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The scaling limit of uniform random plane maps, *via* the Ambjørn–Budd bijection

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Abstract

We prove that a uniform rooted plane map with n edges converges in distribution after a suitable normalization to the Brownian map for the Gromov–Hausdorff topology. A recent bijection due to Ambjørn and Budd allows us to derive this result by a direct coupling with a uniform random quadrangulation with n faces.

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1 Introduction

1.1 Context

The topic of limits of random maps has met an increasing interest over the last two decades, as it is recognized that such objects provide natural model of discrete and continuous 2-dimensional geometries [4, 5]. Recall that a plane map is a cellular embedding of a finite graph (possibly with multiple edges and loops) into the sphere, considered up to orientation-preserving homeomorphisms. By *cellular*, we mean that the faces of the map (the connected components of the complement of edges) are homeomorphic to 2-dimensional open disks. A popular setting for studying scaling limits of random maps is the following. We see a map $\boldsymbol{m}$ as a metric space by endowing the set $V(\boldsymbol{m})$ of its vertices with its natural graph metric $d_{\boldsymbol{m}}$: the graph distance between two vertices is the minimal number of edges of a path linking them. We then choose at random a map of "size" n in a given class and look at the limit as $n \to \infty$ in the sense of the Gromov–Hausdorff topology [13] of the corresponding metric space, once rescaled by the proper factor.

This question first arose in [10], focusing on the class of plane quadrangulations, that is, maps whose faces are of degree 4, and where the size is defined as the number of faces.

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A series of papers, including [21, 14, 23, 15, 9], have been motivated by this question and contributed to its solution, which was completed in [16, 24] by different approaches. Specifically, there exists a random compact metric space $\boldsymbol{S}$ called *the Brownian map* such that, if Q_n denotes a uniform random (rooted) quadrangulation with n faces, then the following convergence holds in distribution for the Gromov–Hausdorff topology on the set of isometry classes of compact metric spaces:

$$\left(\frac{9}{8n}\right)^{1/4} Q_n \xrightarrow[n\to\infty]{(d)} \boldsymbol{S}. \tag{1.1}$$

Here and later in this paper, if $\mathbb{X} = (X, d)$ is a metric space and $a > 0$, we let $a\mathbb{X} = (X, ad)$ be the rescaled space, and we understand a map $\boldsymbol{m}$ as the metric space $(V(\boldsymbol{m}), d_{\boldsymbol{m}})$.

Le Gall [16] also gave a general method to prove such a limit theorem in a broader context, that applies in particular to uniform p-angulations (maps whose faces are of degree p) for any $p \in \{3, 4, 6, 8, 10 \ldots\}$. When this method applies, the scaling factor $n^{-1/4}$ and the limiting metric space $\boldsymbol{S}$ are the same, only the scaling constant $(9/8)^{1/4}$ may differ. One says that the Brownian map possesses a property of universality, and one actually expects the method to work for many more "reasonable" classes of maps. Roughly speaking, this approach relies on two ingredients:

(i) A bijective encoding of the class of maps by a family of labeled trees that converge to the Brownian snake, in which the labels represent the distances to a uniform point of the map.

(ii) A property of invariance under re-rooting for the model under consideration and for the limiting space $\boldsymbol{S}$.

Interestingly enough, as of now, the only known method to derive the invariance under re-rooting of the Brownian map needed in (ii) is by using the convergence of some root-invariant discrete model to the Brownian map, as in (1.1). A robust and widely used bijective encoding in obtaining (i) is the Cori–Vauquelin–Schaeffer bijection [11, 26] and its generalization by Bouttier–Di Francesco–Guitter [7], see for instance [20, 22]. However, this bijection becomes technically uneasy to manipulate when dealing with non-bipartite maps (with the notable exception of triangulations) or maps with topological constraints: see [6] where Beltran and Le Gall consider quadrangulations without vertices of degree 1. Recently, Addario-Berry and Albenque [2] obtained the convergence to the Brownian map for the classes of simple triangulations and simple quadrangulations (maps without loops or multiple edges), by using another bijection due to Poulalhon and Schaeffer [25].

In the present paper, we continue this line of research with another fundamental class of maps, namely uniform random plane maps with a prescribed number of edges. The key to our study is to use a combination of the Cori–Vauquelin–Schaeffer bijection, together with a recent bijection due to Ambjørn and Budd [3], that allows us to couple directly a uniform (pointed) map with n edges and a uniform quadrangulation with n faces, while preserving distances asymptotically. This allows the transfer of known results from uniform quadrangulations to uniform maps, in a way that is comparatively easier than a method based on the Bouttier–Di Francesco–Guitter bijection. However, and this was a bit of a surprise to us, proving the appropriate re-rooting invariance necessary to apply (ii) above does require some substantial work.

We note that our results answer a question asked in the very recent preprint [8]. Let us also mention that, in parallel to our work, Céline Abraham [1] has obtained a similar result to ours for uniform bipartite maps, by using an approach based on the Bouttier–Di Francesco–Guitter bijection.

1.2 Main results

We need to introduce some notation and terminology at this point. If e is an oriented edge of a map, the face that lies to the left of e will be called the face *incident* to e. We denote by e^-, e^+ and $\mathrm{rev}(e)$ the origin, end and reverse of the oriented edge e. It will be convenient to consider *rooted* maps, that is, maps given with a distinguished oriented edge called the *root*, and usually denoted by e_*. The *root vertex* is by definition the vertex e_*^-.

We let $\mathcal{M}_n$ be the set of rooted plane maps with n edges, and $\mathcal{M}_n^\bullet$ be the set of rooted and *pointed* plane maps with n edges, i.e., of pairs $(\boldsymbol{m}, v_*)$ where $\boldsymbol{m} \in \mathcal{M}_n$ and v_* is a distinguished element of $V(\boldsymbol{m})$.

Similarly, we let $\mathcal{Q}_n$ (resp. $\mathcal{Q}_n^\bullet$) be the set of rooted (resp. rooted and pointed) quadrangulations with n faces. We also let $\mathbb{T}_n$ be the set of well-labeled trees with n edges, i.e., of pairs $(\boldsymbol{t}, \boldsymbol{l})$ where $\boldsymbol{t}$ is a rooted plane tree with n edges, and $\boldsymbol{l}$ is an integer-valued label function on the vertices of $\boldsymbol{t}$ that assigns the value 0 to the root vertex of $\boldsymbol{t}$, and such that $|\boldsymbol{l}(u) - \boldsymbol{l}(v)| \leqslant 1$ whenever u and v are neighboring vertices in $\boldsymbol{t}$. Note that we do not require the label function to take positive values, as it is sometimes the case in the literature.

There exists a well-known correspondence, sometimes called the *trivial bijection*, between the sets $\mathcal{M}_n$ and $\mathcal{Q}_n$. Starting from a rooted map $\boldsymbol{m}$, we add a vertex inside each face of $\boldsymbol{m}$, and join this vertex to every corner of the corresponding face by a family of non-crossing arcs. If we remove the relative interiors of the edges of $\boldsymbol{m}$, then the map formed by the added arcs is a quadrangulation $\boldsymbol{q}$, which we can root in a natural way from the root of $\boldsymbol{m}$ by fixing some convention. In this construction, the set of vertices of $\boldsymbol{m}$ is exactly the set $V_0(\boldsymbol{q})$ of vertices of $\boldsymbol{q}$ that are at even distance from the root vertex: this comes from the natural bipartition $V_0(\boldsymbol{q}) \sqcup V_1(\boldsymbol{q})$ of $V(\boldsymbol{q})$ given by the vertices of $\boldsymbol{m}$ and the vertices that are added in the faces of $\boldsymbol{m}$.

