Electronic Journal of Probability

Electron. J. Probab. **19** (2014), no. 71, 1–37.
ISSN: 1083-6489 DOI: 10.1214/EJP.v19-3083

# Euclidean partitions optimizing noise stability*

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**Abstract**

The Standard Simplex Conjecture of Isaksson and Mossel [12] asks for the partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ into $k \leq n+1$ pieces of equal Gaussian measure of optimal noise stability. That is, for $\rho > 0$, we maximize

$$\sum_{i=1}^{k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_{A_i}(x) 1_{A_i}(x\rho + y\sqrt{1-\rho^2}) e^{-(x_1^2+\cdots+x_n^2)/2} e^{-(y_1^2+\cdots+y_n^2)/2} dxdy.$$

Isaksson and Mossel guessed the best partition for this problem and proved some applications of their conjecture. For example, the Standard Simplex Conjecture implies the Plurality is Stablest Conjecture. For $k = 3, n \geq 2$ and $0 < \rho < \rho_0(k,n)$, we prove the Standard Simplex Conjecture. The full conjecture has applications to theoretical computer science [12, 13, 19] and to geometric multi-bubble problems.

**Keywords:** Standard simplex ; plurality ; optimization ; MAX-k-CUT ; Unique Games Conjecture.
**AMS MSC 2010:** 68Q25.
Submitted to EJP on October 20, 2013, final version accepted on May 21, 2014.
Supersedes arXiv:1211.7138v1.

## 1 Introduction

The Standard Simplex Conjecture [12] asks for the partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ into $k \leq n+1$ sets of equal Gaussian measure of optimal noise stability. This Conjecture generalizes a seminal result of Borell, [3, 19], which corresponds to the $k = 2$ case of the Standard Simplex Conjecture. Borell's result says that the two disjoint regions of fixed Gaussian measures $0 < a < 1$ and $1 - a$ and of optimal noise stability must be separated by a hyperplane. Since two disjoint sets of total Gaussian measure $1$ can be described by a single set and its complement, Borell's result can be stated as follows. Let $A \subseteq \mathbb{R}^n$ have Gaussian measure $0 < a < 1$ and let $\rho \in (0,1)$. Then the following quantity, which is referred to as the noise stability of $A$, is maximized when $A$ is a half-space.

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} 1_A(x) 1_A(x\rho + y\sqrt{1-\rho^2}) e^{-(x_1^2+\cdots+x_n^2)/2} e^{-(y_1^2+\cdots+y_n^2)/2} dxdy. \qquad (1.1)$$

*This research is supported by NSF Graduate Research Fellowship DGE-0813964. Part of this work was carried out while visiting the Quantitative Geometry program at MSRI.

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When we say that $A$ is a half-space, we mean that $A$ is the set of points lying on one side of a hyperplane. If $\rho \in (-1, 0)$, then the noise stability (1.1) of $A$ is minimized among all sets of Gaussian measure $a$, when $A$ is a half-space. We can rewrite (1.1) probabilistically as follows. Let $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n) \in \mathbb{R}^n$ be two standard Gaussian random vectors such that $\mathbb{E}(X_i Y_j) = \rho \cdot 1_{(i=j)}$. Then the noise stability (1.1) of $A$ is equal to $\mathbb{P}((X, Y) \in A \times A)$.

For modern proofs of Borell's theorem with additional stability statements, see [18, 7]. In the present work, we prove a specific case of the Standard Simplex Conjecture for $k = 3$, when $0 < \rho < \rho_0(n)$. Already for the case $k = 3$, the methods used in the case $k = 2$ do not seem to apply, so new techniques are required to treat the case $k = 3$. We first discuss consequences of the full conjecture and we then state the conjecture precisely. The Standard Simplex Conjecture appears to be first stated explicitly in [12]. If true, this conjecture implies:

- Optimal hardness results for approximating the MAX-k-CUT problem [12, Theorem 1.13], a generalization of the MAX-CUT problem. (These hardness results are optimal, assuming the Unique Games Conjecture).
- The Plurality is Stablest Conjecture [13],[12][Theorem 1.10], an extension of the Majority is Stablest Conjecture [19] asserting that: the most noise-stable way to determine the winner of an election between $k$ candidates is to take the plurality. (This result assumes that no one person has too much influence over the election's outcome, and each candidate has an equal probability of winning).
- The solution of a multi-bubble problem in Gaussian space [6, 12, 17]: in $\mathbb{R}^n$, minimize the total Gaussian perimeter of $k \leq n + 1$ sets of Gaussian measure $1/k$.

The MAX-k-CUT problem asks for the partition of the vertices of any graph into $k$ sets of maximum total edge perimeter. For the precise statement, see Definition 1.4 below. For a graph on $n$ vertices, the MAX-k-CUT problem cannot be solved time polynomial in $n$, unless P=NP [8]. Yet, we can always find an approximate solution of the MAX-k-CUT problem in time polynomial in $n$ [8]. To create this approximate solution, we label the vertices of the graph by vectors in $\mathbb{R}^n$, solve an appropriate semidefinite program for these vectors, and we then "round" these vectors into $k$ bins. In particular, two vectors are rounded into the same bin if they lie in the same subset of a given partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$. The best way to perform this rounding procedure is then provided by the partition $\{A_i\}_{i=1}^k$ of optimal noise stability. That is, the Standard Simplex Conjecture exactly describes the best way to solve the MAX-k-CUT problem [12, Theorem A.6]. This connection between combinatorial optimization and geometry has been well-studied; see e.g. [20, 2, 14, 16, 4, 10]. For a survey of the complexity theoretic motivation for problems related to the Standard Simplex Conjecture, see [15], where Grothendieck inequalities are emphasized.

The Plurality is Stablest Conjecture for $k = 2$ was proven in [19], where it was found to be a consequence of Borell's theorem, after applying a nonlinear central limit theorem, which is referred to as an invariance principle. For $k = 2$, this problem is known as the Majority is Stablest Theorem. For more on the invariance principle, see also [5]. The invariance principle of [19] is proven by combining the Lindeberg replacement method with the hypercontractive inequality [9]. The Plurality is Stablest Conjecture says that the Plurality function nearly maximizes discrete noise stability over all functions $f \colon \{1, \ldots, k\}^n \to \{1, \ldots, k\}$. In this context, we think of the domain of $f$ as $n$ voters who vote for any one of $k$ candidates. Given $n$ votes $(a_1, \ldots, a_n) \in \{1, \ldots, k\}^n$, the value $f(a_1, \ldots, a_n) \in \{1, \ldots, k\}$ is the winner of the election. The Plurality is Stablest conjecture also assumes that each candidate has an equal probability of winning the election, and no one person has too much influence over the outcome of the election. It turns out that the latter assumption means that the function $f$ can be well approximated by a function $g\colon \mathbb{R}^n \to \{1, \ldots, k\}$. That is, the noise stability of $f$ is close to the sum of noise stabilities of the sets $g^{-1}(1), \ldots, g^{-1}(k)$. This approximation procedure, which uses an invariance principle, shows the equivalence of the Plurality is Stablest Conjecture and Standard Simplex Conjecture [12, Theorems 1.10 and 1.11]. We are therefore partially motivated to solve the Standard Simplex Conjecture to attempt to complete the picture set out by the sequence of works [3, 13, 19, 12].

The problem of minimizing Gaussian perimeter arises as an endpoint case of the Standard Simplex Conjecture. The Standard Simplex Conjecture is a statement involving a sum of terms of the form (1.1), and the Gaussian perimeter is recovered by letting $\rho \to 1^-$.

We now precisely state the Standard Simplex Conjecture. Let $\rho \in (-1, 1)$, $n \geq 1$, $n \in \mathbb{Z}$, let $f\colon \mathbb{R}^n \to \mathbb{R}$ be bounded and measurable. Let $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and define $\gamma_n(y) := e^{-(y_1^2 + \cdots + y_n^2)/2}/(2\pi)^{n/2}$. For $x \in \mathbb{R}^n$, define

$$T_\rho f(x) := \int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y). \tag{1.2}$$

The operator defined by (1.2) is known as the noise operator, or Bonami-Beckner operator, or Ornstein-Uhlenbeck operator. In particular, the Ornstein-Uhlenbeck operator is often written with $\rho = e^{-t}$, $t > 0$, so that $T_{e^{-t}}$ becomes a semigroup.

**Definition 1.1.** *Let $A_1, \ldots, A_k \subseteq \mathbb{R}^n$ be measurable, $k \leq n+1$. We say that $\{A_i\}_{i=1}^k$ is a **partition** of $\mathbb{R}^n$ if $\cup_{i=1}^k A_i = \mathbb{R}^n$, and $\gamma_n(A_i \cap A_j) = 0$ for $i \neq j$, $i, j \in \{1, \ldots, k\}$. Let $\{z_i\}_{i=1}^k$ be the vertices of a regular simplex centered at the origin of $\mathbb{R}^n$. For each $i \in \{1, \ldots, k\}$, define $A_i := \{x \in \mathbb{R}^n\colon \langle x, z_i \rangle = \max_{j \in \{1,\ldots,k\}} \langle x, z_j \rangle\}$, the Voronoi region of $z_i$. We call $\{A_i\}_{i=1}^k$ a **regular simplicial conical partition**.*

**Conjecture 1.2 (Standard Simplex Conjecture, [12]).** *Let $n \geq 2$, let $\rho \in [-1, 1]$, and let $3 \leq k \leq n+1$. Let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$.*

*(a) If $\rho \in (0, 1]$, and if $\gamma_n(A_i) = 1/k$, $\forall\, i \in \{1, \ldots, k\}$, then among all such partitions of $\mathbb{R}^n$, the quantity*

$$J := \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i}(x) T_\rho(1_{A_i})(x) d\gamma_n(x) \tag{1.3}$$

*is maximized by a regular simplicial conical partition.*

*(b) If $\rho \in [-1, 0)$ (with no restriction on the measures of the sets $A_i$, $i \in \{1, \ldots, k\}$), then among all partitions of $\mathbb{R}^n$, the quantity $J$ is minimized by a regular simplicial conical partition.*

The following theorem is our main result.

**Theorem 1.3 (Main Theorem).** *Fix $n \geq 2$, $k = 3$. There exists $\rho_0 = \rho_0(n, k) > 0$ such that Conjecture 1.2 holds for $\rho \in (0, \rho_0)$.*

Theorem 1.3 seems to have no direct relation to Gaussian isoperimetric problems [6], since these problems are implied by letting $\rho \to 1^-$ in Conjecture 1.2. Also, [12, Lemma A.4,Theorem A.6] shows that Theorem 1.3 seems to give no new information about the MAX-k-CUT problem, since in this problem, $\rho < 0$ is most relevant. Surprisingly, our proof strategy does not work for $\rho < 0$, as we show in Theorem 7.4.

Let $X = (X_1, \ldots, X_n), Y = (Y_1, \ldots, Y_n)$ be jointly standard normal $n$-dimensional Gaussian random variables such that the covariances satisfy $\mathbb{E}(X_i Y_j) = \rho \cdot 1_{\{i=j\}}$, $i, j \in$

$\{1,\ldots,n\}$. In [12], the quantity (1.3) is written as $\sum_{i=1}^{k}\mathbb{P}((X,Y)\in A_i\times A_i)$. To see that our formulation of Conjecture 1.2 is equivalent to that of [12], let $A\subseteq\mathbb{R}^n$ and note that

$$\begin{aligned}\int_{\mathbb{R}^n} 1_A T_\rho 1_A d\gamma_n &= \int_{\mathbb{R}^n} 1_A(x)\int_{\mathbb{R}^n} 1_A(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)d\gamma_n(x)\\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} 1_A(x)1_A(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)d\gamma_n(x) = \mathbb{P}((X,Y)\in A\times A).\end{aligned}$$

### 1.1 MAX-k-CUT and the Unique Games Conjecture

We now rigorously describe the complexity theoretic notions referenced above.

**Definition 1.4** (**MAX-k-CUT**). *Let $k,n\in\mathbb{N}$, $k\geq 2$. We define the weighted MAX-k-CUT problem. We are given a symmetric matrix $\{a_{ij}\}_{i,j=1}^{n}$ with $a_{ij}\geq 0$ for all $i,j\in\{1,\ldots,n\}$. The goal of the MAX-k-CUT problem is to find the following quantity:*

$$\max_{c\colon\{1,\ldots,n\}\to\{1,\ldots,k\}} \sum_{\substack{i,j\in\{1,\ldots,n\}:\\ c(i)\neq c(j)}} a_{ij}.$$

**Definition 1.5** (**$\Gamma$-MAX-2LIN(k)**). *Let $k\in\mathbb{N}$, $k\geq 2$. We define the $\Gamma$-MAX-2LIN(k) problem. In this problem, we are given $m\in\mathbb{N}$ and $2m$ variables $x_i\in\mathbb{Z}/k\mathbb{Z}$, $i\in\{1,\ldots,2m\}$. We are also given a matrix $\{a_{ij}\}_{i,j=1}^{2m}$ with $a_{ij}\geq 0$ for all $i,j\in\{1,\ldots,2m\}$. An element $(i,j)$ corresponds to one of $m$ linear equations of the form $x_i-x_j=c_{ij}(\mathrm{mod}\,k)$, $i,j\in\{1,\ldots,2m\}$, $c_{ij}\in\mathbb{Z}/k\mathbb{Z}$. The goal of the $\Gamma$-MAX-2LIN(k) problem is to find the following quantity:*

$$\max_{(x_1,\ldots,x_{2m})\in(\mathbb{Z}/k\mathbb{Z})^{2m}} \sum_{\substack{(i,j)\in E:\\ x_i-x_j=c_{ij}(\mathrm{mod}\,k)}} a_{ij}. \tag{1.4}$$

**Definition 1.6** (**Unique Games Conjecture**, [13]). *For every $\varepsilon\in(0,1)$, there exists a prime number $p(\varepsilon)$ such that no polynomial time algorithm can distinguish between the following two cases, for instances of $\Gamma$-MAX-2LIN($p(\varepsilon)$) with $w=1$:*

*(i)* (1.4) *is larger than $(1-\varepsilon)m$, or*

*(ii)* (1.4) *is smaller than $\varepsilon m$.*

If (1.4) were equal to $m$, then we could find $(x_1,\ldots,x_{2m})$ achieving the maximum in (1.4) by linear algebra. One can therefore interpret the Unique Games Conjecture as an assertion that approximate linear algebra is hard.

**Theorem 1.7.** (**Optimal Approximation for MAX-k-CUT**, [12][Theorem 1.13],[8]). *Let $k\in\mathbb{N}$, $k\geq 2$. Let $\{A_i\}_{i=1}^{k}\subseteq\mathbb{R}^{k-1}$ be a regular simplicial conical partition. Define*

$$\alpha_k := \inf_{-\frac{1}{k-1}\leq\rho\leq 1} \frac{k-k^2\sum_{i=1}^{k}\int_{\mathbb{R}^n}1_{A_i}T_\rho 1_{A_i}d\gamma_n}{(k-1)(1-\rho)} = \inf_{-\frac{1}{k-1}\leq\rho\leq 0} \frac{k-k^2\sum_{i=1}^{k}\int_{\mathbb{R}^n}1_{A_i}T_\rho 1_{A_i}d\gamma_n}{(k-1)(1-\rho)}.$$

*Assume Conjecture 1.2 and the Unique Games Conjecture. Then, for any $\varepsilon>0$, there exists a polynomial time algorithm that approximates MAX-k-CUT within a multiplicative factor $\alpha_k-\varepsilon$, and it is NP-hard to approximate MAX-k-CUT within a multiplicative factor of $\alpha_k+\varepsilon$.*

### 1.2 Plurality is Stablest

We now briefly describe the Plurality is Stablest Conjecture. This Conjecture seems to first appear in [13]. The work [13] emphasizes the applications of this conjecture to MAX-k-CUT and to MAX-2LIN(k).

Let $n \geq 2, k \geq 3$ Let $(W_1, \ldots, W_k)$ be an orthonormal basis for the space of functions $\{g\colon \{1,\ldots,k\} \to [0,1]\}$ equipped with the inner product $\langle g,h\rangle_k := \frac{1}{k}\sum_{\sigma\in\{1,\ldots,k\}} g(\sigma)h(\sigma)$. Assume that $W_1 = 1$. By orthonormality, for every $\sigma \in \{1,\ldots,k\}$, there exists $\widehat{g}(\sigma) \in \mathbb{R}$ such that the following expression holds: $g = \sum_{\sigma\in\{1,\ldots,k\}} \widehat{g}(\sigma)W_\sigma$. Define

$$\Delta_k := \{(x_1,\ldots,x_k) \in \mathbb{R}^k \colon \forall\, 1 \leq i \leq k, 0 \leq x_i \leq 1, \textstyle\sum_{i=1}^k x_i = 1\}.$$

Let $f\colon \{1,\ldots,k\}^n \to \Delta_k$, $f = (f_1,\ldots,f_k)$, $f_i\colon \{1,\ldots,k\}^n \to [0,1]$, $i \in \{1,\ldots,k\}$. Let $\sigma = (\sigma_1,\ldots,\sigma_n) \in \{1,\ldots,k\}^n$. Define $W_\sigma := \prod_{i=1}^n W_{\sigma_i}$, $|\sigma| := |\{i \in \{1,\ldots,n\}\colon \sigma_i \neq 1\}|$. Then for every $\sigma \in \{1,\ldots,k\}^n$ there exists $\widehat{f}_i(\sigma) \in \mathbb{R}$ such that $f_i = \sum_{\sigma\in\{1,\ldots,k\}^n} \widehat{f}_i(\sigma)W_\sigma$, $i \in \{1,\ldots,k\}$. For $\rho \in [-1,1]$ and $i \in \{1,\ldots,k\}$, define

$$T_\rho f_i := \sum_{\sigma\in\{1,\ldots,k\}^n} \rho^{|\sigma|}\widehat{f}_i(\sigma)W_\sigma, \quad T_\rho f := (T_\rho f_1,\ldots,T_\rho f_k) \in \mathbb{R}^k.$$

Let $m \geq 2$, $k \geq 3$. For each $j \in \{1,\ldots,k\}$, let $e_j = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^k$ be the $j^{th}$ unit coordinate vector. Let $\sigma \in \{1,\ldots,k\}^n$. Define $\mathrm{PLUR}_{m,k}\colon \{1,\ldots,k\}^m \to \Delta_k$ such that

$$\mathrm{PLUR}_{m,k}(\sigma) := \begin{cases} e_j & ,\text{if } |\{i\in\{1,\ldots,m\}\colon \sigma_i = j\}| > |\{i\in\{1,\ldots,m\}\colon \sigma_i = r\}|, \\ & \qquad \forall\, r \in \{1,\ldots,k\}\setminus\{j\} \\ \frac{1}{k}\sum_{i=1}^k e_i & ,\text{otherwise} \end{cases}$$

**Conjecture 1.8** (**Plurality is Stablest Conjecture**, [12]). *Let $n \geq 2$, $k \geq 3$, $\rho \in [-\frac{1}{k-1}, 1]$, $\varepsilon > 0$. Let $\langle\cdot,\cdot\rangle$ denote the standard inner product on $\mathbb{R}^n$. Then there exists $\tau > 0$ such that, if $f\colon \{1,\ldots,k\}^n \to \Delta_k$ satisfies $\sum_{\sigma\in\{1,\ldots,k\}^n\colon \sigma_j\neq 1}(\widehat{f}_i(\sigma))^2 \leq \tau$ for all $i \in \{1,\ldots,k\}$, $j \in \{1,\ldots,n\}$, then*

*(a) If $\rho \in (0,1]$, and if $\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n} f(\sigma) = \frac{1}{k}\sum_{i=1}^k e_i$, then*

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma), T_\rho f(\sigma)\rangle \leq \lim_{m\to\infty}\frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m}\langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle + \varepsilon.$$

*(b) If $\rho \in [-1/(k-1), 0)$, then*

$$\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma), T_\rho f(\sigma)\rangle \geq \lim_{m\to\infty}\frac{1}{k^m}\sum_{\sigma\in\{1,\ldots,k\}^m}\langle \mathrm{PLUR}_{m,k}(\sigma), T_\rho(\mathrm{PLUR}_{m,k})(\sigma)\rangle - \varepsilon.$$

### 1.3 A Synopsis of the Main Theorem

We now describe the proof of Theorem 1.3. We first take the derivative $d/d\rho$ of the quantity $J$ defined by (1.3). This procedure is common, and it dates back at least to the the proof of the Log-Sobolev Inequality by Gross [9]. Taking this derivative allows us to relate $J$ to the works [14, 16]. In Section 3, we modify the results of [14, 16] to prove the existence of a partition that maximizes $(d/d\rho)J$. Then, in Section 4, we further modify results of [14, 16] to show that, if $\rho > 0$ is small, then a partition maximizing $(d/d\rho)J$ is close to a partition maximizing $(d/d\rho)|_{\rho=0}J$. And by [14], we know that the partition maximizing $(d/d\rho)|_{\rho=0}J$ is a regular simplicial conical partition, for dimension $n \geq 2$ and $k = 3$ partition elements.

