

Random partitions in statistical mechanics*

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Abstract

We consider a family of distributions on spatial random partitions that provide a coupling between different models of interest: the ideal Bose gas; the zero-range process; particle clustering; and spatial permutations. These distributions are invariant for a “chain of Chinese restaurants” stochastic process. We obtain results for the distribution of the size of the largest component.

Keywords: Spatial random partitions; Bose–Einstein condensation; (inhomogeneous) zero-range process; chain of Chinese restaurants; sums of independent random variables; heavy-tailed variables; infinitely divisible laws..

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1 Introduction

We study systems of random integer partitions that are independent except for a global constraint on their total mass. Such a setting appears, directly or indirectly, in diverse systems of statistical mechanics: the ideal quantum Bose gas, the zero-range process, particle clustering, certain coagulation-fragmentations processes, and some models of spatial permutations. A common feature is the possibility of a Bose–Einstein condensation; namely, under some conditions, a phase transition takes place that is accompanied with the formation of large objects. In the language of probability, a single random variable realizes the large deviation required to satisfy the constraint on its sum; this behavior is a well-known hallmark for sums of heavy-tailed random variables, and in fact many of our results can be read as abstract results for (conditioned) sums of independent random variables.

We introduce the setting in Section 2. The random objects are “spatial partitions”, that is, collections of integer partitions indexed by locations. The distribution has a product structure subject to a global constraint. Two marginals play an important rôle.

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The first marginal deals with the site occupation numbers; the resulting distribution is that of the ideal Bose gas or of the zero-range process. The second marginal deals with integer partitions; the resulting distribution is that of the particle clustering and of spatial permutations. The present study originated in an attempt at unifying the latter two systems, and the links with the former systems were rather unexpected. It is useful to establish connections since many results and properties of one system can be transferred to the others.

The special cases are described in Section 3. The ideal Bose gas can be found in Section 3.1, the zero-range process in Section 3.2, particle clustering in Section 3.3, coagulation-fragmentations processes in Section 3.4, and spatial permutations in Section 3.5.

The measures considered in this article are invariant measures for interesting Markov processes. One process represents customers in a “chain of Chinese restaurants” which combines the usual Chinese restaurant process with the zero-range process. It is described in Section 4.1. This actually holds only when the parameters satisfy certain consistency properties. Another process is a coagulation-fragmentation process which is a variant of the Becker–Döring model, see Section 4.2.

The possible occurrence of Bose–Einstein condensation and its consequences are addressed in Section 5. The relevant asymptotic is the thermodynamic limit of statistical mechanics, and the critical density is given by an explicit formula. We study more specific settings in the last two sections, namely, the case of the trap potential in Section 6 and the case of the square potential in Section 7.

2 Random spatial partitions

2.1 Setting

An *integer partition*¹ λ of the integer $n \in \mathbb{N}$ is a finite decreasing sequence $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$, of varying length k , whose elements add up to n : $\sum_{j=1}^k \lambda_j = n$; one often writes “ $\lambda \vdash n$ ”. We refer to the length k of the sequence as the *number of components* of the partition λ . Every partition is uniquely determined by the numbers $r_j(\lambda) = \#\{i = 1, \dots, k \mid \lambda_i = j\}$, $j \in \mathbb{N}$, which count how many times a given integer j appears in the partition λ ; they are often called *occupation numbers* of the partition. We also define the partition of $n = 0$ as the empty sequence (length 0, all occupation numbers equal to 0).

We are interested in random integer partitions that have additional structure. A *spatial partition* of the integer n is a collection $\boldsymbol{\lambda} = (\lambda_x)_{x \in \mathbb{Z}^d}$ of integer partitions at each site $x \in \mathbb{Z}^d$. That is, each λ_x is a k -tuple $(\lambda_{x1}, \lambda_{x2}, \dots, \lambda_{xk})$, where k varies and where the integers λ_{xj} satisfy

$$\lambda_{x1} \geq \lambda_{x2} \geq \dots \geq \lambda_{xk} \geq 1 \tag{2.1}$$

(and we allow for the “empty” sequence with $k = 0$). The spatial partition $\boldsymbol{\lambda}$ is uniquely determined by the numbers r_{xj} that count how many times a given integer j appears in the partition at site x ,

$$r_{xj} = \#\{i = 1, 2, \dots : \lambda_{xi} = j\}. \tag{2.2}$$

We can form two different types of marginals, summing over integers j or sites x ; this gives rise to two different types of occupation numbers. We let $\boldsymbol{r} = (r_j)_{j \geq 1}$ denote the sums over all the sites, i.e., $r_j = \sum_{x \in \mathbb{Z}^d} r_{xj}$. The site occupation numbers $\boldsymbol{\eta} = (\eta_x)_{x \in \mathbb{Z}^d}$ are given by

$$\eta_x = \sum_{i \geq 1} \lambda_{xi} = \sum_{j \geq 1} j r_{xj}. \tag{2.3}$$

¹When there is no risk of confusion with *set partitions*, we drop the word “integer” in front of “partition”.

Thus for every x , λ_x is a partition of the integer η_x . Sometimes this aspect is stressed and one calls λ a *vector partition* of the vector η , written $\lambda \vdash \eta$ (see e.g. [53]).

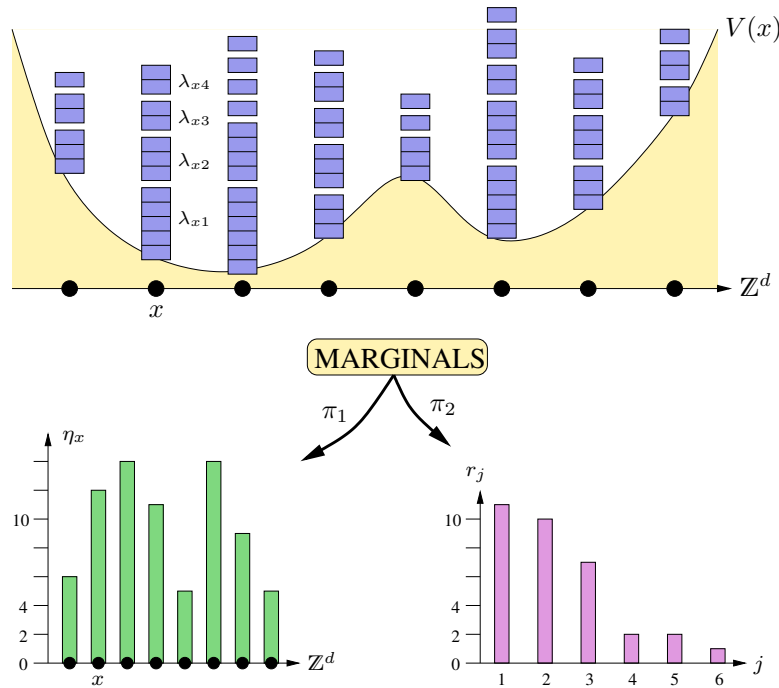


Figure 1: A schematic illustration of spatial partitions and the two relevant marginals.

These definitions are illustrated in Figure 1. The intuition is as follows: we think of $x \in \mathbb{Z}^d$ as sites in space (hence the name *spatial* partitions). “Space” and “site” should be taken loosely — x can be a particle position, a moment vector $x = k$ in a Fourier transformed picture, or a label for an energy level, see the examples and Table 1 in Section 3. Let Λ_n denote the set of spatial partitions such that

$$\sum_{x \in \mathbb{Z}^d, i \geq 1} \lambda_{xi} = \sum_{x \in \mathbb{Z}^d, j \geq 1} j r_{xj} = \sum_{x \in \mathbb{Z}^d} \eta_x = \sum_{j \geq 1} j r_j = n. \tag{2.4}$$

In the language of statistical mechanics, Λ_n is a canonical configuration space with total particle number n . Notice that Λ_n is a countable set. For later purpose we also define $\mathcal{N}_n \subset \mathbb{N}_0^{\mathbb{Z}^d}$ as the set of η 's with $\sum_{x \in \mathbb{Z}^d} \eta_x = n$, and $\mathcal{R}_n \subset \mathbb{N}_0^{\mathbb{N}}$ as the set of r 's with $\sum_{j \in \mathbb{N}} r_j = n$. Let $\pi_1 : \Lambda_n \rightarrow \mathcal{N}_n, \lambda \mapsto \eta(\lambda)$ and $\pi_2 : \Lambda_n \rightarrow \mathcal{R}_n, \lambda \mapsto r(\lambda)$ be the natural projections.

Apart from the space dimension d , the relevant parameters for our probability distribution on Λ_n are the following:

- (i) A potential function $V : \mathbb{R}^d \rightarrow (-\infty, \infty]$.
- (ii) A parameter $\rho \in [0, \infty)$ which represents the density of the system. We set $L^d = n/\rho$.
- (iii) A sequence of non-negative parameters $\theta = (\theta_j)_{j \geq 1}$.

The probability distribution on Λ_n is defined as

$$P_{L,n}(\lambda) = \frac{1}{Z_{L,n}} \prod_{x \in \mathbb{Z}^d} \prod_{j \geq 1} \frac{1}{r_{xj}(\lambda)!} \left(\frac{\theta_j}{j} e^{-jV(x/L)} \right)^{r_{xj}(\lambda)}. \tag{2.5}$$

The number $Z_{L,n}$ is the normalization; it actually depends on V and θ , but we alleviate the notation by neglecting to make it explicit. We assume that $Z_{L,n} < \infty$. The relevant asymptotic is the thermodynamic limit where both n and L tend to infinity, with L such that $n = \rho L^d$. We will propose different interpretations of the measure $\mathbb{P}_{L,n}$ later. One such interpretation is that $\mathbb{P}_{L,n}$ is a canonical Gibbs measure for particles moving in the trap potential $V(x/L)$, forming groups at each site. Particles from different groups do not interact; the parameter θ_j is a Boltzmann weight for intra-group interactions.² See Section 3 for details.

2.2 Marginals and conditional probabilities

An advantage of the probability measure (2.5) is that it allows us to switch between random partitions and sums of independent, *infinitely divisible* random variables. The latter play an important rôle as stationary measures of the zero-range process. These measures arise as marginals of $\mathbb{P}_{L,n}$.

To be sure, sums of independent random variables, corresponding to the occupation numbers r_j , have played a significant rôle in many recent studies of decomposable random structures (see for instance [6]); but, to the best of our knowledge, this connection to the infinitely divisible random variables, corresponding to the site occupation numbers η_x , has not been noticed before. It allows us to deduce limit laws for random partitions from limit laws for sums of independent variables.

Let $\mathbb{P}_{L,n} \circ \pi_1^{-1}$ and $\mathbb{P}_{L,n} \circ \pi_2^{-1}$ be the push-forwards of $\mathbb{P}_{L,n}$ under the projections onto \mathcal{N}_n and \mathcal{R}_n , respectively. Set

$$h_m = \sum_{\lambda \vdash m} \prod_{j \geq 1} \frac{1}{r_j(\lambda)!} \left(\frac{\theta_j}{j}\right)^{r_j(\lambda)}, \tag{2.6}$$

where the first sum is over the partitions λ of the integer m . We can think of h_m as the analogue of the normalization $Z_{L,n}$ for a single site (no product over x , no background potential $V(x/L)$). Later we will discuss the properties of the map $(\theta_j) \mapsto (h_m)$.

Proposition 2.1.

(a) *The measure $\mathbb{P}_{L,n} \circ \pi_1^{-1}$ has the product form*

$$\mathbb{P}_{L,n}(\pi_1^{-1}(\{\eta\})) = \mathbb{P}_{L,n}(\{\lambda : \eta(\lambda) = \eta\}) = \frac{1}{Z_{L,n}} \prod_{x \in \mathbb{Z}^d} (h_{\eta_x} e^{-\eta_x V(x/L)}). \tag{2.7}$$

(b) *The measure $\mathbb{P}_{L,n} \circ \pi_2^{-1}$ is of the Gibbs partition form*

$$\mathbb{P}_{L,n}(\pi_2^{-1}(\{\mathbf{r}\})) = \mathbb{P}_{L,n}(\{\lambda : \mathbf{r}(\lambda) = \mathbf{r}\}) = \frac{1}{Z_{L,n}} \prod_{j \geq 1} \frac{1}{r_j!} \left(\frac{\theta_j}{j} \sum_{x \in \mathbb{Z}^d} e^{-jV(x/L)}\right)^{r_j}. \tag{2.8}$$

Proposition 2.1 has natural probabilistic proof and interpretation, which we defer to Section 2.4. For now, suffice it to say that the first marginal (2.7) arises as the stationary measure of the zero-range process.

In order to recover the full measure from the marginals, we give below the conditional measures $\mathbb{P}_{L,n}(\lambda | \eta(\lambda) = \eta)$ and $\mathbb{P}_{L,n}(\lambda | \mathbf{r}(\lambda) = \mathbf{r})$. Let ν_m be the measure on

²Our interpretation is consistent with the examples given in Section 3 but different from that of Vershik [53] and Pitman [49, Chapter 1.5], who consider $Z_{L,n}$ as a microcanonical rather than a canonical partition function.

integer partitions of m given by

$$\nu_m(\lambda) = \frac{1}{h_m} \prod_{j \geq 1} \frac{1}{r_j(\lambda)!} \left(\frac{\theta_j}{j}\right)^{r_j(\lambda)}. \tag{2.9}$$

The normalization h_m is defined in (2.6). The measure ν_m is the analogue of the measure (2.5) for a single site x . It is an example of a *Gibbs random partition* which we describe in more details in Section 2.3. The next proposition contains expressions of the conditional probabilities.

Proposition 2.2. *The conditional measures are given by*

$$(a) \mathbb{P}_{L,n}(\lambda | \eta(\lambda) = \eta) = \prod_{x \in \mathbb{Z}^d} \nu_{\eta_x}(\lambda_x).$$

$$(b) \mathbb{P}_{L,n}(\lambda | r(\lambda) = r) = \prod_{j \geq 1} \binom{r_j}{\{r_{xj}\}_{x \in \mathbb{Z}^d}} \prod_{x \in \mathbb{Z}^d} p_{xj}^{r_{xj}}, \text{ with } p_{xj} = \frac{\exp(-jV(x/L))}{\sum_{y \in \mathbb{Z}^d} \exp(-jV(y/L))}.$$

We leave the elementary proof to the reader. Notice that $\mathbb{P}_{L,n}(\lambda | \eta)$ does not depend on V , and $\mathbb{P}_{L,n}(\lambda | r)$ does not depend on θ . Part (a) says that, conditioned on the site occupation numbers, the partitions λ_x become independent Gibbs partitions. Part (b) says that given r_j , the r_{xj} 's are multinomial: each of the r_j components of size j chooses the site x with probability p_{xj} .

2.3 Gibbs random partitions

The measure ν_m defined in (2.9) can be viewed as describing Gibbs random partitions, which have been studied in detail before, see [6, 34, 16, 33, 48, 55, 45, 47, 25, 7] and Chapters 1.5 and 2.5 in [49]. The main questions deal with the number of elements and their typical size, for given weights (θ_j) . The probability that the typical size is equal to ℓ is defined by

$$\mathbb{P}_n(X = \ell) = \mathbb{E}_n\left(\frac{\ell R_\ell}{n}\right) = \frac{\theta_\ell h_{n-\ell}}{n h_n}. \tag{2.10}$$

See Section 5 for more discussion about the random variable for the typical size of elements, where the random element is picked with probability that is proportional to its size.

We now study the relation between the θ_j s and the h_n s. This is obviously useful in view of the relation above. But this relation is also conceptually interesting since the θ_j s are related to the Lévy measure of a process on \mathbb{N}_0 given by the h_n s.

Recall that a measure μ is *infinitely divisible* if for all $n \in \mathbb{N}$, there is a measure $\tilde{\mu}_n$ such that $\mu = \tilde{\mu}_n * \dots * \tilde{\mu}_n$ is the n -fold convolution of $\tilde{\mu}_n$. Similarly we say that a sequence $(h_m)_{m \geq 0}$ of non-negative numbers is infinitely divisible if for all $n \in \mathbb{N}$ there is a sequence of non-negative numbers $(h_m^{(n)})$ such that (h_m) is the n -fold convolution of $(h_m^{(n)})_{m \geq 0}$, i.e., $h = h^{(n)} * \dots * h^{(n)}$. There is a rich theory for infinitely divisible measures [39], closely related to the topic of Lévy processes.

Proposition 2.3. *Let $(h_m)_{m \geq 0}$ be a sequence of non-negative numbers such that $h_0 = 1$ and $\sum_m h_m z^m$ has nonzero radius of convergence. Then there is a unique sequence $(\theta_j)_{j \geq 1}$ of real numbers such that Eq. (2.6) holds for all $m \in \mathbb{N}$. Moreover, we have $\theta_j \geq 0$ for all $j \geq 1$ if and only if $(h_m)_{m \geq 0}$ is infinitely divisible.*

Proof. We note the power series identity

$$\sum_{m \geq 0} h_m z^m = \exp\left(\sum_{j \geq 1} \frac{\theta_j}{j} z^j\right) \tag{2.11}$$

(see e.g. [1, Section 3.3]). This identity shows that for a given (h_m) there is a unique sequence of real numbers (θ_j) such that Eq. (2.6) holds.