However, the graph distances in $\boldsymbol{m}$ and those in $\boldsymbol{q}$ are not related in an obvious way, except for the elementary bound

$$d_{\boldsymbol{q}}(u,v) \leqslant 2 d_{\boldsymbol{m}}(u,v) \leqslant \frac{\Delta(\boldsymbol{m})}{2}\, d_{\boldsymbol{q}}(u,v) \qquad u,v \in V(\boldsymbol{m}) = V_0(\boldsymbol{q}),$$

where $\Delta(\boldsymbol{m})$ denotes the largest degree of a face in $\boldsymbol{m}$. On the other hand, it was noticed recently by Ambjørn and Budd [3] that there exists another natural bijection between $\mathcal{M}_n^\bullet \times \{0,1\}$ and $\mathcal{Q}_n^\bullet$, which is much more faithful to graph distances. This bijection is constructed in a way that is very similar to the well-known Cori–Vauquelin–Schaeffer (CVS) bijection between $\mathcal{Q}_n^\bullet$ and $\mathbb{T}_n \times \{0,1\}$, and is in some sense dual to it. For the reader's convenience, we will introduce the two bijections simultaneously in Section 2.

The Ambjørn–Budd (AB) bijection provides a natural coupling between a uniform random element (Q_n, v_*) of $\mathcal{Q}_n^\bullet$, and a uniform random element $(M_n^\bullet, v_*)$ of $\mathcal{M}_n^\bullet$. Using this coupling, it was observed already [3, 8] that the "two-point functions" that govern the limit distribution of the distances between two uniformly chosen points in $M_n^\bullet$ and Q_n coincide. In this paper, we show that much more is true.

Theorem 1.1. *Let (Q_n, v_*) and $(M_n^\bullet, v_*)$ be uniform random elements of $\mathcal{Q}_n^\bullet$ and $\mathcal{M}_n^\bullet$ respectively, that are in correspondence via the Ambjørn–Budd bijection. Then we have the following joint convergence in distribution for the Gromov–Hausdorff topology*

$$\left(\left(\frac{9}{8n}\right)^{1/4} M_n^\bullet, \left(\frac{9}{8n}\right)^{1/4} Q_n\right) \xrightarrow[n\to\infty]{(d)} (\boldsymbol{S}, \boldsymbol{S}),$$

where $\boldsymbol{S}$ is the Brownian map.

A very striking aspect of this is that the scaling constant $(9/8)^{1/4}$ is the same for $M_n^\bullet$ and for Q_n. This implies in particular that

$$n^{-1/4} \mathrm{d}_{\mathrm{GH}}(M_n^\bullet, Q_n) \xrightarrow[n\to\infty]{P} 0$$

where $\mathrm{d_{GH}}$ is the Gromov–Hausdorff distance beween two compact metric spaces, which, to paraphrase the title of [19], says that "the AB bijection is asymptotically an isometry." Although obtaining this scaling constant is theoretically possible using the methods of [22], the computation would be rather involved.

At the cost of an extra "de-pointing lemma," (Proposition 3.1) this will imply the following result.

Corollary 1.2. *Let M_n be a uniformly distributed random variable in $\mathcal{M}_n$. The following convergence in distribution holds for the Gromov–Hausdorff topology*

$$\left(\frac{9}{8n}\right)^{1/4} M_n \xrightarrow[n\to\infty]{(d)} \boldsymbol{S}$$

where $\boldsymbol{S}$ is the Brownian map.

As was pointed to us by Éric Fusy, it is likely that our methods can also be used to prove convergence of uniform (pointed) bipartite maps with n edges. Indeed, following [8], these are in natural correspondence *via* the AB bijection with pointed quadrangulations with no confluent faces (see below for definitions). In turn, the latter are in correspondence *via* the CVS bijection with "very well-labeled trees," which are elements of $\mathbb{T}_n$ in which the labels of two neighboring vertices differ by exactly 1 in absolute value (this has the effect of replacing the scaling constant $(9/8)^{1/4}$ with $2^{-1/4}$). However, checking the details of this approach still requires some work, and we did not pursue this to keep the length of this paper short, and because this result has already been obtained by Abraham [1] using a more "traditional" and robust bijective method.

In Section 2, we present the two abovementionned bijections. Section 3 is devoted to the comparison between the distributions of M_n and $M_n^\bullet$. Section 4 is dedicated to the heart of the proof of Theorem 1.1, and Section 5 proves the key re-rooting identity (ii).

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2 Cori–Vauquelin–Schaeffer and Ambjørn–Budd bijections

In most of this section, we fix an element $(\boldsymbol{q}, v_*) \in \mathcal{Q}_n^\bullet$, and consider one particular embedding of $\boldsymbol{q}$ in the plane. We label the elements of $V(\boldsymbol{q})$ by their distance to v_*, hence letting $\boldsymbol{l}_+(v) = d_{\boldsymbol{q}}(v, v_*)$. Using the bipartite nature of quadrangulations, each quadrangular face is of either one of two types, which are called *simple* and *confluent*, depending on the pattern of labels of the incident vertices. This is illustrated on Figure 1, where the four edges incident to a face of $\boldsymbol{q}$ are represented in thin black lines, and the four corresponding vertices are indicated together with their respective labels. The Cori–Vauquelin–Schaeffer (CVS) bijection consists in adding one extra "red" arc inside each face, linking the vertex with largest label of simple faces to the next one in the face in clockwise order, and the two vertices with larger label in confluent faces. The Ambjørn–Budd (AB) bijection adopts the opposite rules, adding the "green" arcs to $\boldsymbol{q}$.

The connected graphs whose edge-sets are formed by the arcs of either color (red or green) are obviously embedded graphs.

The Cori–Vauquelin–Schaeffer bijection For the CVS bijection, the "red" embedded graph is a plane tree $\boldsymbol{t}$ with n edges, with vertex set $V(\boldsymbol{t}) = V(\boldsymbol{q}) \setminus \{v_*\}$. This tree also inherits a label function, which is simply the label function $\boldsymbol{l}_+$ above. It also inherits a root from the root e_* of $\boldsymbol{q}$, following a convention that we will not need to describe in detail. What is

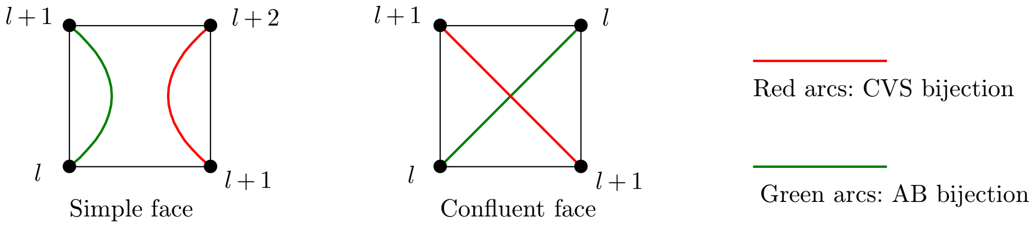

Figure 1: *Convention for adding arcs in the bijections.*

important about the rooting convention, however, is the following. If we are given a vertex v and an oriented edge e in $\boldsymbol{q}$, we say that e *points towards* v if $d_{\boldsymbol{q}}(e^+, v) = d_{\boldsymbol{q}}(e^-, v) - 1$. Then the root vertex of $\boldsymbol{t}$ is equal to e_*^- if e_* points towards v_*, and to e_*^+ otherwise. We let ϵ be respectively equal to 0 or 1 depending on which of these two situations occur.

Remark 2.1. *Throughout this work, we will never consider the root (edge) of the tree; only its* root vertex, *that is, the origin of its root, will be of importance.*

It is then usual to define the label function $\boldsymbol{l}(v) = \boldsymbol{l}_+(v) - \boldsymbol{l}_+(\mathrm{root}(\boldsymbol{t})) = \boldsymbol{l}_+(v) - \boldsymbol{l}_+(e_*^-) - \epsilon$, with values in $\mathbb{Z}$.

Proposition 2.2. *The mapping* $\mathrm{CVS} : \mathcal{Q}_n^\bullet \to \mathbb{T}_n \times \{0,1\}$ *sending the pointed quadrangulation* $(\boldsymbol{q}, v_*)$ *to the pair* $((\boldsymbol{t}, \boldsymbol{l}), \epsilon)$ *as above, is a bijection.*

In the following, we will often omit ϵ from the notation, and will only refer to it when it plays an indispensable role.