So, for small $\rho > 0$, a partition maximizing $(d/d\rho)J$ is close to a regular simplicial conical partition. The structure of the operator $T_\rho$ then permits the exploitation of a feedback loop. This feedback loop says: if our partition maximizes $(d/d\rho)J$ for small $\rho > 0$, and if this partition is close to a regular simplicial conical partition, then this partition is even closer to a regular simplicial conical partition. This feedback loop is investigated in Section 5, especially in the crucial Lemma 6.1. A similar feedback loop was already apparent in [14][Lemma 3.3]. The full argument of Theorem 1.3 is then assembled in Section 7. By using this feedback loop, we show in Theorem 7.1 that a regular simplicial conical partition maximizes $(d/d\rho)J$ for small $\rho > 0$, $k = 3$, $n \geq 2$. Then, the Fundamental Theorem of Calculus allows us to relate $(d/d\rho)J$ to $J$, therefore completing the proof of the main theorem, Theorem 1.3.

Since Lemma 6.1 is rather lengthy and crucial to this investigation, we will further describe the idea behind it. If we know that our partition maximizes $(d/d\rho)J$, and if we also know that this partition is close to a regular simplicial conical partition, then the first variation should immediately tell us that our partition is actually a regular simplicial conical partition. Unfortunately, this intuition does not translate into a proof. The main technical problem is that the sets we are dealing with are unbounded, and we need to know precise information about the Ornstein-Uhlenbeck operator applied to these sets, for points that are very far from the origin. Since the Gaussian measure decays exponentially away from the origin, this means that it becomes hard to say something precise about the points in these sets that are very far from the origin. So, we require very precise estimates of the Ornstein-Uhlenbeck operator, and the errors that it accrues when we evaluate it far from the origin. These estimates are performed in Lemmas 5.2 and 5.3. Unfortunately, to use these estimates effectively, we need to slowly enlarge the regions where we use these estimates. The details of enlarging these regions becomes surprisingly complicated, occupying the seven steps of Lemma 6.1.

In Section 7, we also show the surprising fact that our strategy fails for small negative correlation. That is, for small $\rho < 0$, $(d/d\rho)J$ is not maximized by the regular simplicial conical partition. This result does not confirm or deny Conjecture 1.2 for $\rho < 0$. However, one may interpret from this result that the case of Conjecture 1.2 for $\rho < 0$ could be more difficult than the case $\rho > 0$.

We should also emphasize the lack of symmetrization in the proof of Theorem 1.3. Symmetrization is one of a few general strategies that solves many optimization problems. In our context, symmetrization would appear as follows. Recall the definition of $J$ from (1.3). Suppose we have a partition $\{A_i\}_{i=1}^k \subseteq \mathbb{R}^n$. Change this partition into a "more symmetric" partition $\{\widetilde{A_i}\}_{i=1}^k$ such that $J$ or $(d/d\rho)J$ is larger for $\{\widetilde{A_i}\}_{i=1}^k$. In the proof of the main theorem, it is tempting to use this symmetrization paradigm. The works [3],[19] and [12] use Gaussian symmetrization in a crucial way. However, we find this approach to be less natural for Conjecture 1.2, so we do not explicitly use symmetrization. Nevertheless, symmetry does play a crucial role in our proof, especially in the estimates of Section 4. It should also be noted that the works [14, 16] do not explicitly use symmetrization, and this lack of symmetrization is one of their novel aspects.

### 1.4 Preliminaries

We follow the exposition of [17]. Let $n \geq 1$, $n \in \mathbb{Z}$. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. For $f\colon \mathbb{R}^n \to \mathbb{R}$ measurable, let $\|f\|_{L_2(\gamma_n)} := (\int_{\mathbb{R}^n} |f|^2 \, d\gamma_n)^{1/2}$. Let $L_2(\gamma_n) := \{f\colon \mathbb{R}^n \to \mathbb{R}\colon \|f\|_{L_2(\gamma_n)} < \infty\}$. Let $\ell_2^n$ denote the $\ell_2$ norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $r > 0$, define $B(x, r) := \{y \in \mathbb{R}^n\colon \|x - y\|_{\ell_2^n} < r\}$.

For $f \in L_2(\gamma_n)$, define $T_\rho$ as in (1.2). The operator $T_\rho$ is a parametrization of the Ornstein-Uhlenbeck operator. The operator $T_\rho$ is not a semigroup, but it satisfies $T_{\rho_1} T_{\rho_2} = T_{\rho_1 \rho_2}$, $\rho_1, \rho_2 \in [-1, 1]$, by (1.7) below. We use definition (1.2) since the usual Ornstein-Uhlenbeck operator is only defined for $\rho \in [0, 1]$. Let $\lambda > 0$, $x \in \mathbb{R}$. Recall that the Hermite polynomials of one variable are defined by the generating function

$$e^{\lambda x - \lambda^2/2} =: \sum_{\ell \in \mathbb{N}} \lambda^\ell h_\ell(x). \tag{1.5}$$

Alternatively, one defines the polynomials $H_\ell(x)$ where $h_\ell(x) = 2^{-\ell/2}(\ell!)^{-1} H_\ell(x/\sqrt{2})$. This convention is used in [1], where the orthogonality properties of the Hermite polynomials are derived.

Note that $\int_{\mathbb{R}} h_\ell^2 d\gamma_1 = 1/\ell!$, and $\{\sqrt{\ell!}\, h_\ell\}_{\ell \in \mathbb{N}}$ is an orthonormal basis of $L_2(\gamma_1)$. Recall also that $h_0(x) = 1$ and $h_1(x) = x$. Set $f(x) := e^{\lambda x - \lambda^2/2}$. A routine computation shows that $T_\rho(f)(x) = e^{(\lambda\rho)x - (\lambda\rho)^2/2}$. Indeed

$$\begin{aligned} T_\rho(f)(x) &= \int_{\mathbb{R}^n} e^{\lambda(x\rho + y\sqrt{1-\rho^2}) - \lambda^2/2} d\gamma_1(y) = \int_{\mathbb{R}^n} e^{(\lambda\rho)x + (\lambda\sqrt{1-\rho^2})y - \lambda^2/2 - y^2/2} \frac{dy}{\sqrt{2\pi}} \\ &= e^{(\lambda\rho)x - \lambda^2/2} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(y - \lambda\sqrt{1-\rho^2})^2 + \lambda^2(1-\rho^2)/2} \frac{dy}{\sqrt{2\pi}} = e^{(\lambda\rho)x - \lambda^2/2 + \lambda^2(1-\rho^2)/2} \\ &= e^{(\lambda\rho)x - (\lambda\rho)^2/2}. \end{aligned}$$

Therefore, by (1.5),

$$T_\rho f(x) = \sum_{\ell \in \mathbb{N}} \lambda^\ell \rho^\ell h_\ell(x). \tag{1.6}$$

So, by linearity, $T_\rho h_\ell(x) = \rho^\ell h_\ell(x)$.

We now extend the above observations to higher dimensions. Let $f \in L_2(\gamma_n)$, so that $f = \sum_{\ell \in \mathbb{N}^n} a_\ell h_\ell \sqrt{\ell!}$, $a_\ell \in \mathbb{R}$, where $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ and $h_\ell(x) = \prod_{i=1}^n h_{\ell_i}(x_i)$. Write $|\ell| := \ell_1 + \cdots + \ell_n$ and $\ell! := (\ell_1!) \cdots (\ell_n!)$. Then $T_\rho$ satisfies $T_\rho h_\ell = \rho^{|\ell|} h_\ell$ and for $x \in \mathbb{R}^n$,

$$T_\rho f(x) = \sum_{\ell \in \mathbb{N}^n} \rho^{|\ell|} \sqrt{\ell!}\, h_\ell(x) \left( \int_{\mathbb{R}^n} \sqrt{\ell!}\, h_\ell f d\gamma_n \right) \tag{1.7}$$

Let $\Delta := \sum_{i=1}^n \partial^2 / \partial x_i^2$, and define

$$L := -\Delta + \sum_{i=1}^n x_i \cdot \frac{\partial}{\partial x_i}. \tag{1.8}$$

A well-known calculation shows the following equality, which we prove in the Appendix, Section 9.

$$\frac{d}{d\rho} T_\rho f(x) = \rho^{-1} L T_\rho f(x) = \frac{1}{\rho} \left( \langle x, \nabla T_\rho f(x) \rangle - \Delta T_\rho f(x) \right) \tag{1.9}$$

$$\begin{aligned} &= \frac{1}{\sqrt{1-\rho^2}} \Bigg[ \left\langle x, \int_{\mathbb{R}^n} y f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) \right\rangle \\ &\qquad + \frac{\rho}{\sqrt{1-\rho^2}} \int_{\mathbb{R}^n} \left( \sum_{i=1}^n (1 - y_i^2) \right) f(x\rho + y\sqrt{1-\rho^2}) d\gamma_n(y) \Bigg]. \end{aligned} \tag{1.10}$$

We say that $A \subseteq \mathbb{R}^n$ is a **cone** if $A$ is measurable and $\forall\, t > 0$, $tA = A$.

**Definition 1.9.** *A **simplicial conical partition** $\{A_i\}_{i=1}^k$ is a partition of $\mathbb{R}^n$ together with a simplex $S \subseteq \mathbb{R}^{k-1}$ with $0 \leq k-1 \leq n$ and a rotation $\sigma$ of $\mathbb{R}^n$ such that $0 \in S$ and such that each facet $F_i$ of $\sigma(S \times \mathbb{R}^{n-k+1})$ generates a partition element, i.e. $A_i = \{t\sigma(F_i \times \mathbb{R}^{n-j})\colon t \in [0,\infty)\}$, $i \in \{1,\dots,j+1\}$. Let $k-1 \leq n$ and let $\{z_i\}_{i=1}^k \subseteq \mathbb{R}^n$ be nonzero vectors that do not all lie in a $(k-1)$-dimensional hyperplane. Define a partition such that, for $i \in \{1,\dots,k\}$, $A_i := \{x \in \mathbb{R}^n \colon \langle x, z_i\rangle = \max_{j=1,\dots,k}\langle x, z_j\rangle\}$. Such a partition is called the simplicial conical partition induced by $\{z_i\}_{i=1}^k$.*

*If $\{A_i\}_{i=1}^k$ is a simplicial conical partition induced by the vectors $\{\int_{A_i} x d\gamma_n(x)\}_{i=1}^k$, then we say the partition is a **balanced conical partition**. If $\{z_i\}_{i=1}^k \subseteq \mathbb{R}^n$ are the vertices of a $(k-1)$-dimensional regular simplex in $\mathbb{R}^n$ centered at the origin, then the partition induced by $\{z_i\}_{i=1}^k$ is called a **regular simplicial conical partition**.*

Let $f \in L_2(\gamma_n)$. By Plancherel and (1.7)

$$\int_{\mathbb{R}^n} f T_\rho f d\gamma_n = \sum_{\ell \in \mathbb{N}^n} \rho^{|\ell|} \left| \int_{\mathbb{R}^n} f \sqrt{\ell!} h_\ell d\gamma_n \right|^2. \tag{1.11}$$

Substituting (1.11) into (1.3) gives

$$\sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} T_\rho(1_{A_i}) d\gamma_n = \sum_{i=1}^k \left[ \gamma_n(A_i)^2 + \rho \left\| \int_{A_i} x d\gamma_n(x) \right\|_{\ell_2^n}^2 + \sum_{\substack{\ell \in \mathbb{N}^n \\ |\ell| \geq 2}} \rho^{|\ell|} \left| \int_{A_i} \sqrt{\ell!} h_\ell d\gamma_n \right|^2 \right]. \tag{1.12}$$

Taking the derivative $d/d\rho$ of (1.12) at $\rho = 0$, we get a quantity studied in [14, 16].

$$\frac{d}{d\rho} \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} T_\rho(1_{A_i}) d\gamma_n = \sum_{i=1}^k \left[ \left\| \int_{A_i} x d\gamma_n(x) \right\|_{\ell_2^n}^2 + \sum_{\substack{\ell \in \mathbb{N}^n \\ |\ell| \geq 2}} |\ell|\, \rho^{|\ell|-1} \left| \int_{A_i} \sqrt{\ell!} h_\ell d\gamma_n \right|^2 \right]. \tag{1.13}$$

$$\left.\frac{d}{d\rho}\right|_{\rho=0} \sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} T_\rho(1_{A_i}) d\gamma_n = \sum_{i=1}^k \left\| \int_{A_i} x d\gamma_n(x) \right\|_{\ell_2^n}^2. \tag{1.14}$$

## 2 Noise Stability for Zero Correlation

This section concerns noise stability at the endpoint $\rho = 0$. Specifically, we will investigate the quantity (1.14), which has already been studied in [14, 16]. Using our understanding of (1.14), we will then be able to analyze the left side of (1.13) when $\rho$ is small, using the equality (1.13). Before beginning our discussion, we first need to consider partitions of $\mathbb{R}^n$ within the convex set defined in (2.1). Definition 2.3 provides the metric allowing an assertion that two partitions are close to each other, and Definition 2.4 allows us to discuss the Gaussian measure restricted to hypersurfaces.

**Definition 2.1.** *Let $H := \oplus_{i=1}^k L_2(\gamma_n)$ and define*

$$\Delta_k(\gamma_n) := \{(f_1,\dots,f_k) \in H \colon \forall\, 1 \leq i \leq k, 0 \leq f_i \leq 1, \textstyle\sum_{i=1}^k f_i = 1\}. \tag{2.1}$$

**Definition 2.2.** *Let $\varepsilon \geq 0$. Define*

$$\Delta_k^\varepsilon(\gamma_n) := \{(f_1,\dots,f_k) \in \Delta_k(\gamma_n) \colon \frac{1}{k} - \varepsilon \leq \int_{\mathbb{R}^n} f_i d\gamma_n \leq \frac{1}{k} + \varepsilon\}.$$

**Definition 2.3.** *Define a metric $d_2$ on partitions $\{A_i\}_{i=1}^k, \{C_i\}_{i=1}^k$ of $\mathbb{R}^n$ by the formula*

$$d_2(\{A_i\}_{i=1}^k, \{C_i\}_{i=1}^k) := \inf_{\substack{\sigma \in SO(n) \\ \pi \text{ a permutation}}} \left( \sum_{i=1}^k \left\| 1_{A_i} - 1_{(\sigma C_{\pi(i)})} \right\|_{L_2(\gamma_n)}^2 \right)^{1/2}.$$

**Definition 2.4.** *Let $A \subseteq \mathbb{R}^n$, let $\mathcal{L}$ denote Lebesgue measure on $\mathbb{R}^n$, and define the distance $d(x,y) := \|x-y\|_{\ell_2^n}$, $x,y \in \mathbb{R}^n$. Denote $\delta_A$ as the measure $\mathcal{L}$ on $\mathbb{R}^n$ restricted to $A$. That is, $\delta_A(B) := \liminf_{\delta\to 0} \frac{1}{2\delta}\mathcal{L}\{y \in \mathbb{R}^n \colon \exists\, x \in A\cap B$ with $d(x,y) < \delta\}$, $B \subseteq \mathbb{R}^n$. Also, we denote $\gamma_n(\delta_A) := \liminf_{\delta\to 0} \frac{1}{2\delta}\gamma_n\{y \in \mathbb{R}^n \colon \exists\, x \in A$ with $d(x,y) < \delta\}$.*

The next two lemmas are derived from [14]. Lemma 2.6 is a quantitative variant of Lemma 2.5, and it will be further improved in Lemma 2.8 below. In particular, Lemma 2.6 says that, if the first variation condition for achieving the optimum value of (2.2) is nearly satisfied, then the partition is close to being simplicial.

**Lemma 2.5.** *[14, Lemma 3.3, Corollary 3.4] Let $n \geq 2$ and let $\{B_i\}_{i=1}^3$ be a regular simplicial conical partition of $\mathbb{R}^n$. Then $(1_{B_1}, 1_{B_2}, 1_{B_3})$ uniquely achieves the following supremum, up to orthogonal transformation.*

$$\sup_{(f_1,f_2,f_3)\in\Delta_3(\gamma_n)} \sum_{i=1}^{3} \left\| \int_{\mathbb{R}^n} x f_i(x) d\gamma_n(x) \right\|_{\ell_2^n}^2. \tag{2.2}$$

**Lemma 2.6.** *Let $n \geq 2$ and let $\{B_i\}_{i=1}^3, \{C_i\}_{i=1}^2 \subseteq \mathbb{R}^n$ be regular simplicial conical partition. Let $\{A_i\}_{i=1}^3 \subseteq \mathbb{R}^n$ be a simplicial conical partition. Let $z_i := \int_{A_i} x d\gamma_n(x)$, and let $v_{ij} \in S^{n-1} \cap A_i \cap A_j \cap \mathrm{span}\{z_i, z_j\}$. If $|\langle z_i - z_j, v_{ij}\rangle| \leq \varepsilon < 10^{-16}$ $\forall\, i,j \in \{1,2,3\}$, $i \neq j$, and if $d_2(\{A_i\}_{i=1}^3, \{C_1, C_2, \emptyset\}) > 1/100$, then $d_2(\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3) \leq \sqrt{6\varepsilon}$.*

*Proof.* For $i,j \in \{1,2,3\}$, let $0 \leq \alpha_i \leq \pi$ such that $A_i$ is a cone with angle $\alpha_i$. Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ be a reflection that fixes $A_i \cap A_j$. Without loss of generality, $\sigma(A_j) \subseteq A_i$. Then $z_i - z_j = \int_{A_i\setminus\sigma(A_j)} x d\gamma_n(x)$ and $\|z_i - z_j\|_2 = \sin((\alpha_i - \alpha_j)/2)/\sqrt{2\pi}$. Let $0 \leq \theta \leq \pi$ such that $\|z_i - z_j\|_2 \cos(\theta) = \langle z_i - z_j, v_{ij}\rangle$. Then either $\|z_i - z_j\|_2 \leq \sqrt{\varepsilon/18\pi}$, or $|\cos\theta| \leq \sqrt{18\pi\varepsilon}$. In the first case, $\alpha_i - \alpha_j \leq \sqrt{\varepsilon}$. So, to complete the proof, it suffices to show that the second case does not occur. We find a contradiction by assuming that the second case occurs.

If $|\cos\theta| \leq \sqrt{18\pi\varepsilon}$, then since $\theta = (\alpha_i - \alpha_j)/2$, we must have $|\alpha_i - \alpha_j - \pi| < 18\sqrt{\varepsilon}$, so $\pi - 18\sqrt{\varepsilon} < \alpha_i - \alpha_j < \pi + 18\sqrt{\varepsilon}$, i.e. $\pi - 18\sqrt{\varepsilon} < \alpha_i \leq \pi$ and $\alpha_j \leq 18\sqrt{\varepsilon}$. Then for $r \neq i,j$, $r \in \{1,2,3\}$, we have $\alpha_r = 2\pi - \alpha_i - \alpha_j > 2\pi - \pi - 18\sqrt{\varepsilon} > \pi - 18\sqrt{\varepsilon}$. Since $\alpha_i, \alpha_j > \pi - 18\sqrt{\varepsilon}$, we conclude that $d_2(\{A_i\}_{i=1}^3, \{C_1, C_2, \emptyset\}) < 18\varepsilon^{1/4} < 1/100$, a contradiction. □

We require the ensuing explicit calculation from [14] in Lemma 2.8 below. This calculation is reduced to a computation of Lagrange Multipliers in [14, Corollary 3.4]. For any $(f_1, \ldots, f_k) \in \Delta_k(\gamma_n)$, define $\psi_0(f_1, \ldots, f_k) := \sum_{i=1}^k \left\| \int_{\mathbb{R}^n} x f_i(x) d\gamma_n(x) \right\|_{\ell_2^n}^2$.

**Lemma 2.7.** *[14, Corollary 3.4]*

$$\sup_{(f_1,f_2)\in\Delta_2(\gamma_n)} \psi_0(f_1,f_2) = \frac{1}{\pi}, \qquad \sup_{(f_1,f_2,f_3)\in\Delta_3(\gamma_n)} \psi_0(f_1,f_2,f_3) = \frac{9}{8\pi}.$$

The following Lemma is a quantitative improvement of Lemmas 2.5 and 2.6. Combining Lemma 2.8 with (1.14) will show that an optimizer of $(d/d\rho)\sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} T_\rho 1_{A_i} d\gamma_n$ is almost simplicial conical for small $\rho > 0$.