Let R denote the radius of convergence of the series $\sum h_m z^m$. Fix $z \in (0, R)$, and let $C^{(z)} = \sum_{m \geq 0} h_m z^m$ and $p_m^{(z)} = C^{-1} h_m z^m$. One can check that (h_m) is infinitely divisible if and only if (p_m) is. Furthermore, $(p_m^{(z)})$ defines a probability measure on \mathbb{N}_0 with cumulant generating function

$$\log\left(\sum_{m \geq 0} p_m^{(z)} e^{tm}\right) = \sum_{j \geq 1} \frac{\theta_j}{j} z^j (e^{tj} - 1). \tag{2.12}$$

Here, t satisfies $z e^t < R$. We recognize the Lévy-Khinchin representation for an infinitely divisible measure in the special case of a measure on \mathbb{N}_0 ; it follows from classical results (see e.g. Chapter 18 in [39]) that $(p_m^{(z)})$ is infinitely divisible if and only if $\theta_j \geq 0$ for all $j \geq 1$. If this is the case, the weights $\alpha_j^{(z)} = \frac{\theta_j}{j} z^j$ define a measure on \mathbb{N} , the Lévy measure of $(p_m^{(z)})$. \square

Another question of interest is the relation between the asymptotic behaviors of the sequences (θ_m) and (h_m) as $m \rightarrow \infty$. This has been investigated in the references cited at the beginning of this subsection. Relevant results can also be found in the probability literature on the relation between the tails of an infinitely divisible measure and its Lévy measure, e.g. in [30, 31]. Here we quote a result of Embrechts and Hawkes [30] on the tail equivalence of an infinitely divisible measure and its Lévy measure; equivalently, on the relation between the tails of (h_m) and (θ_j/j) . Recall that the convolution between two sequences is defined by $(a * b)_n = \sum_{j=0}^n a_j b_{n-j}$.

Theorem 2.4. (Embrechts and Hawkes [30]) *Let $(p_n)_{n \geq 0}$ define an infinitely divisible law on \mathbb{N}_0 with Lévy measure $(\alpha_j)_{j \geq 1}$. Suppose that $\alpha_j > 0$ for all $j \geq 1$. Let $\bar{\alpha}_j = \alpha_j / (\sum_{k \geq 1} \alpha_k)$. The following are equivalent as $n \rightarrow \infty$:*

- (i) $(p * p)_n = 2(1 + o(1))p_n$ and $p_{n+1}/p_n \rightarrow 1$.
- (ii) $(\bar{\alpha} * \bar{\alpha})_n = 2(1 + o(1))\bar{\alpha}_n$ and $\bar{\alpha}_{n+1}/\bar{\alpha}_n \rightarrow 1$.
- (iii) $p_n = (1 + o(1))\alpha_n$ and $\alpha_{n+1}/\alpha_n \rightarrow 1$.

We note that the Lévy measure of an integer-valued random variable has always finite mass, hence $\sum_{j \geq 1} \alpha_j < \infty$.

A probability measure on \mathbb{N} satisfying $(p * p)_n \sim 2p_n$ is called *discrete subexponential*. This property suggests that, if two independent variables are conditioned so their sum takes some big value, one variable will take a small value and the other variable will take the appropriate big value. Discrete subexponential variables are necessarily heavy-tailed, that is, $\sum_n p_n z^n$ has radius of convergence 1. We can apply this theorem if $\sum_j \theta_j/j < \infty$ and $\theta_{j+1}/\theta_j \rightarrow 1$. Let $p_m = h_m \exp(-\sum_{j \geq 1} \theta_j/j)$. The condition (ii) becomes

$$\sum_{j=1}^{n-1} \frac{\theta_j}{j} \frac{\theta_{n-j}}{n-j} = 2 \sum_{j=1}^{n/2} \frac{\theta_j}{j} \frac{\theta_{n-j}}{n-j} = 2(1 + o(1)) \frac{\theta_n}{n} \sum_{j \geq 1} \frac{\theta_j}{j}. \tag{2.13}$$

The first equality is valid if n is even, there is an unimportant correction in the case of n odd. An immediate consequence of Theorem 2.4 and of the dominated convergence theorem is the following.

Theorem 2.5. *Assume that $\theta_{n+1}/\theta_n \rightarrow 1$ and that $\theta_{n-j}/\theta_n \leq c_j$ for $1 \leq j \leq \frac{n}{2}$ and all n , with c_j satisfying $\sum \theta_j c_j/j < \infty$. Then, as $n \rightarrow \infty$,*

$$h_n = (1 + o(1)) \frac{\theta_n}{n} \exp\left(\sum_{j \geq 1} \frac{\theta_j}{j}\right).$$

Notice that c_j is necessarily greater or equal to 1, which requires $\sum_j \theta_j/j < \infty$. The theorem was actually proposed in [16] but the connection with infinitely divisible laws and with Theorem 2.4 had not been noticed. It applies in particular to stretched exponential weights, $\theta_j/j = \exp(-j^\gamma)$ with $0 < \gamma < 1$.

2.4 Gibbs partitions and Poisson random variables

We conclude this section by explaining in more details the probabilistic picture behind Proposition 2.1. To this aim we generalize a well-known relationship between Gibbs partitions and Poisson variables, see for example Eq. (1.53) in [49] or the *conditioning relation* (3.1) in [6].

We assume that there exists $z > 0$ such that for all $L > 0$,

$$\sum_{x \in \mathbb{Z}^d} \sum_{j \geq 1} \frac{\theta_j}{j} z^j e^{-jV(x/L)} < \infty. \tag{2.14}$$

Let $(\Omega, \mathcal{F}, \mathbb{P}_L^z)$ be a probability space and let $(R_{xj})_{x \in \mathbb{Z}^d, j \in \mathbb{N}}$ be a family of independent Poisson random variables,

$$R_{xj} \sim \text{Poiss}\left(\frac{\theta_j}{j} z^j e^{-jV(x/L)}\right). \tag{2.15}$$

\mathbb{P}_L^z is the *grand-canonical measure*. The occupation numbers are the random variables

$$H_x := \sum_{j \in \mathbb{N}} j R_{xj}, \quad R_j := \sum_{x \in \mathbb{Z}^d} R_{xj}. \tag{2.16}$$

We also define the total number of particles by

$$N = \sum_{x \in \mathbb{Z}^d} \sum_{j \in \mathbb{N}} j R_{xj} = \sum_{x \in \mathbb{Z}^d} H_x = \sum_{j \in \mathbb{N}} j R_j. \tag{2.17}$$

A moment of thought shows that the law $\mathbb{P}_{L,n}$ is recovered by conditioning on the event $\sum_{x,j} j R_{xj} = n$,

$$\mathbb{P}_{L,n}(\lambda) = \mathbb{P}_L^z\left(\forall x \in \mathbb{Z}^d \forall j \in \mathbb{N} : R_{xj} = r_{xj}(\lambda) \mid N = n\right), \tag{2.18}$$

and the normalization satisfies

$$z^n Z_{L,n} = \mathbb{P}_L^z(N = n) \times \exp\left(\sum_{x \in \mathbb{Z}^d} \sum_{j \in \mathbb{N}} \frac{\theta_j}{j} z^j e^{-jV(x/L)}\right). \tag{2.19}$$

Note that in Eq. (2.18) the right-hand side is independent of z ; this is related to the invariance of the measure $\mathbb{P}_{L,n}$ under rescalings $\theta_j \rightarrow \theta_j z^j$. Under the measure \mathbb{P}_L^z , the random variables R_j are independent Poisson variables

$$R_j \sim \text{Poiss}\left(\frac{\theta_j}{j} z^j \sum_{x \in \mathbb{Z}^d} e^{-jV(x/L)}\right), \tag{2.20}$$

and the H_x are independent variables with cumulant generating function

$$\log \mathbb{E}_L^z[e^{tH_x}] = \sum_{j \in \mathbb{N}} \frac{\theta_j}{j} z^j e^{-jV(x/L)} (e^{jt} - 1) \quad (t \in \mathbb{R}). \tag{2.21}$$

Put differently, H_x is an integer-valued, infinitely divisible random variable with Lévy measure $\nu(j) = (\theta_j/j)z^j \exp(-jV(x/L))$, compare with the proof of Proposition 2.3. Furthermore,

$$\mathbb{P}_L^z(H_x = m) = h_m z^m e^{-mV(x/L)} e^{-\sum_{j \geq 1} \frac{\theta_j}{j} z^j \exp(-jV(x/L))}, \tag{2.22}$$

with h_m as defined in Eq. (2.6). Indeed,

$$\begin{aligned} \mathbb{P}_L^z(H_x = m) &= \sum_{\lambda \vdash m} \prod_{j \geq 1} \mathbb{P}_L^z(R_{xj} = r_j(\lambda)) \\ &= \sum_{\lambda \vdash m} \prod_{j \geq 1} \left\{ \frac{1}{r_j(\lambda)!} \left(\frac{\theta_j}{j} z^j e^{-jV(\frac{x}{L})} \right)^{r_j(\lambda)} e^{-\frac{\theta_j}{j} z^j \exp(-jV(\frac{x}{L}))} \right\}. \end{aligned} \tag{2.23}$$

Proof of Proposition 2.1. For (a), we note that

$$\begin{aligned} \mathbb{P}_{L,n}(\pi_1^{-1}(\{\boldsymbol{\eta}\})) &= \mathbb{P}_L^z(\forall x \in \mathbb{Z}^d : H_x = \eta_x \mid N = n) \\ &= \frac{1}{\mathbb{P}_L^z(\sum_x H_x = n)} \prod_{x \in \mathbb{Z}^d} \mathbb{P}_L^z(H_x = \eta_x) \end{aligned} \tag{2.24}$$

and we conclude with Eqs. (2.22) and (2.19). For (b), we observe that

$$\begin{aligned} \mathbb{P}_{L,n}(\pi_2^{-1}(\{\boldsymbol{r}\})) &= \mathbb{P}_L^z(\forall j \in \mathbb{N} : R_j = r_j \mid N = n) \\ &= \frac{1}{\mathbb{P}_L^z(\sum_j j R_j = n)} \prod_{j \in \mathbb{N}} \mathbb{P}_L^z(R_j = r_j) \end{aligned} \tag{2.25}$$

and conclude with Eqs. (2.21) and (2.19). □

3 Relationship with existing models

Our setting is closely related to several models of interest, namely the ideal Bose gas, the zero-range process, particle clustering, coagulation-fragmentation, Becker-Döring, spatial permutations, and population genetics. The relations are explained in this section. Each situation comes with its own language; the keywords and their correspondence are summarized in Table 1.

Zero-range	particle	site	-
Chinese restaurant	customer	restaurant	table
ideal Bose gas, spatial permutation	particle	Fourier mode, energy level	cycle
particle clustering, nucleation	particle	site in space	cluster (droplet)
population genetics	individual	colony	same-allele group within colony

Table 1: Language of the different models and their correspondence.

3.1 Ideal Bose gas

Although it was not fully appreciated at the time, Bose-Einstein condensation is the first description of a phase transition in statistical mechanics. The ideal Bose gas is a quantum system whose description involves a complex Hilbert space and the Schrödinger equation; but its equilibrium state is a probability distribution of occupation numbers of Fourier modes. It fits our setting, by choosing $V(x) = \beta \|x\|^2$ and $\theta_j \equiv 1$; β is the inverse temperature. With the change of variables $k = \frac{2\pi}{L}x$, writing λ_{kj} instead of λ_{xj} , $\mathbb{P}_{L,n}$ becomes a probability measure on spatial partitions (λ_{kj}) . Summing over j and writing η_k instead of $\eta_x = \eta_{kL}$, we get the familiar probability measure on

occupation numbers

$$\mathbb{P}_{L,n}(\pi_1^{-1}(\{\boldsymbol{\eta}\})) = \frac{1}{Z_{L,n}} \prod_{k \in \frac{2\pi}{L}\mathbb{Z}^d} e^{-\beta \|k\|^2 \eta_k}. \quad (3.1)$$

Summing over k we get that the marginal $\mathbb{P}_{L,n} \circ \pi_2^{-1}$ is the probability distribution of cycle lengths associated with the ideal Bose gas [52, 10]. Thus $\mathbb{P}_{L,n}$ provides a coupling of the distribution of momenta and cycle lengths for the ideal Bose gas. This generalizes to independent bosons in a trap, when the weight $\exp(-\beta \sum_k \|k\|^2 \eta_k)$ is replaced with $\exp(-\beta \sum_{r \in \mathcal{I}} E_r \eta_r)$, where \mathcal{I} is a countable index set replacing \mathbb{Z}^d and E_r ($r \in \mathcal{I}$) are the eigenvalues of the Schrödinger operator in the trap. Results in this case were recently obtained in [20] (see also [18]).

The interacting Bose gas is much more complicated and it does not fit the present setting. But a partial mean-field approach for the dilute regime suggests that interactions can be approximated by cycle weights θ_j [14].

3.2 Zero-range process

This describes a system of classical particles with stochastic dynamics. There are n particles moving on the sites $\{1, \dots, L\}$ and we let $\boldsymbol{\eta} \in \mathcal{H}_n$ denote the occupations of the sites. The dynamics is as follows. A particle exits the site x at rate $g(\eta_x)$, where g is a given function $\mathbb{N} \rightarrow (0, \infty)$, and it chooses a new site uniformly at random among the neighbors. As it turns out, the spatial dimension does not appear in the stationary measure, so the model is usually studied in $d = 1$. The invariant measure is

$$\mathbb{P}_{L,n}(\boldsymbol{\eta}) = \frac{1}{Z_{L,n}} \prod_{x=1}^L h_{\eta_x}, \quad (3.2)$$

where the function h_k is related to the rates g by

$$h_k = \prod_{i=1}^k \frac{1}{g(i)}. \quad (3.3)$$

It fits the setting studied in this article by choosing the potential V such that $e^{-V(s)} = \chi_{[0,1]}(s)$; see Eq. (2.7).

It was noticed by Evans [36] that for certain rates g , the system possesses a critical density where a sort of Bose–Einstein condensation takes place. See also [41, 5, 4, 22] for further studies. Variants of the model allow for motion on graphs other than \mathbb{Z}^d and hopping mechanisms different from the simple random walk, see [36, 54, 40] and the references therein. When e^{-V} differs from the characteristic function, the marginal (2.7) is the stationary measure of an *inhomogeneous* zero-range process, compare with Eq. (2.4) of [40].

3.3 Particle clustering

The measure $\mathbb{P}_{L,n}$ describes approximately the droplet size distributions for a system of interacting particles in the canonical Gibbs ensemble, see [51, 43] and the references therein. Let v be a pair potential with a finite range $R > 0$. Thus particles at mutual distance larger than R do not interact. With each configuration $(x_1, \dots, x_n) \in [0, L]^{dn}$ we associate a graph by drawing a line between x_i, x_j whenever $|x_i - x_j| \leq R$, and we call $N_k(\mathbf{x})$ the number of connected components having k particles. We have $\sum_{k=1}^n k N_k(\mathbf{x}) = n$. We put on $[0, L]^{dn}$ the canonical Gibbs measure at inverse temperature $\beta > 0$. If we neglect the constraint that particles belonging to different connected components have mutual distance larger than R , the probability of seeing a

given droplet size distribution (N_k) becomes approximately

$$\prod_{k=1}^n \frac{1}{N_k!} \left(L^d Z_k(\beta) \right)^{N_k}, \tag{3.4}$$

where $Z_j(\beta)$ is a partition function over droplet-internal degrees of freedom [43]. Eq. (3.4) corresponds to a measure of the form (2.8) with $\sum_x \exp(-V(x/L))$ replaced by L^d , and $\theta_j/j = Z_j(\beta)$.

3.4 Coagulation-fragmentation processes and Becker–Döring equations

The Becker–Döring system of coupled ordinary differential equations [9] is a popular model for the dynamics of nucleation, with interesting long-time behavior [8]. A natural stochastic variant of the model is the following. Let $(a_j)_{j \geq 1}$ and $(b_j)_{j \geq 1}$ be positive numbers such that for all j , $\frac{\theta_1 \theta_j a_j}{j} = \frac{\theta_{j+1} b_{j+1}}{j+1}$. We define a continuous-time Markov chain with state space $\{\lambda : \lambda \vdash n\}$ and two types of transition:

- *Coagulation*: a monomer decides to join a j -cluster. The transition $(r_1, r_j, r_{j+1}) \rightarrow (r_1 - 1, r_j - 1, r_{j+1} + 1)$ happens at rate $a_j r_1 r_j / L^d$ if $j \neq 1$, and $a_1 r_1 (r_1 - 1) / L^2$ for $j = 1$.
- *Fragmentation*: a monomer decides to depart from a j -cluster, resulting in the transition $(r_1, r_{j-1}, r_j) \rightarrow (r_1 + 1, r_{j-1} + 1, r_j - 1)$. This happens at rate $b_j r_j$.

One can check through detailed balance that the marginal (2.8) with the square potential $\exp(-V(s)) = \mathbf{1}_{[0,1]}(s)$ is a stationary measure of this process. The dynamics is a stochastic version of the Becker–Döring equations much in the same way as the Marcus–Lushnikov coalescent is a stochastic version of the Smoluchowski coagulation equations [2]. The model can be easily generalized to allow for joining and departure of groups of size larger than one, and falls into the class of well-studied coagulation-fragmentation processes, see e.g. [29, 12]. In Section 4.2 we propose a “spatial” version of the process for which our measure $\mathbb{P}_{L,n}$ is stationary.