The Ambjørn–Budd bijection On the other hand, the "green" embedded graph formed following the rules of the AB bijection is a plane map $\boldsymbol{m}$ with n edges, but with vertex-set equal to $V(\boldsymbol{q}) \setminus V_{\max}(\boldsymbol{q})$, where $V_{\max}(\boldsymbol{q})$ is the set of vertices v of $\boldsymbol{q}$ that are local maxima of the function $\boldsymbol{l}_+$, i.e., such that $d_{\boldsymbol{q}}(u, v_*) = d_{\boldsymbol{q}}(v, v_*) - 1$ for every neighbor u of v. Note that $V_{\max}(\boldsymbol{q})$ really depends on the pointed map $(\boldsymbol{q}, v_*)$ rather than on $\boldsymbol{q}$ alone, but we nevertheless adopt this shorthand notation for convenience. One should note that the distinguished vertex $v_* \in V(\boldsymbol{q})$ is never a local maximum of $\boldsymbol{l}$ (it is indeed the global minimum!), so that it is an element of $V(\boldsymbol{m})$, also naturally distinguished.

By the Euler formula, this implies that $\boldsymbol{m}$ has $\#V_{\max}(\boldsymbol{q})$ faces. One can be more precise by saying that when embedding $\boldsymbol{m}$ and $\boldsymbol{q}$ jointly in the plane as in the above construction, each face of $\boldsymbol{m}$ contains exactly one of the vertices of $V_{\max}(\boldsymbol{q})$. Finally, we can use the root e_* of $\boldsymbol{q}$ to root the map $\boldsymbol{m}$ according to some convention that we will not describe fully, but for which the root vertex of $\boldsymbol{m}$ is equal to e_*^+ if e_* points towards v_*, and to e_*^- otherwise. We let ϵ be equal to 0 or 1 accordingly. See Figure 3 for an example of both bijections.

Proposition 2.3. *The mapping* $\mathrm{AB} : \mathcal{Q}_n^\bullet \to \mathcal{M}_n^\bullet \times \{0,1\}$ *sending the pointed quadrangulation* $(\boldsymbol{q}, v_*)$ *to the pair* $((\boldsymbol{m}, v_*), \epsilon)$ *as above, is a bijection.*

Again, we will usually omit ϵ from the notation. The map $\boldsymbol{m}$ also inherits the labeling function $\boldsymbol{l}_+$ from the quadrangulation $\boldsymbol{q}$, but contrary to what happens for the CVS bijection, this information turns out to be redundant thanks to the remarkable identity

$$d_{\boldsymbol{m}}(v, v_*) = d_{\boldsymbol{q}}(v, v_*) = \boldsymbol{l}_+(v), \qquad v \in V(\boldsymbol{m}) = V(\boldsymbol{q}) \setminus V_{\max}(\boldsymbol{q}). \tag{2.1}$$

In fact, we are going to make this identity slightly more precise by showing that $\boldsymbol{q}$ and $\boldsymbol{m}$ actually "share" some specific geodesics to v_*. In order to specify the exact meaning of this, we need a couple extra definitions. Let e be an oriented edge in $\boldsymbol{q}$, and let f be the face incident to e. We say that e is *special* if the green arc associated with f by the AB bijection is incident to the same two vertices as e (in particular, f must be a simple face). In this case, we let $\tilde{e}$ be this green arc. On the above picture of a simple face, the face is incident to exactly one special edge, which is the one on the left, oriented from top to bottom. More generally, we use the following definition:

Definition 2.4. *If $c = (e_1, e_2, \ldots, e_k)$ is a chain of oriented edges in $\boldsymbol{q}$, in the sense that $e_i^+ = e_{i+1}^-$ for every $i \in \{1, 2 \ldots, k-1\}$, and if all these oriented edges are special, then we say that the chain c is special and we let $\tilde{c} = (\tilde{e}_1, \ldots, \tilde{e}_k)$ be the corresponding chain in $\boldsymbol{m}$.*

Next, if e is an edge of $\boldsymbol{q}$, we can canonically give it an orientation so that it points towards v_*. Then, among all geodesic chains $(e, e_1, \ldots, e_k)$ from e^- to v_* with first step e (so that $k = d_{\boldsymbol{q}}(e^-, v_*) - 1$), there is a distinguished one, called the *left-most geodesic to v_* with first step* e, which is the one for which the clockwise angular sector between e_i and e_{i+1}, and excluding e_{i+1}, contains only edges pointing towards $e_i^+ = e_{i-1}^-$, with the convention that $e_0 = e$. We let $\gamma(e)$ be this distinguished geodesic, and $\hat{\gamma}(e) = (e_1, e_2, \ldots, e_k)$ be the same path, with the first step removed. This is illustrated in the following picture, where two corresponding steps of the geodesic $\gamma(e)$ are depicted.

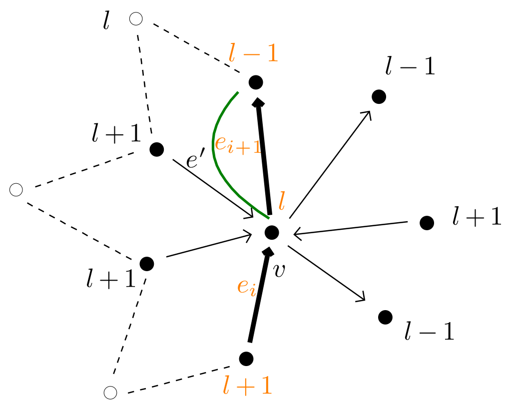

Figure 2: *Two consecutive steps of a left-most geodesic.*

Proposition 2.5. *Let e be an oriented edge of $\boldsymbol{q}$ that points toward v_*. Then the chain $\hat{\gamma}(e)$ is special.*

Proof. Here the reader might want to use Figure 2 to follow the details of this proof. Fix $i \in \{0, 1, \ldots, k-1\}$ and let $v = e_i^+ = e_{i+1}^-$.

Consider the last edge e' before e_{i+1} in clockwise order around v. Then by definition of the left-most geodesic, e' must be pointing towards v. Then the face incident to v that has the sector between e' and e_{i+1} as a corner is necessarily a simple face, and the vertex of this face that is diagonally opposed to v must have label equal to the label $l = d_{\boldsymbol{q}}(v, v_*)$ of v (since the other two labels must be $l + 1 = d_{\boldsymbol{q}}((e')^-, v_*)$ and $l - 1 = d_{\boldsymbol{q}}(e_{i+1}^+, v)$. Therefore, e_{i+1} is the special edge incident to this simple face.

Since by hypothesis $e = e_0$ is pointing towards v_*, this implies by our argument that e_1 is special, and we can conclude by an induction argument. □

Proposition 2.5 has an apparently anecdotal consequence on which, in fact, most of this work relies. Let e, e' be two oriented edges of $\boldsymbol{q}$ pointing towards v_*. The two left-most geodesics $\gamma(e) = (e_0, e_1, \ldots, e_k)$ and $\gamma(e') = (e'_0, e'_1, \ldots, e'_{k'})$ share a maximal common suffix, say $e_{k-r+1} = e'_{k'-r+1}, \ldots, e_k = e'_{k'}$ where $r \geqslant 0$ is the largest possible. (Note that r may be equal to $k+1$ or $k'+1$, in the case where one geodesic is entirely a suffix of the other one.) But then it always holds that $e^+_{k-r} = (e'_{k'-r})^+$, so that the sequence $(e_0, e_1, \dots, e_{k-r}, \mathrm{rev}(e'_{k'-r}), \mathrm{rev}(e'_{k'-r-1}), \dots, \mathrm{rev}(e'_1), \mathrm{rev}(e'_0))$ is a chain, with total length that we denote by $d^\circ_{\boldsymbol{q}}(e, e')$. Recall that $\Delta(\boldsymbol{m})$ denotes the largest face degree of $\boldsymbol{m}$.