**Lemma 2.8.** *Let $\varepsilon > 0$, $n \geq 2$. Let $\{A_i\}_{i=1}^3$ be a partition of $\mathbb{R}^n$, and let $\{B_i\}_{i=1}^3$ be a regular simplicial conical partition of $\mathbb{R}^n$. Assume that $\varepsilon < 1/100$ and*

$$\psi_0(1_{A_1}, 1_{A_2}, 1_{A_3}) > \sup_{(f_1,f_2,f_3)\in\Delta_3(\gamma_n)} \psi_0(f_1,f_2,f_3) - \varepsilon. \tag{2.3}$$

*Then*

$$d_2(\{A_i\}_{i=1}^3, \{B_i\}_{i=1}^3) \leq 6\varepsilon^{1/8}. \tag{2.4}$$

*Proof.* Assume that (2.3) holds. For $i\in\{1,2,3\}$, let $z_i:=\int_{A_i}xd\gamma_n(x)$, $w_i:=\int_{B_i}xd\gamma_n(x)$. We may assume that, for all $i,j\in\{1,2,3\}$ with $i\neq j$, $\langle z_i,z_j\rangle<0$. To see this, we argue by contradiction. Suppose there exist $i,j\in\{1,2,3\}$, $i\neq j$ with $\langle z_i,z_j\rangle\geq 0$. For $p\in\{1,2,3\}$, $p\neq i,j$, let $A_p'':=A_p$, let $A_i'':=A_i\cup A_j$, and let $A_j'':=\emptyset$. For $p\in\{1,2,3\}$, let $z_p'':=\int_{A_p''}xd\gamma_n(x)$. Then

$$\sum_{p=1}^{3}\|z_p''\|_{\ell_2^n}^2-\sum_{p=1}^{3}\|z_p\|_{\ell_2^n}^2=\|z_i+z_j\|_{\ell_2^n}^2-\|z_i\|_{\ell_2^n}^2-\|z_j\|_{\ell_2^n}^2\geq 0.$$

Rewriting this inequality using the definition of $\psi_0$,

$$\psi_0(1_{A_1},1_{A_2},1_{A_3})\leq\psi_0(1_{A_1''},1_{A_2''},1_{A_3''}). \tag{2.5}$$

Since $\{A_p''\}_{p=1}^3$ is a partition of $\mathbb{R}^n$ with at most two nonempty elements, Lemma 2.7 says

$$\left(\sup_{(f_1,f_2,f_3)\in\Delta_3(\gamma_n)}\psi_0(f_1,f_2,f_3)\right)-\psi_0(1_{A_1''},1_{A_2''},1_{A_3''})\geq\frac{1}{8\pi}>10^{-2}. \tag{2.6}$$

Combining (2.5) and (2.6) contradicts (2.3). Therefore, $\langle z_i,z_j\rangle<0$ for all $i,j\in\{1,2,3\}$.

We now claim that, for each pair $i,j\in\{1,2,3\}$ with $i\neq j$, we have

$$\max_{p\in\{i,j\}}\|z_p\|_{\ell_2^n}^2\geq 1/16. \tag{2.7}$$

We again argue by contradiction. Suppose there exist $i,j\in\{1,2,3\}$ with $i\neq j$ and $\max_{p\in\{i,j\}}\|z_p\|_{\ell_2^n}^2<1/16$. Let $p\in\{1,2,3\}$, $p\neq i,j$. Then $\|z_p\|_{\ell_2^n}^2\leq 1/(2\pi)$ with equality if and only if $1_{A_p}$ is a half-space whose boundary contains the origin of $\mathbb{R}^n$. This follows immediately from rearrangement. Observe, if $z_p\neq 0$,

$$\begin{aligned}\|z_p\|_{\ell_2^n}^2&=\left\langle z_p,\int_{A_p}xd\gamma_n(x)\right\rangle\leq\left\langle z_p,\int_{A_p\cap\{x\colon\langle x,z_p\rangle\geq 0\}}xd\gamma_n(x)\right\rangle\\&\leq\left\langle z_p,\int_{\{x\colon\langle x,z_p\rangle\geq 0\}}xd\gamma_n(x)\right\rangle\\&=\left\langle\int_{A_p}xd\gamma_n(x),\int_{\{x\colon\langle x,z_p\rangle\geq 0\}}xd\gamma_n(x)\right\rangle\leq\left\|\int_{\{x\colon\langle x,z_p\rangle\geq 0\}}xd\gamma_n(x)\right\|_{\ell_2^n}^2=\frac{1}{2\pi}.\end{aligned}$$

Therefore,

$$\psi_0(1_{A_1},1_{A_2},1_{A_3})\leq 1/8+1/(2\pi)\leq 1/\pi.$$

This inequality contradicts (2.3) as in (2.6), since $\sup_{(f_1,f_2,f_3)\in\Delta_k(\gamma_n)}\psi_0(f_1,\ldots,f_k)=9/(8\pi)$, using Lemma 2.7. We conclude that (2.7) holds.

Define $\delta$ such that

$$\delta:=\max_{i,j\in\{1,2,3\},i\neq j}\gamma_n(\{x\in\mathbb{R}^n\colon\langle z_i-z_j,x\rangle\leq 0\}\cap A_i). \tag{2.8}$$

Fix $i,j\in\{1,2,3\}$ such that $\delta=\gamma_n(\{x\in\mathbb{R}^n\colon\langle z_i-z_j,x\rangle\leq 0\}\cap A_i)$. We want to find a bound on $\delta$. Let $0<h$ such that $\int_0^h d\gamma_1=\delta$. Now, define $\{A_r'\}_{r=1}^3$ such that $A_p'=A_p$ for $p\neq i,j$, $A_i'=A_i\setminus(A_i\cap\{x\in\mathbb{R}^n\colon\langle z_i-z_j,x\rangle\leq 0\})$ and

$$A_j'=A_j\cup(A_i\cap\{x\in\mathbb{R}^n\colon\langle z_i-z_j,x\rangle\leq 0\}).$$

Let $z_p' := \int_{A_p'} x d\gamma_n(x)$, $p = 1,2,3$. Then

$$\begin{aligned}
&\sum_{p=1}^{3} \left\| z_p' \right\|_{\ell_2^n}^2 - \sum_{p=1}^{3} \left\| z_p \right\|_{\ell_2^n}^2 \\
&\quad = 2\left\langle \int_{\{y:\ \langle z_i - z_j, y\rangle \le 0\}\cap A_i} y d\gamma_n(y), z_j - z_i \right\rangle + 2\left\| \int_{\{y:\ \langle z_i - z_j, y\rangle \le 0\}\cap A_i} y d\gamma_n(y) \right\|_{\ell_2^n}^2 \\
&\quad \ge 2\left\langle \int_{\{y:\ -h \le \langle z_i - z_j, y\rangle \le 0\}} y d\gamma_n(y), z_j - z_i \right\rangle = 2\left\| z_i - z_j \right\|_{\ell_2^n} \int_0^h y d\gamma_1(y) \overset{(2.7)}{>} \delta^2/3.
\end{aligned} \tag{2.9}$$

Here we used rearrangement and the inequality $\|z_i - z_j\|_{\ell_2^n} > (\max_{p\in\{i,j\}} \|z_p\|_{\ell_2^n}^2)^{1/2}$, which itself uses $\langle z_i, z_j\rangle < 0$.

By (2.3) and (2.9), $\delta^2 < 3\varepsilon$, i.e.

$$\delta < \sqrt{3\varepsilon}. \tag{2.10}$$

Now, for $p \in \{1,2,3\}$, let $\widetilde{A_p} := \{x \in \mathbb{R}^n : \langle x, z_p\rangle = \max_{j=1,2,3}\langle x, z_j\rangle\}$ and let $\widetilde{z_p} := \int_{\widetilde{A_p}} x d\gamma_n(x)$. By (2.10) and (2.8),

$$d_2(\{A_i\}_{i=1}^3, \{\widetilde{A_i}\}_{i=1}^3) \le 3\sqrt{2}\,\varepsilon^{1/4}. \tag{2.11}$$

For $p \in \{1,2,3\}$, let $y_p := \widetilde{z_p} - z_p \in \mathbb{R}^n$, so that $\|y_p\|_2 \le 3\sqrt{2}\,\varepsilon^{1/4}$ by (2.11) and Hilbert space duality. Let $x \in \mathbb{R}^n$. Then for $i,j \in \{1,2,3\}$, $i \neq j$,

$$\langle \widetilde{z_i} - \widetilde{z_j}, x\rangle = \langle z_i - z_j, x\rangle + \langle y_i - y_j, x\rangle. \tag{2.12}$$

For $i,j \in \{1,2,3\}$, $i \neq j$, let $v_{ij} = S^{n-1} \cap \widetilde{A_i} \cap \widetilde{A_j} \cap \mathrm{span}\{\widetilde{z_i}\}_{i=1}^3$. By definition of $v_{ij}$ and $\{\widetilde{A_i}\}_{i=1}^3$, $\langle z_i - z_j, v_{ij}\rangle = 0$. So, by (2.12), $|\langle \widetilde{z_i} - \widetilde{z_j}, v_{ij}\rangle| \le 3\sqrt{2}\,\varepsilon^{1/4}$, implying that $d_2(\{\widetilde{A_i}\}_{i=1}^3, \{B_i\}_{i=1}^3) \le 3 \cdot 2^{3/4}\varepsilon^{1/8}$, by Lemma 2.6. This inequality together with (2.11) and the triangle inequality for $d_2$ prove (2.4). □

## 3 The First Variation

Recall (1.8). The following existence argument which uses convexity is a variant of [14, Lemma 3.1] and [16, Lemma 2.1].

**Lemma 3.1** (**First Variation**). *Let $\rho \in (0,1)$. Then $\exists$ a partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ such that*

$$\sum_{i=1}^k \int_{\mathbb{R}^n} 1_{A_i} \frac{d}{d\rho} T_\rho 1_{A_i} d\gamma_n = \sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)} \sum_{i=1}^k \int_{\mathbb{R}^n} f_i \frac{d}{d\rho} T_\rho f_i d\gamma_n \tag{3.1}$$

*Also, for each $i \in \{1,\ldots,k\}$, the following containment holds, less sets of $\gamma_n$ measure zero:*

$$A_i \supseteq \{x \in \mathbb{R}^n : LT_\rho 1_{A_i}(x) > LT_\rho 1_{A_j}(x), \forall\, j \neq i, j \in \{1,\ldots,k\}\}. \tag{3.2}$$

*Proof.* We show that (1.3) is maximized over $\Delta_k(\gamma_n)$, which contains the set of partitions of $\mathbb{R}^n$. Note that $\Delta_k(\gamma_n) \subseteq H$ is norm closed, convex, and norm bounded. Therefore, $\Delta_k(\gamma_n)$ is weakly closed. Also, $\Delta_k(\gamma_n)$ is weakly compact by the Banach-Alaoglu Theorem. Using (1.9), define $\psi_\rho : \Delta_k(\gamma_n) \to \mathbb{R}$ by

$$\psi_\rho(g_1,\ldots,g_k) := \frac{d}{d\rho} \sum_{i=1}^k \int_{\mathbb{R}^n} g_i T_\rho g_i d\gamma_n := \rho^{-1} \sum_{i=1}^k \int_{\mathbb{R}^n} g_i L T_\rho g_i d\gamma_n. \tag{3.3}$$

By (1.12), $\psi_\rho$ is an exponentially decaying sum of uniformly bounded weakly continuous functions. Therefore, $\psi_\rho$ is weakly continuous on the weakly compact set $\Delta_k(\gamma_n)$. So there exists $(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)$ that maximizes $\psi_\rho$.

Since $\rho\in(0,1]$, (1.13) implies: $\forall\, f\in L_2(\gamma_n)$, $\int fLT_\rho f d\gamma_n\geq 0$. We now apply this fact to see that $\psi_\rho$ is convex. Let $\lambda\in[0,1]$, $(g_1,\ldots,g_k),(h_1,\ldots,h_k)\in\Delta_k(\gamma_n)$. Then

$$
\begin{aligned}
&\lambda\psi_\rho(g_1,\ldots,g_k)+(1-\lambda)\psi_\rho(h_1,\ldots,h_k)-\psi_\rho(\lambda g_1+(1-\lambda)h_1,\ldots,\lambda g_k+(1-\lambda)h_k)\\
&=\frac{1}{\rho}\sum_{i=1}^{k}\bigg[\lambda\int_{\mathbb{R}^n} g_iLT_\rho g_i+(1-\lambda)\int_{\mathbb{R}^n} h_iLT_\rho h_i\\
&\qquad\qquad -\int_{\mathbb{R}^n}(\lambda g_i-(1-\lambda)h_i)LT_\rho(\lambda g_i-(1-\lambda)h_i)\bigg]\\
&=\lambda(1-\lambda)\int_{\mathbb{R}^n}(g_i-h_i)LT_\rho(g_i-h_i)\geq 0.
\end{aligned}
$$

Since $\psi_\rho$ is convex on $\Delta_k(\gamma_n)$, $\psi_\rho$ achieves its maximum at an extreme point of $\Delta_k(\gamma_n)$. Therefore, there exists a partition $\{A_i\}_{i=1}^k$ of $\mathbb{R}^n$ such that $(1_{A_1},\ldots,1_{A_k})\in\Delta_k(\gamma_n)$ maximizes $\psi_\rho$ on $\Delta_k(\gamma_n)$ [16, Lemma 2.1]. Specifically, it is noted in [16, Lemma 2.1] that the extreme points of $\Delta_k(\gamma_n)$ are exactly the partitions of $\mathbb{R}^n$ into $k$ pieces.

We now prove (3.2) by contradiction. By the Lebesgue density theorem [21][1.2.1, Proposition 1], we may assume that, for all $i\in\{1,\ldots,k\}$, if $y\in A_i$, then we have $\lim_{r\to 0}\gamma_n(A_i\cap B(y,r))/\gamma_n(B(y,r))=1$. Suppose there exist $j,m\in\{1,\ldots,k\}$ and there exists $y\in\mathbb{R}^n$, $r>0$ such that $y\in A_j$, $\gamma_n(B(y,r)\cap A_j)>0$ and $LT_\rho 1_{A_j}(y)<LT_\rho 1_{A_m}(y)$. By (1.2),

$$T_\rho 1_{A_j}(x)=\int_{\mathbb{R}^n}1_{A_j}(y)e^{-\|y-x\rho\|_2^2/[2(1-\rho^2)]}\frac{dy}{(2\pi(1-\rho^2))^{n/2}}.$$

So, $LT_\rho 1_{A_j}=\rho(d/d\rho)T_\rho 1_{A_j}(x)$ is a continuous function of $x$.

Therefore, there exists a ball $B(y,r)$, $r>0$ such that $\gamma_n(B(y,r)\cap A_j)>0$ and such that

$$\sup_{x\in B(y,r)}LT_\rho 1_{A_j}(x)<\inf_{x\in B(y,r)}LT_\rho 1_{A_m}(x).$$

Let $\phi(x):=1_{B(y,r)\cap A_j}(x)$. For $\lambda\in[0,1]$, note that

$$\big(1_{A_1},\ldots,1_{A_j}-\lambda\phi,\ldots,1_A+\lambda\phi,\ldots,1_{A_k}\big)\in\Delta_k(\gamma_n). \tag{3.4}$$

However,

$$
\begin{aligned}
&\frac{d}{d\lambda}\bigg|_{\lambda=0}\psi_\rho(1_{A_1},\ldots,1_{A_j}-\lambda\phi,\ldots,1_{A_m}+\lambda\phi,\ldots,1_{A_k})\\
&\qquad=\frac{2}{\rho}\int\phi(x)LT_\rho(1_{A_m}-1_{A_j})(x)d\gamma_n(x)>0.
\end{aligned}
\tag{3.5}
$$

But (3.5) contradicts the maximality of $(1_{A_1},\ldots,1_{A_k})$ on $\Delta_k(\gamma_n)$, so (3.2) holds.

□

## 4 Perturbative Estimates

Recalling (3.3), the following estimates allow us to relate $\psi_\rho$ to $\psi_0$ for small $\rho>0$, for simplicial conical partitions. In particular, we make a close examination of the two quantities of (1.10). Since lemma 4.2 gives precise estimates of the two quantities of (1.10), combining Lemma 4.2 with (3.2) gives precise geometric information about a partition $\{A_i\}_{i=1}^k\subseteq\mathbb{R}^n$ optimizing noise stability. In particular, to see one way that we will apply Lemma 4.2, see (7.20) below. However, note that (7.20) below does not give sufficiently precise information to identify the sets optimizing noise stability. So, the real need for Lemma 4.2 will occur in the proof of the Main Lemma 6.1, where the precise estimate (6.7) is used.

**Lemma 4.1.** *Let $A \subseteq \mathbb{R}^n$ be a cone. Then*

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{n}(1-y_i^2)\right)1_A(y)d\gamma_n(y) = 0.$$

*Proof.* The assertion follows by standard equalities for the moments of a Gaussian random variable. Let $\alpha > 0$. Define $f(\alpha)$ by the formula

$$f(\alpha) := \int_{\mathbb{R}^n} 1_A(y)e^{-\alpha(y_1^2+\cdots+y_n^2)/2}\frac{dy}{(2\pi)^{n/2}}.$$

By changing variables, $f(\alpha) = \alpha^{-n/2}\int_{\mathbb{R}^n} 1_A(y)d\gamma_n(y)$. So,

$$-\frac{1}{2}\int_{\mathbb{R}^n}\left(\sum_{i=1}^{n} y_i^2\right)1_A(y)d\gamma_n(y) = \left.\frac{df(\alpha)}{d\alpha}\right|_{\alpha=1} = -\frac{n}{2}\int_{\mathbb{R}^n} 1_A(y)d\gamma_n(y).$$

□

**Lemma 4.2.** *Fix $k = 3$, $n \geq 2$, $\rho \in (0,1)$. Let $\{C_i\}_{i=1}^k \subseteq \mathbb{R}^2$ be a simplicial conical partition. Let $\{B_i\}_{i=1}^k := \{C_i \times \mathbb{R}^{n-2}\}_{i=1}^k$. Fix $i,j \in \{1,\ldots,k\}$. Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ denote reflection across $B_i \cap B_j$. Assume that $B_i = \sigma B_j$ and that $B_i \subseteq \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Let $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, $e_1, e_2 \in \mathbb{R}^n$. For $p \in \{1,\ldots,k\}$, let $z_p := \int_{B_p} x d\gamma_n(x)$. Note that $\mathrm{span}\{z_i, z_j\} = \mathrm{span}\{e_1, e_2\}$. Let $n_j \in \mathbb{R}^n$ be the interior unit normal of $B_j$ so that $n_j$ is normal to the face $(\partial B_j) \setminus (\partial B_i)$, and let $n_i \in \mathbb{R}^n$ be the interior unit normal of $B_i$ so that $n_i$ is normal to the face $(\partial B_i) \setminus (\partial B_j)$.*

*(i) If $x \in B_i \cap \{x \in \mathbb{R}^n : \langle x, n_j\rangle \leq 0\}$, then*

$$\frac{1}{\rho}\left\langle x, \nabla T_\rho(1_{B_i} - 1_{B_j})(x)\right\rangle \geq 2x_1\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right) + \langle x, n_i\rangle\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right). \tag{4.1}$$

*(ii) If $x \in B_i \cap \{x \in \mathbb{R}^n : \langle x, n_j\rangle \geq 0\}$, then*

$$\frac{1}{\rho}\left\langle x, \nabla T_\rho(1_{B_i} - 1_{B_j})(x)\right\rangle \geq 2x_1\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right). \tag{4.2}$$

*(iii) For $x \in B_i$,*

$$\left|\int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right| \leq \frac{\rho}{\sqrt{1-\rho^2}}(\sqrt{6} + (n-1)\sqrt{2})x_1. \tag{4.3}$$