3.5 Spatial permutations

Models of spatial permutations are motivated by Feynman’s approach to the Bose gas and by Sütő’s work on cycles [52]. A more general framework was proposed in [13], which was studied further in [15, 17]. Spatial permutations involve a distribution jointly over points in \mathbb{R}^d and over permutations of these points, with penalties that discourage long jumps. More precisely, the probability space is $\Lambda^n \times \mathcal{S}_n$, where Λ is the box $[0, L]^d$ (with periodic boundary conditions), n is the number of points, and \mathcal{S}_n is the group of permutations of n elements. Let $\xi : \mathbb{R}^d \rightarrow [0, \infty]$ and define

$$Z_{L,n} = \sum_{\sigma \in \mathcal{S}_n} \int_{\Lambda^n} dx_1 \dots dx_n \prod_{j=1}^n e^{-\xi(x_j - x_{\sigma(j)})}. \tag{3.5}$$

The probability element for having points x_1, \dots, x_n and permutation σ is then

$$\frac{1}{Z_{L,n}} \left(\prod_{j=1}^n e^{-\xi(x_j - x_{\sigma(j)})} \right) dx_1 \dots dx_n. \tag{3.6}$$

The marginal obtained after summing over permutations is a permanent point process, which we do not discuss here despite interest of its own. We rather focus on properties of permutation cycles. We make the extra assumption that $e^{-\xi}$ has non-negative Fourier transform, which we write e^{-V} , with V a real function on \mathbb{R}^d . Then

the marginal over the cycle lengths, after integrating over positions and summing over compatible permutations, is precisely the marginal (2.8). This follows from the Poisson summation formula, see [13] for details. Notice that the special case $\xi(x) = \beta\|x\|^2$ corresponds to the homogeneous ideal Bose gas described in Section 3.1.

3.6 Population genetics

Consider n individuals that carry different alleles of a given gene, and live in different locations or *colonies* x . Inside each colony, we group individuals that carry the same allele. This gives rise to a spatial partition. In the Ewens case $\theta_j \equiv \theta$, one might imagine that our measure is stationary for a population model that includes migration (with some colonies possibly more attractive than others) and mutation. Remember that the Ewens sampling formula appears naturally in the infinite alleles model with mutation rate θ ; see [28] and references therein for more background.

4 Stochastic processes

Here we propose two continuous-time Markov processes with state space Λ_n that have $\mathbb{P}_{L,n}$ as a reversible (hence stationary) measure. The first process combines the zero-range process with the Chinese restaurant process; it has the nice structural property that the vector of site occupation numbers $\eta(\lambda(t))$ is Markovian (regardless of the starting point $\lambda(0)$) and evolves as the zero-range process. However we are able to define the Chinese restaurant part only when the weights θ_j are j -independent. For non-constant θ_j 's, we replace the Chinese restaurant step by instant reshuffling. The second process is a coagulation-fragmentation process that is very natural from the point of view of the Becker–Döring model of nucleation explained in Section 3.4.

4.1 Chain of Chinese restaurants

Recall that, in the Chinese restaurant process, customers enter the restaurant one by one. The $(n + 1)$ th customer sits next to an existing customer with probability $\frac{1}{n+\theta}$, and starts a new table with probability $\frac{\theta}{n+\theta}$. The table occupation is a random partition and the distribution after n customers is given by the Ewens measure. That is, take $\theta_j \equiv \theta$ in Eq. (2.9). We refer to Chapter 3 in [49] for background and details.

We adapt the dynamics to spatial partitions as follows. To each site $x \in \mathbb{Z}^d$ is associated a restaurant. The total number of customers is n and it is conserved. Let η_x denote the number of customers at site x , and $\lambda_x \vdash \eta_x$ denote the table occupation. We consider a continuous-time Markov dynamics where

- A customer exits restaurant x at rate $g(\eta_x)$; he is chosen uniformly among the η_x customers at x .
- The new restaurant y is chosen with probability $t(x, y)$. It may depend on L . We assume that $\sum_y t(x, y) = 1$, and that

$$e^{-V(x/L)} t(x, y) = e^{-V(y/L)} t(y, x). \tag{4.1}$$

- In the restaurant y , the new customer sits next to another customer with probability $\frac{1}{\eta_y + \theta}$, and at an empty table with probability $\frac{\theta}{\eta_y + \theta}$.

With h_n defined in (2.6), we take the rate to be $g(n) = h_{n-1}/h_n$ (with $g(0) = 0$). One can check that $h_n = \theta(\theta + 1) \dots (\theta + n - 1)/n!$ and that $g(n) = \frac{n}{\theta + n - 1}$. A possible choice for $t(x, y)$ is $e^{-V(y/L)} / \sum_z e^{-V(z/L)}$ (with $y = x$ being allowed).

It is clear that this defines a Markov process on spatial partitions. Let $q(\lambda, \lambda')$ denote the rate of the transition $\lambda \mapsto \lambda'$. In order to write it explicitly, observe that it is zero

unless there exist $x, y \in \mathbb{Z}^d$ and $j, k \in \mathbb{N}$ (with $k \neq 0$) such that

$$\begin{aligned} r_{xj}(\boldsymbol{\lambda}') &= r_{xj}(\boldsymbol{\lambda}) - 1, & r_{y,k+1}(\boldsymbol{\lambda}') &= r_{yk}(\boldsymbol{\lambda}) - 1, \\ r_{x,j-1}(\boldsymbol{\lambda}') &= r_{x,j-1}(\boldsymbol{\lambda}) + 1, & r_{y,k}(\boldsymbol{\lambda}') &= r_{yk}(\boldsymbol{\lambda}) - 1 \end{aligned} \tag{4.2}$$

and all other occupation numbers are r_{zi} unchanged. In this case, the rate is

$$q(\boldsymbol{\lambda}, \boldsymbol{\lambda}') = g(\eta_x(\boldsymbol{\lambda})) t(x, y) p_-(j, \lambda_x) p_+(k, \lambda_y). \tag{4.3}$$

Here, the probability that the customer leaving restaurant x leaves a j -table is

$$p_-(j; \lambda_x) = \frac{j r_{xj}}{\eta_x} \tag{4.4}$$

and the probability that the customer joins a k -table ($k \geq 1$) or opens a new table ($k = 0$) is

$$p_+(k; \lambda_y) = \frac{k r_{yk}}{\eta_y + \theta}, \quad p_+(0; \lambda_y) = \frac{\theta}{\eta_x + \theta}. \tag{4.5}$$

The stationary measure of this process is precisely the measure defined in Eq. (2.5). Indeed, it satisfies the detailed balance condition.

Lemma 4.1. *The rate of Eq. (4.3) satisfies the detailed balance condition*

$$\mathbb{P}_{L,n}(\boldsymbol{\lambda}) q(\boldsymbol{\lambda}, \boldsymbol{\lambda}') = \mathbb{P}_{L,n}(\boldsymbol{\lambda}') q(\boldsymbol{\lambda}', \boldsymbol{\lambda}).$$

Proof. We only need to consider the case $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$. When the customer changes restaurants, we have to check that

$$\begin{aligned} \mathbb{P}_{L,n}(\boldsymbol{\lambda}) g(\eta_x(\boldsymbol{\lambda})) t(x, y) p_-(j; \lambda_x) p_+(k; \lambda_y) \\ = \mathbb{P}_{L,n}(\boldsymbol{\lambda}') g(\eta_y(\boldsymbol{\lambda}')) t(y, x) p_+(k+1; \lambda'_y) p_-(j-1; \lambda'_x). \end{aligned} \tag{4.6}$$

Using (4.1), one can check that for all $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$ and all $\boldsymbol{\eta}$, we have the identities

$$\begin{aligned} \mathbb{P}_{L,n}(\pi_1^{-1}(\{\boldsymbol{\eta}\})) g(\eta_x) t(x, y) &= \mathbb{P}_{L,n}(\pi_1^{-1}(\{\boldsymbol{\eta} - \delta_x + \delta_y\})) g(\eta_y + 1) t(y, x), \\ \nu_{\eta_x}(\lambda_x) p_-(j; \lambda_x) &= \nu_{\eta_x-1}(\lambda'_x) p_+(j-1; \lambda'_x), \\ \nu_{\eta_y}(\lambda_y) p_+(k; \lambda_y) &= \nu_{\eta_y+1}(\lambda'_y) p_-(k+1; \lambda'_y). \end{aligned} \tag{4.7}$$

The detailed balance property now follows from Proposition 2.2. □

Since the state space Λ_n is infinite, we need to check that the Markov process is non-explosive; to this aim we need the following lemma.

Lemma 4.2. *We have*

$$\sum_{\substack{\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda_n \\ \boldsymbol{\lambda} \neq \boldsymbol{\lambda}'}} \mathbb{P}_{L,n}(\boldsymbol{\lambda}) q(\boldsymbol{\lambda}, \boldsymbol{\lambda}') < \infty. \tag{4.8}$$

Proof. We have

$$\begin{aligned} \sum_{\boldsymbol{\lambda}'} q(\boldsymbol{\lambda}, \boldsymbol{\lambda}') &= \sum_{x,y \in \mathbb{Z}^d} \sum_{j,k \in \mathbb{N}} g(\eta_x(\boldsymbol{\lambda})) t(x, y) p_-(j, \lambda_x) p_+(k, \lambda_y) \\ &= \sum_x g(\eta_x(\boldsymbol{\lambda})) \\ &= \sum_x \frac{\eta_x}{\theta + \eta_x - 1} \\ &\leq \frac{n}{\theta}. \end{aligned} \tag{4.9}$$

□

Define $q(\lambda, \lambda) = -\sum_{\lambda' \in \Lambda_n, \lambda' \neq \lambda} q(\lambda, \lambda')$ and $Q = (q(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda_n}$. For later purpose we note

$$q(\lambda, \lambda) = -g(\eta_x(\lambda)) \sum_{y \neq x} t(x, y) - g(\eta_x(\lambda)) t(x, x) \sum_{k \neq j} p_+(k; \lambda_x^{-j}). \tag{4.10}$$

Consider the backward Kolmogorov equations

$$\frac{\partial P_t}{\partial t}(\lambda, \lambda') = (QP_t)(\lambda, \lambda') = \sum_{\lambda'' \in \Lambda_n} q(\lambda, \lambda'') P_t(\lambda'', \lambda'), \tag{4.11}$$

with the standard initial condition $P_0(\lambda, \lambda') = \delta_{\lambda, \lambda'}$.

Proposition 4.3. *There is a unique Markov semi-group $(P_t(\lambda, \lambda'))_{t \geq 0}$ solving the backwards Kolmogorov equations for the rates (4.3). It defines a Markov process $(\lambda(t))_{t \geq 0}$ with state space Λ_n , which has $\mathbb{P}_{L,n}$ as a reversible (hence stationary) measure.*

Note that (P_t) defines a strongly continuous contraction semi-group in $\ell^\infty(\Lambda_n)$ with infinitesimal generator Q : $P_t = \exp(tQ)$.

Proof. This follows from Lemmas 4.1 and 4.2, and the non-explosion criterion given in [21], Corollary 3.8. □

Proposition 4.4. *Let $(\lambda(t))_{t \geq 0}$ be the Markov process from Proposition 4.3, with arbitrary initial law. Then the marginal $(\eta(\lambda(t)))_{t \geq 0}$ is a Markov process with transitions $(\eta_x, \eta_y) \rightarrow (\eta_x - 1, \eta_y + 1)$ ($x \neq y$, all other occupation numbers unchanged) happening at rate $g(\eta_x)p(x, y)$.*

Hence the site occupation numbers evolve according to a (possibly inhomogeneous) zero-range process; the total rate for a customer to leave a restaurant is $g(\eta_x)(1 - p(x, x))$.

Proof. Observe

$$\sum_{\lambda' \vdash \eta'} q(\lambda, \lambda') = g(\eta_x(\lambda)) t(x, y) \sum_{j,k} p_-(j; \lambda_x) p_+(k; \lambda_y) = g(\eta_x(\lambda)) t(x, y) \tag{4.12}$$

if $\eta' = \eta(\lambda) - \delta_x + \delta_y$, and in view of Eq. (4.10),

$$\sum_{\lambda \vdash \eta} q(\lambda, \lambda) = g(x, \eta_x) t(x, x) \sum_{k \neq j} p_+(k; \lambda_x^{-j}) + q(\lambda, \lambda) = -g(x, \eta_x) \sum_{y \neq x} t(x, y). \tag{4.13}$$

Let $\tilde{q}(\eta, \eta') := g(\eta_x) t(x, y)$ if $\eta' = \eta - \delta_x + \delta_y$ for some $x, y \in \mathbb{Z}^d$ with $x \neq y$, $\tilde{q}(\eta, \eta') = 0$ if $\eta' \neq \eta$ is not of the form just described, $\tilde{q}(\eta, \eta) := 1 - \sum_{\eta' \in \mathcal{N}_n, \eta' \neq \eta} \tilde{q}(\eta, \eta')$, and $\tilde{Q} := (\tilde{q}(\eta, \eta'))_{\eta, \eta' \in \mathcal{N}_n}$. An argument similar to the one used in the proof of Proposition 4.3 shows that $\tilde{P}_t := \exp(t\tilde{Q})$ uniquely defines a Markov semi-group on \mathcal{N}_n , which is an inhomogeneous zero-range process. Eqs. (4.12) and (4.13) become

$$\sum_{\lambda' \vdash \eta'} q(\lambda, \lambda') = \tilde{q}(\eta(\lambda), \eta') \tag{4.14}$$

for all $\lambda \in \Lambda_n, \eta' \in \mathcal{N}_n$. The latter identity can be conveniently recast in operator language: set $R(\lambda, \eta) := \mathbf{1}_{\{\lambda \vdash \eta\}} = \delta_{\eta, \eta(\lambda)}$. We have $QR = R\tilde{Q}$, hence $P_t R = R\tilde{P}_t$ for all $t \geq 0$, i.e.,

$$\sum_{\lambda' \vdash \eta'} P_t(\lambda, \lambda') = \tilde{P}_t(\eta(\lambda), \eta'). \tag{4.15}$$

This condition is sufficient to ensure that $\eta(\lambda(t))_{t \geq 0}$ is Markovian with transition rates $\tilde{q}(\lambda, \lambda')$, see Theorem 4 in [19] (the theorem is formulated for finite state spaces but holds in our set-up as well). □

Thus we have constructed a process combining the zero-range and Chinese restaurant process for which the measures $\mathbb{P}_{L,n}$ are reversible. Finally, let us address the non-Ewens case where the parameters (θ_j) depend on j . There exist Markov processes such that the measure (2.5) is stationary. For instance, take a zero-range process on the (η_x) , together with an “instant reshuffling” of the partitions at the two locations where a customer has been lost or gained. The transition rate of the process is

$$g(\eta_x) t(x, y) \nu_{\eta_x-1}(\lambda'_x) \nu_{\eta_y+1}(\lambda'_y).$$

One can check that Propositions 4.3 and 4.4 remain true. It would be nice to discover other, more natural Markov processes.

4.2 Coagulation-fragmentation processes

For our second process, let $(a_j)_{j \geq 1}$ and $(b_j)_{j \geq 2}$ be positive sequences such that for all j ,

$$a_j \frac{\theta_j}{j} \theta_1 = b_{j+1} \frac{\theta_{j+1}}{j+1}. \tag{4.16}$$

In addition to (4.1), we assume that for some $z > 0$ satisfying Eq. (2.14), we have

$$\sum_{x,y \in \mathbb{Z}^d} e^{-V(x/L)} t(x, y) < \infty, \quad \sum_{x \in \mathbb{Z}^d, j \geq 2} b_j \frac{\theta_j}{j} z^j e^{-jV(x/L)} < \infty. \tag{4.17}$$

As the process presented here is a generalization of the stochastic version of the Becker-Döring model presented in Sec. 3.4, we use the vocabulary of nucleation/particle clustering and we speak of particles and clusters (or droplets) instead of customers or restaurants. *Monomers* are clusters of size 1. We allow three types of transitions:

- *Coagulation* at a given site x : a j -cluster ($j \geq 1$) and a 1-cluster coagulate to a $(j+1)$ -cluster, i.e., $(r_{x,j-1}, r_{j,x}, r_{x,j+1}) \rightarrow (r_{x,j-1}-1, r_{x,j}-1, r_{x,j+1}+1)$. This occurs at rate $a_j r_{xj}(\boldsymbol{\lambda}) r_{x1}(\boldsymbol{\lambda})$ if $j \neq 1$, and at rate $a_1 r_{x1}(\boldsymbol{\lambda})(r_{x1}(\boldsymbol{\lambda}) - 1)$ if $j = 1$.
- *Fragmentation* at a given site x : a monomer departs from a j -cluster ($j \geq 2$). The transition $(r_{x1}, r_{x,j-1}, r_{x,j}) \rightarrow (r_{x1}+1, r_{x,j-1}+1, r_{x,j}-1)$ occurs at rate $b_j r_{xj}(\boldsymbol{\lambda})$.
- *Jump of a monomer* from site x to site $y \neq x$: the transition $(r_{x1}, r_{y1}) \rightarrow (r_{x1}-1, r_{y1}+1)$ occurs at rate $r_{x1}(\boldsymbol{\lambda}) t(x, y)$.