Corollary 2.6. *Let $v, v' \in V(\boldsymbol{m}) = V(\boldsymbol{q}) \setminus V_{\max}(\boldsymbol{q})$ be given, and let e, e' be two oriented edges in $\boldsymbol{q}$ both pointing towards v_*, and such that $e^- = v$ and $(e')^- = v'$. Then it holds that*

$$d_{\boldsymbol{m}}(v, v') \leqslant d^\circ_{\boldsymbol{q}}(e, e') + \Delta(\boldsymbol{m}).$$

Proof. We assume that $e \neq e'$ to avoid trivialities. By Proposition 2.5, the geodesics $\hat\gamma(e)$ and $\hat\gamma(e')$ are special, so that there are paths in $\boldsymbol{m}$ starting from e^+ and $(e')^+$ with edges $(\tilde e_1, \dots, \tilde e_k)$ and $(\tilde e'_1, \dots, \tilde e'_{k'})$ respectively. But then the maximal suffix shared by these paths has the same length as the one shared by $\gamma(e)$ and $\gamma(e')$. Therefore, we can join e^+ and $(e')^+$ in $\boldsymbol{m}$ by a path of length $d^\circ_{\boldsymbol{q}}(e, e') - 2$. Now by construction of the AB bijection, the edge e lies in a single face of $\boldsymbol{m}$, so that we can join e^- to e^+ with a path of length at most $\Delta(\boldsymbol{m})/2$. The same is true for the extremities of e', which concludes. ☐

3 Comparing pointed and non-pointed maps

Let M_n be a uniformly distributed random variable in $\mathcal{M}_n$, and let $(M^\bullet_n, v_*)$ be a uniformly distributed random variable in $\mathcal{M}^\bullet_n$. The superscript in $M^\bullet_n$ is here to indicate that, even after forgetting the distinguished vertex v_*, it does not have same distribution as M_n. Rather, it holds that

$$P(M^\bullet_n = \boldsymbol{m}) = \frac{\#V(\boldsymbol{m})}{\#\mathcal{M}^\bullet_n}, \qquad \boldsymbol{m} \in \mathcal{M}_n. \tag{3.1}$$

Note that, by contrast, if (Q_n, v_*) is a uniformly distributed random variable in $\mathcal{Q}^\bullet_n$, then Q_n is indeed uniform in $\mathcal{Q}_n$ since a quadrangulation with n faces has $n+2$ vertices, so that pointing such a quadrangulation does not introduce a bias. The goal of this subsection is to obtain the following comparison theorem for the laws of M_n and $M^\bullet_n$. Let μ_n be the law of M_n and $\mu^\bullet_n$ be the law of $M^\bullet_n$. We let $\|\cdot\|$ denote the total variation norm of signed measures.

Proposition 3.1. *It holds that $\|\mu_n - \mu^\bullet_n\| \to 0$ as $n \to \infty$.*

Proof. By (3.1), one has

$$\|\mu_n - \mu^\bullet_n\| = \sum_{\boldsymbol{m} \in \mathcal{M}_n} \left| \frac{1}{\#\mathcal{M}_n} - \frac{\#V(\boldsymbol{m})}{\#\mathcal{M}^\bullet_n} \right|.$$

Now recall that

$$\#\mathcal{M}_n = \#\mathcal{Q}_n = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}, \qquad \#\mathcal{M}^\bullet_n = \frac{1}{2}\#\mathcal{Q}^\bullet_n = \frac{3^n}{n+1}\binom{2n}{n},$$

where we used the trivial graph bijection between a rooted map with n edges and a rooted quadrangulation with n faces on the one hand, and the AB bijection on the other hand. This implies that

$$\|\mu_n - \mu^\bullet_n\| = \frac{1}{\#\mathcal{M}_n} \sum_{\boldsymbol{m} \in \mathcal{M}_n} \left| \frac{2\,\#V(\boldsymbol{m})}{n+2} - 1 \right| = E\left[\left| \frac{2\,\#V(M_n)}{n+2} - 1 \right| \right]. \tag{3.2}$$

To show that this vanishes as $n \to \infty$, we compute the first two moments of $\#V(M_n)$. Note that by the trivial graph bijection, $\#V(M_n)$ has same distribution as the number of vertices at even distance from the root vertex e_*^- in a uniform rooted quadrangulation Q_n. By an obvious symmetry argument, this implies that

$$E\big[\#V(M_n)\big] = \frac{1}{2}E\big[\#V(Q_n)\big] = \frac{n+2}{2}. \tag{3.3}$$

For the second moment, we use the CVS bijection again. Select a uniform random vertex v_* among the $n+2$ elements of $V(Q_n)$ and let $((T_n,\ell_n),\epsilon) = \mathrm{CVS}(Q_n,v_*)$. Since $\ell_n(v) = d_{Q_n}(v,v_*) - d_{Q_n}(e_*^-,v_*) - \epsilon$ for every $v \in V(Q_n)$, we have that the vertices v at even distance from e_*^- are those for which $\ell_n(v)+\epsilon$ is even. So

$$\begin{aligned} E\big[\#V(M_n)^2\big] &= E\left[\sum_{u,v\in V(T_n)\cup\{v_*\}} \mathbf{1}_{\{\ell_n(u)+\epsilon\equiv\ell_n(v)+\epsilon\equiv 0[\mathrm{mod}\,2]\}}\right] \\ &= (n+2)^2\,P\big(\ell_n(U)+\epsilon\equiv\ell_n(U')+\epsilon\equiv 0[\mathrm{mod}\,2]\big). \end{aligned}$$

where U, U' are uniformly chosen in $V(T_n)\cup\{v_*\}$ conditionally given T_n and independently of (ℓ_n,ϵ).

Plainly, the probability under consideration is equivalent to the same quantity where U, U' are instead chosen uniformly in $V(T_n)$. Furthermore, conditionally given T_n, U, U', the labels along the branch from U to U' in T_n form a random walk with i.i.d. steps that are uniform in $\{-1,0,1\}$, and thus the parity of the labels follow an irreducible Markov chain with values in {0,1} with transition matrix $\begin{pmatrix}1/3 & 2/3\\ 2/3 & 1/3\end{pmatrix}$ and stationary measure $(1/2,1/2)$. It follows that the probability that $\ell_n(U)$ and $\ell_n(U')$ have same parity is a function of $d_{T_n}(U,U')$ with limit $1/2$ at infinity, while the probability that $\ell_n(U)+\epsilon$ is even is exactly $1/2$ since ϵ is a $\mathrm{Bernoulli}(1/2)$ random variable independent of (T_n,ℓ_n,U,U'). On the other hand, it is classical that $d_{T_n}(U,U')/\sqrt{2n}$ converges to a Rayleigh distribution as $n\to\infty$, so that $d_{T_n}(U,U')$ converges to ∞ in probability. These facts easily entail that $P(\ell_n(U)+\epsilon\equiv\ell_n(U')+\epsilon\equiv 0[\mathrm{mod}\,2])$ converges to $1/4$ as $n\to\infty$. Consequently,

$$E\big[\#V(M_n)^2\big] = \frac{n^2}{4}\big(1+o(1)\big), \qquad \text{as } n\to\infty. \tag{3.4}$$

Together, equations (3.3) and (3.4) imply that $2\,\#V(M_n)/n$ converges to 1 in L^2, which entails the result by (3.2). □

We deduce a bound in probability for $\Delta(M_n^\bullet)$: Theorem 3 of Gao and Wormald [12] shows that if $\Delta^V(M_n)$ denotes the largest degree of a vertex in M_n, then

$$P(\Delta^V(M_n) > \log n) \to 0 \qquad \text{as } n\to\infty.$$

From the obvious fact that the dual map of M_n has same distribution as M_n, the same is true if we replace $\Delta^V(M_n)$ with $\Delta(M_n)$. By Proposition 3.1 we conclude that the same holds for $M_n^\bullet$.