*(iv) For $x \in B_i$ with $x_1 > \sqrt{n}\sqrt{1-\rho^2}/\rho$,*

$$\int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) \geq 0. \tag{4.4}$$

*Proof of (i).* Below, we use differentiation in the distributional sense. Let $x \neq 0$. For $x \in (\partial B_i) \cap B_j$, $\nabla 1_{B_i}(x) = e_1$ since $B_i \subseteq \{x \in \mathbb{R}^n : x_1 \geq 0\}$, and for $x \in (\partial B_i) \setminus B_j$, $\nabla 1_{B_i}(x) = n_i$. Similarly, for $x \in (\partial B_j) \cap B_i$, $-\nabla 1_{B_j}(x) = e_1$, and for $x \in (\partial B_j) \setminus B_i$, $-\nabla 1_{B_j}(x) = -n_j$. Then

$$\begin{aligned}\frac{1}{\rho}\nabla T_\rho(1_{B_i} - 1_{B_j})(x) &= T_\rho(\nabla(1_{B_i} - 1_{B_j}))(x)\\ &= T_\rho[2(e_1)\delta_{B_i\cap B_j} + n_i\delta_{(\partial B_i)\setminus B_j} + (-n_j)\delta_{(\partial B_j)\setminus B_i}](x)\\ &\overset{(1.2)}{=} 2e_1\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right) + n_i\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right) + (-n_j)\gamma_n\left(\delta_{\frac{((\partial B_j)\setminus B_i)-x\rho}{\sqrt{1-\rho^2}}}\right).\end{aligned} \tag{4.5}$$

Here we used

$$\int_{\mathbb{R}^n} 1_A(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) = \int_{\mathbb{R}^n} 1_{A-x\rho}(y\sqrt{1-\rho^2})d\gamma_n(y)$$
$$= \int_{\mathbb{R}^n} 1_{(A-x\rho)/\sqrt{1-\rho^2}}(y)d\gamma_n(y).$$

Let $x$ with $x \in B_i$ and $\langle x, (-n_j)\rangle \geq 0$. Then (4.5) immediately proves (4.1). □

*Proof of (ii).* Let $x \in B_i \cap \{x \in \mathbb{R}^n : \langle x, n_j\rangle \geq 0\}$. By reflecting across $B_i \cap B_j$,

$$\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right) \geq \gamma_n\left(\delta_{\frac{((\partial B_j)\setminus B_i)-x\rho}{\sqrt{1-\rho^2}}}\right). \tag{4.6}$$

Define

$$w := n_i\gamma_n\left(\delta_{\frac{((\partial B_i)\setminus B_j)-x\rho}{\sqrt{1-\rho^2}}}\right) + (-n_j)\gamma_n\left(\delta_{\frac{((\partial B_j)\setminus B_i)-x\rho}{\sqrt{1-\rho^2}}}\right).$$

By (4.6), $w$ is in the convex hull of $e_1$ and $n_i$. In particular, $\langle x, w\rangle \geq 0$, since $x \in B_i$. Combining $\langle x, w\rangle \geq 0$ with (4.5) proves (4.2). □

*Proof of (iii).* By reflecting across $B_i \cap B_j$,

$$x \in B_i \cap B_j \quad \Longrightarrow \quad \int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) = 0.$$

So, a derivative bound gives (4.3). Specifically, we apply the Fundamental Theorem of Calculus to the following identity, with $\|(1-y_2^2)y_1\|_{L_2(\gamma_n)} = \sqrt{2}$ and $\|y_1^3+3y_1\|_{L_2(\gamma_n)} = \sqrt{6}$.

$$\frac{\partial}{\partial x_1}\int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)$$
$$= \frac{\rho}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^n}\left(-3y_1 - y_1^3 + \sum_{\ell\neq 1}(1-y_\ell^2)y_1\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y).$$

□

*Proof of (iv).* Let $x$ with $x_1 > \sqrt{n}\sqrt{1-\rho^2}/\rho$ and consider the following cone

$$A := \{0\} \cup \left\{y \in \mathbb{R}^n : y \neq 0 \,\wedge\, \sqrt{n}\frac{y}{\|y\|_2} \in \frac{B_i - x\rho}{\sqrt{1-\rho^2}}\right\}.$$

By Lemma 4.1, $\int_{\mathbb{R}^n} 1_A(y)\sum_{\ell=1}^{n}(1-y_\ell^2)d\gamma_n(y) = 0$. If $d(x, \partial B_i) \geq \sqrt{n}\sqrt{1-\rho^2}/\rho$, then $A = \mathbb{R}^n$ and $1_A(y)\sum_{\ell=1}^{n}(1-y_\ell^2)1_{B_i^c}(x\rho + y\sqrt{1-\rho^2}) \leq 0$, so

$$\int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)$$
$$\geq \int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)1_{B_i}(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) \geq \int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)d\gamma_n(y) = 0.$$

So, it remains to consider the case $d(x, \partial B_i) < \sqrt{n}\sqrt{1-\rho^2}/\rho$. In this case $A \neq \mathbb{R}^n$.

Since $x_1 > \sqrt{n}\sqrt{1-\rho^2}/\rho$, we have $\sum_{\ell=1}^{n}(1-y_\ell^2)1_{B_j}(x\rho+y\sqrt{1-\rho^2}) \leq 0$. Also, we have $\sum_{\ell=1}^{n}(1-y_\ell^2)1_{A^c}(y)1_{B_i}(x\rho+y\sqrt{1-\rho^2}) \geq 0$, $\sum_{\ell=1}^{n}(1-y_\ell^2)1_{A}(y)1_{B_i^c}(x\rho+y\sqrt{1-\rho^2}) \leq 0$, so

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)(1_{B_i}-1_{B_j})(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\quad\geq \int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)1_{B_i}(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\quad\geq \int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)1_{A}(y)1_{B_i}(x\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\\
&\quad\geq \int_{\mathbb{R}^n}\left(\sum_{\ell=1}^{n}(1-y_\ell^2)\right)1_{A}(y)d\gamma_n(y)=0.
\end{aligned}$$

□

## 5 Iterative Estimates

The following estimates control the errors that appear in the proof of Theorem 1.3. Being rather technical in nature, this section could be skipped on a first reading.

**Lemma 5.1.** *For $\ell=(\ell_1,\ldots,\ell_n)\in\mathbb{N}^n$ and $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$,*

$$h_\ell(x)\sqrt{\ell!} \leq |\ell|^n\, 3^{|\ell|}\prod_{i=1}^{n}\max\{1,|x_i|^{\ell_i}\}.$$

*Proof.* Let $\ell\in\mathbb{N}$.

$$\sum_{\ell=0}^{\infty}\lambda^\ell h_\ell(x) \overset{(1.5)}{=} e^{\lambda x-\lambda^2/2} = \sum_{p=0}^{\infty}\frac{x^p}{p!}\lambda^p\sum_{q=0}^{\infty}\frac{(-1)^q(\lambda)^{2q}(1/2)^q}{q!} = \sum_{\ell=0}^{\infty}\lambda^\ell\sum_{m=0}^{\lfloor \ell/2\rfloor}\frac{x^{\ell-2m}(-1)^m2^{-m}}{m!(\ell-2m)!}.$$

Here we let $p+2q=\ell$, $m=q$. In particular,

$$h_\ell(x)=\sum_{m=0}^{\lfloor \ell/2\rfloor}\frac{x^{\ell-2m}(-1)^m2^{-m}}{m!(\ell-2m)!}. \tag{5.1}$$

Using Stirling's formula, $\sqrt{2\pi}\ell^{\ell+1/2}e^{-\ell}\leq \ell!\leq e\,\ell^{\ell+1/2}e^{-\ell}$. Let $\ell,m$ such that $m\in\{0,\ldots,\ell/2\}$, $\ell\geq 1$. Note that $\lim_{x\to 0^+}x^x=1$ and $\min_{x\in[0,1]}x^x>2/3$. Also, $m+\ell-2m=\ell-m\geq \ell/2$. For $m\neq 0$, write $m=\ell j$, $j\in[1/\ell,1/2]$. Note that $\max\{m,\ell-2m\}\geq \ell/3$. Then

$$\begin{aligned}
&\frac{\sqrt{\ell!}}{m!(\ell-2m)!}\\
&\leq \frac{\sqrt{e}\,\ell^{(1/2)\ell+1/4}e^{-\ell/2}}{2\pi m^{m+1/2}e^{-m}(\ell-2m)^{\ell-2m+1/2}e^{-(\ell-2m)}} = \frac{\sqrt{e}}{2\pi}\frac{\ell^{(1/2)\ell+1/4}}{m^{m+1/2}(\ell-2m)^{\ell-2m+1/2}}e^{\ell/2-m}\\
&=\frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}}{m^m(\ell-2m)^{\ell-2m}}e^{\ell/2-m}\\
&=\frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}}{(\ell j)^{\ell j}(\ell(1-2j))^{\ell(1-2j)}}e^{\frac{\ell}{2}-m}\\
&=\frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}e^{\ell/2-m}}{\ell^{\ell(1-j)}j^{\ell j}(1-2j)^{\ell(1-2j)}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{\ell^{\ell/2}e^{\ell/2-m}}{\ell^{\ell/2}\ell^{\frac{\ell}{2}(1-2j)}j^{\ell j}(1-2j)^{\ell(1-2j)}} \\
&= \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{1}{j^{\ell j}(1-2j)^{\ell(1-2j)}}\frac{e^{\frac{\ell}{2}(1-2j)}}{\ell^{\frac{\ell}{2}(1-2j)}} = \frac{\sqrt{e}}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{(e/\ell)^{\frac{\ell}{2}(1-2j)}}{j^{\ell j}(1-2j)^{\ell(1-2j)}} \\
&\le \frac{e}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{1}{j^{\ell j}(1-2j)^{\ell(1-2j)}} \le \frac{e}{2\pi}\frac{\ell^{1/4}}{\sqrt{m}\sqrt{\ell-2m}}\frac{1}{(2/3)^{2\ell}} \\
&\le \frac{e\sqrt{3}}{\ell^{1/4}2\pi}\frac{1}{(2/3)^{2\ell}} = \frac{e\sqrt{3}}{\ell^{1/4}2\pi}(9/4)^{\ell} \le \ell^{-1/4}(9/4)^{\ell}.
\end{aligned}
$$

Here we used $(e/\ell)^{(\ell/2)(1-2j)} \le \sqrt{e}$ for $\ell = 1, 2$.

Also, for $m = 0$ we have $\frac{\sqrt{\ell!}}{m!(\ell-2m)!} = 1$, and for $m = \ell/2$ we have

$$
\begin{aligned}
\frac{\sqrt{\ell!}}{m!(\ell-2m)!} &= \frac{\sqrt{\ell!}}{(\ell/2)!} \le \frac{\sqrt{e}\ell^{\ell/2+1/4}e^{-\ell/2}}{\sqrt{2\pi}(\ell/2)^{\ell/2+1/2}e^{-\ell/2}} = \sqrt{e}\frac{\ell^{1/4}}{\sqrt{2\pi}\ell^{1/2}2^{-\ell/2}2^{-1/2}} \\
&= \sqrt{\frac{e}{\pi}}\ell^{-1/4}2^{\ell/2} \le \ell^{-1/4}2^{\ell/2}.
\end{aligned}
$$

So, combining the above estimates with (5.1),

$$
\begin{aligned}
|h_\ell(x)\sqrt{\ell!}| &\le \sum_{m=0}^{\lfloor \ell/2 \rfloor} \ell^{-1/4}(9/4)^{\ell}\,|x|^{\ell-2m} \le \sum_{m=0}^{\lfloor \ell/2 \rfloor} \ell^{-1/4}(9/4)^{\ell}\max\{1, |x|^{\ell-2m}\} \\
&\le \ell\,\ell^{-1/4}(9/4)^{\ell}\max\{1, |x|^{\ell}\} \le \ell\,3^{\ell}\max\{1, |x|^{\ell}\}.
\end{aligned}
$$

Therefore, for $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$,

$$
h_\ell(x)\sqrt{\ell!} \le \ell_1 \cdots \ell_n 3^{\ell_1+\cdots+\ell_n}\prod_{i=1}^{k}\max\{1, |x_i|^{\ell_i}\} \le |\ell|^n\,3^{|\ell|}\prod_{i=1}^{n}\max\{1, |x_i|^{\ell_i}\}.
$$

□

The following Lemma uses standard tail bounds for a Gaussian random variable. We therefore omit the proof.

**Lemma 5.2.** *Let $\eta > 0, t > 0$, and let $n \ge 2$. Then*

$$
\left|\int_{[-\eta,\eta]\times[t,\infty]\times\mathbb{R}^{n-2}} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\le|\ell|\le 3}} \prod_{i=1}^{n} |y_i|^{\ell_i}\,d\gamma_n(y)\right| \le 3000n^3\eta(t^2+2)e^{-t^2/2},
$$

$$
\begin{aligned}
\left|\int_{B(0,t)^c} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\le|\ell|\le 3}} \prod_{i=1}^{n} |y_i|^{\ell_i}\,d\gamma_n(y)\right| &\le 4n^3 2^{-n/2}(\Gamma(n/2))^{-1}(n+2)!(t^{n+1}+1)e^{-t^2/2} \\
&\le 100(n+2)!(t^{n+1}+1)e^{-t^2/2}.
\end{aligned}
$$

The following Lemma says, if $\int_{\mathbb{R}^n} xf(x)d\gamma_n(x)$ is parallel to the $x_1$-axis, then the quantity $(d/d\rho)T_\rho f(x)$ should be bounded by a constant multiplied by $|x_1| + O(\rho)$. The precise error term (5.2) will be needed in Lemma 6.1 to determine the size of the function $(d/d\rho)T_\rho(1_{A_i} - 1_{A_j})$. The error term (5.2) will be estimated by Lemma 5.2, and the resulting estimate will be introduced into (3.2).

**Lemma 5.3.** *Let $\rho \in (-1,1)$, $n \geq 2$. Suppose $f \in L_2(\gamma_n)$ with $\int_{\mathbb{R}^n} y_2 f(y) d\gamma_n(y) = 0$. Let $x_1 \geq 0$ and $x_2 \geq 0$. Then*

$$\left|\frac{d}{d\rho}T_\rho f(x_1,x_2,0,\ldots,0)\right| \leq \Big(|x_1| + 2\rho(|x_2|^2 + (n+1)\rho|x_2| + |x_1x_2| + 2n)\Big) \cdot \sup_{\substack{t_1\in[0,x_1],\, t_2\in[0,x_2]\\ \eta\in[0,\rho]}} \left|\int_{\mathbb{R}^n} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^n |y_i|^{\ell_i} f((t_1,t_2,0,\ldots,0)\eta + y\sqrt{1-\eta^2}) d\gamma_n(y)\right|. \tag{5.2}$$

*Proof.* By integrating by parts, note that

$$\frac{d}{d\rho}\int_{\mathbb{R}^n} y_2 f(y\sqrt{1-\rho^2})d\gamma_n(y) = \frac{\rho}{1-\rho^2}\int_{\mathbb{R}^n} y_2((n+1)-y_2^2)f(y\sqrt{1-\rho^2})d\gamma_n(y) - \sum_{i\neq 2}\frac{\rho}{1-\rho^2}\int_{\mathbb{R}^n} y_i^2 y_2 f(y\sqrt{1-\rho^2})d\gamma_n(y).$$

So, using $\int_{\mathbb{R}^n} y_2 f(y)d\gamma_n(y) = 0$ and the Fundamental Theorem of Calculus,

$$\begin{aligned}&\int_{\mathbb{R}^n} y_2 f(y\sqrt{1-\rho^2})d\gamma_n(y)\\ &\leq \frac{\rho^2}{1-\rho^2}\sup_{\eta\in[0,\rho]}\left(\int_{\mathbb{R}^n} y_2((n+1)-y_2^2)f(y\sqrt{1-\eta^2})d\gamma_n(y) - \sum_{i\neq 2}\int_{\mathbb{R}^n} y_i^2 y_2 f(y\sqrt{1-\eta^2})d\gamma_n(y)\right).\end{aligned} \tag{5.3}$$

By integrating by parts again, note that

$$\begin{aligned}&\frac{\partial}{\partial x_2}\int_{\mathbb{R}^n} y_2 f((0,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\\ &\quad= \frac{\rho}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^n} f((0,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})(y_2^2-1)d\gamma_n(y).\end{aligned} \tag{5.4}$$

Applying the Fundamental Theorem of Calculus to (5.4) and then using (5.3),

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} y_2 f((0,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right|\\ &\leq |x_2|\sup_{t\in[0,x_2]}\left|\frac{\rho}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^n} f((0,t,0,\ldots,0)\rho + y\sqrt{1-\rho^2})(y_2^2-1)d\gamma_n(y)\right|\\ &+ \frac{\rho^2}{1-\rho^2}\sup_{\eta\in[0,\rho]}\left(\int_{\mathbb{R}^n} y_2((n+1)-y_2^2)f(y\sqrt{1-\eta^2})d\gamma_n(y) - \sum_{i\neq 2}\int_{\mathbb{R}^n} y_i^2 y_2 f(y\sqrt{1-\eta^2})d\gamma_n(y)\right).\end{aligned} \tag{5.5}$$

By integrating by parts as before,

$$\begin{aligned}&\frac{\partial}{\partial x_1}[x_2\int_{\mathbb{R}^n} y_2 f((x_1,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y)]\\ &\quad= x_2\frac{\rho}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^n} y_2 y_1 f((x_1,x_2,0,\ldots,0)\rho + y\sqrt{1-\rho^2})d\gamma_n(y).\end{aligned} \tag{5.6}$$

Combining (1.10), (5.5) and (5.6),

$$\begin{aligned}|(d/d\rho)T_\rho f(x)| &\leq |x_1|\left|\int_{\mathbb{R}^n} y_1 f((x_1,x_2,0,\ldots,0)\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\right| \\ &+|x_2|^2\sup_{t\in[0,x_2]}\left|\frac{\rho}{\sqrt{1-\rho^2}}\int_{\mathbb{R}^n} f((0,t,0,\ldots,0)\rho+y\sqrt{1-\rho^2})(y_2^2-1)d\gamma_n(y)\right| \\ &+\frac{\rho^2|x_2|}{1-\rho^2}\sup_{\eta\in[0,\rho]}\left(\int_{\mathbb{R}^n} y_2((n+1)-y_2^2)f(y\sqrt{1-\eta^2})d\gamma_n(y)\right. \\ &\qquad\qquad \left. -\sum_{i\neq 2}\int_{\mathbb{R}^n} y_i^2 y_2 f(y\sqrt{1-\eta^2})d\gamma_n(y)\right) \\ &+|x_1x_2|\sup_{t\in[0,x_1]}\frac{\rho}{\sqrt{1-\rho^2}}\left|\int_{\mathbb{R}^n} y_2y_1 f((t,x_2,0,\ldots,0)\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\right| \\ &+\frac{\rho}{\sqrt{1-\rho^2}}\left|\int_{\mathbb{R}^n}(\sum_{i=1}^n(y_i^2-1))f((x_1,x_2,0,\ldots,0)\rho+y\sqrt{1-\rho^2})d\gamma_n(y)\right|.\end{aligned} \tag{5.7}$$

We then deduce (5.2) from (5.7).

□

## 6 The Main Lemma

Lemma 6.1 below represents the main tool in the proof of the main theorem. As depicted in Figure 1, Lemma 6.1 says that, if an optimal partition is close to being simplicial conical, then it is actually much closer to being simplicial conical. So, this Lemma can be understood as a feedback loop, or as a contractive mapping type of argument. We first give an intuitive sketch of the proof of the Lemma. Let $\rho > 0$. We begin with a partition $\{A_p\}_{p=1}^3 \subseteq \mathbb{R}^n$ maximizing noise stability (1.3). We assume that there are disjoint sets $\{D_p\}_{p=1}^3$ that resemble a simplicial conical partition, as in the left side of Figure 1. We also assume that $A_p \supseteq D_p$ for all $p = 1, 2, 3$. We then find a sequence of sets $\{D_{p,1}\}_{p=1}^3$, $\{D_{p,2}\}_{p=1}^3$, ... $\{D_{p,R}\}_{p=1}^3$ such that $D_{p,r} \subseteq D_{p,r+1}$ for all $1 \leq p \leq 3$, for all $r \geq 1$. This sequence of sets is chosen so that the following implication can be proven:

$$A_p \supseteq D_{p,r} \quad \Longrightarrow \quad A_p \supseteq D_{p,r+1}. \tag{6.1}$$

In order to prove (6.1), we need to show: if $A_p \supseteq D_{p,r}$, then we can get sufficiently strong estimates on $LT_\rho 1_{D_{p,r+1}}$ such that (3.2) can be verified on $A_p$ for each $p = 1, 2, 3$. For example, in Step 1 of the proof of Lemma 6.1, the estimate (6.10) eventually implies (6.14). And (6.14) says that $A_p$ must contain more points than the initial information that we assumed in (6.4).

