(\tau)$ show that

$$\mathbb{V}\mathrm{ar}\log \mathcal{Z}^{\varphi}(\tau) \leq C'\tau^{\frac{2}{3}},$$

for some $C' > 0$ and $\tau > 0$ large enough. The lower bound follows from Proposition 2.3

$$\begin{aligned}\mathbb{P}\left\{\log \mathcal{Z}^{\varphi}(\tau) - \mathbb{E}\log \mathcal{Z}^{\varphi}(\tau) \geq c\tau^{1/3}\right\} &\geq \overline{\lim_n}\, \mathbb{P}\left\{\log \mathcal{Z}_n^{\varphi}(\tau) - \mathbb{E}\log \mathcal{Z}_n^{\varphi}(\tau) \geq c\tau^{1/3}\right\}\\ &\geq \overline{\lim_n}\, \mathbb{P}\left\{\log \mathcal{Z}_n(\tau) - \mathbb{E}\log \mathcal{Z}_n(\tau) \geq c\tau^{1/3} - 2K\right\} \;\geq\; \delta,\end{aligned}$$

for suitable $c, \delta > 0$ and all n and τ large enough. This completes the proof of the theorem. □

4 Facts about the polygamma functions

We collect here some basic facts about the polygamma functions. Recall that $\Psi_0 = \Gamma'/\Gamma$ and $\Psi_n = \Psi'_{n-1}$ for $n \geq 1$. These functions satisfy

$$\Psi_n(x) = (-1)^{n+1} n! \sum_{k\geq 0}(x+k)^{-n-1} \qquad \text{for } x > 0 \text{ and } n \geq 1. \tag{4.1}$$

Ψ_0 is strictly increasing and Ψ_1 is positive and strictly decreasing. For $x > 0$ and $n \geq 1$

$$\frac{(n-1)!}{x^n} \leq |\Psi_n(x)| \leq \frac{(n-1)!}{x^n} + \frac{n!}{x^{n+1}}. \tag{4.2}$$

For Ψ_0 and Ψ_1 we have the following asymptotics for large $x > 0$:

$$\Psi_0(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right), \qquad \Psi_1(x) = \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right). \tag{4.3}$$

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