Lemma 3.2. *It holds that as $n\to\infty$,*

$$P(\Delta(M_n^\bullet) > \log n) \to 0.$$

4 Encoding with processes and convergence results

We now proceed by following the general approach introduced by Le Gall [14, 16], which we mentioned in the Introduction. It first requires to code maps with stochastic processes. Let (Q_n,v_*) be a uniform random element of $\mathcal{Q}_n^\bullet$, and let $((T_n,\ell_n),\epsilon) = \mathrm{CVS}(Q_n,v_*)$

and $((M_n^\bullet, v_*), \epsilon) = \mathrm{AB}(Q_n, v_*)$. Since CVS and AB are bijections, the random variables (T_n, l_n) and $(M_n^\bullet, v_*)$ are respectively uniform in $\mathbb{T}_n$ and $\mathcal{M}_n^\bullet$, while ϵ is uniform in {0,1} and independent of (T_n, ℓ_n) and $(M_n^\bullet, v_*)$. Note that our conventions imply that the variable ϵ is indeed the same in the images by the two bijections.

4.1 Coding with discrete processes

For $i \in \{0, 1, \dots, 2n\}$ we let c_i be the i-th corner of T_n in contour order, starting from the root corner, so in particular $c_0 = c_{2n}$. We extend this to a sequence $(c_i, i \in \mathbb{Z})$ by $2n$-periodicity. Let also v_i be the vertex of T_n that is incident to c_i. The contour and label functions of (T_n,ℓ_n) are defined by

$$C_n(i) = d_{T_n}(v_i, v_0), \qquad L_n(i) = \ell_n(v_i), \qquad i \in \{0, 1, \dots, 2n\},$$

and these functions are extended to continuous functions $[0, 2n] \to \mathbb{R}$ by linear interpolation between integer coordinates. Now recall that the sets $V(T_n)$ and $V(Q_n) \setminus \{v_*\}$ are identified by the CVS bijection, so that we can view v_i, $0 \leqslant i \leqslant 2n$ as elements of $V(Q_n)$. With this identification we let

$$D_n(i, j) = d_{Q_n}(v_i, v_j), \qquad i, j \in \{0, 1, \dots, 2n\},$$

and we extend D_n to a continuous function $[0, 2n]^2 \to \mathbb{R}$ by linear interpolation between integer coordinates, successively on each coordinate.

We now recall how the mapping CVS^{-1} is constructed. Starting from a given plane embedding of T_n, we add the extra vertex v_* arbitrarily in the unique face f of the map T_n, and declare it to be incident to a unique corner that we denote by c_∞. Next, for every $i \in \mathbb{Z}$ we let $s(i) = \inf\{j > i : L_n(j) = L_n(i)-1\}$, which we call the successor of i. Note that $s(i) = \infty$ if $L_n(i) = \min L_n$. The successor of the corner c_i is then $s(c_i) = c_{s(i)}$ by definition. The construction then consists in drawing an arc e_i from c_i to $s(c_i)$ for every $i \in \{0, 1, \dots, 2n-1\}$, in such a way that these arcs do not cross each other, and that the relative interior of e_i is contained in f. This construction uniquely defines a map, which is Q_n, and this map is pointed at v_* (here again, we will not specify the rooting convention). By construction, there is a one-to-one correspondence between the corners c_i of T_n and the edges e_i of Q_n. It turns out that the natural orientation of e_i obtained in the construction (that is, from v_i to $v_{s(i)}$) coincides with the orientation that we introduced above for quadrangulations, namely, e_i points towards v_* in Q_n. Consequently, the oriented paths following the arcs are geodesics towards v_*. See Figure 3.

We let, for i, $j \in \{0, 1, \dots, 2n\}$,

$$D_n^\circ(i, j) = L_n(i) + L_n(j) - 2\max(\check{L}_n(i, j), \check{L}_n(j, i)) + 2.\mathbf{1}_{\{i \not\dashrightarrow j \text{ and } j \not\dashrightarrow i\}},$$

where $\check{L}_n(i, j) = \inf\{L_n(k) : i \leqslant k \leqslant j\}$ if $i \leqslant j$, $\check{L}_n(i, j) = \inf\{L_n(k) : k \in [0, j] \cup [i, 2n]\}$ if $i > j$, and the notation $i \not\dashrightarrow j$ means that $\{s^k(i) : k \geq 0\} \cap (j + 2n\mathbb{Z}) = \varnothing$. The somehow unusual indicator in this definition only serves the purpose to match our definition of $d_{\boldsymbol{q}}^\circ$.

Lemma 4.1. *Let e be an edge of Q_n, and let c be the corner of T_n such that e is the arc linking c with $s(c)$. Let $k = d_{\boldsymbol{q}}(e^-, v_*) - 1$ and for $0 \leqslant i \leqslant k$ let $e_{(i)}$ be the arc going from $s^i(c)$ to $s^{i+1}(c)$. Then the chain $(e_{(0)}, e_{(1)}, \dots, e_{(k)})$ is the left-most geodesic to v_* with first step e. Consequently,*

$$D_n^\circ(i, j) = d_{Q_n}^\circ(e_i, e_j), \qquad i, j \in \{0, 1, \dots, 2n\}.$$

Proof. Fix $i \in \{0, 1, \dots, k\}$. By construction, every arc between $e_{(i)}$ and $e_{(i+1)}$ in the clockwise order around $e_{(i)}^+$ is necessarily pointing toward $e_{(i)}^+$. The first claim easily follows. The second claim follows by noticing that the event $\{i \not\dashrightarrow j \text{ and } j \not\dashrightarrow i\}$ says that neither of the left-most geodesic to v_* with first steps e_i or e_j is a suffix of the other. ☐

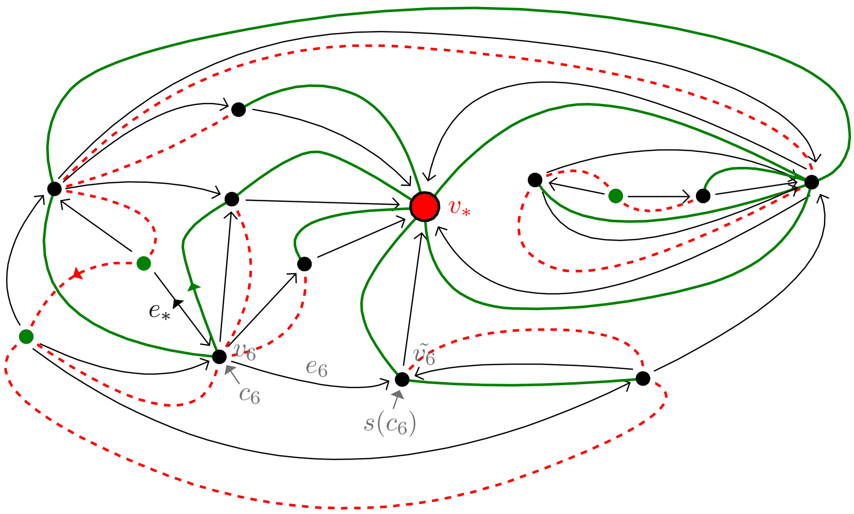

Figure 3: *The two bijections, and some notation. The three green vertices correspond to the three faces of the map obtained by the* AB *bijection.*

We now define a function $\tilde{D}_n$ similar to D_n but associated with the map $M_n^\bullet$. Recall that e_i is the arc of Q_n from the corner c_i of T_n to $s(c_i)$. We let $\tilde{v}_i = e_i^+$ so that for every $i \in \{0, 1, \dots, 2n\}$, $\tilde{v}_i$ is always an element of $V(M_n^\bullet)$. Set

$$\tilde{D}_n(i,j) = d_{M_n^\bullet}(\tilde{v}_i, \tilde{v}_j), \qquad i, j \in \{0, 1, \dots, 2n\}.$$

We also extend $\tilde{D}_n$ to a continuous function $[0, 2n]^2 \to \mathbb{R}$ as we did for D_n. Clearly, the set $\{\tilde{v}_i : i \in \{0, 1, \dots, 2n\}\}$ is equal to $V(M_n^\bullet)$, so that $(\{0, 1, \dots, 2n\}, \tilde{D}_n)$ is a pseudo-metric space isometric to $(V(M_n^\bullet), d_{M_n^\bullet})$ through the mapping $i \mapsto \tilde{v}_i$. Combining Corollary 2.6 and Lemma 4.1, we obtain the bound

$$\tilde{D}_n(i,j) \leqslant D_n^\circ(i,j) + \Delta_n, \qquad i, j \in \{0, 1, \dots, 2n\}, \tag{4.1}$$

where $\Delta_n := \Delta(M_n^\bullet)$, and this remains true for every $s, t \in [0, 2n]$ in place of i, j.