Finally, we need to choose our sets $\{D_{p,r}\}_{p=1}^3$ appropriately so that, after finitely many implications of the form (6.1), we eventually get the conclusion (6.5). That is, the three sets $\{D_{p,R}\}_{p=1}^3$ resemble the right side of Figure 1, and $A_p \supseteq D_{p,R}$ for each $p = 1, 2, 3$. Thus concludes our description of the main strategy of the proof. Within the proof itself, the sets $\{D_{p,1}\}_{p=1}^3$, $\{D_{p,2}\}_{p=1}^3$, ... will not be explicitly defined. However, portions of these sets will be defined at the end of every Step of the proof. In particular, examine the sets defined by the following sequence of assertions: (6.4), (6.14), (6.19), (6.24), (6.34), (6.44), (6.51), (6.57), and finally (6.5).

Unfortunately, there are many technical obstacles that stand in the way of bringing this strategy to fruition. The first minor issue is that we cannot control small rotations of our sets. At every step of the proof, we therefore need to redefine our simplicial sets $B_i, B_j$ to account for these small rotations. However, the main technical issue is that it is not at all obvious how to choose the sets $\{D_{p,r}\}_{p=1}^{3}$ for $r=1,2,3,\ldots$ such that (6.1) can be proven for each $r=1,2,3,\ldots$. Moreover, the simplest choice of these sets, namely dilations of the sets depicted in Figure 1, do not produce satisfactory estimates.

Ultimately, the sequence of sets defined by (6.14), (6.19), (6.24), (6.34), (6.44), (6.51), (6.57) succeeds in proving the sequence of implications (6.1) for $r=1,2,3,\ldots$. Lemma 5.3 allows us to control the errors from our estimates, and we then make around seven modifications of the same error estimate within Lemma 6.1. This error estimate allows us to apply Lemma 4.2, so that we can improve our knowledge of the optimal partition $\{A_i\}_{i=1}^{k}$ via (3.2).

It would be preferable to write Lemma 6.1 as seven applications of a single Lemma, however the statement of such a Lemma would perhaps be so long and convoluted that its application would become opaque. We therefore use the longer presentation below in the hope of providing greater clarity. Finally, in the statement of Lemma 6.1 below, note that the plane $\Pi$ exists independently of $i,j\in\{1,\ldots,k\}$.

**Lemma 6.1.** *Fix $n\geq 2$, $k=3$. Let $0<\eta<\rho<e^{-20(n+1)^{10^{12}n^{3}(n+2)!}}$. Let $\{A_i\}_{i=1}^{k}$ be a partition of $\mathbb{R}^n$ such that (3.2) holds. Let $\Pi\subseteq\mathbb{R}^n$ be a fixed 2-dimensional plane such that $0\in\Pi$. Assume that, for each pair $i,j\in\{1,2,\ldots,k\}$ with $i\neq j$, there exists $\lambda'>0$ and there exists a regular simplicial conical partition $\{B_p'\}_{p=1}^{k}\subseteq\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} y(1_{A_i}(y)-1_{A_j}(y))d\gamma_n(y)=\lambda'\int_{\mathbb{R}^n} y(1_{B_i'}(y)-1_{B_j'}(y))d\gamma_n(y), \tag{6.2}$$

*such that*

$$\int_{B_p'} x d\gamma_n(x)\in\Pi,\ \forall\, p\in\{i,j\}, \tag{6.3}$$

*and such that*

$$\begin{aligned}&\{x\in B_i'\cup B_j' \colon 1_{A_i}(x)-1_{A_j}(x)\neq 1_{B_i'}(x)-1_{B_j'}(x)\}\\ &\quad\subseteq\{x\in B_i'\cup B_j' \colon |d(x,(\partial B_i')\cup(\partial B_j'))|<\eta\vee\|x\|_2\geq\sqrt{-2\log\eta}+(\rho+\eta)\sqrt{-2\log\rho}\}.\end{aligned} \tag{6.4}$$

*Then, for each pair $i,j\in\{1,2,\ldots,k\}$ with $i\neq j$, there exists $\lambda''>0$ and there exists a regular simplicial conical partition $\{B_p''\}_{p=1}^{k}\subseteq\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} y(1_{A_i}(y)-1_{A_j}(y))d\gamma_n(y)=\lambda''\int_{\mathbb{R}^n} y(1_{B_i''}(y)-1_{B_j''}(y))d\gamma_n(y)$, such that $\int_{B_p''} x d\gamma_n(x)\in\Pi$, $\forall\, p\in\{i,j\}$, and such that*

$$\begin{aligned}&\{x\in B_i''\cup B_j'' \colon 1_{A_i}(x)-1_{A_j}(x)\neq 1_{B_i''}(x)-1_{B_j''}(x)\}\\ &\quad\subseteq\{x\in B_i''\cap B_j'' \colon |d(x,(\partial B_i'')\cup(\partial B_j''))|<\rho\eta\vee\|x\|_2\geq\sqrt{-2\log(\rho\eta)}+1\}.\end{aligned} \tag{6.5}$$

*Proof.* Fix $i,j\in\{1,2,\ldots,k\}$ with $i\neq j$. By applying a rotation to $\mathbb{R}^n$, we assume that $B_i'\cap B_j'\subseteq\{x\in\mathbb{R}^n\colon x_1=0\}$ and $B_i'\subseteq\{x\in\mathbb{R}^n\colon x_1\geq 0\}$. Assume that (6.4) and (6.2) hold. Let $n_i'\in\mathbb{R}^n$ denote the interior unit normal of $B_i'$ such that $n_i'$ is normal to $(\partial B_i')\setminus B_j'$, and let $n_j'\in\mathbb{R}^n$ denote the interior unit normal of $B_j'$ such that $n_j'$ is normal to $(\partial B_j')\setminus B_i'$. Define $B_i,B_j$ such that

$$\begin{aligned}B_i=B_{i,\frac{2\eta}{\sqrt{-2\log\eta}}}&:=B_i'\cup\{x\in\mathbb{R}^n\colon x_1\geq 0\wedge\langle n_i',x/\|x\|_2\rangle\geq -2\eta/\sqrt{-2\log\eta}\},\\ B_j=B_{j,\frac{2\eta}{\sqrt{-2\log\eta}}}&:=B_j'\cup\{x\in\mathbb{R}^n\colon x_1\leq 0\wedge\langle n_j',x/\|x\|_2\rangle\geq -2\eta/\sqrt{-2\log\eta}\}.\end{aligned} \tag{6.6}$$

Let $x=(x_1,\ldots,x_n)\in B_i\cup B_j$. If $x_1<\sqrt{n}\sqrt{1-\rho^2}/\rho$, then

$$\gamma_n\left(\delta_{\frac{(B_i\cap B_j)-x\rho}{\sqrt{1-\rho^2}}}\right)\geq\frac{e^{-n/2}}{\sqrt{2\pi}}\int_{\sqrt{n}}^{\infty}e^{-t^2/2}dt/\sqrt{2\pi}\geq\frac{e^{-n/2}}{2\pi}\frac{1}{2\sqrt{n}}e^{-n/2}\geq\frac{1}{100\sqrt{n}}e^{-n}.$$

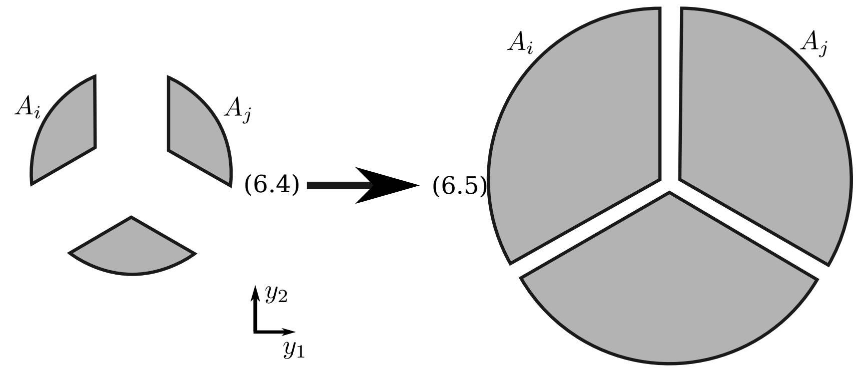


Figure 1: Depiction of Lemma 6.1

So, using Lemma 4.2, (1.9), and $\rho < 10^{-5}n^{-3/2}e^{-n}$, if $x \in B_i \cup B_j$ then

$$\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x) \geq \begin{cases} \frac{1}{9}\left|x_1\right| e^{-x_1^2\rho^2/(2(1-\rho^2))} & , \left|x_1\right| \leq 1 \,\vee\, x_2 \geq 0 \\ \frac{1}{9}\frac{\left|x_1\right|}{\rho\left|x_2\right|} e^{-\|x\|_2^2\rho^2/(2(1-\rho^2))} & , \rho x_2 \leq -1/\sqrt{3}. \end{cases} \quad (6.7)$$

Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ be a rotation such that the $x_1$-axis is fixed. For any such rotation, let

$$g(x) = g_\sigma(x) := 1_{A_i}(\sigma x) - 1_{A_j}(\sigma x) - (1_{B_i}(\sigma x) - 1_{B_j}(\sigma x)). \quad (6.8)$$

By (6.2), and since $B_i \cup B_j$ is symmetric with respect to reflection across $B_i \cap B_j \subseteq \{x \in \mathbb{R}^n \colon x_1 = 0\}$, $\exists\ \lambda > 0$ such that $\int_{\mathbb{R}^n} y(1_{A_i}(y) - 1_{A_j}(y))d\gamma_n(y) = \lambda\int_{\mathbb{R}^n} y(1_{B_i}(y) - 1_{B_j}(y))d\gamma_n(y)$. So $\int_{\mathbb{R}^n} y_2 g(y)d\gamma_n(y) = 0$, for all such rotations $\sigma$. For all $x \in \mathbb{R}^n$, and for all rotations $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ fixing the $x_1$-axis,

$$\left|LT_\rho(1_{A_i}-1_{A_j})(\sigma x) - LT_\rho(1_{B_i}-1_{B_j})(\sigma x)\right| \leq \left|LT_\rho g(x)\right|. \quad (6.9)$$

**Step 1. An estimate for large $x$.**

Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ be any rotation fixing the $x_1$-axis. By (6.8), $|g| \leq 2$. Applying (6.4) and (6.6) and the inclusion-exclusion principle, $g = 0$ on the set

$$\{y \in \mathbb{R}^n \colon d(\sigma y - \rho x, (\partial B_i') \cup (\partial B_j')) > \eta + 3\eta$$
$$\wedge \left\|\sigma y - \rho x\right\|_2 \leq \sqrt{-2\log\eta} + (\rho+\eta)\sqrt{-2\log\rho}\}.$$

Let $x \in \mathbb{R}^n$ with $\|x\|_2^2 \leq -4\log(\eta\rho)$. Since $0 < \eta < \rho$, we have $\rho\left\|x\right\|_2 \leq -4\rho\log(\eta\rho) \leq -8\rho\log\eta$. By (6.4), (6.3) and the inclusion-exclusion principle, $g \neq 0$ only on the following sets: $\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, (\partial B_i') \cup (\partial B_j'))| \leq 4\eta/\sqrt{1-\rho^2}\}$ and $\{y \in \mathbb{R}^n \colon \left\|\sigma y - \rho x\right\|_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\}$. Then Lemma 5.2 says

$$\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)] \\ t_2\in[\min(x_2,0),\max(x_2,0)] \\ \alpha\in[0,\rho]}} \left|\int_{\mathbb{R}^n} \sum_{\substack{\ell\in\mathbb{N}^n\colon \\ 0\leq|\ell|\leq 3}} \prod_{i=1}^n \left|y_i\right|^{\ell_i} g((t_1,t_2,0,\ldots,0)\alpha + y\sqrt{1-\alpha^2})d\gamma_n(y)\right|$$
$$\leq 500000n^3 4\eta + 200(n+2)!((-2(1-2\rho)\log\eta)^{(n+1)/2}+1)\eta^{1-2\rho}.$$

Using Lemma 5.3 and (6.9),

$$\begin{aligned} &\|x\|_2^2 \leq -4\log(\rho\eta) \,\wedge\, |x_1| \geq (\rho\eta)^{1/3} \\ &\Longrightarrow \rho^{-1}\left|LT_\rho(1_{A_i}-1_{A_j})(x_1,x_2,0,\ldots,0) - LT_\rho(1_{B_i}-1_{B_j})(x_1,x_2,0,\ldots,0)\right| \\ &\quad\leq [\left|x_1\right| + 2\rho(\left|x_2\right|^2 + (n+1)\rho\left|x_2\right| + \left|x_1x_2\right| + 2n)] \\ &\qquad\cdot [500000n^3 4\eta + 200(n+2)!((-2(1-2\rho)\log\eta)^{(n+1)/2}+1)\eta^{1-2\rho}] \\ &\quad< 10^7(n+2)!(-2\log\eta)^{(n+5)/2}\eta^{1-2\rho}. \end{aligned} \quad (6.10)$$

Also, by (6.7), and using that $0<\eta<\rho<10^{-5}n^{-3/2}e^{-n}$,

$$\begin{aligned}&\|x\|_2^2\leq -4\log(\rho\eta)\,\wedge\, x\in B_i\cup B_j\\&\quad\Longrightarrow \rho^{-1}\mathrm{sign}(x_1)\cdot(LT_\rho(1_{B_i}-1_{B_j})(x))>\begin{cases}\frac{1}{9}\left|x_1\right|(\rho\eta)^{2\rho^2/(1-\rho^2)} & ,\left|x_1\right|\leq 1\,\vee\, x_2\geq 0\\ \frac{1}{9}\frac{\left|x_1\right|}{\rho\left|x_2\right|}(\rho\eta)^{2\rho^2/(1-\rho^2)} & ,\rho x_2\leq -1/\sqrt{3}\end{cases}.\end{aligned}\tag{6.11}$$

Combining (6.10) and (6.11), using (6.8) and $0<\eta<\rho<e^{-20(n+1)^{10^{12}(n+2)!}}$,

$$\begin{aligned}&\left|x_1\right|\geq(\rho\eta)^{1/3}\wedge\|x\|_2^2\leq -4\log(\rho\eta)\wedge x\in B_i\cup B_j\\&\quad\Longrightarrow \rho^{-1}\left|LT_\rho(1_{A_i}-1_{A_j})(x)-LT_\rho(1_{B_i}-1_{B_j})(x)\right|\leq\eta^{4/5}\\&\quad\wedge\,\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)\geq(\rho\eta)^{2\rho^2/(1-\rho^2)}(\rho\eta)^{1/3}\min\left(1,\frac{1}{\rho\sqrt{-4\log(\eta\rho)}}\right).\end{aligned}\tag{6.12}$$

By (6.12),

$$\left|x_1\right|\geq(\rho\eta)^{1/3}\,\wedge\,\|x\|_2^2\leq -4\log(\rho\eta)\,\wedge\, x\in B_i\cup B_j\Longrightarrow\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i}-1_{A_j})(x)>0.\tag{6.13}$$

Finally, applying (6.13) to Lemma 3.1 for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, and using (6.6) together with the inclusion-exclusion principle,

$$\left|x_1\right|\geq(\rho\eta)^{1/3}\,\wedge\,\|x\|_2^2\leq -4\log(\rho\eta)\,\wedge\, x\in B_i'\cup B_j'\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\tag{6.14}$$

**Step 2. An estimate for small $x$.**

Let $\sigma\colon\mathbb{R}^n\to\mathbb{R}^n$ be any rotation fixing the $x_1$-axis. For $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ with $\|x\|_2^2\leq -2\log\rho$, we have $\rho\left\|x\right\|_2\leq\rho\sqrt{-2\log\rho}$. Suppose also that $\left|x_1\right|\leq\eta$ and $x\in B_i\cup B_j$. By (6.6), (6.4), (6.14), (6.3) and the inclusion-exclusion principle, $g\neq 0$ only on the following sets:

$$\begin{aligned}&\{y\in\mathbb{R}^n\colon\left|d(\sigma y-\rho x,(B_i\cap B_j)\cup[(B_i\cup B_j)\setminus(B_i'\cup B_j')])\right|\leq\eta/\sqrt{1-\rho^2}\},\\&\{y\in\mathbb{R}^n\colon\left|d(\sigma y-\rho x,(B_i\cap B_j)\cup[(B_i\cup B_j)\setminus(B_i'\cup B_j')])\right|\leq(\rho\eta)^{1/4}\\&\qquad\qquad\wedge\left\|\sigma y-\rho x\right\|_2\geq\sqrt{-2\log\eta}\},\\&\{y\in\mathbb{R}^n\colon\ \left\|\sigma y-\rho x\right\|_2\geq(1+1/10)\sqrt{-3\log(\eta\rho)}/\sqrt{1-\rho^2}\}.\end{aligned}$$

We then apply Lemma 5.2 to get

$$\begin{aligned}&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n\colon\\ 0\leq\left|\ell\right|\leq 3}}\prod_{i=1}^n\left|y_i\right|^{\ell_i}g((t_1,t_2,0,\ldots,0)\alpha+y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\&\quad\leq 500000n^3\eta+500000n^3(\rho\eta)^{1/4}(-2(1-\rho)^2\log\eta+1)\eta^{(1-\rho)^2}\\&\qquad+200(n+2)!((-3\log(\rho\eta))^{(n+1)/2}+1)(\rho\eta)^{3/2}+1600(n+2)!2\eta/\sqrt{-2\log\eta}.\end{aligned}$$

So, using Lemma 5.3, (6.9) and $0<\eta<\rho<e^{-20(n+1)^{10^{12}n^3(n+2)!}}$,

$$
\begin{aligned}
&\rho^{3/4}\eta\leq|x_1|\leq\eta\,\wedge\,\|x\|_2^2\leq-2\log\rho\,\wedge\,x\in B_i\cup B_j\\
&\Longrightarrow\rho^{-1}\left|LT_\rho(1_{A_i}-1_{A_j})(x_1,x_2,0,\ldots,0)-LT_\rho(1_{B_i}-1_{B_j})(x_1,x_2,0,\ldots,0)\right|\\
&\quad\leq[|x_1|+2\rho(|x_2|^2+(n+1)\rho\,|x_2|+|x_1x_2|+2n)]\\
&\qquad\cdot\Big[500000n^33\eta+500000n^3(\rho\eta)^{1/4}(-2(1-\rho)^2\log\eta+1)\eta^{(1-\rho)^2}\\
&\qquad\quad+200(n+2)!((-3\log(\rho\eta))^{(n+1)/2}+1)(\rho\eta)^{3/2}+1600(n+2)!2\eta/\sqrt{-2\log\eta}\Big]\\
&\quad<\frac{1}{10}\eta\rho^{3/4}.
\end{aligned}
\tag{6.15}
$$

Also, by (6.7),

$$
\begin{aligned}
\eta\rho^{3/4}\leq|x_1|\leq\eta\,\wedge\,\|x\|_2^2\leq-2\log\rho\,\wedge\,x\in B_i\cup B_j&\\
\Longrightarrow\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>\frac{1}{10}\,|x_1|\,.&
\end{aligned}
\tag{6.16}
$$

Combining (6.15) and (6.16), and using (6.8),

$$
\eta\rho^{3/4}\leq|x_1|\leq\eta\,\wedge\,\|x\|_2^2\leq-2\log\rho\,\wedge\,x\in B_i\cup B_j\Longrightarrow\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i}-1_{A_j})(x)>0.
\tag{6.17}
$$

Similarly, by (6.15) and (6.8), we have the following estimate.

$$
\eta\leq|x_1|\leq(\rho\eta)^{1/3}\wedge\|x\|_2^2\leq1\wedge x\in B_i\cup B_j\Longrightarrow\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i}-1_{A_j})(x)>0.
\tag{6.18}
$$

Finally, applying (6.17) to Lemma 3.1 for all $i',j'\in\{1,\ldots,k\}$, and using (6.6) together with the inclusion-exclusion principle, (6.13) and (6.18),

$$
\eta\rho^{3/4}\leq|x_1|\leq\eta\wedge\|x\|_2^2\leq-2\log\rho\wedge x\in B_i'\cup B_j'\Longrightarrow\mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.
\tag{6.19}
$$

**Step 3. An estimate for intermediate values of $x$.**

In summary, (6.19) and (6.14) improve our initial assumption (6.4). We now repeat the above procedure with the improved assumptions. Before continuing, we need to redefine $B_i,B_j$. Via (6.6), let

$$
B_i:=B_{i,\min\left(\frac{2\rho^{3/4}\eta}{\sqrt{-2\log\rho}},\frac{2\eta}{\sqrt{-2\log\eta}}\right)},\;B_j:=B_{j,\min\left(\frac{2\rho^{3/4}\eta}{\sqrt{-2\log\rho}},\frac{2\eta}{\sqrt{-2\log\eta}}\right)}
\tag{6.20}
$$

Let $x$ with $\|x\|_2^2\leq-4\log\rho$, $\eta\rho\leq|x_1|\leq\eta$. Let $B:=(B_i\cap B_j)\cup[(B_i\cup B_j)\setminus(B_i'\cup B_j')]$. Suppose $x\in B_i\cup B_j$ also. By (6.19), (6.4), (6.14), (6.3) and the inclusion-exclusion principle, $g\neq0$ only on the following sets

$$
\begin{aligned}
&\{y\in\mathbb{R}^n\colon|d(\sigma y-\rho x,B)|\leq\eta\rho^{3/4}/\sqrt{1-\rho^2}\},\\
&\{y\in\mathbb{R}^n\colon|d(\sigma y-\rho x,B)|\leq\eta/\sqrt{1-\rho^2},\|\sigma y-\rho x\|_2>\sqrt{-2\log\rho}/\sqrt{1-\rho^2}\},\\
&\{y\in\mathbb{R}^n\colon|d(\sigma y-\rho x,B)|\leq(\rho\eta)^{1/4}/\sqrt{1-\rho^2},\|\sigma y-\rho x\|_2\geq\sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},\\
&\{y\in\mathbb{R}^n\colon\;\|\sigma y-\rho x\|_2>(1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.
\end{aligned}
$$

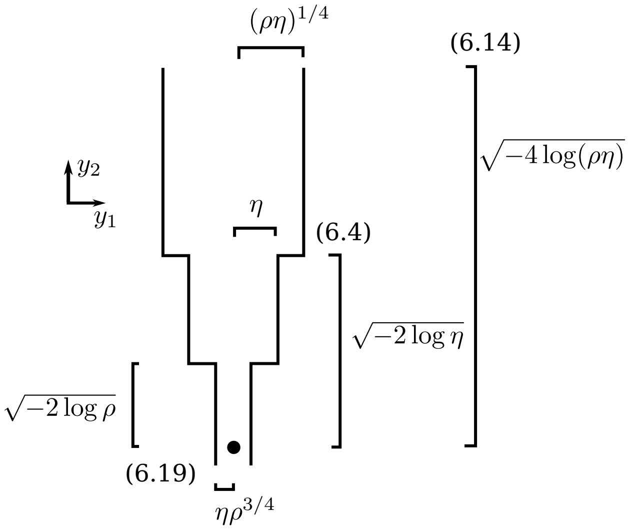


Figure 2: Integration regions where $g\neq 0$, near $B_i'\cap B_j'$ for (6.21).