There are many possible generalizations: for example, we could allow for the jumping of whole clusters rather than only monomers, or allow two groups of size larger than one to coalesce, or take coagulation-fragmentation rates proportional to powers of the occupation numbers r_{xj}^γ , see [29]. We stick to the process presented here for notational simplicity, and also because it is closest in spirit to the original Becker-Döring model of nucleation [9] set in the framework of kinetic gas theory: monomers are mobile particles of a dilute gas, clusters of size $j \geq 2$ are small chunks of condensed precipitate and they are deemed immobile (or at least very slow compared to gas particles).

Proposition 4.5. *The transition rates specified above define uniquely a Markov semigroup with state space Λ_n . The measure $\mathbb{P}_{L,n}$ is reversible, hence stationary, for the associated Markov process.*

Proof. We leave the proof of the detailed balance conditions to the reader and look at the non-explosion criterion from Lemma 4.2. Let $z > 0$ be such that Eq. (2.14) holds. The contribution to $\sum_{\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'} \mathbb{P}_L^z(\boldsymbol{\lambda}) q(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ from coagulation of j -clusters ($j \neq 1$) with

monomers is estimated as

$$\begin{aligned} \sum_{\lambda \in \cup_n \Lambda_n} \mathbb{P}_L^z(\lambda) \sum_{x \in \mathbb{Z}^d, j \in \mathbb{N}} a_j r_{xj}(\lambda) r_{x1}(\lambda) &= \sum_{x \in \mathbb{Z}^d, j \in \mathbb{N}} a_j \left(\theta_1 z e^{-V(x/L)} \right) \left(\frac{\theta_j}{j} z^j e^{-jV(x/L)} \right) \\ &= \sum_{x \in \mathbb{Z}^d, j \geq 1} b_{j+1} \frac{\theta_{j+1}}{j+1} z^{j+1} e^{-(j+1)V(x/L)} < \infty. \end{aligned} \quad (4.18)$$

We have used that under \mathbb{P}_L^z the R_{xj} s are independent Poisson variables, and we have used the assumption (4.16). The contributions of coagulation of monomers, fragmentation, and random walk can be evaluated in a similar way. We conclude as in Lemma 4.2 and Proposition 4.3. \square

It is natural to ask about the evolution of the marginals $\eta(\lambda(t))$ and $r(\lambda(t))$. Let us first look at the site occupation numbers. The rate for a particle to leave a site depends on the number of monomers at this site, i.e., it is not a function of the total number of particles at the site alone. Therefore $\eta(\lambda(t))$ is not Markovian. However, it is natural to approximate the process by replacing the jump rate $r_{x1}t(x, y)$ by the conditional expectation

$$t(x, y) \mathbb{E}_{L, n}(r_{x1}(\lambda) | \eta_x(\lambda) = \eta_x) = t(x, y) \sum_{r_{x1} \geq 1} r_{x1} \nu_{\eta_x}(r_{x1}) = \frac{\theta_1 h_{\eta_x - 1}}{h_{\eta_x}} t(x, y), \quad (4.19)$$

which is the transition rate for a zero-range process (for the last equality, see Proposition 2.1 in [33]). Now look at the cluster size counts $r_j = \sum_x r_{xj}$. Remember that the conditional law of r_{xj} is binomial with r_j trials and success probability $p_{xj} \propto \exp(-jV(x/L))$. Again, $r(\lambda(t))_{t \geq 0}$ will not be Markovian, but it is natural to approximate it by a process with effective coagulation rate

$$\sum_{x \in \mathbb{Z}^d} a_j \mathbb{E}_{L, n} [r_{x1}(\lambda) r_{xj}(\lambda) | r_1(\lambda) = r_1, r_j(\lambda) = r_j] = a_j r_1 r_j \sum_{x \in \mathbb{Z}^d} p_{x1} p_{xj}, \quad (4.20)$$

for $j \neq 1$. Similar expressions hold for coagulation of two monomers and for the effective fragmentation rate. In the case of square traps, $\exp(-V(x/L)) = \mathbf{1}_{[0, L]^d}(x)$ with $L \in \mathbb{N}$, we have $p_{xj} = L^{-d} \mathbf{1}_{[0, L]^d}(x)$ and the effective rates are exactly those of the Becker–Döring process from Section 3.4.

5 Condensation

We study the asymptotic behavior in the thermodynamic limit $n \rightarrow \infty, L \rightarrow \infty$ at fixed density $\rho = n/L^d$. The model may undergo a *phase transition* when the density varies. Above a certain *transition density* ρ_c , a *condensation* occurs that is characterized by the presence of large components.

There are two natural definitions of condensation, one in terms of site occupation numbers (number of customers of a restaurant), used in the zero-range process, and the other in terms of random partitions (number of occupants of a table). We shall actually observe that both definitions are equivalent. The results of this section can be found in Theorems 5.2 and 5.3. The expressions are familiar for the zero-range process and for spatial permutations. The motivation of this section is to present a novel, unified perspective.

Condensation is measured by the following “order parameters”:

$$\nu_\infty = \lim_{K \rightarrow \infty} \lim_{\substack{n \rightarrow \infty \\ \rho L^d = n}} \mathbb{E}_{L,n} \left(\frac{1}{n} \sum_{j > K} j R_j \right) \tag{5.1}$$

$$\mu_\infty = \lim_{K \rightarrow \infty} \lim_{\substack{n \rightarrow \infty \\ \rho L^d = n}} \mathbb{E}_{L,n} \left(\frac{1}{n} \sum_{x: \eta_x > K} \eta_x(\lambda) \right). \tag{5.2}$$

(The existence of the limits over n is proved in the proof of Theorem 5.3 below. The existence of the limits over K is guaranteed by monotonicity.) We say that condensation occurs if $\mu_\infty > 0$ or $\nu_\infty > 0$, and refer to μ_∞ or ν_∞ as the *condensate fraction*. In principle, it might happen that $\nu_\infty < \mu_\infty$. But Theorem 5.2 below shows that $\nu_\infty = \mu_\infty$; the two definitions of condensation are equivalent.

Under some assumptions, we prove the existence of a transition density ρ_c such that the order parameters are zero for $\rho \leq \rho_c$ and positive for $\rho > \rho_c$. In order to define the transition density, let

$$\rho(z) = \sum_{j \geq 1} \theta_j z^j \int_{\mathbb{R}^d} e^{-jV(x)} dx. \tag{5.3}$$

Let $z_c \in (0, \infty]$ be the radius of convergence of the power series above. The *transition density* is

$$\rho_c = \rho(z_c). \tag{5.4}$$

It is possible that ρ_c is infinite, meaning that no transition takes place. In the case $z_c = 1$ we get

$$\rho_c = \sum_{j \geq 1} \theta_j \int_{\mathbb{R}^d} e^{-jV(x)} dx. \tag{5.5}$$

The transition density may be finite for two reasons (or a combination of both). First, a trap such as $V(x) = \|x\|^\delta$ yields $\int e^{-jV} \sim j^{-d/\delta}$, which is summable if $\delta < d$. This is the case of the ideal Bose gas of Section 3.1, where $\theta_j \equiv 1$ and $\delta = 2$. Eq. (5.5) is then the famous formula derived by Einstein in 1924. Second, the parameters θ_j may be summable. This is the case of the invariant measure of the zero-range process of Section 3.2 in the regime studied by Evans [36]. There is no confining potential here, $\int e^{-jV} \equiv 1$. A very different situation is particle clustering, see Section 3.3, where the parameters θ_j are given by certain cluster integrals [44]. The general form of (5.3) with $\theta_j \neq 1$ and $V \neq 0$ appeared in the context of spatial random permutations with cycle weights [14, 15] discussed in Section 3.5. A variant of the alternative expression (5.3) for zero-range processes with disorder can be found e.g. in [40] (Eq. (4.2)).

If we are given parameters h_n instead of the θ_j , we can consider the alternative expression

$$\rho(z) = \int_{\mathbb{R}^d} \frac{\sum_{n \geq 1} n h_n z^n e^{-nV(x)}}{\sum_{n \geq 0} h_n z^n e^{-nV(x)}} dx, \tag{5.6}$$

where the integrand is by definition equal to 0 when $V(x) = \infty$. Indeed, the right side of Eq. (5.6) is equal to

$$\int_{\mathbb{R}^d} z \frac{\partial}{\partial z} \log \left(\sum_{k \geq 0} h_k z^k e^{-kV(x)} \right) dx,$$

and using (2.11) with $z \mapsto z e^{-V(x)}$ we get $\rho(z)$ in (5.3). The expression (5.6) is the density-activity series for the (inhomogeneous) zero-range processes.

We make the following assumptions throughout this section.

Assumption 5.1. Assume that $V : \mathbb{R}^d \rightarrow [0, \infty]$ is continuous at 0, with $V(0) = 0$, and that for every $j \geq 1$, we have

$$\frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} e^{-jV(x/L)} \rightarrow \int_{\mathbb{R}^d} e^{-jV(x)} dx$$

as $L \rightarrow \infty$.

It follows from this assumption that the series $\sum_{j \geq 1} \frac{\theta_j}{j} z^j$ has the same radius of convergence z_c as the series (5.3) for $\rho(z)$. The next theorem claims that ρ_c is indeed the transition density.

Theorem 5.2 (Condensation). Under Assumption 5.1, we have

$$\nu_\infty = \mu_\infty = \max\left(0, 1 - \frac{\rho_c}{\rho}\right).$$

The proof of this theorem can be found immediately after Theorem 5.3 below.

It is possible to provide more detailed results about “typical sizes” of components. Choose a customer at random, and consider the size of the element of the partition that he belongs to. Informally, we have

$$\text{Prob}(\text{typical size of element is } j) = \mathbb{E}_{L,n} \left(\frac{jR_j}{n} \right). \tag{5.7}$$

Indeed, there are jR_j customers in tables with j people. Similarly, let N_k denote the number of restaurants with k people:

$$N_k(\omega) = \#\{x \in \mathbb{Z}^d : H_x(\omega) = k\}. \tag{5.8}$$

The probability that a random customer finds himself in a restaurant with k people is

$$\text{Prob}(\text{typical site occupation is } k) = \mathbb{E}_{L,n} \left(\frac{N_k}{n} \right). \tag{5.9}$$

The next theorem gives the asymptotic behavior of typical sizes. In order to state it, we introduce $z_0(\rho)$ as the unique solution of the equation $\rho(z) = \rho$ when $\rho < \rho_c$; we set $z_0(\rho) = z_c$ when $\rho \geq \rho_c$.

Theorem 5.3 (Typical sizes). Under Assumption 5.1, we have for all $j \geq 1$ and $k \geq 0$,

- (a) $\frac{jR_j}{n} \xrightarrow{p} \frac{1}{\rho} \theta_j z_0(\rho)^j \int_{\mathbb{R}^d} e^{-jV(x)} dx,$
- (b) $\frac{N_k}{n} \xrightarrow{p} \frac{1}{\rho} \int_{\mathbb{R}^d} \frac{k h_k z_0(\rho)^k \exp(-kV(x))}{\sum_{j \geq 0} h_j z_0(\rho)^j \exp(-jV(x))} dx.$

Here, \xrightarrow{p} denotes the convergence in probability as $n, L \rightarrow \infty$ with $\rho L^d = n$.

The proof of this theorem can be found at the end of the section. We first use it to prove Theorem 5.2.

Proof of Theorem 5.2. We have

$$\begin{aligned} \nu_\infty &= 1 - \lim_{K \rightarrow \infty} \lim_{L, n \rightarrow \infty} \sum_{j=1}^K \mathbb{E}_{L,n} \left(\frac{jR_j}{n} \right) \\ &= 1 - \frac{1}{\rho} \sum_{j \geq 1} \theta_j z_0(\rho)^j \int_{\mathbb{R}^d} e^{-jV(x)} dx \end{aligned} \tag{5.10}$$

and the definitions of ρ_c and $z_0(\rho)$ imply that $\nu_\infty = \max\left(0, 1 - \frac{\rho_c}{\rho}\right)$. The proof for μ_∞ is similar and is based on the alternative formula (5.6) for $\rho(z)$. \square

The rest of this section is devoted to the proof of Theorem 5.3. The method relies on the conditioning relation (2.18); we first prove limit laws for the grand-canonical measures P_L^z and then show that the conditioning merely picks the right activity $z = z_0(\rho)$. Technicalities arise because the grand-canonical probability to have exactly n particles goes to 0, and because the power series at finite L need not converge at $z = z_c$.

Lemma 5.4 below and the limit law for the R_j 's are closely related to large deviations results [11, 52, 10, 41, 44]; we provide a complete proof because our setting is more general.

Lemma 5.4.

$$\lim_{\substack{n \rightarrow \infty \\ \rho L^d = n}} \frac{1}{n} \log Z_{L,n} = \frac{1}{\rho} \sum_{j \geq 1} \frac{\theta_j}{j} z_0(\rho)^j \int_{\mathbb{R}^d} e^{-jV(x)} dx - \log z_0(\rho).$$

Proof. Using Eq. (2.19) but neglecting the probability term, we get

$$\frac{1}{L^d} \log Z_{L,n} \leq \sum_{j \geq 1} \frac{\theta_j}{j} z^j \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} e^{-jV(x/L)} - \rho \log z. \tag{5.11}$$

For any $z < z_c$, we have by dominated convergence

$$\limsup_{\substack{n \rightarrow \infty \\ \rho L^d = n}} \frac{1}{L^d} \log Z_{L,n} \leq \sum_{j \geq 1} \frac{\theta_j}{j} z^j \int_{\mathbb{R}^d} e^{-jV(x)} dx - \rho \log z. \tag{5.12}$$

The bound extends to $z = z_c$ by continuity. The equation of Lemma 5.4 holds at least as an upper bound.

For the lower bound, we start with Proposition 2.1 (b). We have

$$Z_{L,n} \geq \prod_{j=1}^n \frac{1}{r_j!} \left(\frac{\theta_j}{j} \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} e^{-jV(x/L)} \right)^{r_j} \tag{5.13}$$

for any choice of non-negative integers (r_j) such that $\sum j r_j = n$. Let us introduce the following entropy function on sequences $\mathbf{a} = (a_j)$ such that $a_j \geq 0$ and $\sum_j j a_j \leq 1$:

$$I(\mathbf{a}) = - \sum_{j \geq 1} a_j \log \left(\frac{e \theta_j}{j a_j \rho} \int_{\mathbb{R}^d} e^{-jV(x)} dx \right). \tag{5.14}$$

We have

$$\frac{1}{n} \log Z_{L,n} \geq -I\left(\frac{1}{n} r_1, \frac{1}{n} r_2, \dots\right) + \sum_{j \geq 1} \frac{r_j}{n} \log \frac{\frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} e^{-jV(x/L)}}{\int_{\mathbb{R}^d} e^{-jV(x)} dx} - \frac{1}{n} \sum_{j=1}^n \log \frac{r_j!}{(r_j/e)^{r_j}}. \tag{5.15}$$

Let a_j be equal to the right side of Theorem 5.3 (a). We choose a sequence $\mathbf{r}^{(n)}$ such that $\sum j r_j^{(n)} = n$, $\frac{1}{n} r_j^{(n)} \leq 2a_j$, and $\frac{1}{n} r_j^{(n)} \rightarrow a_j$ as $n \rightarrow \infty$. By continuity of V around 0, we have

$$\begin{aligned} \text{const } j^{-d} &\leq \int e^{-jV(x)} dx \leq \int e^{-V(x)} dx, \\ \text{const } j^{-d} &\leq \frac{1}{L^d} \sum_x e^{-jV(x/L)} \leq \frac{1}{L^d} \sum_x e^{-V(x/L)} \leq \text{const}, \end{aligned} \tag{5.16}$$

where the constants do not depend on L, j . The logarithm in the first sum in Eq. (5.15) is less than a constant times j , so the summand is less than a constant times $j a_j$. We can

then use the dominated convergence theorem and the first sum in Eq. (5.15) vanishes in the limit $n \rightarrow \infty$. The last sum also vanishes: Using $1 \leq \frac{r_j!}{(r_j/e)^{r_j}} \leq Cr_j$, we have for any K

$$\frac{1}{n} \sum_{j=1}^K \log \frac{r_j!}{(r_j/e)^{r_j}} \leq \frac{1}{n} \sum_{j=1}^K \log Cn, \tag{5.17}$$

which goes to 0 as $n \rightarrow \infty$. Further, using $\log s \leq s$,

$$\frac{1}{n} \sum_{j=K+1}^n \log \frac{r_j!}{(r_j/e)^{r_j}} \leq \frac{1}{n} \sum_{j=K+1}^n \log Cr_j \leq \frac{C}{K} \sum_{j=K+1}^n \frac{jr_j}{n} \leq \frac{C}{K}. \tag{5.18}$$

This is arbitrarily small by choosing K large enough. Finally, recall that I is continuous on the set of non-negative sequences (b_j) satisfying $\sum j b_j \leq 1$ [8]. Then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log Z_{L,n} \geq -I(\mathbf{a}), \tag{5.19}$$

and a calculation shows that $-I(\mathbf{a})$ is equal to the right side of the expression in Lemma 5.4. \square

Recall the definition of \mathbb{E}_L^z given in Section 2.4.