4.2 Scaling limits and proof of Theorem 1.2

We now introduce renormalized versions of our encoding processes. Namely, for $s, t \in [0, 1]$, let

$$C_{(n)}(s) = \frac{C_n(2ns)}{\sqrt{2n}}, \quad L_{(n)}(s) = \left(\frac{9}{8n}\right)^{1/4} L_n(2ns), \quad D_{(n)}(s,t) = \left(\frac{9}{8n}\right)^{1/4} D_n(2ns, 2nt)$$

and define $D_{(n)}^\circ(s,t)$ and $\tilde{D}_{(n)}(s,t)$ similarly to $D_{(n)}$ by replacing D_n with D_n° and $\tilde{D}_n$. The main result of [16, 24] (which implies (1.1)) shows that one has the following convergence in distribution as $n \to \infty$ in $\mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1], \mathbb{R}) \times \mathcal{C}([0,1]^2, \mathbb{R})$:

$$(C_{(n)}, L_{(n)}, D_{(n)}) \longrightarrow (\boldsymbol{e}, Z, D), \tag{4.2}$$

where $(\boldsymbol{e}, Z)$ is a pair of stochastic processes sometimes called the head of the Brownian snake, and D is a random pseudo-distance on $[0, 1]$ defined from $(\boldsymbol{e}, Z)$ as follows. Define two pseudo-distances on $[0, 1]$ by the formulas

$$d_{\boldsymbol{e}}(s,t) = \boldsymbol{e}(s) + \boldsymbol{e}(t) - 2 \min\{\boldsymbol{e}(u) : s \wedge t \leqslant u \leqslant s \vee t\}$$

and

$$d_Z(s,t) = Z(s) + Z(t) - 2\max(\check{Z}(s,t), \check{Z}(t,s)),$$

where similarly as for the definition of D_n° we let $\check{Z}(s,t) = \min\{Z(u) : s \leqslant u \leqslant t\}$ if $s \leqslant t$, and $\check{Z}(s,t) = \min\{Z(u) : u \in [s,1] \cup [0,t]\}$ otherwise. Then D is the largest pseudo-distance d on $[0,1]$ that satisfies the following two properties:

$$\{d_{\boldsymbol{e}} = 0\} \subset \{d = 0\} \qquad \text{and} \qquad d \leqslant d_Z. \tag{4.3}$$

At this point, we recall that the Brownian map $\boldsymbol{S}$ is the quotient space $[0,1]/\{D = 0\}$, endowed with the (true) distance function induced by D on this set, which we still denote by D.

We would like to study the joint convergence of (4.2) with $\tilde{D}_{(n)}$, and show that the limit of the latter is D as well. To this end, we proceed in three steps.

First step: tightness We observe that (4.2) implies that $D^\circ_{(n)}$ converges (jointly) to d_Z. On the other hand, the bound (4.1) combined with Lemma 3.2 easily implies that the laws of $\tilde{D}_{(n)}$, $n \geqslant 1$, form a relatively compact family of probability measures on $\mathcal{C}([0,1]^2, \mathbb{R})$, by repeating the argument of [14]. Indeed, for every $\delta > 0$, let

$$\omega(\tilde{D}_{(n)}, \delta) = \sup\left\{ |\tilde{D}_{(n)}(s,t) - \tilde{D}_{(n)}(s',t')| : |s-s'| \vee |t-t'| \leqslant \delta \right\}$$

be the modulus of continuity of $\tilde{D}_{(n)}$ evaluated at δ, so by the triangle inequality and (4.1), we have

$$\begin{aligned} \omega(\tilde{D}_{(n)}, \delta) &\leqslant 2 \sup\left\{ \tilde{D}_{(n)}(s,s') : |s-s'| \leqslant \delta \right\} \\ &\leqslant 2 \sup\left\{ D^\circ_{(n)}(s,s') : |s-s'| \leqslant \delta \right\} + \frac{\Delta_n}{(8n/9)^{1/4}}. \end{aligned}$$

It follows from Lemma 3.2 and the convergence in distribution (4.2) that for every $\varepsilon > 0$,

$$\limsup_{n\to\infty} P\big(\omega(\tilde{D}_{(n)}, \delta) \geqslant \varepsilon\big) \leqslant P\big(2\sup\{d_Z(s,s') : |s-s'| \leqslant \delta\} \geqslant \varepsilon\big)$$

and the a.s. continuity of Z implies that this converges to 0 as $\delta \to 0$. Since $\tilde{D}_{(n)}(0,0) = 0$, this entails the requested tightness result.

Hence, up to extraction of a subsequence (n_k), we may assume that

$$\big(C_{(n)}, L_{(n)}, D^\circ_{(n)}, D_{(n)}, \tilde{D}_{(n)}, n^{-1/4}\Delta_n\big) \longrightarrow \big(\boldsymbol{e}, Z, d_Z, D, \tilde{D}, 0\big) \tag{4.4}$$

in distribution, where $\tilde{D}$ is a random continuous function on $[0,1]^2$. In order to simplify the arguments to follow, we apply the Skorokhod representation theorem, and assume that *the underlying probability space is chosen so that this convergence holds almost surely rather than in distribution*. Until the end of the paper, all the convergences as $n \to \infty$ are understood to take place along this subsequence (n_k).

Second step: bound on $\tilde{D}$ It is not difficult to check that $\tilde{D}$ is a pseudo-distance, because $\tilde{D}_{(n)}$ is already symmetrical and satisfies the triangle inequality, and because $\tilde{D}_{(n)}(s,s) = 0$ as soon as s is in $\{k/2n : k \in \{0,1,\ldots,2n\}\}$. Let us prove that $\tilde{D}$ satisfies the properties appearing in (4.3). First, assume that $d_{\boldsymbol{e}}(s,t) = 0$. Then it is elementary to see that there are sequences of integers i_n, j_n such that $i_n/2n$ and $j_n/2n$ respectively converge to s and t, and such that $v_{i_n} = v_{j_n}$. As a consequence, it holds that $\tilde{v}_{i_n}$ and $\tilde{v}_{j_n}$ lie in the same face or in two adjacent faces of $M_n^\bullet$, and therefore are at distance at most Δ_n in $M_n^\bullet$. Consequently, one has that

$$\tilde{D}(s,t) = \lim_{n\to\infty} \tilde{D}_{(n)}\left(\frac{i_n}{2n}, \frac{j_n}{2n}\right) = \lim_{n\to\infty} \left(\frac{9}{8n}\right)^{1/4} d_{M_n^\bullet}(\tilde{v}_{i_n}, \tilde{v}_{j_n}) = 0,$$

as wanted. Finally, the bound $\tilde{D} \leqslant d_Z$ is a simple consequence of (4.1) and (4.4).