We then apply Lemma 5.2 to get

$$\begin{aligned}\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}&\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}}\prod_{i=1}^{n}|y_i|^{\ell_i}\,g((t_1,t_2,0,\ldots,0)\alpha+y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\ &\leq \eta\rho^{3/4}500000n^3+500000n^3\eta(-2(1-\sqrt{2}\rho)^2\log\rho+1)\rho^{(1-\sqrt{2}\rho)^2}\\ &\quad+500000n^3(\eta\rho)^{1/4}(-2(1-\sqrt{2}\rho)^2\log\eta+1)\eta^{(1-\sqrt{2}\rho)^2}\\ &\quad+200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2}\\ &\quad+1600(n+2)!\min(2\rho^{3/4}\eta/\sqrt{-2\log\rho},2\eta/\sqrt{-2\log\eta}).\end{aligned}\tag{6.21}$$

Applying (6.21) to Lemma 5.3, using (6.8) and $\eta<\rho<e^{-20(n+1)^{10^{12}n^3(n+2)!}}$,

$$\begin{aligned}&\|x\|_2^2\leq -4\log(\rho)\ \wedge\ \eta\rho\leq|x_1|\leq\eta\ \wedge x\in B_i\cup B_j\\ &\quad\Longrightarrow \rho^{-1}\left|LT_\rho(1_{A_i}-1_{A_j})(x)-LT_\rho(1_{B_i}-1_{B_j})(x)\right|<\frac{1}{10}\rho\eta.\end{aligned}\tag{6.22}$$

Also, by (6.7),

$$\rho\eta\leq|x_1|\leq\eta\wedge\|x\|_2^2\leq-4\log\rho\wedge x\in B_i\cup B_j\Longrightarrow \rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>\frac{1}{10}\rho\eta.\tag{6.23}$$

So, combining (6.22), (6.23) for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, Lemma 3.1, (6.20), and by applying the inclusion-exclusion principle, (6.13) and (6.18),

$$\rho\eta\leq|x_1|\leq\eta\wedge\|x\|_2^2\leq-4\log\rho\wedge x\in B_i'\cup B_j'\Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\tag{6.24}$$

**Step 4. Iterating the estimate for intermediate values of $x$.**

The estimate (6.24) now has a cascading effect on the estimates below. From (6.24),

$$\rho^{.9}\eta\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq-4\log\rho\ \wedge\ x\in B_i'\cup B_j'\Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.$$

This estimate can be iterated on itself. Let $K \in \mathbb{N}$, $K \geq 1$, and let $M \in \mathbb{N}$ with $0 \leq M \leq \sqrt{K}$. Suppose $\rho^{.9K} > \eta^{1/5}$. We prove by induction on $K$ and $M$ that

$$\begin{aligned} 2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.25}$$

We already verified the case $M = 0, K = 1$. We assume that, for $0 \leq m < M$,

$$\begin{aligned} 2^{m^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{m+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.26}$$

Assume also that, for $M \leq m \leq \sqrt{K-1}$ and $K \geq 1$,

$$\begin{aligned} 2^{m^2}\eta\rho^{.9(K-1)} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{m+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.27}$$

We will conclude that (6.26) holds for $m = M$, i.e.

$$\begin{aligned} 2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.28}$$

We repeat the calculations (6.20) through (6.24). Redefine $B_i, B_j$ so that

$$B_i := B_{i,\min\left(\frac{2\eta\rho^{.9K}}{\sqrt{-4\log\rho}},\frac{2\eta}{\sqrt{-2\log\eta}}\right)}, \quad B_j := B_{j,\min\left(\frac{2\eta\rho^{.9K}}{\sqrt{-4\log\rho}},\frac{2\eta}{\sqrt{-2\log\eta}}\right)}. \tag{6.29}$$

If $M > 0$, we use (6.26) for $m = M - 1$. For any $M \geq 0$, we use (6.27) for $M \leq m \leq \sqrt{K}$. Let $x, M$ with $\|x\|_2^2 \leq -2^{M+2}\log\rho \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$, $2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta$, $x \in B_i \cup B_j$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Combining (6.26), (6.27), (6.4), (6.14), (6.3) and the inclusion-exclusion principle, $g \neq 0$ only on the following sets

$$\begin{aligned}
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9K}/\sqrt{1-\rho^2}\},\\
&\cup_{M\leq m\leq\lfloor\sqrt{K-1}\rfloor}\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq 2^{m^2}\eta\rho^{.9(K-1)}/\sqrt{1-\rho^2},\\
&\qquad\qquad \|\sigma y - \rho x\|_2 > \min(m,1)\cdot\sqrt{-2^{m+1}\log\rho}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq \eta, \|\sigma y - \rho x\|_2 > \sqrt{-2^{\lfloor\sqrt{K-1}\rfloor+2}\log\rho}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/4}/\sqrt{1-\rho^2}, \|\sigma y - \rho x\|_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},\\
&\{y \in \mathbb{R}^n \colon \|\sigma y - \rho x\|_2 > (1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.
\end{aligned}$$

We then apply Lemma 5.2 to get

$$\begin{aligned}
&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n\colon\\ 0\leq|\ell|\leq 3}}\prod_{i=1}^n |y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha + y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\
&\quad\leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9K}500000n^3\\
&\quad\quad + \eta(-(1-\rho)^2 2^{\lfloor\sqrt{K-1}\rfloor+2}\log\rho + 1)\rho^{(1-\rho)^2 2^{\lfloor\sqrt{K-1}\rfloor+1}}500000n^3\\
&\quad\quad + 500000n^3\sum_{m=M}^{\lfloor\sqrt{K-1}\rfloor}\eta\rho^{.9(K-1)}(-(1-\rho)^2 2^{m+1}\log\rho + 1)2^{m^2}\rho^{(1-\rho)^2 2^m\cdot\min(m,1)}\\
&\quad\quad + 500000n^3(\eta\rho)^{1/4}(-2(1-\sqrt{2}\rho)^2\log\eta + 1)\eta^{(1-\sqrt{2}\rho)^2}\\
&\quad\quad + 200(n+2)!((-3\log(\eta\rho))^{(n+1)/2} + 1)(\rho\eta)^{3/2}\\
&\quad\quad + 1600(n+2)!\min(2\eta\rho^{.9K}/\sqrt{-4\log\rho}, 2\eta/\sqrt{-2\log\eta}).
\end{aligned} \tag{6.30}$$

Applying (6.30) to Lemma 5.3, using (6.8), $\eta < \rho < e^{-20(n+1)^{10^{12}n^3(n+2)!}}$, and $\rho^{.9K} > \eta^{1/5}$,

$$\begin{aligned} &\|x\|_2^2 \leq -2^{M+2}\log(\rho) \,\wedge\, 2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, x \in B_i \cup B_j \\ &\qquad \Longrightarrow \rho^{-1}\left|LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x)\right| < \frac{1}{10}2^{M^2}\eta\rho^{.9K}. \end{aligned} \tag{6.31}$$

Also, by (6.7),

$$\begin{aligned} &2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i \cup B_j \\ &\qquad \Longrightarrow \rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > \frac{1}{10}2^{M^2}\eta\rho^{.9K}. \end{aligned} \tag{6.32}$$

So, combining (6.31), (6.32) for all $i', j' \in \{1, \ldots, k\}$, $i' \neq j'$, Lemma 3.1, (6.29), and by applying the inclusion-exclusion principle, (6.13) and (6.18),

$$\begin{aligned} &2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ &\qquad \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.33}$$

Thus, the inductive step is completed.

Let $K \in \mathbb{N}$ with $-2\log\eta \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$. Then (6.33) and (6.4) say that

$$2^K\eta\rho^{.9K} \leq |x_1| \leq 1 \,\wedge\, \|x\|_2^2 \leq -2\log\eta \,\wedge\, x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \tag{6.34}$$

**Step 5. Another iterative estimate, now for larger values of $x$.**

We perform another induction, though this time we hold $K$ fixed and use the additional ingredient (6.34). Let $M, R \in \mathbb{N}$ with $0 \leq M \leq \sqrt{K}$, $R \geq 0$ such that $\rho^{.9(K+R)} > \eta^{1/5}$. We will induct on $M$ and $R$. We assume that, for $0 \leq m < M$,

$$\begin{aligned} &2^{m^2}\eta\rho^{.9(K+R)} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{m+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ &\qquad \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.35}$$

We know that the case $R = 0, 0 \leq M \leq \sqrt{K}$ of (6.35) holds by (6.28). We therefore assume that $R \geq 1$. Assume also that, for $M \leq m \leq \sqrt{K}$,

$$\begin{aligned} &2^{m^2}\eta\rho^{.9(K+R-1)} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{m+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ &\qquad \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.36}$$

We will conclude that (6.35) holds for $m = M$, i.e.

$$\begin{aligned} &2^{M^2}\eta\rho^{.9(K+R)} \leq |x_1| \leq \eta \,\wedge\, \|x\|_2^2 \leq -2^{M+2}\log\rho \,\wedge\, x \in B_i' \cup B_j' \\ &\qquad \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \end{aligned} \tag{6.37}$$

Redefine $B_i, B_j$ so that

$$B_i := B_{i,\min\left(\frac{2\eta\rho^{.9(K+R)}}{\sqrt{-4\log\rho}}, \frac{2\eta}{\sqrt{-2\log\eta}}\right)}, \quad B_j := B_{j,\min\left(\frac{2\eta\rho^{.9(K+R)}}{\sqrt{-4\log\rho}}, \frac{2\eta}{\sqrt{-2\log\eta}}\right)}. \tag{6.38}$$

If $M > 0$, we use (6.35) for $m = M - 1$. For any $M \geq 0$, we also use (6.36) for $M \leq m \leq \sqrt{K}$. Let $x, M$ with $\|x\|_2^2 \leq -2^{M+2}\log\rho \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$, $2^{M^2}\eta\rho^{.9K} \leq |x_1| \leq \eta$, $x \in B_i \cup B_j$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Combining (6.35), (6.36), (6.4), (6.14), (6.3), and the fact that $-2\log\eta \leq -2^{\lfloor\sqrt{K}\rfloor+2}\log\rho \leq -4\log\eta$, we conclude that $g \neq 0$ only on the following sets:

$$\begin{aligned}&\{y\in\mathbb{R}^n\colon |d(\sigma y-\rho x,B)|\leq \min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9(K+R)}/\sqrt{1-\rho^2}\},\\ &\cup_{M\leq m\leq\lfloor\sqrt{K}\rfloor}\{y\in\mathbb{R}^n\colon |d(\sigma y-\rho x,B)|\leq 2^{m^2}\eta\rho^{.9(K+R-1)}/\sqrt{1-\rho^2},\\ &\qquad\qquad \|\sigma y-\rho x\|_2>\min(m,1)\cdot\sqrt{-2^{m+1}\log\rho}/\sqrt{1-\rho^2}\},\\ &\{y\in\mathbb{R}^n\colon |d(\sigma y-\rho x,B)|\leq(\rho\eta)^{1/4}/\sqrt{1-\rho^2},\|\sigma y-\rho x\|_2\geq\sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},\\ &\{y\in\mathbb{R}^n\colon\ \|\sigma y-\rho x\|_2>(1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.\end{aligned}$$

We then apply Lemma 5.2 to get

$$\begin{aligned}&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}}\prod_{i=1}^n|y_i|^{\ell_i}\,g((t_1,t_2,0,\ldots,0)\alpha+y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\ &\quad\leq\min(M,1)\cdot 2^{(M-1)^2}\eta\rho^{.9(K+R)}500000n^3\\ &\qquad+500000n^3\sum_{m=M}^{\lfloor\sqrt{K}\rfloor}\eta\rho^{.9(K+R-1)}(-(1-\rho)^2 2^{m+1}\log\rho+1)2^{m^2}\rho^{(1-\rho)^2 2^m}\\ &\qquad+500000n^3(\eta\rho)^{1/4}(-2(1-\sqrt{2}\rho)^2\log\eta+1)\eta^{(1-\sqrt{2}\rho)^2}\\ &\qquad+200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2}\\ &\qquad+1600(n+2)!\min(2\eta\rho^{.9(K+R)}/\sqrt{-4\log\rho},2\eta/\sqrt{-2\log\eta}).\end{aligned}\tag{6.39}$$

Applying (6.39) to Lemma 5.3, using (6.8), $\eta<\rho<e^{-20(n+1)^{10^{12}n^3(n+2)!}}$, and also $\rho^{.9(K+R)}>\eta^{1/5}$,

$$\begin{aligned}&\|x\|_2^2\leq -2^{M+2}\log(\rho)\ \wedge\ 2^{M^2}\eta\rho^{.9(K+R)}\leq|x_1|\leq\eta\ \wedge\ x\in B_i\cup B_j\\ &\quad\Longrightarrow\rho^{-1}\left|LT_\rho(1_{A_i}-1_{A_j})(x)-LT_\rho(1_{B_i}-1_{B_j})(x)\right|<\frac{1}{10}2^{M^2}\eta\rho^{.9(K+R)}.\end{aligned}\tag{6.40}$$

Also, by (6.7),

$$\begin{aligned}&2^{M^2}\eta\rho^{.9(K+R)}\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq -2^{M+2}\log\rho\ \wedge\ x\in B_i\cup B_j\\ &\quad\Longrightarrow\rho^{-1}\text{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>\frac{1}{10}2^{M^2}\eta\rho^{.9(K+R)}.\end{aligned}\tag{6.41}$$

So, combining (6.40), (6.41) for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, Lemma 3.1, (6.38), and by applying the inclusion-exclusion principle, (6.13) and (6.18),

$$\begin{aligned}&2^{M^2}\eta\rho^{.9(K+R)}\leq|x_1|\leq\eta\ \wedge\ \|x\|_2^2\leq -2^{M+2}\log\rho\ \wedge\ x\in B_i'\cup B_j'\\ &\quad\Longrightarrow\text{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\end{aligned}\tag{6.42}$$

Thus, the inductive step is completed. Let $M=\lfloor\sqrt{K}\rfloor$. Let $R\in\mathbb{N}$ such that $\eta^{1/5}\leq\rho^{.9(K+R)}\leq\eta^{1/5}\rho^{-.9}$. If no such $R$ exists, then $\rho^{.9K}<\eta^{1/5}$, so $\rho^{.45K}<\eta^{1/10}$, and (6.44) below holds by combining (6.34) and (6.4). Otherwise, $R\geq 0$, so (6.42) and (6.4) say that

$$2^K\eta^{6/5}\rho^{-.9}\leq|x_1|\leq 1\wedge\|x\|_2^2\leq -2\log\eta\wedge x\in B_i'\cup B_j'\Longrightarrow\text{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.\tag{6.43}$$

Since $\eta^{1/5}\leq\rho^{.9(K+R)}\leq\eta^{1/5}\rho^{-.9}$, note that $\eta^{1/10}\leq\rho^{.45(K+R)}\leq\eta^{1/10}\rho^{-.45}$, so for $R\geq 2$, we have $2^K\eta^{6/5}\rho^{-.9}\leq 2^K\rho^{.45K}\rho^{.45R}\eta^{11/10}\rho^{-.9}<\eta^{11/10}$. If $R=1$, and if $K\geq$

2, note that $2^{K}\eta^{6/5}\rho^{-.9} \leq 2^{K}\rho^{.2K}\rho^{.25K}\rho^{.45}\eta^{11/10}\rho^{-.9} < \eta^{11/10}$. If $R=0$, $K\geq 3$ then $2^{K}\eta^{6/5}\rho^{-.9} \leq 2^{K}\rho^{.1K}\rho^{.35K}\eta^{11/10}\rho^{-.9} < \eta^{11/10}$. If $1\leq R+K\leq 3$, then $(1/5)\log\eta \leq 3\log\rho$ and $2\log\rho \leq (1/5)\log\eta$, so (6.30) directly implies (6.44). More specifically, by (6.30), Lemma 5.3,(6.13) and (6.18), $\mathrm{sign}(x_1)\cdot(1_{A_i}-1_{A_j})(x)>0$ for $x\in B_i'\cup B_j'$ with $\|x\|_2^2\leq -2\log\eta$ and $\eta\rho^{.9K}\rho^{.9}\leq |x_1|\leq \eta$. Now, $\rho^{.45(K+R)}\leq \eta^{1/10}\rho^{-.45}$, so $\eta\rho^{.9K}\rho^{.9} = \eta\rho^{.45K}\rho^{.45K}\rho^{.9}\leq \eta\rho^{.45K}\rho^{.45(K+R)}\leq \eta^{11/10}$.

In the latter case, (6.44) follows, and in the former cases, (6.43) implies

$$\eta^{11/10}\leq |x_1|\leq 1 \,\wedge\, \|x\|_2^2\leq -2\log\eta \,\wedge\, x\in B_i'\cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0. \tag{6.44}$$

In all cases, (6.44) holds. We can finally use (6.44) to conclude the proof.