Lemma 5.5. *For any $z < z_c$, we have*

- (a) $\lim_{L \rightarrow \infty} \mathbb{E}_L^z \left(\frac{jR_j}{L^d} \right) = \theta_j z^j \int_{\mathbb{R}^d} e^{-jV(x)} dx.$
- (b) $\lim_{L \rightarrow \infty} \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) = \int_{\mathbb{R}^d} \frac{h_k z^k e^{-kV(x)}}{\sum_{j \geq 1} h_j z^j e^{-jV(x)}} dx.$

Proof. We have

$$\mathbb{E}_L^z \left(\frac{jR_j}{L^d} \right) = \theta_j z^j \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} e^{-jV(x)}, \tag{5.20}$$

and the limit (a) exists by Assumption 5.1. For the limit (b), we have

$$\begin{aligned} \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) &= \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} \frac{h_k z^k e^{-kV(x/L)}}{\sum_j h_j z^j e^{-jV(x/L)}} \\ &= \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} h_k z^k e^{-kV(x/L)} \exp \left\{ - \sum_j \frac{\theta_j}{j} z^j e^{-jV(x/L)} \right\}. \end{aligned} \tag{5.21}$$

We can expand the last exponential so as to get a convergent series of terms of the form $e^{-nV(x/L)}$ with $n \in \mathbb{N}$. The Riemann sums converge in the limit $L \rightarrow \infty$ by Assumption 1, and the result follows from dominated convergence. \square

Lemma 5.6. *Let $z < z_c$ and $a > 0$. There exist L_0 and $b > 0$ such that for all $L \geq L_0$, we have*

- (a) $\mathbb{P}_L^z \left(\frac{1}{L^d} |N_k - \mathbb{E}_L^z N_k| > a \right) \leq e^{-bL^d},$
- (b) $\mathbb{P}_L^z \left(\frac{1}{L^d} |R_j - \mathbb{E}_L^z R_j| > a \right) \leq e^{-bL^d}.$

Proof. We bound the cumulant generating functions and then deduce bounds with the help of Markov's inequality. Since random variables at different locations are independent, we have

$$\begin{aligned} \frac{1}{L^d} \log \mathbb{E}_L^z \left[e^{t(N_k - \mathbb{E}_L^z N_k)} \right] &= \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} \log \mathbb{E}_L^z \left[\exp \left(t \mathbf{1}_{\{H_x(\omega)=k\}} - t \mathbb{P}_L^z(H_x = k) \right) \right] \\ &= \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} \left\{ -t \mathbb{P}_L^z(H_x = k) + \log \left(1 + (e^t - 1) \mathbb{P}_L^z(H_x = k) \right) \right\}. \end{aligned} \tag{5.22}$$

Using $\log(1 + s) \leq s$, we find

$$\begin{aligned} \frac{1}{L^d} \log \mathbb{E}_L^z \left[e^{t(N_k - \mathbb{E}_L^z N_k)} \right] &\leq (e^t - 1 - t) \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d} \mathbb{P}_L^z(H_x = k) \\ &= (e^t - 1 - t) \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right). \end{aligned} \tag{5.23}$$

The latter expectation converges by Lemma 5.5 (b) so that everything is bounded by $c(z)t^2$ for $t \leq 1$ with $c(z)$ depending on z only. Then

$$\begin{aligned} \mathbb{P}_L^z \left(\frac{1}{L^d} (N_k - \mathbb{E}_L^z N_k) > a \right) &= \mathbb{P}_L^z \left(e^{t(N_k - \mathbb{E}_L^z N_k)} > e^{taL^d} \right) \\ &\leq e^{-taL^d} \mathbb{E}_L^z \left(e^{t(N_k - \mathbb{E}_L^z N_k)} \right) \\ &\leq e^{-L^d (ta - c(z)t^2)}. \end{aligned} \tag{5.24}$$

This is smaller than $\frac{1}{2} e^{-bL^d}$ for some $b > 0$. A similar bound can be found for $\mathbb{P}_L^z(\frac{1}{L^d}(N_k - \mathbb{E}_L^z N_k) < -a)$ and this completes the proof of (a). The same method can be used to prove the claim (b) for R_j . \square

Proof of Theorem 5.3. We only prove the claim (b), as the proof for (a) is the same. Let m_k be the limit in (b). We have

$$\begin{aligned} \mathbb{P}_{L,n} \left(\left| \frac{N_k}{L^d} - m_k \right| > \varepsilon \right) &\leq \mathbb{P}_{L,n} \left(\left| \frac{N_k}{L^d} - \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) \right| > \varepsilon - \left| m_k - \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) \right| \right) \\ &= \frac{\mathbb{P}_L^z \left(\left| \frac{N_k}{L^d} - \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) \right| > \varepsilon - \left| m_k - \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) \right| \right)}{\mathbb{P}_L^z \left(\sum_{j \geq 1} j R_j = n \right)}. \end{aligned} \tag{5.25}$$

Then

$$\mathbb{P}_L^z \left(\sum_{j \geq 1} j R_j = n \right) = z^n Z_{L,n} \exp \left\{ - \sum_{x \in \mathbb{Z}^d} \sum_{j \geq 1} \frac{\theta_j}{j} z^j e^{-jV(x/L)} \right\}. \tag{5.26}$$

We choose z close to $z_0(\rho)$ in such a way that

$$\begin{aligned} \left| m_k - \int_{\mathbb{R}^d} \frac{h_k z^k e^{-kV(x)}}{\sum_{j \geq 1} h_j z^j e^{-jV(x)}} dx \right| &\leq \frac{\varepsilon}{2}, \\ \left| \sum_{j \geq 1} \frac{\theta_j}{j} (z^j - z_0(\rho)^j) \int_{\mathbb{R}^d} e^{-jV(x)} dx - \rho \log \frac{z}{z_0(\rho)} \right| &\leq \frac{a}{2}. \end{aligned} \tag{5.27}$$

(If $\rho < \rho_c$, we can choose $z = z_0(\rho)$; if $\rho \geq \rho_c$, we need to choose z close enough to z_c .) From Lemma 5.6 there exist $b > 0$ and L_0 such that for all $L > L_0$, we have

$$\mathbb{P}_L^z \left(\left| \frac{N_k}{L^d} - \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) \right| > \varepsilon - \left| m_k - \mathbb{E}_L^z \left(\frac{N_k}{L^d} \right) \right| \right) \leq e^{-bL^d}. \tag{5.28}$$

It follows from Lemma 5.4 that there exists L_1 such that if $L > L_1$, we have

$$\mathbb{P}_L^z \left(\sum_{j \geq 1} j R_j = n \right) \geq e^{-aL^d}. \tag{5.29}$$

Choosing $a < b$, we get the claim. □

6 Trap potentials

In this section we investigate the shape and location of the condensate for a class of potential functions that have a unique minimum at the origin; square traps are treated in Section 7. Theorem 6.2 states that the condensate is located at the origin (the trap’s minimum), and Theorems 6.4–6.7 provide the fluctuations of the occupation of the origin. The fluctuations are governed by infinitely divisible laws that need not be normal or α -stable, as expected for sums of independent random variables that are not identically distributed [39]. Theorems 6.2, 6.4 and 6.5 generalize known results [15, 18, 20], Theorems 6.6 and 6.7 are new. A graphical overview is given in Figure 2. Section 6.3 discusses the distribution of the partition elements at the origin: the condensate can concentrate on a single large element or be distributed according to some non-trivial law, e.g. a Poisson-Dirichlet distribution.

6.1 Macroscopic occupation of the origin

In this subsection we consider the marginal measure on $\boldsymbol{\eta}$ and we establish that the excess mass is concentrated at $x = 0$. We also prove a central limit theorem.

Assumption 6.1. *We assume that the potential function $V : \mathbb{R}^d \rightarrow [0, \infty]$ satisfies the following properties:*

- (i) $V(x) = \|x\|^\delta(1 + o(1))$ around the origin with $0 < \delta < d$.
- (ii) $V(x) > b > 0$ for all $\|x\| > 1$.
- (iii) For every $a > 0$, there exists a constant C_a such that

$$\frac{1}{L^d} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} e^{-aV(x/L)} < C_a \int_{\mathbb{R}^d} e^{-aV(x)} dx < \infty;$$

we also assume that the left side converges to $\int e^{-aV}$ as $L \rightarrow \infty$.

For the weights, we assume that $\theta_j > 0$ for all j , that $\lim_{j \rightarrow \infty} \frac{1}{j} \log \theta_j = 0$, and that

$$\sum_{j \geq 1} \frac{\theta_j}{j^{d/\delta}} < \infty. \tag{6.1}$$

Notice that if $V(x) \approx c\|x\|^\delta$ with $c > 0$ around the origin, there is no loss in generality in taking $c = 1$. Indeed, let $L' = c^{-1/\delta}L$ and $V'(x) = V(c^{-1/\delta}x)$. Then $V'(x) \approx \|x\|^\delta$ around 0 and the probability can be written in terms of L' and V' by replacing $e^{-jV(x/L)}$ by $e^{-jV'(x/L')}$.

It follows from our assumptions that $\int_{\mathbb{R}^d} \exp(-jV(x))dx$ is bounded from above and below by a constant times $j^{-d/\delta}$, whence $z_c = 1$ and $\rho_c < \infty$. Additional regularity conditions on the θ_j s, formulated as conditions on the tails of the h_n s, will be imposed in the theorems. They are loosely related to regularity conditions for heavy-tailed random variables [31] and can be checked, in part, with the help of Theorem 2.5.

The assumption $\delta < d$ will be important in our proofs. We suspect that a condition on δ is necessary, as for large δ the potential may be too shallow in order to confine the

condensate to a single site; in fact the limit $\delta \rightarrow \infty$ corresponds, formally, to the square traps from Section 7, where all sites are alike and the condensate chooses uniformly among them. It is not clear how large δ should be in order for this to occur.

Recall that H_0 is the random variable for the total occupation of the origin.

Theorem 6.2. *Suppose that Assumption 6.1 holds true, and that there exist constants $C, c \geq 0$ and $a \leq \frac{1}{2}(1 - \frac{\delta}{d})$ such that for all $n \geq 1$,*

$$C^{-1} e^{-cn^a} \leq h_n \leq C e^{cn^a}.$$

Assume also that $\rho > \rho_c$. Then as $n, L \rightarrow \infty$ with fixed $\rho = n/L^d$, we have the convergence in distribution

$$\frac{1}{L^d} H_0 \xrightarrow{d} \rho - \rho_c.$$

The theorem applies to algebraically decaying weights $\theta_j = j^{-\gamma}$, $\gamma > 0$, by Theorem 2.5. It also applies to stretched exponential weights $\theta_j/j = e^{-j^\gamma}$ and algebraically growing weights $\theta_j = j^\gamma$ with $\gamma > 0$ small enough. In the latter case, we indeed have

$$h_n = (1 + o(1)) cn^{-(\gamma+2)/[2(\gamma+1)]} \exp(Cn^{\gamma/(1+\gamma)}) \tag{6.2}$$

for suitable constants c, C . Eq. (6.2) was proven by Erlihson and Granovsky, see Eq. (4.66) in [34]. It also follows from results proven independently in [33]. (The proof in [33] was given for specific weights that satisfy $\theta_j = (1 + o(1))j^\gamma$, but it can be extended with the help of known results on polylogarithms [37, Chapter VI.8].)

For the proof of Theorem 6.2 we follow [18, 15], see also [20]. First we express the canonical expectations with respect to the grand-canonical measure at $z = z_c = 1$ with the help of the conditioning relation (2.18). A difficulty here is that $\sum_j \theta_j/j$ can be infinite, in which case $\mathbb{P}_L^{z_c}(H_0 = \infty) = 1$. Therefore we give a special treatment to $x = 0$. Let

$$M(\omega) = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} H_x(\omega) = \sum_{j \geq 1} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} j R_{xj}(\omega) \tag{6.3}$$

be the number of particles not at 0.³ Let $\rho_c^L = \frac{1}{L^d} \mathbb{E}_L^{z_c}(M)$; this is approximately equal to the critical density:

$$\rho_c^L = \frac{1}{L^d} \mathbb{E}_L^{z_c}(M) = \frac{1}{L^d} \sum_{j \geq 1} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \theta_j e^{-jV(x/L)}, \tag{6.4}$$

which converges to ρ_c by dominated convergence.

Lemma 6.3. *Under the same assumptions as in Theorem 6.2, there exist $C, c > 0$ such that for all $L > 0$ and all $B > 0$,*

$$\mathbb{P}_L^{z_c} \left(\left| \frac{1}{L^d} M - \rho_c^L \right| \geq B \right) \leq C \exp(-c B L^{(d-\delta)/2}).$$

If in addition $\sum_j j \theta_j / j^{d/\delta} < \infty$, the same estimate holds with exponent $\min(d/2, d - \delta)$ instead of $(d - \delta)/2$.

As we shall see later, the additional condition means that M has normal fluctuations of order $L^{d/2}$; if the confining potential is quite shallow, $\delta > d/2$, our large deviations estimate kicks in at fluctuations of the order of L^δ only.

³A careful inspection of the definitions shows that the measure μ_Λ used by Buffet and Pulé [18] and Betz and Ueltschi [15] is, in our notation, the law of M under $\mathbb{P}_L^{z_c}$. The variable M is also used by Chatterjee and Diaconis [20].

Proof. Recall that the cumulant generating function of a Poisson variable $N \sim \text{Poiss}(\lambda)$ is $\log \mathbb{E} e^{tN} = \lambda(e^t - 1)$. Let $t = t_L \in \mathbb{R}$ with $t_L^2 = O(L^{d-\delta})$. We have

$$\begin{aligned} \log \mathbb{E}_L^{z_c} \left[e^{t_L(\frac{1}{L^d}M - \rho_c^L)} \right] &= \sum_{j \geq 1} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \log \mathbb{E}_L^{z_c} \left[e^{j t_L \frac{1}{L^d} (R_{xj}(\omega) - \mathbb{E}_L^{z_c} R_{xj})} \right] \\ &= \sum_{j \geq 1} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \frac{\theta_j}{j} e^{-jV(x/L)} \left(e^{j t_L / L^d} - 1 - j \frac{t_L}{L^d} \right). \end{aligned} \tag{6.5}$$

Using $|e^s - 1 - s| \leq \frac{1}{2}s^2 e^{|s|}$, the interior sum is bounded by

$$\frac{1}{2} \theta_j \frac{1}{L^d} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} e^{-\frac{1}{2}jV(x/L)} \left[\frac{j t_L^2}{L^d} e^{-j(\frac{1}{2}V(x/L) - \frac{1}{L^d}t_L)} \right]. \tag{6.6}$$

Because of Assumption 6.1 (i), there exists $b' > 0$ such that $V(x/L) \geq b' L^{-\delta}$ for all non-zero $x \in \mathbb{Z}^d$. Since $t_L \ll L^{d-\delta}$, the square bracket is bounded, for large L , by

$$j \frac{t_L^2}{L^d} \exp\left(-\frac{b'}{4L^\delta}j\right) \leq \frac{t_L^2}{L^d} \times \frac{4L^\delta}{b'e}. \tag{6.7}$$

Then there is a constant $C' > 0$ such that

$$\left| \log \mathbb{E}_L^{z_c} \left[e^{t_L(\frac{1}{L^d}M - \rho_c^L)} \right] \right| \leq C' \frac{t_L^2}{L^{d-\delta}} \sum_{j \geq 1} \frac{\theta_j}{j^{d/\delta}} = O(1). \tag{6.8}$$

Using Markov's inequality,

$$\mathbb{P}_L^{z_c} \left(\frac{1}{L^d}M - \rho_c^L \geq B \right) \leq e^{-t_L B} \mathbb{E}_L^{z_c} \left[e^{t_L(\frac{1}{L^d}M - \rho_c^L)} \right], \tag{6.9}$$

and a similar bound for the probability that $\frac{1}{L^d}M - \rho_c^L \leq -B$ (use a negative t_L).

If $\sum_j j \theta_j / j^{d/\delta} < \infty$, we choose $t_L = O(L^{d-\delta})$ to bound the square bracket in (6.6) by a constant times j . The right side in (6.8) is then replaced by $\text{const}(t_L^2/L^d) \sum j \theta_j / j^{d/\delta}$ and it is bounded when $t_L = O(L^{d/2})$. \square

Proof of Theorem 6.2. We have

$$\mathbb{P}_L^{z_c} \left(\sum_{j=1}^n j R_{0j} = \ell \right) = h_\ell \exp\left(-\sum_{j=1}^n \frac{\theta_j}{j}\right) \tag{6.10}$$

so that

$$\mathbb{P}_{L,n}(H_0 = \ell) = \frac{\mathbb{P}_L^{z_c}(\sum_{j=1}^n j R_{0j} = \ell) \mathbb{P}_L^{z_c}(M = n - \ell)}{\mathbb{P}_L^{z_c}(\sum_{j=1}^n j R_{0j} + M = n)}, \tag{6.11}$$

$$\mathbb{E}_{L,n} \left[e^{t(n-H_0)} \right] = \frac{\mathbb{E}_L^{z_c} [h_{n-M} e^{tM}]}{\mathbb{E}_L^{z_c} [h_{n-M}]}. \tag{6.12}$$

We use the convention that $h_m = 0$ when $m < 0$.