From this and the definition of D as the largest pseudo-distance satisfying (4.3), we obtain that $\tilde{D} \leqslant D$. On the other hand, let s_* be the (a.s. unique [18]) point at which Z attains its minimum. Taking a sequence (i_n) such that $\tilde{v}_{i_n} = v_*$, it is not difficult to see, using the convergence of $L_{(n)}$ to Z, that $i_n/2n$ must converge to s_*. Therefore, by choosing other sequences (j_n) such that $j_n/2n$ converges, it follows from (2.1) that, almost surely,

$$\tilde{D}(s_*, s) = D(s_*, s) = Z_s - Z_{s_*} \qquad \text{for every } s \in [0,1]. \tag{4.5}$$

Third step: re-rooting argument The final crucial property on which the proof relies is that if U_1, U_2 are independent random variables in $[0,1]$ that are also independent of all the previously considered random variables, then

$$\tilde{D}(U_1, U_2) \overset{(d)}{=} \tilde{D}(s_*, U_1). \tag{4.6}$$

The proof of this re-rooting identity is a bit long so that we postpone it to the next Section. Let us see how this concludes the proof of Theorem 1.2. Observe that D also satisfies property (4.6) (which can be obtained using the fact that quadrangulations are invariant under re-rooting, see [16]). Given this and (4.5), we deduce that

$$E\big[\tilde{D}(U_1,U_2)\big] = E\big[\tilde{D}(s_*,U_1)\big] = E\big[D(s_*,U_1)\big] = E\big[D(U_1,U_2)\big],$$

which entails that $\tilde{D}(U_1, U_2) = D(U_1, U_2)$ a.s., since we already know that $\tilde{D} \leqslant D$. By Fubini's theorem, this shows that a.s. $\tilde{D}$ and D agree on a dense subset of $[0,1]^2$, hence everywhere by continuity. The convergence (4.4) can thus in part be rewritten

$$\big(C_{(n)}, L_{(n)}, D_{(n)}, \tilde{D}_{(n)}\big) \xrightarrow[n\to\infty]{(d)} \big(\boldsymbol{e}, Z, D, D\big), \tag{4.7}$$

from which it is easy to deduce Theorem 1.1, using the fact that the Gromov–Hausdorff distance between two metric spaces is bounded by the distortion of any correspondence between these spaces, see for instance Section 3.3 in [17]. Namely, we can assume again that (4.7) holds almost surely rather than in distribution by a further use of the Skorokhod representation theorem. Then, if we denote by $\mathbf{p} : [0,1] \to \boldsymbol{S}$ the canonical projection, we note that the sets $\big\{(v_{\lfloor 2nt\rfloor}, \mathbf{p}(t)) : t \in [0,1]\big\}$ and $\big\{(\tilde{v}_{\lfloor 2nt\rfloor}, \mathbf{p}(t)) : t \in [0,1]\big\}$ are correspondences between, on the one hand, the metric spaces $(V(Q_n) \setminus \{v_*\}, (9/8n)^{1/4} d_{Q_n})$ and $(V(M_n^\bullet), (9/8n)^{1/4} d_{M_n^\bullet})$, and the Brownian map $(\boldsymbol{S}, D)$ on the other hand. Moreover, their distortions are bounded from above by

$$\sup_{s,t\in[0,1]} \big|D_{(n)}(\lfloor 2ns\rfloor/2n, \lfloor 2nt\rfloor/2n) - D(s,t)\big|$$

and

$$\sup_{s,t\in[0,1]} \big|\tilde{D}_{(n)}(\lfloor 2ns\rfloor/2n, \lfloor 2nt\rfloor/2n) - D(s,t)\big|,$$

which both converge to 0 almost surely. This, and the obvious fact that the Gromov-Hausdorff distance between $(V(Q_n), d_{Q_n})$ and $(V(Q_n) \setminus \{v_*\}, d_{Q_n})$ is at most 1, imply Theorem 1.1. Corollary 1.2 follows by Proposition 3.1.

5 Proof of the re-rooting identity

It remains to prove (4.6). This again relies on a limiting argument. Namely, recall that the distinguished point v_* in $M_n^\bullet$ is a uniformly chosen element of $V(M_n^\bullet)$. Therefore, if V_1 and V_2 are two other such elements, chosen independently, and independently of v_*, then it holds trivially that

$$d_{M_n^\bullet}(V_1, V_2) \stackrel{(d)}{=} d_{M_n^\bullet}(v_*, V_1).$$

On the other hand, let (i_n) be a sequence of integers such that $\tilde{v}_{i_n} = v_*$, so that $i_n/2n \to s_*$. If U_1, U_2 are uniform on $[0,1]$ as above, then they naturally code the vertices $\tilde{v}_{\lfloor 2nU_1 \rfloor}, \tilde{v}_{\lfloor 2nU_2 \rfloor}$, and so by (4.4) we have that

$$\left(\frac{9}{8n}\right)^{1/4} d_{M_n^\bullet}(v_*, \tilde{v}_{\lfloor 2nU_1 \rfloor}) \underset{n\to\infty}{\longrightarrow} \tilde{D}(s_*, U_1)$$

and

$$\left(\frac{9}{8n}\right)^{1/4} d_{M_n^\bullet}(\tilde{v}_{\lfloor 2nU_1 \rfloor}, \tilde{v}_{\lfloor 2nU_2 \rfloor}) \underset{n\to\infty}{\longrightarrow} \tilde{D}(U_1, U_2).$$

Therefore, (4.6) would follow directly if the vertices $\tilde{v}_{\lfloor 2nU_1 \rfloor}$ and $\tilde{v}_{\lfloor 2nU_2 \rfloor}$ were uniform in $V(M_n^\bullet)$. Unfortunately, the probability that $\tilde{v}_{\lfloor 2nU_1 \rfloor}$ is equal to a given vertex v of $M_n^\bullet$ is proportional to the number of edges e of $Q_n$pointing towards v_* such that $e^+ = v$. Using the construction of the AB bijection, one can see that this number of edges is precisely the degree of v in $M_n^\bullet$, but we leave this as an exercise to the reader as we are not going to use it explicitly.

On the other hand, (4.6) will follow if $\tilde{v}_{\lfloor 2nU_1 \rfloor}$ can be coupled with a uniformly chosen vertex V_1 in $M_n^\bullet$ in such a way that $d_{M_n^\bullet}(\tilde{v}_{\lfloor 2nU_1 \rfloor}, V_1) = o(n^{1/4})$ almost surely, possibly along a subsequence of (n_k). This is what we now demonstrate, except that the vertex V_1 that we will produce (denoted by v_{j_n} below) will be uniform on $V(M_n^\bullet) \setminus \{v_*\}$ rather than on $V(M_n^\bullet)$. This distinction is of course of no importance.

First recall that $V(M_n^\bullet) = V(Q_n) \setminus V_{\max}(Q_n)$ where $V_{\max}(Q_n)$ was defined in Section 2 as the set of vertices of Q_n whose neighbors are all closer to v_*. With the usual identification of vertices of $V(Q_n) \setminus \{v_*\}$ with $V(T_n)$, we can view the vertices $V_{\max}(Q_n)$ as a subset of $V(T_n)$.

Lemma 5.1. *A vertex $v \in V(T_n)$ is an element of $V_{\max}(Q_n)$ if and only if its label is a local maximum in T_n in the broad sense. Namely, for every vertex u adjacent to v in T_n, it holds that $\ell_n(u) \leqslant \ell_n(v)$.*

Proof. Let $l = \ell_n(v)$. Assume first that one of the neighbors u of v has a label $l+1$. Let c be the last corner of u before visiting v in contour order. Then the successor $s(c)$ in the CVS bijection is by construction a corner incident to v, so that u and v are adjacent in Q_n, but u is further away from v_* than v, so that $v \notin V_{\max}(Q_n)$. Conversely, if a vertex u adjacent to v has label l or $l-1$, consider the maximal subtree of T_n that contains u but not v. Then clearly every corner incident to a vertex in this subtree with label $l+1$ cannot be linked by an arc to v. Moreover, by construction, every corner of v is linked to a vertex with label $l-1$. So if v is a local maximum in T_n in the broad sense, v has no neighbors in Q_n that are further away from v_* than v, so $v \in V_{\max}(Q_n)$. □

If $(\boldsymbol{t}, \boldsymbol{l})$ is a labeled tree, we will let $V_{\max}(\boldsymbol{t}, \boldsymbol{l})$ be the set of vertices of $\boldsymbol{t}$ that are local maxima of $\boldsymbol{l}$ in the broad sense, so the last lemma states that $V_{\max}(Q_n) = V_{\max}(T_n, \ell_n)$.