**Step 6. Using Step 5 to get an estimate for large values of $x$.**

Redefine $B_i, B_j$ so that

$$B_i := B_{i,2\eta^{11/10}/\sqrt{-2\log\eta}},\quad B_j := B_{j,2\eta^{11/10}/\sqrt{-2\log\eta}}. \tag{6.45}$$

Let $\sigma\colon\mathbb{R}^n\to\mathbb{R}^n$ be any rotation fixing the $x_1$-axis. Let $x$ with $\|x\|_2^2\leq -4\log(\eta\rho)\leq -8\log\eta$ and $\eta^{21/20}\rho^{1/2}\leq|x_1|\leq(\eta\rho)^{1/4}$. Let $B:=(B_i\cap B_j)\cup[(B_i\cup B_j)\setminus(B_i'\cup B_j')]$. Suppose $x\in B_i\cup B_j$ also. Combining (6.44), (6.14), and (6.3), $g\neq 0$ only on the following sets:

$$\begin{aligned}
&\{y\in\mathbb{R}^n\colon |d(\sigma y-\rho x,B)|\leq \eta^{11/10}/\sqrt{1-\rho^2}\},\\
&\{y\in\mathbb{R}^n\colon |d(\sigma y-\rho x,B)|\leq (\rho\eta)^{1/4}/\sqrt{1-\rho^2}, \|\sigma y-\rho x\|_2\geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},\\
&\{y\in\mathbb{R}^n\colon\ \|\sigma y-\rho x\|_2 > (1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.
\end{aligned}$$

We then apply Lemma 5.2 to get

$$\begin{aligned}
&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}}\left|\int_{\mathbb{R}^n}\sum_{\substack{\ell\in\mathbb{N}^n\colon\\ 0\leq|\ell|\leq 3}}\prod_{i=1}^{n}|y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha+y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\
&\quad\leq 500000n^3\eta^{11/10}+500000n^3(\rho\eta)^{1/4}(-2(1-2\rho)^2\log\eta+1)\eta^{(1-2\rho)^2}\\
&\qquad+200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2}+1600(n+2)!2\eta^{11/10}/\sqrt{-2\log\eta}.
\end{aligned} \tag{6.46}$$

Applying (6.46) to Lemma 5.3, using (6.8) and $\eta<\rho<e^{-20(n+1)10^{12}n^3(n+2)!}$,

$$\begin{aligned}
&\|x\|_2^2\leq -4\log(\eta\rho)\,\wedge\,\eta^{21/20}\rho^{1/2}\leq|x_1|\leq(\eta\rho)^{1/4}\,\wedge\, x\in B_i\cup B_j\\
&\quad\Longrightarrow \rho^{-1}\left|LT_\rho(1_{A_i}-1_{A_j})(x)-LT_\rho(1_{B_i}-1_{B_j})(x)\right|<\frac{1}{10}\eta^{21/20}\rho^{1/2}.
\end{aligned} \tag{6.47}$$

Also, by (6.7),

$$\begin{aligned}
&\eta^{21/20}\rho^{1/2}\leq|x_1|\leq(\eta\rho)^{1/4}\,\wedge\,\|x\|_2^2-4\log(\eta\rho)\,\wedge\, x\in B_i\cup B_j\\
&\quad\Longrightarrow \rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>\frac{1}{10}\eta^{21/20}\rho^{1/2}.
\end{aligned} \tag{6.48}$$

So, combining (6.47), (6.48) for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, Lemma 3.1, (6.45), and by applying the inclusion-exclusion principle, (6.13) and (6.18),

$$\begin{aligned}
&\eta^{21/20}\rho^{1/2}\leq|x_1|\leq(\eta\rho)^{1/4}\,\wedge\,\|x\|_2^2\leq -4\log(\eta\rho)\,\wedge\, x\in B_i'\cup B_j'\\
&\quad\Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0.
\end{aligned} \tag{6.49}$$

So, (6.49) and (6.14) say that

$$\eta^{21/20}\rho^{1/2}\leq|x_1|\,\wedge\,\|x\|_2^2\leq -4\log(\eta\rho)\,\wedge\, x\in B_i'\cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0. \tag{6.50}$$

Finally, we use (6.50) in place of (6.44) and repeat the computations (6.46) through (6.49) to get

$$\eta\rho \leq |x_1| \,\wedge\, \|x\|_2^2 \leq -4\log(\eta\rho) \,\wedge\, x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \quad (6.51)$$

In conclusion, (6.5) follows from (6.51) and (6.2), letting $B_i'' := B_i'$ and $B_j'' := B_j'$.

**Step 7. Completing the proof.**

For completeness, we derive (6.51). Redefine $B_i, B_j$ so that

$$B_i := B_{i,2\eta^{21/20}\rho^{1/2}/\sqrt{-4\log(\eta\rho)}}, \quad B_j := B_{j,2\eta^{21/20}\rho^{1/2}/\sqrt{-4\log(\eta\rho)}}. \quad (6.52)$$

Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ be any rotation fixing the $x_1$-axis. Let $x$ with $\|x\|_2^2 \leq -4\log(\eta\rho) \leq -8\log\eta$ and $\eta\rho \leq |x_1| \leq (\eta\rho)^{1/3}$. Let $B := (B_i \cap B_j) \cup [(B_i \cup B_j) \setminus (B_i' \cup B_j')]$. Suppose $x \in B_i \cup B_j$ also. Combining (6.50), (6.14), and (6.3), $g \neq 0$ only on the following sets:

$$\{y \in \mathbb{R}^n\colon |d(\sigma y - \rho x, B)| \leq \eta^{21/20}\rho^{1/2}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n\colon |d(\sigma y - \rho x, B)| \leq (\rho\eta)^{1/3}/\sqrt{1-\rho^2}, \|\sigma y - \rho x\|_2 \geq \sqrt{-2\log\eta}/\sqrt{1-\rho^2}\},$$
$$\{y \in \mathbb{R}^n\colon\ \|\sigma y - \rho x\|_2 > (1+1/10)\sqrt{-3\log(\rho\eta)}/\sqrt{1-\rho^2}\}.$$

We then apply Lemma 5.2 to get

$$\begin{aligned}&\sup_{\substack{t_1\in[\min(x_1,0),\max(x_1,0)]\\ t_2\in[\min(x_2,0),\max(x_2,0)]\\ \alpha\in[0,\rho]}} \left|\int_{\mathbb{R}^n} \sum_{\substack{\ell\in\mathbb{N}^n:\\ 0\leq|\ell|\leq 3}} \prod_{i=1}^n |y_i|^{\ell_i}\, g((t_1,t_2,0,\ldots,0)\alpha + y\sqrt{1-\alpha^2})d\gamma_n(y)\right|\\ &\leq 500000n^3\eta^{21/20}\rho^{1/2} + 500000n^3(\rho\eta)^{1/3}(-2(1-2\rho)^2\log\eta + 1)\eta^{(1-2\rho)^2}\\ &\quad + 200(n+2)!((-3\log(\eta\rho))^{(n+1)/2}+1)(\rho\eta)^{3/2} + 1600(n+2)!2\eta^{6/5}\rho^{1/2}/\sqrt{-4\log(\eta\rho)}.\end{aligned} \quad (6.53)$$

Applying (6.53) to Lemma 5.3, using (6.8) and $\eta < \rho < e^{-20(n+1)10^{12}n^3(n+2)!}$,

$$\begin{aligned}&\|x\|_2^2 \leq -4\log(\eta\rho) \,\wedge\, \eta\rho \leq |x_1| \leq (\eta\rho)^{1/3} \,\wedge\, x \in B_i \cup B_j\\ &\Longrightarrow \rho^{-1}\left|LT_\rho(1_{A_i} - 1_{A_j})(x) - LT_\rho(1_{B_i} - 1_{B_j})(x)\right| < \frac{1}{10}\eta\rho.\end{aligned} \quad (6.54)$$

Also, by (6.7),

$$\begin{aligned}&\eta\rho \leq |x_1| \leq (\eta\rho)^{1/3} \,\wedge\, \|x\|_2^2 \leq -4\log(\eta\rho) \,\wedge\, x \in B_i \cup B_j\\ &\Longrightarrow \rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i} - 1_{B_j})(x) > \frac{1}{10}\eta\rho.\end{aligned} \quad (6.55)$$

So, combining (6.54), (6.55) for all $i', j' \in \{1,\ldots,k\}$, $i' \neq j'$, Lemma 3.1, (6.52), and by applying the inclusion-exclusion principle, (6.13) and (6.18),

$$\eta\rho \leq |x_1| \leq (\eta\rho)^{1/3} \wedge \|x\|_2^2 \leq -4\log(\eta\rho) \wedge x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \quad (6.56)$$

Then, (6.56) and (6.14) say that

$$\eta\rho \leq |x_1| \,\wedge\, \|x\|_2^2 \leq -4\log(\eta\rho) \,\wedge\, x \in B_i' \cup B_j' \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x) - 1_{A_j}(x)) > 0. \quad (6.57)$$

Finally, (6.51) follows from (6.57), completing the proof. □

## 7 Proof of the Main Theorem

We now combine the Lemmas of the previous sections, as described in Section 1. The main effort involves verifying the assumption of the Main Lemma 6.1. Once this is done, Lemma 6.1 can be iterated infinitely many times to complete the proof.

**Theorem 7.1.** *Fix $k=3$, $n\geq 2$. Define $\Delta_k(\gamma_n)$ as in Definition 2.1 and define $\psi_\rho$ as in* (3.3)*. Let $\{C_i\}_{i=1}^k\subseteq\mathbb{R}^n$ be a regular simplicial conical partition. Then there exists $\rho_0=\rho_0(n,k)>0$ such that, for all $\rho\in(0,\rho_0)$, $(1_{C_1},\ldots,1_{C_k})$ uniquely achieves the following supremum, up to rotation*

$$\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)}\rho^{-1}\sum_{i=1}^k\int_{\mathbb{R}^n}f_iLT_\rho f_id\gamma_n=\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)}\psi_\rho(f_1,\ldots,f_k).$$

*Proof.* Within the proof, we will assert that $\rho>0$ satisfies several upper bounds, and then at the end of the proof, we will define $\rho_0$ as the minimum of these upper bounds. By Lemma 3.1, let $\{A_i\}_{i=1}^k$ be a partition of $\mathbb{R}^n$ such that

$$\psi_\rho(1_{A_1},\ldots,1_{A_k})=\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)}\psi_\rho(f_1,\ldots,f_k). \tag{7.1}$$

By (1.12), write

$$\rho^{-1}\sum_{i=1}^k\int_{\mathbb{R}^n}1_{A_i}LT_\rho 1_{A_i}d\gamma_n=\sum_{i=1}^k\sum_{\ell\in\mathbb{N}^n}|\ell|\left|\int_{\mathbb{R}^n}1_{A_i}\sqrt{\ell!}h_\ell d\gamma_n\right|^2\rho^{|\ell|-1}. \tag{7.2}$$

**Step 1. The partition $\{A_i\}_{i=1}^k$ is close to being simplicial.**

For $i\in\{1,\ldots,k\}$, let $z_i:=\int_{A_i}xd\gamma_n(x)\in\mathbb{R}^n$. Subtracting the $|\ell|=1$ term from both sides of (7.2), treating the remaining terms as error terms, and using that $\|1_{A_i}\|_{L_2(\gamma_n)}\leq 1$ for all $i=1,\ldots,k$,

$$\left|\rho^{-1}\sum_{i=1}^k\int_{\mathbb{R}^n}1_{A_i}LT_\rho 1_{A_i}d\gamma_n-\sum_{i=1}^k\|z_i\|_{\ell_2^n}^2\right|\leq 3k\rho. \tag{7.3}$$

Therefore,

$$\sum_{i=1}^k\|z_i\|_{\ell_2^n}^2\overset{(1.14)}{=}\psi_0(1_{A_1},\ldots,1_{A_k})\overset{(7.3)}{\geq}\psi_\rho(1_{A_1},\ldots,1_{A_k})-3k\rho\overset{(7.1)}{\geq}\psi_\rho(1_{B_1},\ldots,1_{B_k})-3k\rho$$
$$\overset{(7.3)}{\geq}\psi_0(1_{B_1},\ldots,1_{B_k})-6k\rho\overset{(\text{Lemma 2.5})}{=}\sup_{(f_1,\ldots,f_k)\in\Delta_k(\gamma_n)}\psi_0(f_1,\ldots,f_k)-6k\rho.$$

**Step 2. Applying a small rotation.**

For $i\in\{1,\ldots,k\}$, let $w_i:=\int_{B_i}xd\gamma_n(x)$. Let $\rho>0$ such that $6k\rho<10^{-2}$. Then by Lemma 2.8,

$$d_2(\{A_i\}_{i=1}^k,\{B_i\}_{i=1}^k)<6(6k\rho)^{1/8}, \tag{7.4}$$

$$\inf_{\sigma\in SO(n)}\left(\sum_{i=1}^k\|\sigma z_i-w_i\|_{\ell_2^n}^2\right)^{1/2}<6(6k\rho)^{1/8}. \tag{7.5}$$

Note that (7.5) follows from (7.4) by Hilbert space duality and since the set of functions $\{x_i\}_{i=1}^n$ are contained in the orthonormal basis $\{h_\ell\sqrt{\ell!}\}_{\ell\in\mathbb{N}^n}$ of $L_2(\gamma_n)$.

Let $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, let $i,j\in\{1,\ldots,k\}$, $i\neq j$, and write the following equality of $L_2$ functions

$$1_{A_i}(x)-1_{A_j}(x)=:\sum_{\ell\in\mathbb{N}^n}c_\ell h_\ell(x)\sqrt{\ell!}. \tag{7.6}$$

Let $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$. By applying an orthogonal change of coordinates to $\{A_p\}_{p=1}^k$, we may assume that $c_\ell = 0$ when $|\ell| = 1$, $\ell_1 = 0$. By (1.9) and (1.6), write

$$\rho^{-1} L T_\rho (1_{A_i} - 1_{A_j})(x) = \sum_{\ell \in \mathbb{N}^n} c_\ell \, |\ell| \, \rho^{|\ell|-1} h_\ell(x) \sqrt{\ell!}. \tag{7.7}$$

Since $d_2(\{A_p\}_{p=1}^k, \{B_p\}_{p=1}^k) < 6(6k\rho)^{1/8}$, there exists $\{B_p''\}_{p=1}^k$ a regular simplicial conical partition, such that $(\sum_{p=1}^k \|1_{A_p} - 1_{B_p''}\|_{L_2(\gamma_n)}^2)^{1/2} < 6(6k\rho)^{1/8}$. In particular, by Hilbert space duality,

$$\left\| \int_{\mathbb{R}^n} x(1_{A_i}(x) - 1_{A_j}(x) - (1_{B_i''}(x) - 1_{B_j''}(x))) d\gamma_n(x) \right\|_{\ell_2^n} < 6(6k\rho)^{1/8}, \tag{7.8}$$

$$\left\| \int_{\mathbb{R}^n} x(1_{(A_i \cup A_j)^c}(x) - 1_{(B_i'' \cup B_j'')^c}(x)) d\gamma_n(x) \right\|_{\ell_2^n} < 6(6k\rho)^{1/8}. \tag{7.9}$$

Since $k = 3$, and since $\sum_{p=1}^k \int_{A_p} x d\gamma_n(x) = \int_{\mathbb{R}^n} x d\gamma_n(x) = 0$, there exists a 2-dimensional plane $\Pi \subseteq \mathbb{R}^n$ such that $0 \in \Pi$ and such that, for all $p \in \{1, \ldots, k\}$, $\int_{A_p} x d\gamma_n(x) \in \Pi$. Without loss of generality, $\Pi$ contains the $x_1$ and $x_2$ axes.

Let $\{B_p'\}_{p=1}^k$ be a regular simplicial conical partition such that

$$\left( \sum_{p=1}^k \|1_{B_p'} - 1_{B_p''}\|_{L_2(\gamma_n)}^2 \right)^{1/2} < 10(6k\rho)^{1/16}, \tag{7.10}$$

such that for fixed $i \neq j$, $i, j \in \{1, \ldots, k\}$ and for some $\lambda' \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} x(1_{A_i}(x) - 1_{A_j}(x)) d\gamma_n(x) = \lambda' \int_{\mathbb{R}^n} x(1_{B_i'}(x) - 1_{B_j'}(x)) d\gamma_n(x), \tag{7.11}$$

and such that

$$\int_{\mathbb{R}^n} x(1_{(B_i' \cup B_j')^c}) d\gamma_n(x) \in \Pi. \tag{7.12}$$

Such $\{B_p'\}_{p=1}^k$ exists by (7.8), letting $\rho > 0$ such that $\rho < (10000k)^{-8}$, so that

$$\left\| \int_{\mathbb{R}^n} x(1_{B_i''}(x) - 1_{B_j''}(x)) d\gamma_n(x) \right\|_{\ell_2^n} = 3\sqrt{2}/(4\sqrt{\pi}).$$

$$\left\| \int_{\mathbb{R}^n} x(1_{(B_i'' \cup B_j'')^c}(x)) d\gamma_n(x) \right\|_{\ell_2^n} = \left\| \int_{\mathbb{R}^n} x(1_{B_i''}(x)) \right\|_{\ell_2^n} = \sqrt{6}/(4\sqrt{\pi}).$$

So, by the triangle inequality applied to (7.8), and (7.9),

$$\left\| \int_{\mathbb{R}^n} x(1_{A_i}(x) - 1_{A_j}(x)) d\gamma_n(x) \right\|_{\ell_2^n} > 3\sqrt{2}/(4\sqrt{\pi}) - 10^{-2} > 1/3. \tag{7.13}$$

$$\left\| \int_{\mathbb{R}^n} x(1_{(A_i \cup A_j)^c}(x)) d\gamma_n(x) \right\|_{\ell_2^n} > \sqrt{6}/(4\sqrt{\pi}) - 10^{-2} > 1/3. \tag{7.14}$$

Specifically, we first apply a rotation to $\{B_p''\}_{p=1}^k$ such that (7.11) holds. Then, by (7.9), we then apply another rotation that fixes the $x_1$ axis, so that (7.12) holds. By (7.8), (7.9), (7.13) and (7.14), each of these two rotations can be chosen so that a given unit vector is moved in $\mathbb{R}^n$ a distance not more than $12(6k\rho)^{1/8}$. And since we are rotating three polygonal cones with two facets each, (7.10) holds.

Using (7.10) and the triangle inequality,

$$(\sum_{p=1}^{k}\|1_{A_p}-1_{B_p'}\|_{L_2(\gamma_n)}^2)^{1/2}<20(6k\rho)^{1/16}. \tag{7.15}$$

Also, using that $c_\ell=0$ for $|\ell|=1,\ell_1=0$, (7.11) implies that $B_i'\cap B_j'\subseteq\{x\in\mathbb{R}^n\colon x_1=0\}$, and we may assume that $B_i'\subseteq\{x\in\mathbb{R}^n\colon x_1\geq 0\}$.

Let $n_i'\in\mathbb{R}^n$ denote the interior unit normal of $B_i'$ such that $n_i'$ is normal to $(\partial B_i')\setminus B_j'$, and let $n_j'\in\mathbb{R}^n$ denote the interior unit normal of $B_j'$ such that $n_j'$ is normal to $(\partial B_j')\setminus B_i'$. Then, define $B_i,B_j$ such that

$$\begin{aligned}B_i&:=B_i'\cup\{x\in\mathbb{R}^n\colon x_1\geq 0\wedge\langle n_i',x/\|x\|_2\rangle\geq -4\rho^{21/20}/\sqrt{-3\log\rho}\},\\ B_j&:=B_j'\cup\{x\in\mathbb{R}^n\colon x_1\leq 0\wedge\langle n_j',x/\|x\|_2\rangle\geq -4\rho^{21/20}/\sqrt{-3\log\rho}\}.\end{aligned} \tag{7.16}$$

Since $B_i\cup B_j$ is symmetric with respect to reflection across $B_i\cap B_j=B_i'\cap B_j'$, equation (7.11) implies that there is a $\lambda>0$ such that

$$\int_{\mathbb{R}^n}x(1_{A_i}(x)-1_{A_j}(x))d\gamma_n(x)=\lambda\int_{\mathbb{R}^n}x(1_{B_i}(x)-1_{B_j}(x))d\gamma_n(x). \tag{7.17}$$

**Step 3. An estimate for small $x$.**

Combining (1.6), (1.9), and (7.17), there exists $|b_1|<50(6k\rho)^{1/16}$ (by Hilbert space duality, eqrefthree7.93 and (7.16)) such that

$$\rho^{-1}LT_\rho(1_{A_i}-1_{A_j})(x)-\rho^{-1}LT_\rho(1_{B_i}-1_{B_j})(x)-x_1b_1=:\sum_{\ell\in\mathbb{N}^n\colon|\ell|\geq 2}b_\ell\,|\ell|\,\rho^{|\ell|-1}h_\ell(x)\sqrt{\ell!}. \tag{7.18}$$

Choose $\rho_1$ so that $0<\rho<\rho_1$ implies that

$$100k^{1/16}\sum_{m=2}^{\infty}m(m+n-1)^n\rho^{m-2}m^n3^m(-\log\rho^3)^{m/2}<\rho^{-1/80}/20.$$

Recall that the number of $\ell\in\mathbb{N}^n$ such that $|\ell|=m$ is equal to $\frac{m+n-1!}{m!(n-1)!}\leq(m+n-1)^n$. Note that, $|b_\ell|<100k^{1/16}\rho^{1/16}$, for all $\ell\in\mathbb{N}^n$, $|\ell|\geq 2$, by Hilbert space duality. Let $x\in\mathbb{R}^n$ with $\|x\|_2^2\leq-\log\rho^3$. By (7.18), Lemma 5.1,

$$\begin{aligned}&\left|\rho^{-1}LT_\rho(1_{A_i}-1_{A_j})(x)-\rho^{-1}LT_\rho(1_{B_i}-1_{B_j})(x)-x_1b_1\right|\\ &\quad\leq 100k^{1/8}\rho^{17/16}\sum_{\ell\in\mathbb{N}^n\colon|\ell|\geq 2}|\ell|\,\rho^{|\ell|-2}\,|h_\ell(x)|\,\sqrt{\ell!}\\ &\quad\leq 100k^{1/8}\rho^{17/16}\sum_{\ell\in\mathbb{N}^n\colon|\ell|\geq 2}|\ell|\,\rho^{|\ell|-2}\,|\ell|^n\,3^{|\ell|}\prod_{i=1}^{n}\max\{1,|x_i|^{\ell_i}\}\\ &\quad\leq 100k^{1/8}\rho^{17/16}\sum_{m=2}^{\infty}m(m+n-1)^n\rho^{m-2}m^n3^m(-\log\rho^3)^{m/2}\leq\rho^{21/20}/20.\end{aligned} \tag{7.19}$$