We show that $\mathbb{E}_{L,n} \left[\exp\left(\frac{t}{L^d}(n - H_0)\right) \right] \rightarrow e^{t\rho_c}$ for any t . Convergence in distribution follows from the pointwise convergence of the Laplace transform and we get the claim of the theorem. It is enough to show that for any $\varepsilon > 0$, we have

$$\frac{\mathbb{E}_L^{z_c} [h_{n-M} \mathbf{1}_{|M/L^d - \rho_c^L| > \varepsilon}]}{\mathbb{E}_L^{z_c} [h_{n-M}]} \rightarrow 0. \tag{6.13}$$

We have

$$\begin{aligned} \frac{\mathbb{E}_L^{z_c} [h_{n-M} \mathbf{1}_{|M/L^d - \rho_c^L| > \varepsilon}]}{\mathbb{E}_L^{z_c} [h_{n-M}]} &\leq \frac{\max_{j \leq n} h_j}{\min_{j \leq n} h_j} \frac{\mathbb{P}_L^{z_c} (|\frac{1}{L^d} M - \rho_c^L| > \varepsilon)}{\mathbb{P}_L^{z_c} (M \leq n)} \\ &\leq \frac{\max_{j \leq n} h_j}{\min_{j \leq n} h_j} C e^{-c\varepsilon L^{(d-\delta)/2}}. \end{aligned} \tag{6.14}$$

The last inequality follows from Lemma 6.3. The last term vanishes indeed as $L \rightarrow \infty$. \square

6.2 Fluctuations of the condensate

We now study the fluctuations of the condensate. The goal is to find the correct scaling α and the limiting random variable X such that

$$\frac{H_0 - L^d(\rho - \rho_c^L)}{L^\alpha} \xrightarrow{d} X. \tag{6.15}$$

For simplicity, we fix the potential $V(x) = c\|x\|^\delta$ in this subsection. It turns out that the phase diagram of the fluctuations is very rich. It is pictured in Fig. 2. We only provide partial results, see the regions in dark colors. The lightly colored region is left to future studies; it would certainly be interesting to know what happens there.

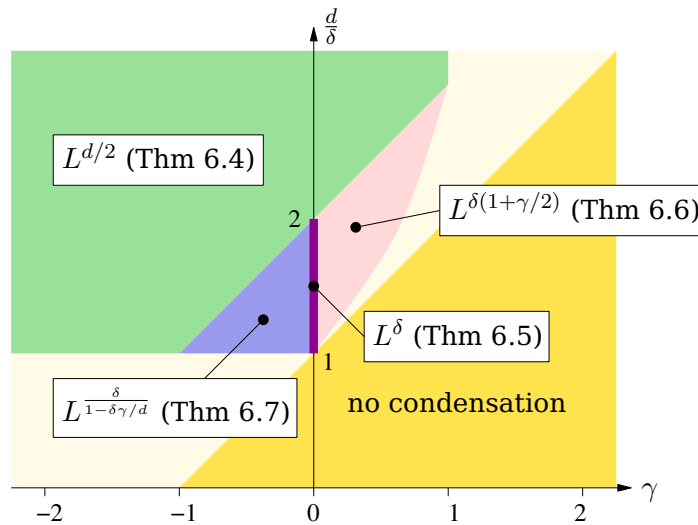


Figure 2: Phase diagrams of the fluctuations of the condensate for the weights $\theta_j = j^\gamma$ and the potential $V(x) = c\|x\|^\delta$. Results for the dark regions are stated in Theorems 6.4 to 6.7. For $\frac{d}{\delta} \leq \gamma + 1$, there is no condensation. The light region remains to be investigated.

The first result deals with normal fluctuations. Let

$$\sigma_c^2 = \sum_{j \geq 1} j \theta_j \int_{\mathbb{R}^d} e^{-jV(x)} dx. \tag{6.16}$$

The first result holds in the case where the variance above is finite.

Theorem 6.4 (Central limit theorem for H_0). *Assume that $V(x) = \|x\|^\delta$, $\rho > \rho_c$ and $\sum_{j \geq 1} j \theta_j / j^{d/\delta} < \infty$. In addition, assume that there exist $C, c > 0$, $a < \min(\frac{1}{2}, 1 - \frac{\delta}{d})$,*

and $b > a + \max(\frac{1}{2}, \frac{\delta}{d})$ such that for all n ,

$$C^{-1} e^{-cn^a} \leq h_n \leq C e^{cn^a},$$

$$\lim_{n \rightarrow \infty} \max_{m \in [-n^b, n^b]} \frac{h_{n+m}}{h_n} = 1.$$

Then, as $L, n \rightarrow \infty$ with $n = \rho L^d$,

$$\frac{H_0 - L^d(\rho - \rho_c^L)}{\sigma_c L^{d/2}} \xrightarrow{d} \mathcal{N}(0, 1). \tag{6.17}$$

The conditions on the h_n s can be easily verified for many given (θ_j) with the help of Theorem 2.5 and Eq. (6.2). In the case of stretched exponential weights $\theta_j = e^{-j^\alpha}$, where $h_n \approx e^{-n^\alpha}$, we have $h_{n+m}/h_n \approx e^{-\alpha m/n^{1-\alpha}}$ so we need $b < 1 - \alpha$. This is possible only if $\alpha < \frac{1}{2} \min(\frac{1}{2}, 1 - \frac{\delta}{d})$. For larger α , or for weights $\theta_j = j^\gamma$ with large γ , we expect a deterministic shift in the average occupation number of the origin, small compared to n but large compared to \sqrt{n} , and then normal fluctuations around the shifted mean; compare with Theorem 7.1 (c).

Proof. This is similar to Theorem 6.2. Since $H_0 = \rho L^d - M$, it is enough to show that

$$\mathbb{E}_{L,n} \left(\exp \left\{ iu \frac{M - L^d \rho_c^L}{\sigma_c L^{d/2}} \right\} \right) \rightarrow e^{-u^2/2}. \tag{6.18}$$

As in Eq. (6.12), and again using the convention $h_m = 0$ when $m < 0$, we have

$$\begin{aligned} \mathbb{E}_{L,n} \left(\exp \left\{ iu \frac{M - L^d \rho_c^L}{\sigma_c L^{d/2}} \right\} \right) &= \frac{\mathbb{E}_L^{z_c} \left(h_{n-M} \exp \left\{ iu \frac{M - L^d \rho_c^L}{\sigma_c L^{d/2}} \right\} \right)}{\mathbb{E}_L^{z_c} (h_{n-M})} \\ &= \mathbb{E}_L^{z_c} \left(\exp \left\{ iu \frac{M - L^d \rho_c^L}{\sigma_c L^{d/2}} \right\} \right) + \mathbb{E}_L^{z_c} \left(\left[\frac{h_{n-M}}{\mathbb{E}_L^{z_c} (h_{n-M})} - 1 \right] \exp \left\{ iu \frac{M - L^d \rho_c^L}{\sigma_c L^{d/2}} \right\} \right). \end{aligned} \tag{6.19}$$

We have

$$\mathbb{E}_L^{z_c} \left[\exp \left(iu \frac{M - \rho_c^L L^d}{\sigma_c L^{d/2}} \right) \right] = \exp \left\{ \sum_{j \geq 1} \frac{\theta_j}{j} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} e^{-jV(x/L)} \left[e^{ij \frac{u}{\sigma_c L^{d/2}}} - 1 - ij \frac{u}{\sigma_c L^{d/2}} \right] \right\}. \tag{6.20}$$

The absolute value of the square bracket is less than $j^2 u^2 / \sigma_c^2 L^d$. From our assumptions and dominated convergence, we get

$$\lim_{L \rightarrow \infty} \mathbb{E}_{L,n} \left(\exp \left\{ iu \frac{M - L^d \rho_c^L}{\sigma_c L^{d/2}} \right\} \right) = e^{-u^2/2}. \tag{6.21}$$

It remains to verify that the second term in (6.19) vanishes in the limit $L \rightarrow \infty$.

Let $\alpha = \min(\frac{d}{2}, d - \delta)$ and $\nu < \alpha - ad$. As in the proof of Theorem 6.2, it follows from Lemma 6.3 that

$$\mathbb{E}_L^{z_c} \left(\left[\frac{h_{n-M}}{\mathbb{E}_L^{z_c} (h_{n-M})} - 1 \right] \mathbf{1}_{|\frac{1}{L^d} M - \rho_c^L| > L^{-\nu}} \right) \leq \text{const } e^{2cn^a - cL^{\alpha-\nu}} \rightarrow 0. \tag{6.22}$$

We also have

$$(1 + o(1)) \min_{m, |m - L^d \rho_c^L| < L^{d-\nu}} h_{n-m} \leq \mathbb{E}_L^{z_c} (h_{n-M}) \leq (1 + o(1)) \max_{m, |m - L^d \rho_c^L| < L^{d-\nu}} h_{n-m}. \tag{6.23}$$

We choose $\nu > d(1 - b)$, which is possible by the assumptions of the theorem. Then $\mathbb{E}_L^{z_c} (h_{n-M}) = h_{n - \rho_c^L L^d} (1 + o(1))$ and the second term in (6.19) vanishes in the limit $L \rightarrow \infty$. \square

The central limit theorem applies to the ideal Bose gas in dimensions $d \geq 5$. It was noted by Buffet and Pulé that the fluctuations were non-normal in $d = 3$, of order L^2 [18]. We propose here three theorems about non-normal fluctuations larger than $L^{d/2}$. They always apply to situations where the variance σ_c^2 defined in (6.16) is infinite. In order to guess the scale of the fluctuations, we observe that

$$\mathbb{E}_L^{z_c} \left((M - \mathbb{E}_L^{z_c} M)^2 \right) = \sum_{j \geq 1} j \theta_j \sum_{x \in \mathbb{Z}^d \setminus \{0\}} e^{-jV(x/L)} = O(L^{d+\delta}). \tag{6.24}$$

This can be shown by controlling the sum over j as in the proof of Lemma 6.3.

The first theorem deals with the Ewens case where the weights θ_j are constant. The critical density is finite and the variance infinite when $1 < \frac{d}{\delta} - \gamma \leq 2$. Let $(G_x)_{x \in \mathbb{Z}^d}$ be i.i.d. Gamma variables with parameters $(\theta, 1)$ (that is, the probability density function is $\Gamma(\theta)^{-1} t^{\theta-1} e^{-t}$ for $t \geq 0$).

Theorem 6.5. *Suppose that $V(x) = c\|x\|^\delta$ with $\frac{d}{2} < \delta < d$, and that $\theta_j \equiv \theta$ is constant. Then if $\rho > \rho_c$, we have*

$$\frac{H_0 - (\rho - \rho_c^L)L^d}{L^\delta} \xrightarrow{d} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \frac{\theta - G_x}{\|x\|^\delta}.$$

When $\theta = 1$, G_x is exponential with mean 1 and we recover the result from [18, 20]. The limiting random variable of Theorem 6.5 requires some explanations (see [20] for a closely related situation). Each G_x has mean θ . Consider

$$Y_\Lambda = \sum_{x \in \Lambda} \frac{\theta - G_x}{\|x\|^\delta}. \tag{6.25}$$

Since $2\delta > d$ and $\sum_{x \in \mathbb{Z}^d \setminus \{0\}} \|x\|^{-2\delta} < \infty$, a standard theorem applies that concerns random series [27, Chapter 2.5]: For every increasing sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of domains with $\cup_n \Lambda_n = \mathbb{Z}^d$, the limit $Y(\omega) = \lim_{n \rightarrow \infty} Y_{\Lambda_n}(\omega)$ exists almost surely. The dominated convergence theorem further shows that the limiting random variable has characteristic function

$$\mathbb{E}(e^{iuY}) = \exp \left\{ \theta \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \left[\frac{iu}{\|x\|^\delta} - \log \left(1 + \frac{iu}{\|x\|^\delta} \right) \right] \right\}. \tag{6.26}$$

It follows in particular that the law of Y is independent of the precise choice of the sequence of domains. The infinite sum in Theorem 6.5 is by definition a variable equal to Y in law.

Theorem 6.6. *Suppose that $V(x) = \|x\|^\delta$, $\theta_j = j^\gamma$ with $\gamma > 0$, and that $(1 + \gamma)(1 + \frac{1}{2}\gamma) < \frac{d}{\delta} < \gamma + 2$. Then if $\rho > \rho_c$, we have*

$$\frac{H_0 - (\rho - \rho_c^L)L^d}{L^{\delta(1+\gamma/2)}} \xrightarrow{d} \left(\Gamma(\gamma + 2) \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|x\|^{\delta(\gamma+2)}} \right)^{1/2} \mathcal{N}(0, 1).$$

Next, let $b_{\gamma,\delta,d}$ be equal to

$$b_{\gamma,\delta,d} = -\frac{\Gamma(1 + \gamma)}{|\gamma|} \int_{\mathbb{R}^d} \left((1 + \|x\|^\delta)^{|\gamma|} - \|x\|^{\delta|\gamma|} - |\gamma| \|x\|^{-\delta(1+\gamma)} \right) dx. \tag{6.27}$$

The integrand behaves like $\|x\|^{-\delta(2+\gamma)}$ for large $\|x\|$ and like $\|x\|^{-\delta(1+\gamma)}$ for small $\|x\|$, so the integral is well-defined. It follows from $(1 + \|x\|^\delta)^{|\gamma|} \leq \|x\|^{\delta|\gamma|} (1 + |\gamma| \|x\|^{-\delta})$ that $b_{\gamma,\delta,d}$ is positive. We define Z to be the stable random variable whose Laplace transform for $t \geq 0$ is

$$\mathbb{E} e^{tZ} = e^{b_{\gamma,\delta,d} t^{|\gamma|+d/\delta}}. \tag{6.28}$$

Theorem 6.7. *Suppose that $V(x) = \|x\|^\delta$, $\theta_j = j^\gamma$ with $\gamma < 0$, and that $1 < \frac{d}{\delta} < \gamma + 2$. Then if $\rho > \rho_c$, we have*

$$\frac{H_0 - (\rho - \rho_c^L)L^d}{L^\alpha} \xrightarrow{d} Z,$$

where $\alpha = \frac{\delta}{1 - \delta\gamma/d}$.

Remark 6.8. *If $\delta \geq d$, an analogous result for the fluctuations of M in the grand-canonical ensemble at $z = z_c$ still holds, but we can no longer use Lemma 6.3 in order to pass from the grand-canonical to the canonical ensemble. We suspect that Theorem 6.7 can be shown with an adequate replacement of Lemma 6.3 (e.g., algebraic deviations bounds based on finite moments), but leave the proof or disproof as an open problem.*

We prove these theorems by showing convergence of Laplace or Fourier transforms. Let

$$I_s(z) = \sum_{j \geq 1} j^s \theta_j z^j. \tag{6.29}$$

We have

$$\log \mathbb{E}_L^{z_c} \left[e^{t(M - \mathbb{E}_L^{z_c} M)} \right] = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \log \mathbb{E}_L^{z_c} \left[e^{t(H_x - \mathbb{E}_L^{z_c} H_x)} \right], \tag{6.30}$$

$$\log \mathbb{E}_L^{z_c} \left[e^{t(H_x - \mathbb{E}_L^{z_c} H_x)} \right] = I_{-1}(e^{t-V(x/L)}) - I_{-1}(e^{-V(x/L)}) - tI_0(e^{-V(x/L)}). \tag{6.31}$$

Combining Eq. (6.31) with asymptotic expansions of $I_{-1}(e^{-\mu})$, $I_0(e^{-\mu})$ around $\mu = 0$, we obtain two lemmas on single-site fluctuations.

Lemma 6.9. *Suppose that $V(x) = \|x\|^\delta$ and that $\theta_j = \theta$ for all j . Then under $\mathbb{P}_L^{z_c}$, for every fixed $x \in \mathbb{Z}^d \setminus \{0\}$, we have*

$$\frac{H_x - \mathbb{E}_L^{z_c} H_x}{L^\delta} \xrightarrow{d} \frac{1}{\|x\|^\delta} (\theta - G_x),$$

where G_x is a Gamma random variable with parameter $(\theta, 1)$.