Now let $N_0 = 0$ and, for $j \in \{1, 2, \dots, 2n\}$, let N_j be the number of vertices in $\{v_0, v_1, \dots, v_{j-1}\}$ that do not belong to $V_{\max}(T_n, \ell_n)$. Note that $N_{2n} = \#V(T_n) - \#V_{\max}(T_n, \ell_n) = \#V(M_n^\bullet) - 1$ (the -1 comes from the fact that $V(T_n) = V(M_n^\bullet) \setminus \{v_*\}$). Fix $t \in [0,1]$ and let

$i = \lfloor 2nt \rfloor$. Let also $v(0), v(1), \ldots, v(h) = v_i$ be the spine consisting of the ancestors of v_i in T_n indexed by their heights, so that $v(0) = v_0$ is the root vertex of T_n and $h = C_n(i)$ is the height of v_i. Note that the vertices $v_0, v_1, \ldots, v_{i-1}, v_i$ are the vertices contained in the subtrees of T_n rooted on $v(0), v(1), \ldots, v(h)$ that lie to the left of the spine, and more specifically, between the root corner c_0 and the corner c_i of T_n. We let $T(0), T(1), \ldots, T(h)$ be these trees, ordered by size, that is, in such a way that $n_0 \geqslant n_1 \ldots \geqslant n_h$ where $n_j = \#E(T(j))$ (we arbitrarily choose in case of ties). Note that $T(j)$ is naturally rooted at the first corner of a vertex $v(k_j)$ visited by the contour exploration of T_n. For $j > h$, we set $n_j = 0$.

We also let L_j be the label function ℓ_n restricted to $T(j)$, and shifted by the label of the root, so that $L_j(u) = \ell_n(u) - \ell_n(v(k_j))$ for $u \in V(T(j))$.

We then note two important facts:

1. Conditionally given $(n_0, n_1, \ldots)$, the labeled trees $(T(0), L_0), \ldots, (T(h), L_h)$ are independent uniform elements of $\mathbb{T}_{n_0}, \mathbb{T}_{n_1}, \ldots, \mathbb{T}_{n_h}$, where $h = \max\{j : n_j > 0\}$.
2. For every $\varepsilon > 0$, there exists $K > 0$ such that, for sufficiently large n, $P(n_0 + n_1 + \cdots + n_K < n(t - \varepsilon)) < \varepsilon$.

The first property is easy. To see why the second is true, note that the contour processes of $T(0), T(1), \ldots, T(h)$ are the excursions of $\big(C_n(s) - \inf\{C_n(u) : s \leqslant u \leqslant i\}, 0 \leqslant s \leqslant i\big)$. The convergence of the rescaled contour function $C_{(n)}$ to the normalized Brownian excursion $\boldsymbol{e}$ then easily implies that for every $j \geqslant 0$, the j+1-th longest of these excursions (the one coding $T(j)$) converges uniformly to the $j + 1$-th longest excursion of $\boldsymbol{e}$ above the process $(\inf\{\boldsymbol{e}(u) : s \leqslant u \leqslant t\}, 0 \leqslant s \leqslant t)$. Note that this excursion is unambiguously defined. This implies that n_j/n converges to the length of the $j+1$-th longest excursion. By standard properties of Brownian motion, these excursion lengths sum to t, and this implies the wanted result.

Now since the label functions L_j are just shifted versions of ℓ_n, note that

$$\left| N_i - \sum_{j=0}^{h} \Gamma_j \right| \leq h,$$

where $\Gamma_j := \#V(T(j)) - \#V_{\max}(T(j), L_j)$. Since $h = C_n(i)$ converges after renormalization by $\sqrt{2n}$ to $\boldsymbol{e}(t)$, we obtain that h/n converges to 0 in probability. Also, conditionally given $n_1, n_2, \ldots$, point 1. above implies that the random variables Γ_j, $j \geqslant 0$, are independent and, by Lemma 5.1, Γ_j has the same distribution as $V(M^\bullet_{n_j}) - 1$. But the L^2 convergence of $2\,\#V(M_n)/n$ to 1 established in the proof of Proposition 3.1 entails that $2(\#V(M^\bullet_n) - 1)/n$ also converges to 1 in probability, by Proposition 3.1. Fix $\varepsilon > 0$, K as in point 2. above, and N such that $n \geq N$ implies that both the conclusion of point 2. and

$$P(|2(\#V(M^\bullet_n) - 1)/n - 1| > \varepsilon/t) < \varepsilon/(K+1)$$

hold. Observe that if both $\sum_{j=0}^K n_j \geqslant n(t-\varepsilon)$ and $\sum_{j=0}^K n_j(1-\varepsilon/t) \leqslant 2\sum_{j=0}^K \Gamma_j \leqslant \sum_{j=0}^K n_j(1+\varepsilon/t)$ hold, then, on the one hand, $2\sum_{j=0}^h \Gamma_j \geqslant 2\sum_{j=0}^K \Gamma_j \geqslant n\,(t - 2\varepsilon)$ and, on the other hand, $2\sum_{j=0}^h \Gamma_j \leqslant 2\sum_{j=0}^K \Gamma_j + 2\sum_{j=K+1}^h n_j \leqslant n\,(t+2\varepsilon)$, because it always holds that $\Gamma_j \leqslant n_j$ and $\sum_{j=0}^h n_j \leqslant nt$. As a result,

$$\begin{aligned} P\left(\left|\frac{2}{n}\sum_{j=0}^h \Gamma_j - t\right| \geqslant 2\varepsilon\right) &\leqslant P\left(\sum_{j=0}^K n_j < n(t-\varepsilon)\right) + \sum_{j=0}^K P\left(\left|\frac{2\,\Gamma_j}{n_j} - 1\right| > \frac{\varepsilon}{t}\right) \\ &\leqslant 2\varepsilon + (K+1)P(n_K < N). \end{aligned}$$

The last inequality is obtained by conditioning on n_j and treating separately whether $n_j \geq N$ or $n_j < N$. As n_K/n converges to a non-degenerate random variable, it follows that $2N_i/n$ converges in probability to t.

Since this is valid for every $t \in [0,1]$, standard monotony arguments entail that

$$\left(\frac{2N_{\lfloor 2nt \rfloor}}{n}, 0 \leqslant t \leqslant 1\right) \underset{n\to\infty}{\longrightarrow} \mathrm{Id}_{[0,1]} .$$

in probability for the uniform norm. Upon further extraction from (n_k), we can in fact assume that this convergence holds a.s.

Now let U_1 be uniform in $[0,1]$ as above, and let j_n be the first integer j such that $N_j > U_1 \times N_{2n}$. By definition, the vertex v_{j_n} is uniformly distributed in $V(T_n) \setminus V_{\max}(T_n, \ell_n) = V(M_n^\bullet) \setminus \{v_*\}$. On the other hand, the previous convergence implies that $j_n/2n \to U_1$. Consequently, since v_{j_n} is at distance at most Δ_n from $\tilde{v}_{j_n}$ in M_n,

$$\left(\frac{9}{8n}\right)^{1/4} d_{M_n^\bullet}(v_{j_n}, \tilde{v}_{\lfloor 2nU_1 \rfloor}) \leq \left(\frac{9}{8n}\right)^{1/4} \left(\tilde{D}_n(j_n, \lfloor 2nU_1 \rfloor) + \Delta_n\right) \longrightarrow \tilde{D}(U_1, U_1) = 0,$$

where the last convergence comes from (4.4), and this is what we needed to conclude.

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