Proof. For $\theta_j \equiv \theta$ we have $I_{-1}(z) = -\theta \log(1 - z)$. As $\mu \rightarrow 0$,

$$\begin{aligned} I_{-1}(e^{-\mu}) &= -\theta \log(1 - e^{-\mu}) = -\theta \log \mu + \theta \frac{\mu}{2} + O(\mu^2) \\ I_0(e^{-\mu}) &= \frac{\theta}{1 - e^{-\mu}} = \theta \mu^{-1} + o(\mu^{-1}) \end{aligned} \tag{6.32}$$

hence as long as $t \rightarrow 0$, $t < \mu$,

$$I_{-1}(e^{t-\mu}) - I_{-1}(e^{-\mu}) - tI_0(e^{-\mu}) = -\theta \log\left(1 - \frac{t}{\mu}\right) + \theta \frac{t}{\mu} + o(1). \tag{6.33}$$

Applying this to $\mu = V(x/L)$ and $t = sL^{-\delta}$, we get

$$\log \mathbb{E}_L^{z_c} \left[e^{sL^{-\delta}(H_x - \mathbb{E}_L^{z_c} H_x)} \right] = -\theta \left(\log\left(1 - \frac{s}{\|x\|^\delta}\right) - \frac{s}{\|x\|^\delta} \right) + o(1). \tag{6.34}$$

On the other hand,

$$\log \mathbb{E} \left[e^{s(\theta - G)} \right] = \theta(s - \log(1 + s)). \tag{6.35}$$

The result follows with the help of the usual continuity theorems (convergence of Laplace transforms implies convergence in distribution). \square

Lemma 6.10. *Suppose that $V(x) = \|x\|^\delta$ and $\theta_j = j^\gamma$ with $\gamma > 0$. Then under $\mathbb{P}_L^{z_c}$, for every fixed $x \in \mathbb{Z}^d \setminus \{0\}$,*

$$\frac{H_x - \mathbb{E}_L^{z_c} H_x}{L^{\delta(\gamma+2)/2}} \xrightarrow{d} \frac{(\Gamma(\gamma + 2))^{1/2}}{\|x\|^{\delta(\gamma+2)/2}} \mathcal{N}(0, 1). \tag{6.36}$$

Proof. Consider first the case $\gamma \notin \mathbb{N}$, and slightly modified weights corresponding to

$$I_0(e^{-\mu}) = \frac{\Gamma(\gamma + 1)}{(1 - e^{-\mu})^{\gamma+1}} - \Gamma(\gamma + 1). \tag{6.37}$$

The corresponding weights satisfy $\theta_j = j^\gamma(1 + o(1))$ [33]. We note the identity

$$I_{-1}(e^{t-\mu}) - I_{-1}(e^{-\mu}) - tI_0(e^{-\mu}) = \int_0^t (I_0(e^{s-\mu}) - I_0(e^{-\mu})) ds. \tag{6.38}$$

For $\mu \rightarrow 0$, we have

$$I_0(e^{-\mu}) = \Gamma(\gamma + 1)\mu^{-\gamma-1} + O(\mu^{-\gamma}). \tag{6.39}$$

More precisely, $I_0(e^{-\mu})$ has an asymptotic expansion of the form $\mu^{-\gamma-1} \sum_{n \geq 0} a_n \mu^n$. As $t, \mu \rightarrow 0$ with $t = o(\mu)$,

$$\begin{aligned} \int_0^t ((\mu - s)^{-\gamma-1} - \mu^{-\gamma-1}) ds &= \frac{1}{\gamma} ((\mu - t)^{-\gamma} - \mu^{-\gamma}) - t\mu^{-\gamma-1} \\ &= (1 + o(1)) \frac{1}{2}(\gamma + 1)\mu^{-\gamma-2}t^2. \end{aligned} \tag{6.40}$$

The other terms of the asymptotic expansion — which can be pushed to arbitrarily high order — can be evaluated in a similar way and are found to give contributions of smaller order. Choosing $t = sL^{-\delta(\gamma+2)/2}$ and $\mu = \|x\|^\delta/L^\delta$, we get

$$\lim_{L \rightarrow \infty} \log \mathbb{E}_L^{z_c} \left[\exp \left(\frac{s(H_x - \mathbb{E}_L^{z_c} H_x)}{L^{\delta(\gamma+2)/2}} \right) \right] = \frac{1}{2} \Gamma(\gamma + 2) s^2 \|x\|^{-\delta(\gamma+2)} \tag{6.41}$$

and we recognize the Laplace transform of a normal random variable. This proves the lemma for the modified weights from [33]. For the original weights $\theta_j = j^\gamma$, we recognize in $I_{-1}(z)$ a *dilogarithm*, and the results still apply because the dilogarithm and the function (6.37) have similar asymptotic expansions [37, Chapter VI.8]. \square

Now we come to the proofs of the theorems. The proof of Theorem 6.5 is based on Lemma 6.9 and is analogous to the proofs in [18, 20] and to the proof of Theorem 6.6, and it is therefore omitted. In contrast with algebraically growing weights no additional condition on γ is needed in order to go from the grand-canonical to the canonical ensemble. This is because $h_n = \theta(\theta + 1) \cdots (\theta + n - 1)/n!$ is a slowly varying function of n .

Proof of Theorem 6.6. We treat first the fluctuations of M in the grand-canonical ensemble at $z = z_c$. To this aim we look at the Fourier transform of the law of $L^{-\delta(\gamma+2)}(M - \mathbb{E}_L^{z_c} M)$ under $\mathbb{P}_L^{z_c}$; it is given by a sum over x analogous to Eq. (6.30). We split the sum in two parts. Revisiting the proof of Lemma 6.10, we find that for suitable c and ε , all $\mu \leq \varepsilon^\delta$, and all $|t| \leq \frac{1}{2}\mu$,

$$|I_{-1}(e^{it-\mu}) - I_{-1}(e^{-\mu}) - itI_0(e^{-\mu})| \leq (\gamma + 1)|t|^2 \mu^{-\gamma-2}. \tag{6.42}$$

We apply this inequality to $\mu = \|x\|^\delta/L^\delta$ with $\|x\| \leq \varepsilon L$, and $t = s/L^{\delta(\gamma+2)}$, with L large enough so that $|s|L^{-\delta(\gamma+2)} \leq \frac{1}{2}L^{-\delta}$. Then

$$\left| \log \mathbb{E}_L^{z_c} \left[e^{isL^{-\delta(\gamma+2)/2}(H_x - \mathbb{E}_L^{z_c} H_x)} \right] \right| \leq (\gamma + 1) \frac{s^2}{\|x\|^{\delta(\gamma+2)}}. \tag{6.43}$$

We have $\delta(\gamma + 2) > d$, so the right side can be summed over $x \in \mathbb{Z}^d \setminus \{0\}$. From Lemma 6.10 and dominated convergence, we get

$$\lim_{L \rightarrow \infty} \sum_{\substack{x \in \mathbb{Z}^d \setminus \{0\} \\ \|x\| \leq \varepsilon L}} \log \mathbb{E}_L^{z_c} \left[e^{isL^{-\delta(\gamma+2)/2}(H_x - \mathbb{E}_L^{z_c} H_x)} \right] = -\frac{s^2}{2} \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \frac{\Gamma(\gamma + 2)}{\|x\|^{\delta(\gamma+2)}}, \tag{6.44}$$

which is the logarithm of the Fourier transform of the limiting normal random variable. It remains to show that the sum over $\|x\| \geq \varepsilon L$ does not contribute. Notice that

$$\sum_{j \geq 1} j \theta_j \int_{\|x\| > \varepsilon} e^{-jV(x)} dx < \infty. \tag{6.45}$$

Arguments similar to the proof of Lemma 6.3 show that

$$\frac{1}{L^d} \mathbb{E}_L^{z_c} \left[\sum_{\substack{x \in \mathbb{Z}^d \\ \|x\| \geq \varepsilon L}} (H_x - \mathbb{E}_L^{z_c} H_x)^2 \right] = O(1). \tag{6.46}$$

Write M_ε for the sum of the H_x over $\|x\| \geq \varepsilon L$. We have, for every $\kappa > 0$,

$$\mathbb{P}_L^{z_c} \left(\frac{M_\varepsilon}{L^{\delta(\gamma+2)/2}} \geq \kappa \right) \leq \kappa^{-2} L^{-\delta(\gamma+2)} \mathbb{E}_L^{z_c} \left[(M_\varepsilon - \mathbb{E}_L^{z_c} M_\varepsilon)^2 \right] = O(L^{d-\delta(\gamma+2)}) \rightarrow 0. \tag{6.47}$$

It follows that $\mathbb{E}_L^{z_c} [\exp(is(M_\varepsilon - \mathbb{E}_L^{z_c} M_\varepsilon)L^{-\delta(\gamma+2)/2})] \rightarrow 1$: the sites $\|x\| \geq 1$ do not contribute to the Fourier transform. Thus we have proven that M fluctuates as $L^{\delta(\gamma+2)/2}$ times the limiting Gaussian variable. We conclude with the help of Eq. (6.12) as in the proof of Theorems 6.2 and Theorems 6.4. Let $\varepsilon_L = L^\beta$. The condition $\beta + \frac{1}{2}(d+\delta) > d \frac{\gamma}{\gamma+1}$ ensures that $|M - \rho_c^L L^d| \geq \varepsilon_L L^{(d+\delta)/2}$ does not contribute; the condition $\delta(1 + \frac{1}{2}\gamma) \leq \beta + \frac{1}{2}(d+\delta) < \frac{d}{\gamma+1}$ ensures that the h_n is approximately constant on the scale of fluctuations and in the interval $|M - \rho_c^L L^d| \leq \varepsilon_L L^d$. Such a β exists in the region of parameters considered here. \square

Proof of Theorem 6.7. We proceed as in the proofs of Theorems 6.2 and Theorems 6.4. Observe that by Theorem 2.5, $h_n = (1 + o(1))\theta_n/n = (1 + o(1))n^{-|\gamma|-1}$ is a regularly varying function of n . We can invoke Eq. (6.12) and prove that

$$\mathbb{E}_L^{z_c} \left(e^{-\frac{t}{L^\alpha}(M - \mathbb{E}_L^{z_c} M)} \right) \rightarrow \mathbb{E}(e^{tZ}) \tag{6.48}$$

for each fixed t . Using Eqs (6.30) and (6.31), we have

$$\log \mathbb{E}_L^{z_c} \left(e^{-\frac{t}{L^\alpha}(M - \mathbb{E}_L^{z_c} M)} \right) = \sum_{x \in \mathbb{Z}^d \setminus \{0\}} \left(I_{-1} \left(e^{-\frac{t}{L^\alpha} - V(x/L)} \right) - I_{-1} \left(e^{-V(x/L)} \right) + \frac{t}{L^\alpha} I_0 \left(e^{-V(x/L)} \right) \right). \tag{6.49}$$

Fix $\varepsilon > 0$. Just as in the proof of Theorem 6.6, one can show that sites x with $\|x\| \geq \varepsilon L$ are irrelevant. Eq. (6.39) stays true [37, Chapter VI.8]. Therefore as $\varepsilon \rightarrow 0$, uniformly in L , Eq. (6.49) is equal to

$$\begin{aligned} & - (1 + o(1)) \frac{\Gamma(1 + \gamma)}{|\gamma|} \sum_{\substack{x \in \mathbb{Z}^d \setminus \{0\} \\ \|x\| \leq \varepsilon L}} \left[\left(\frac{t}{L^\alpha} + \frac{\|x\|^\delta}{L^\delta} \right)^{|\gamma|} - \left(\frac{\|x\|^\delta}{L^\delta} \right)^{|\gamma|} - |\gamma| \frac{t}{L^\alpha} \left(\frac{\|x\|^\delta}{L^\delta} \right)^{|\gamma|-1} \right] \\ & = - (1 + o(1)) \frac{\Gamma(1 + \gamma)}{|\gamma|} \sum_{\substack{x \in \mathbb{Z}^d \setminus \{0\} \\ \|x\| \leq \varepsilon L}} \frac{1}{L^{\alpha|\gamma|}} \left[\left(t + \left\| \frac{x}{L^{1-\frac{\alpha}{\delta}}} \right\|^\delta \right)^{|\gamma|} - \left\| \frac{x}{L^{1-\frac{\alpha}{\delta}}} \right\|^{\delta|\gamma|} - |\gamma| t \left\| \frac{x}{L^{1-\frac{\alpha}{\delta}}} \right\|^{-\delta(1+\gamma)} \right]. \end{aligned} \tag{6.50}$$

We recognize a Riemann sum in the equation above, and this converges as $L \rightarrow \infty$ to

$$-\frac{\Gamma(1+\gamma)}{|\gamma|} \int_{\mathbb{R}^d} \left[(t + \|x\|^\delta)^{|\gamma|} - \|x\|^{\delta|\gamma|} - |\gamma| t \|x\|^{-\delta(1+\gamma)} \right] dx = t^{|\gamma|+d/\delta} b_{\gamma,\delta,d}. \quad (6.51)$$

This proves (6.48), and Theorem 6.7. □

6.3 Size of partition elements

In this section we have so far discussed the occupation number at $x = 0$. We now turn to the partition structure of the condensate. Again we consider the potential $V(x) = \|x\|^\delta$ and the weights $\theta_j = j^\gamma$. In view of Proposition 2.2 (a), and since by Theorem 6.2 we have $H_0/L^d \rightarrow \rho - \rho_c$ when $\rho > \rho_c$, it is natural to assume that the limiting structure of the partition λ_0 under $\mathbb{P}_{L,n}$ is the same as that of a single-site Gibbs partition ν_m when $m \sim (\rho - \rho_c)L^d \rightarrow \infty$. We do not wish to carry out the proof here (see however the proof of Proposition 7.2 in Section 7) and we base our discussion on the assumption that this is true.

Recall the definition (5.1) of ν_∞ , and let

$$\nu_{\text{macro}} = \lim_{\varepsilon \searrow 0} \lim_{\substack{n \rightarrow \infty \\ \rho L^d = n}} \mathbb{E}_{L,n} \left(\frac{1}{n} \sum_{j > \varepsilon n} j R_j \right). \quad (6.52)$$

be the fraction of particles in partition components of macroscopic size, i.e., of the order of n . It is clear that $\nu_\infty \geq \nu_{\text{macro}}$. There are three phases in the (γ, ρ) -plane, as illustrated in Fig. 3. For $\rho < \rho_c$, we have $\nu_\infty = \nu_{\text{macro}} = 0$ by Theorem 5.2.

For $\rho > \rho_c$, the site 0 is macroscopically occupied and the partition structure results from the Gibbs measure on partitions, ν_m . There are three possibilities, depending on the parameter γ :

- $\gamma < 0$ The partition displays a unique large element that contains all the indices save a (random) finite number [16]. This gives a phase with macroscopic elements.
- $\gamma = 0$ (Uniform random permutations or the Ewens measure.) The number of partition elements is logarithmic, they have typically a size proportional to m and their joint law is asymptotically a Poisson-Dirichlet law, as for the lengths of cycles in uniform random permutations; see e.g. [6]. Here also, $\nu_{\text{macro}} = \nu_\infty$.
- $\gamma > 0$ It was proved in [33, 25] that typical elements have size of order $m^{1/(\gamma+1)}$, with Gamma distribution. Moreover the number of elements divided by $m^{\gamma/(\gamma+1)}$ converges in law to some constant [34, 33], and the limit behavior of the partition is described by a deterministic limit shape [35, 23]: there is a function $w(x)$ such that for all $x > 0$, we have the convergence in law under ν_m :

$$\frac{1}{m^{\gamma/(\gamma+1)}} \#\{j : \lambda_j \geq x m^{1/(\gamma+1)}\} \rightarrow w(x). \quad (6.53)$$

Remark 6.11. *The Poisson-Dirichlet process appearing for $\gamma = 0$ has been generalized and belongs to a two-parameter family introduced in [50], called two-parameter Poisson-Dirichlet or Pitman-Yor processes. They can be constructed using subordinators or stick-breaking schemes [50] and can appear as limits of more general Gibbs partitions where the weight is allowed to depend explicitly on the number of cycles [49].*

7 Square potentials

This last section is devoted to square traps, $e^{-V(x)} = \mathbf{1}_{(-\frac{1}{2}, \frac{1}{2}]^d}(x)$. There is not much spatial structure left, as the L^d available sites are all alike with indifferent location. But

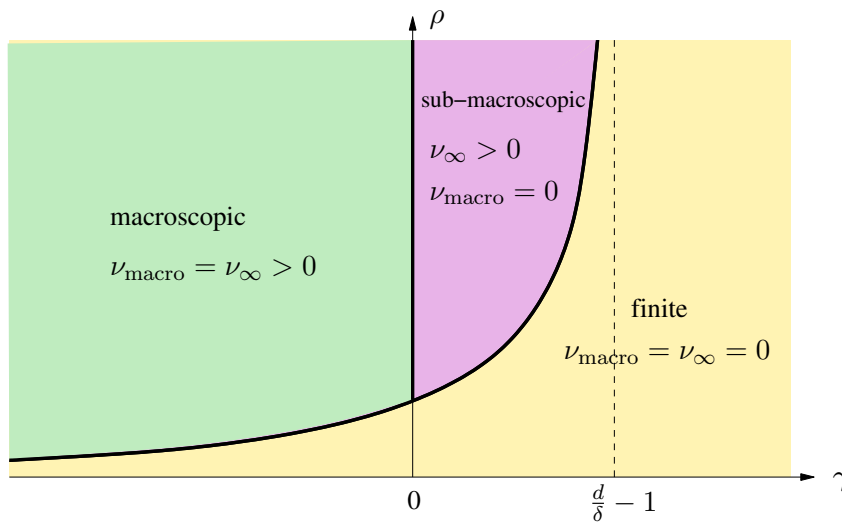


Figure 3: Phase diagram for the family of measures with $V = \|x\|^\delta$ and $\theta_j = j^\gamma$. The transition line bordering the finite phase is given by the transition density $\rho = \rho_c$ defined in (5.5).

this case is actually very interesting mathematically as it involves conditioned random variables. And it is relevant in mathematical physics because it describes the stationary measure of the zero-range process, and certain equilibrium measures in droplet formations.

The results of Theorems 5.2 and 5.3 are valid here. In the case $z_c = 1$, the transition density (5.5) takes the simple form

$$\rho_c = \sum_{j \geq 1} \theta_j. \tag{7.1}$$

We refer to our previous discussion of the zero-range process for background and literature behind this formula. In contrast to the confining potentials of Section 6, condensation does not take place at $x = 0$ but on a random location picked uniformly at random. On that site, the partition has one large element. Instead of H_0 , we consider here the following random variables:

$$\begin{aligned} M_L &= \max_{x \in \mathbb{Z}^d, \|x\|_1 \leq \frac{L}{2}} H_x, \\ T_L &= \max\{j \in \mathbb{N} : R_j \geq 1\}. \end{aligned} \tag{7.2}$$

The main theorem of this section is due to Nagaev [46], and to Armendáriz, Grosskinsky, and Loulakis [4], see Theorem 7.1 below. It concerns the random variable M_L and we complement it with a result for T_L . Before this we discuss the grand-canonical measure $\mathbb{P}_L^{z_c}$ and the connection with Cramér series. At the end of the section, we comment on the relation with surface tension and the critical point of fluids.

For concreteness we focus on algebraic and stretched exponential weights, $\theta_j = j^{-\alpha}$ with $\alpha > 1$ or $\theta_j = j \exp(-j^\gamma)$ with $0 < \gamma < 1$. Under $\mathbb{P}_L^{z_c}$ the random variables H_x are i.i.d. and their distribution satisfies

$$\mathbb{P}_L^{z_c}(H_x = n) = h_n \exp\left(-\sum_{j \geq 1} \theta_j/j\right) = \frac{\theta_n}{n} (1 + o(1)). \tag{7.3}$$

The last relation follows from Theorem 2.5. The mean and variance of H_x are

$$\begin{aligned} \mathbb{E}_L^{z_c}(H_x) &= \sum_{j \geq 1} \theta_j = \rho_c, \\ \mathbb{E}_L^{z_c}([H_x - \mathbb{E}_L^{z_c}(H_x)]^2) &= \sum_{j \geq 1} j\theta_j = \sigma_c^2. \end{aligned} \tag{7.4}$$

The variance is consistent with Eq. (6.24).

It turns out that the limit law for stretched exponential weights with fast decay ($\frac{1}{2} \leq \gamma < 1$) involves a deterministic shift Δ_L . In order to define it, consider first the cumulant generating function

$$\varphi(t) = \log \mathbb{E}_L^{z_c}(e^{tH_x}) = \sum_{j \geq 1} \frac{\theta_j}{j} (e^{tj} - 1). \tag{7.5}$$

It is finite for $t \leq 0$. Given $\tau < 0$, let t_τ be the unique solution of the equation $\varphi'(t) = \rho_c + \tau$. Then as $\tau \nearrow 0$, the Legendre transform $\tilde{\varphi}(\tau) = \sup_{t \in \mathbb{R}} (t\tau - \varphi(t))$ has an asymptotic expansion of the form

$$-\tilde{\varphi}(\tau) = \varphi(t_\tau) - (\rho_c + \tau)t_\tau = -\frac{\tau^2}{2\sigma_c^2} + \tau^3 \sum_{k \geq 0} \lambda_k \tau^k. \tag{7.6}$$

The coefficients λ_k are rational functions of the cumulants and they form the *Cramér series* of H_x [42]. In a more general context, the series provides corrections to the central limit theorem when looking at *moderate deviations* that are larger than \sqrt{n} but smaller than n ; see Chapter 7 in [42]. For heavy-tailed variables, the Cramér series diverges and we truncate it at $t := \lfloor (1 - \gamma)^{-1} - 2 \rfloor + 1$. Let

$$f_L(\Delta) = ((\rho - \rho_c)L^d - \Delta)^\gamma + \frac{\Delta^2}{2\sigma_c^2(L^d - 1)} - \frac{\Delta^3}{(L^d - 1)^2} \sum_{k=0}^t \lambda_k \left(\frac{\Delta}{L^d - 1}\right)^k. \tag{7.7}$$

It follows from the results of Nagaev [46] that for $\rho > \rho_c$ and $\frac{1}{2} \leq \gamma < 1$,

$$\mathbb{P}_L^{z_c} \left(\sum_{x \in \mathbb{Z}^d, \|x\|_1 \leq L/2} H_x = \rho L^d \right) = (1 + o(1)) L^d \exp \left(- \min_{\Delta > 0} f_L(\Delta) \right). \tag{7.8}$$

The interpretation of Eq. (7.8) is that one of the L^d sites captures almost all of the excess particles $(\rho - \rho_c)L^d$, the remaining $L^d - 1$ sites have a small excess Δ of particles, and the shift Δ minimizes the free energy penalty $f_L(\Delta)$. In order to compute Δ , it is convenient to change variables and write $\Delta = a(\rho - \rho_c)L^d$; setting the derivative equal to zero yields the equation

$$\frac{\gamma \sigma_c^2 (L^d - 1)}{((\rho - \rho_c)L^d)^{2-\gamma}} = a(1 - a)^{1-\gamma} \left(1 - \sigma_c^2 \sum_{k=0}^t (k+3) \lambda_k \left(\frac{(\rho - \rho_c)L^d a}{L^d - 1} \right)^{k+2} \right). \tag{7.9}$$

It has a solution $a \sim \gamma \sigma_c^2 L^{d(\gamma-1)} (\rho - \rho_c)^{\gamma-2}$. The minimizer Δ_L satisfies

$$\Delta_L = (1 + o(1)) \gamma \sigma_c^2 (\rho - \rho_c)^{\gamma-1} L^{d\gamma}. \tag{7.10}$$

When $0 < \gamma < \frac{1}{2}$, the shift $\Delta_L \ll \sqrt{L^d}$ is negligible and Eq. (7.8) is replaced by the simpler relation [46]

$$\mathbb{P}_L^{z_c} \left(\sum_{x \in \mathbb{Z}^d, \|x\|_1 \leq L/2} H_x = \rho L^d \right) \sim L^d \mathbb{P}_L^{z_c}(H_x = (\rho - \rho_c)L^d) \sim \frac{\theta_{(\rho - \rho_c)L^d}}{\rho - \rho_c}. \tag{7.11}$$

(The last identity follows from Theorem 2.5). Eq. (7.11) also holds in the case of the algebraic weights $\theta_j = j^\alpha$ with $\alpha > 1$ [26].

Before formulating the theorem on M_L we recall the definition of α -stable random variables. They are needed in the case of slowly decaying algebraic weights ($1 < \alpha < 2$). They are conveniently defined via their Laplace transform: For $t \geq 0$,

$$\mathbb{E}[e^{-tY}] = \exp\left(\frac{1}{\alpha} \int_0^\infty (e^{-tu} - 1 + tu) \frac{du}{u^{1+\alpha}}\right) = \exp\left(\frac{1}{\alpha} M(\alpha)t^\alpha\right), \quad (7.12)$$

where $M(\alpha) = \int_0^\infty (e^{-y} - 1 + y)y^{-1-\alpha}dy > 0$. The random variable Y has zero expectation $\mathbb{E}(Y) = 0$; its right tail is $\mathbb{P}(Y \geq x) = (1 + o(1))\alpha^{-1}x^{-\alpha}$ as $x \rightarrow \infty$, and its left tail $\mathbb{P}(Y \leq -x)$ is exponentially small [39].

Recall that $\rho_c = \sum_{j \geq 1} \theta_j$ and $\sigma_c^2 = \sum_{j \geq 1} j\theta_j$. The results from conditioned sums of random variables can be formulated in our context as follows.

Theorem 7.1 (Nagaev [46], Armendáriz, Grosskinsky, Loulakis [3, 4]). *Assume that $\rho > \rho_c$.*

(a) *If $\theta_j = j^{-\alpha}$ with $1 < \alpha < 2$, then*

$$M_L = (\rho - \rho_c)L^d - L^{d/\alpha}Y_L,$$

where Y_L converges in law to the α -stable variable defined in (7.12).

(b) *If $\theta_j = j^{-\alpha}$ with $\alpha > 2$ or $\theta_j = \exp(-j^\gamma)$ with $0 < \gamma < \frac{1}{2}$, then*

$$M_L = (\rho - \rho_c)L^d + \sigma_c L^{d/2} Y_L$$

where Y_L converges in law to a standard normal variable.

(c) *If $\theta_j = j \exp(-j^\gamma)$ with $\frac{1}{2} \leq \gamma < 1$, then*

$$M_L = (\rho - \rho_c)L^d - \Delta_L + \sigma_c L^{d/2} Y_L$$

where Y_L converges in law to a standard normal variable, and Δ_L is the deterministic shift of order $L^{d\gamma} \geq L^{d/2}$ defined in (7.10).

The next proposition is a variant of the non-spatial Theorem 3.2 in [16].

Proposition 7.2. *Suppose that $\rho > \rho_c$ and that the weights are from one of the cases in Theorem 7.1. Let $(N_j)_{j \geq 1}$ be independent Poisson random variables with parameters θ_j/j . Then $\sum_{j \geq 1} jN_j$ is almost surely finite, and under $\mathbb{P}_{L,n}$ we have the convergence in law*

$$M_L - T_L \xrightarrow{d} \sum_{j \geq 1} jN_j. \quad (7.13)$$

The proof is given at the end of this section. Proposition 7.2 implies that for all sequences $a_L \rightarrow \infty$, $\mathbb{P}_{L,n}(M_L - T_L \geq a_L) \rightarrow 0$. Therefore T_L and M_L have the same fluctuations, and we have proved the following:

Theorem 7.3. *Theorem 7.1 holds true for T_L as well as M_L .*

Thus we have proved a limit law for the largest component of a Gibbs partition of n with probability weights

$$\frac{1}{Z_{L,n}} \prod_{j \geq 1} \frac{1}{r_j!} \left(L^d \frac{\theta_j}{j}\right)^{r_j} \quad (7.14)$$

(see Proposition 2.1), and Eqs. (7.8) and (7.11) provide, up to the factor $\exp(-L^d \sum_j \theta_j/j)$, the precise asymptotics of the partition function $Z_{L,n}$.

From a mathematical point of view, Theorem 7.3 is new⁴ and it highlights similarities between limit laws for the Gibbs partition (7.14) and sums of i.i.d. variables. These similarities might seem surprising at first, but we have given a natural explanation by providing a coupling (the measure $\mathbb{P}_{L,n}$) to a model with i.i.d. variables (the H_x s).

From a physical point of view, the coupling $\mathbb{P}_{L,n}$ establishes an explicit relation between stationary measures of the zero-range process on the one hand and the ideal Bose gas and classical droplet models on the other hand. The former is often used for driven systems that are *out of equilibrium*, the latter belong to *equilibrium* statistical mechanics. It is amusing to note, moreover, that the intricacies of the large deviations of stretched exponential variables with $\gamma \geq \frac{1}{2}$ have a natural physical interpretation in the droplet model, and the non-normal fluctuations are related to the critical point of fluids.

Indeed, it is common to model the free energy of a droplet of volume j as a bulk term proportional to j plus a surface term j^γ with $\gamma = (d - 1)/d$ [51]. Assume that the bulk term is zero and set $\mu_{\text{sat}} = 0$ (the chemical potential at which the gas saturates). Then $p(\mu) = \sum_j (\theta_j/j) e^{\mu j}$ is the pressure of the gas, and Eq. (7.6) gives an asymptotic expansion of the free energy

$$\begin{aligned} f(\rho) &= \sup_{\mu} (\mu\rho - p(\mu)) \\ &= f(\rho_{\text{sat}}) + \frac{1}{2\sigma^2}(\rho - \rho_{\text{sat}})^2 - \sum_{k \geq 0} \lambda_k (\rho - \rho_{\text{sat}})^{3+k}, \end{aligned} \tag{7.15}$$

as $\rho \nearrow \rho_{\text{sat}}$. The variance σ^2 is proportional to the isothermal compressibility κ_T of the saturated gas [51]: A formal manipulation in thermodynamics notation shows that

$$\frac{1}{\kappa_T} = -V \frac{\partial p}{\partial V} \Big|_T = V \frac{\partial^2}{\partial V^2} \{Vf(\frac{N}{V})\} = \rho^2 \frac{\partial^2 f}{\partial \rho^2}(\rho) \tag{7.16}$$

which converges to $\frac{\rho_{\text{sat}}^2}{\sigma^2}$ as $\rho \nearrow \rho_{\text{sat}}$. Here, V is the volume, $N = \rho V$ the particle number, and $F = Vf(N/V)$ the extensive free energy. Eq. (7.7) is up to an additive constant the free energy of supersaturated gas in coexistence with a large droplet; the shift $\Delta_L \propto L^{d-1} \propto n^{(d-1)/d}$ is natural because surface tension tends to make the condensate droplet smaller. This gives a physical interpretation to the large deviations for stretched exponential variables with $\gamma = (d - 1)/d \geq 1/2$.

The anomalous fluctuations arise when the compressibility $\kappa_T \propto \sigma^2$ becomes infinite, i.e., at the critical point of the fluid. In the Fisher droplet model, this is achieved by assigning to each droplet a “surface” free energy more general than $j^{(d-1)/d}$, with temperature-dependent parameters; see the review [51] and the references therein.

Proof of Proposition 7.2. Observe that for $k \geq 0$,

$$\mathbb{P}\left(\sum_{j \geq 1} jN_j = k\right) = h_k e^{-\sum_{j \geq 1} \theta_j/j} \tag{7.17}$$

(with $h_0 = 1$). By Eq. (2.11) the right side adds up to 1 when k is summed over $k \geq 0$, so that $\sum_{j \geq 1} jN_j$ is almost surely finite. Let S_n be the size of the largest partition element, at the site of higher occupation. Clearly $S_L \leq T_L \leq M_L$; we show below that $S_L = T_L$ with high probability. Fix $k \in \mathbb{N}_0$ and let $\delta > 0$ with $\delta < 1 - \rho_c/\rho$. Choose n large enough so that $\delta n - k > k$. Using Proposition 2.2, we get

$$\mathbb{P}_{L,n}(S_L = M_L - k) = \sum_{m \geq \delta n} \frac{\theta_{m-k}}{m-k} \frac{h_k}{h_m} \mathbb{P}_{L,n}(M_L = m) + \mathbb{P}_{L,n}(S_L = M_L - k, M_L \leq \delta n), \tag{7.18}$$

⁴Bogachev and Zeindler prove the existence of a giant cycle of size of order $(1 - \rho_c/\rho)n$ for a related model of random permutations, but without results on its fluctuations [17, Corollary 5.14].

hence

$$\begin{aligned} & \left| \mathbb{P}_{L,n}(S_L = M_L - k) - \mathbb{P}\left(\sum_{j \geq 1} jN_j = k\right) \right| \\ & \leq 2\mathbb{P}_{L,n}(M_L \leq \delta n) + \sup_{m \geq \delta n} \left| \frac{\theta_{m-k} h_k}{m-k h_m} - h_k \exp\left(-\sum_{j \geq 1} \frac{\theta_j}{j}\right) \right|. \end{aligned} \quad (7.19)$$

The first term vanishes because $\frac{1}{n}M_L \rightarrow 1 - \frac{\rho_c}{\rho} > \delta$; the second term vanishes because (θ_j/j) is discrete subexponential (Theorems 2.4 and 2.5). Thus we have shown that for all $k \geq 0$,

$$\lim_{\substack{n \rightarrow \infty \\ \rho L^d = n}} \mathbb{P}_{L,n}(S_L = M_L - k) = \mathbb{P}\left(\sum_{j \geq 1} jN_j = k\right), \quad (7.20)$$

that is, $M_L - S_L$ converges in law to $\sum_{j \geq 1} jN_j$. It follows in particular that $\frac{1}{n}S_n \rightarrow 1 - \frac{\rho_c}{\rho}$. To conclude, we note that since $S_L \leq T_L$, we have

$$\mathbb{P}_{L,n}(T_L \neq S_L) \leq \mathbb{P}_{L,n}(T_L > S_L \mid S_L \geq \delta n) + \mathbb{P}_{L,n}(S_L < \delta n). \quad (7.21)$$

The last term vanishes because $\frac{1}{n}S_L \rightarrow 1 - \frac{\rho_c}{\rho}$ and $\delta < 1 - \rho_c/\rho$. The first term in the right side is bounded by the conditional probability that there is a second site, in addition to the most occupied one, whose occupation is at least δn . With the help of Eqs. (7.8) and (7.11) one can show that the probability of the latter event goes to zero. Hence $\mathbb{P}_{L,n}(T_L \neq S_L) \rightarrow 0$ and we conclude with the help of Eq. (7.20). \square

In forthcoming work [32] we will introduce analytic continuations, known as *Lindelöf integrals*, of the grand canonical ensembles studied in this paper and, by methods of Hardy space analysis and contour deformation, show how the results of this section may be extended to a broader class of decaying weights in which the weights θ_j arise as the values, at positive integers j , of a function $p(\xi)$ analytic for $\xi \in \mathbb{C} \setminus (-\infty, 0]$.

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