From Lemma 4.2 and (1.9), for $x=(x_1,\ldots,x_n)$, with $B_i\cap B_j\subseteq\{x\in\mathbb{R}^n\colon x_1=0\}$,

$$x\in B_i\cup B_j\;\wedge\;\|x\|_2^2\leq-\log\rho^3\Longrightarrow\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{B_i}-1_{B_j})(x)>(1/10)\,|x_1|\,. \tag{7.20}$$

Then (7.19) and (7.20) show that

$$|x_1|>\rho^{21/20}\;\wedge\;\|x\|_2^2\leq-\log\rho^3\;\wedge\;x\in B_i\cup B_j\Longrightarrow\rho^{-1}\mathrm{sign}(x_1)\cdot LT_\rho(1_{A_i}-1_{A_j})(x)>0. \tag{7.21}$$

By (7.1), Lemma 3.1, and by applying (7.21) for all $i',j'\in\{1,\ldots,k\}$, $i'\neq j'$, along with the inclusion-exclusion principle,

$$|x_1|>\rho^{21/20}\wedge\|x\|_2^2\leq-\log\rho^3\wedge x\in B_i'\cup B_j'\Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0. \quad (7.22)$$

From (7.22),

$$\begin{aligned} x\in B_i'\cup B_j' \wedge |d(x,(\partial B_i')\cup(\partial B_j'))|>\rho^{21/20}\wedge\|x\|_2^2\leq-\log\rho^3 \\ \Longrightarrow \mathrm{sign}(x_1)\cdot(1_{A_i}(x)-1_{A_j}(x))>0. \end{aligned} \quad (7.23)$$

**Step 4. Applying the Main Lemma.**

Recall that there exists a 2-dimensional plane $\Pi\subseteq\mathbb{R}^n$ such that $0\in\Pi$ and such that, for all $p\in\{1,\ldots,k\}$, $\int_{A_p}xd\gamma_n(x)\in\Pi$. Define

$$S:=\mathrm{span}\left\{\int_{\mathbb{R}^n}(1_{B_i'}(x)-1_{B_j'}(x))xd\gamma_n(x),\int_{\mathbb{R}^n}(1_{B_i'}(x)-1_{(B_i'\cup B_j')^c}(x))xd\gamma_n(x)\right.$$
$$\left.,\int_{\mathbb{R}^n}(1_{B_j'}(x)-1_{(B_i'\cup B_j')^c}(x))xd\gamma_n(x)\right\}.$$

Note that $S$ is a 2-dimensional plane and $0\in S$. By (7.11), $\int_{\mathbb{R}^n}(1_{B_i'}(x)-1_{B_j'}(x))d\gamma_n(x)\in\Pi$. Moreover, since $\{B_i',B_j',(B_i'\cup B_j')^c\}$ is a regular simplicial conical partition,

$$S=\mathrm{span}\left\{\int_{B_i'}xd\gamma_n(x),\int_{B_j'}xd\gamma_n(x),\int_{(B_i'\cup B_j')^c}xd\gamma_n(x)\right\}.$$

From (7.12), $S$ and $\Pi$ are 2-dimensional planes that both contain the linearly independent vectors $\int_{(B_i'\cup B_j')^c}xd\gamma_n(x)$ and $\int_{\mathbb{R}^n}(1_{B_i'}(x)-1_{B_j'}(x))xd\gamma_n(x)$. We therefore conclude that $S=\Pi$. In particular,

$$\int_{B_p'}xd\gamma_n(x)\in\Pi,\ \forall\, p\in\{i,j\}. \quad (7.24)$$

Let $\rho_0:=\min(\rho_1,10^{-2}/6k,e^{-20(n+1)^{10^{12}n^3(n+2)!}})$. Using (7.23), (7.11) and (7.24), we can iteratively apply Lemma 6.1 an infinite number of times. In particular, any time we know the conclusion (6.5), we use (6.5) in the assumption (6.4). That is, we first apply Lemma 6.1 with $\eta=\rho^{21/20}$. In this case, since $\eta=\rho^{21/20}$, (7.23) implies (6.4), (7.24) implies (6.3), and (7.17) implies (6.2). Now, using the conclusion (6.5) of Lemma 6.1, we can then apply Lemma 6.1 with $\eta=\rho^{1+21/20}$. Once again, using the conclusion (6.5) of Lemma 6.1, we can apply Lemma 6.1 with $\eta=\rho^{2+21/20}$, and so on. Repeating this process infinitely many times shows that there exists a regular simplicial conical partition $\{C_i\}_{i=1}^k$ which is equal to $\{A_i\}_{i=1}^k$. □

The Main Theorem now follows from Theorem 7.1 and the Fundamental Theorem of Calculus.

**Theorem 7.2** (**Main Theorem**). *Let $n\geq 2,k=3$. There exists $\rho_0=\rho_0(n,k)>0$ such that Conjecture 1.2 holds for $\rho\in(0,\rho_0)$. Moreover, up to orthogonal transformation, the regular simplicial conical partition uniquely achieves the maximum of* (1.3) *in Conjecture 1.2.*

*Proof.* Choose $\rho_0$ via Theorem 7.1 and let $0<\rho<\rho_0$. Let $\{B_i\}_{i=1}^k\subseteq\mathbb{R}^n$ be a regular simplicial conical partition. By Theorem 7.1 and the fact that $\Delta_k^0(\gamma_n)\subseteq\Delta_k(\gamma_n)$,

$$\psi_\rho(1_{B_1},\ldots,1_{B_k})=\sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)}\psi_\rho(f_1,\ldots,f_k). \quad (7.25)$$

Let $(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)$. By (1.12), $\sum_{i=1}^k\int_{\mathbb{R}^n}f_iT_0f_id\gamma_n=k(1/k^2)=1/k$. By the Fundamental Theorem of Calculus and (7.25),

$$\sum_{i=1}^k\int_{\mathbb{R}^n}f_iT_\rho f_id\gamma_n=\int_0^\rho\left[\frac{d}{d\alpha}\sum_{i=1}^k\int_{\mathbb{R}^n}f_iT_\alpha f_id\gamma_n\right]d\alpha+\frac{1}{k}=\int_0^\rho\psi_\alpha(f_1,\ldots,f_k)d\alpha+\frac{1}{k}$$
$$\leq\int_0^\rho\psi_\alpha(1_{B_1},\ldots,1_{B_k})d\alpha+\frac{1}{k}=\int_0^\rho\left[\frac{d}{d\alpha}\sum_{i=1}^k\int_{\mathbb{R}^n}1_{B_i}T_\alpha 1_{B_i}d\gamma_n\right]d\alpha+\frac{1}{k}$$
$$=\sum_{i=1}^k\int_{\mathbb{R}^n}1_{B_i}T_\rho 1_{B_i}d\gamma_n.$$

□

By using the invariance principle [12, Theorem 1.10,Theorem 3.6,Theorem 7.1,Theorem 7.4] which transfers results from partitions of Euclidean space to low-influence discrete functions, Theorem 7.2 implies a weak form of the Plurality is Stablest Conjecture. While the following result is quite far from Conjecture 1.8 and might not be of immediate use to complexity theory, it is included to indicate a possible application of Theorem 7.2. Essentially, if we modify the exact application of the invariance principle that is used in [12, Theorem 7.1], then Conjecture 1.8 follows. However, by avoiding [12, Theorem 7.1], we must make very restrictive assumptions on the function $f$ in Conjecture 1.8. Nevertheless, [12, Theorem 7.4] shows that the class of functions $f$ described in Corollary 7.3 is nonempty.

Note that the most straightforward application of Theorem 7.2 only gives vacuous cases of Conjecture 1.8, in which $0<\rho<\rho_0(n,k)$. In particular, since Theorem 7.2 requires $0<\rho<\rho_0(n,k)$, by (1.12) we must take $\varepsilon<3k\rho$ to get a nontrivial statement in Conjecture 1.8. In this case, the invariance principle [12, Theorem 3.6] gives $\tau$ with $\log\tau=-C(\log(\varepsilon))^2(1/\varepsilon)$, so that $\tau$ becomes a function of $\rho$. Since we provide a $\rho$ with inverse exponential dependence on $n$, then $\tau$ also has inverse exponential dependence on $n$. Thus, no function $f$ can satisfy the assumptions of Conjecture 1.8 in this case. To avoid this issue, we modify Conjecture 1.8 as follows.

**Corollary 7.3** (Weak Form of Plurality is Stablest)**.** *Let $\rho_0(n,k)$ be given by Theorem 7.2. Fix $n\geq 2$, $k=3$, and Let $N:=\log\log\log\log\log(n)\geq 1$. Let $0<\rho<\rho_0(N,k)<1/2$, $\varepsilon>0$, $\tau=\tau(\varepsilon,k)>0$. Let $f\colon\{1,\ldots,k\}^n\to\Delta_k$ with $\sum_{\sigma\in\{1,\ldots,k\}^n\colon\sigma_j\neq 0}(\widehat{f_i}(\sigma))^2\leq\tau$ for all $i\in\{1,\ldots,k\}$, $j\in\{1,\ldots,n\}$. Assume that there exists $0<m<N$ and $g\colon\mathbb{R}^m\to\Delta_k$ with $\int_{\mathbb{R}^m}gd\gamma_m=\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}f(\sigma)$, and such that*

$$\left|\int_{\mathbb{R}^n}\langle g,T_\rho g\rangle d\gamma_n-\frac{1}{k^n}\sum_{\sigma\in\{1,\ldots,k\}^n}\langle f(\sigma),T_\rho f(\sigma)\rangle\right|<\varepsilon.$$

*Then part (a) of Conjecture 1.8 holds. From [12][Theorem 7.4], this class of $f$ is nontrivial.*

Unfortunately, the proof of Theorem 7.2 fails for small negative $\rho$, as we now show.

**Theorem 7.4.** *Fix $k=3$, $n\geq 2$. Define $\Delta_k^0(\gamma_n)$ as in Definition 2.2 and define $\psi_\rho$ as in* (3.3). *Let $\{B_i\}_{i=1}^k\subseteq\mathbb{R}^n$ be a regular simplicial conical partition. Then there exists $\rho_2=\rho_2(n,k)>0$ such that, for $\rho\in(-\rho_2,0)$, $(1_{B_1},\ldots,1_{B_k})$ does not achieve the following supremum.*

$$\sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)}\rho^{-1}\sum_{i=1}^k\int_{\mathbb{R}^n}f_iLT_\rho f_id\gamma_n=\sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)}\psi_\rho(f_1,\ldots,f_k).$$

*Proof.* Let $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$. Fix $i,j \in \{1,\ldots,k\}, i \neq j$. Let $\sigma\colon \mathbb{R}^n \to \mathbb{R}^n$ denote reflection across $B_i \cap B_j$. Since $B_i = \sigma(B_j)$, by (3.5), it suffices to find $i,j \in \{1,\ldots,k\}$ and $x \in B_i$ such that $\rho^{-1}LT_\rho 1_{B_i}(x) < \rho^{-1}LT_\rho 1_{B_j}(x)$. By replacing $\{B_i\}_{i=1}^k$ with $\{\tau B_i\}_{i=1}^k$ for $\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ a rotation, we may assume that $\mathrm{span}\{z_i\}_{i=1}^k = \mathrm{span}\{e_1, e_2\}$. Moreover, we may assume $B_i \cap B_j \subseteq \{x \in \mathbb{R}^n : x_1 = 0\}$ and $B_i \subseteq \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Let $y := (\sqrt{3}/2)e_1 + (1/2)e_2$, $\widetilde{y} := -(1/2)e_1 + (\sqrt{3}/2)e_2$. Fix $x \in B_i$ with $\langle x, \widetilde{y}\rangle > 0$ also fixed. From (4.5) and the fact that $\rho < 0$, there exists $c = c(\langle x, \widetilde{y}\rangle) > 0$ such that

$$\left\langle x, \frac{1}{\rho}\nabla T_\rho(1_{B_i} - 1_{B_j})(x)\right\rangle = -\langle x, \widetilde{y}\rangle(c + O(e^{-\langle x,y\rangle^2/2})). \tag{7.26}$$

For $x \in \mathbb{R}^n$ with $\langle x, \widetilde{y}\rangle = 0$, we have, as in Lemma 5.2, and Lemma 4.1,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right)\right| \\
&\qquad \leq 2\left|\int_{B(0,\rho\|x\|_2)}\sum_{i=1}^n (1-y_i^2)d\gamma_n(y)\right| \leq 200(n+1)!((\rho\|x\|_2)^n + 1)e^{-\rho^2\|x\|_2^2}.
\end{aligned}$$

So, a derivative bound as in the proof of (4.3) shows

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right)\right| \\
&\qquad \leq \rho\langle x, \widetilde{y}\rangle 200(n+2)! + 200(n+1)!((\rho\|x\|_2)^n + 1)e^{-\rho^2\|x\|_2^2}.
\end{aligned} \tag{7.27}$$

Then,

$$\begin{aligned}
\rho^{-1}LT_\rho(1_{B_i} - 1_{B_j})(x) &\overset{(1.9)}{=} \frac{1}{\rho}(\langle x, \nabla T_\rho(1_{B_i} - 1_{B_j})(x)\rangle - \Delta T_\rho(1_{B_i} - 1_{B_j})(x)) \\
&= \langle x, T_\rho(\nabla(1_{B_i} - 1_{B_j}))(x)\rangle \\
&\qquad + \frac{\rho}{1-\rho^2}\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)(1_{B_i} - 1_{B_j})(x\rho + y\sqrt{1-\rho^2})\right)d\gamma_n(y).
\end{aligned} \tag{7.28}$$

So, choose $\rho < (c/8)(200(n+2)!)^{-1}$, then choose $\langle x, y\rangle$ sufficiently large, and then combine (7.26),(7.27) and (7.28) to get

$$\rho^{-1}LT_\rho(1_{B_i} - 1_{B_j})(x) < -\langle x, \widetilde{y}\rangle\frac{c}{4}.$$

□

## 8 Open Problems

There are two problems that are left open in this work. First, Conjecture 1.2 remains entirely open for $k \geq 4$ partition elements. Some of the results of this work hold for the case $k \geq 4$, and some do not. The first variation in Lemma 3.1 holds for all $k \geq 4$. Strictly speaking, the argument of Lemma 3.1 may not hold for $\rho < 0$ since it is not clear whether or not the functional (3.3) is convex. Also, the technical error estimate from Lemma 5.3 holds. One of the main issues for the case $k \geq 4$ is that Lemma 2.5 is no longer available. Moreover, the stability estimate in Lemma 2.8 would be needed for $k \geq 4$. The following conjecture summarizes the main technical issue in proving an analogue of Lemma 2.5 for $k = 4$, $n = 3$. If we could have a stability estimate for Conjecture 8.1 below, resembling the estimate of Lemma 2.8, then in principle the proof of the Main Lemma, Lemma 6.1 would go through, and therefore Theorem 7.2 would hold for $k \geq 4$ as well. Before we state the conjecture, recall Definition 2.2, (3.3) and (1.14).

**Conjecture 8.1.** *Let $k = 4$, $n = 3$. Suppose $\{A_i\}_{i=1}^k \subseteq \mathbb{R}^n$ satisfies*

$$\psi_0(1_{A_1}, \ldots, 1_{A_k}) = \sup_{(f_1,\ldots,f_k)\in\Delta_k^0(\gamma_n)} \psi_0(f_1, \ldots, f_k).$$

*Then $\{A_i\}_{i=1}^k$ is a simplicial conical partition.*

This result is known to be true if we replace $\Delta_k^0(\gamma_n)$ with $\Delta_k(\gamma_n)$, by [14, Lemma 3.3]. However, the volume constraint of $\Delta_k^0(\gamma_n)$ causes difficulties for the methods of [14, 16].

The second problem that remains open is Conjecture 1.2 for $\rho < 0$ or for $\rho$ positive and much larger than $0$. We have already discussed the issues for $\rho < 0$ in Theorem 7.4, where it is shown that our proof strategy surprisingly fails for $\rho < 0$. For $\rho$ with, e.g. $\rho \in (1/2, 1)$, the error bounds that we use in the proof of Theorem 7.2 seem to break down, especially when we apply Lemma 6.1, Lemma 4.2, and (4.3). There is nothing special about our choice of $1/2$ here, other than that it is a positive number that is sufficiently far from $0$. So, it seems that our method is not applicable for $\rho \in (1/2, 1)$. For example, Lemma 5.3 has an error term which is estimated by Lemma 5.2. However, the error estimate of Lemma 5.2 grows exponentially in $n$. And to compensate for this error, we need to choose $\rho$ to decrease exponentially in $n$. Even before we apply the Main Lemma 6.1, there is also a loss in (7.19), where we essentially need a very specific $L_\infty$ bound on the Gaussian heat kernel (or Mehler kernel). We used the rather crude method of bounding each Hermite polynomial separately in Lemma 5.1, and then summing up these polynomials. In principle, both of these losses could be avoided with dimension independent error estimates in Lemmas 5.3 and Lemma 5.2. However, this seems to be a difficult task.

However, since the case $\rho \in (1/2, 1)$ relates to geometric multi-bubble problems, whereas the case of small $\rho$ seems to concern entirely different geometric information, it is unclear whether or not a single method could simultaneously solve or interpolate between different values of $\rho$ in Conjecture 1.2.

Finally, a new open problem has emerged subsequent to this work. It turns out that if we modify the measure restriction in Conjecture 1.2 in any way, then the analogue of Conjecture 1.2 is false [11]. To be precise, in the statement of Conjecture 1.2, let $(a_1, \ldots, a_k)$ with $0 < a_i < 1$ for all $i = 1, \ldots, k$, and such that $\sum_{i=1}^k a_i = 1$. Assume that $(a_1, \ldots, a_k) \neq (1/k, \ldots, 1/k)$. Then, the partition $\{A_i\}_{i=1}^k \subseteq \mathbb{R}^n$ which optimizes the noise stability (1.3) subject to the constraint $\gamma_n(A_i) = a_i$ for all $i = 1, \ldots, k$ is not any translation of a regular simplicial conical partition. In fact, the optimal partition $\{A_i\}_{i=1}^k$ has essentially no elementary description using simplices. See [11, Theorem 2.6] for a precise statement. So, for example, the following question is open.

**Question 8.2.** *Let $\rho > 0$, $k = 3$, $n = 2$, and let $(a_1, a_2, a_3) \in (0, 1)^3$ with $\sum_{i=1}^3 a_i = 1$. What is the partition $\{A_i\}_{i=1}^3$ maximizing the noise stability (1.3) subject to the constraint $\gamma_2(A_i) = a_i$ for all $i = 1, 2, 3$?*

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**Acknowledgments.** Thanks to Assaf Naor for guidance and encouragement, and for helpful comments concerning Corollary 7.3. Thanks to Elchanan Mossel for helpful comments concerning the details of Corollary 7.3 and the vacuous cases of Conjecture 1.8. Thanks to Oded Regev for reading the manuscript thoroughly and providing helpful comments. Thanks also to the anonymous reviewers' thorough reading and helpful comments.

## 9 Appendix: Differentiating the Ornstein Uhlenbeck Semigroup

We prove (1.9) and (1.10). Let $\rho \in (-1, 1)$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$. In the following calculations, we use integration by parts freely, and we use differentiation in the distributional sense. We first calculate derivatives of $T_\rho f(x)$ with respect to $x \in \mathbb{R}^n$, and then we calculate the derivative of $T_\rho f(x)$ with respect to $\rho$.

$$\begin{aligned}
\frac{\partial}{\partial x_i}T_\rho f(x) &\overset{(1.2)}{=} \int \frac{\partial}{\partial x_i}[f(x\rho + y\sqrt{1-\rho^2})]d\gamma_n(y) \\
&= \int \frac{\partial f(x\rho + y\sqrt{1-\rho^2})}{\partial z_i} d\gamma_n(y)\rho \\
&= \int \frac{\partial}{\partial y_i}[f(x\rho + y\sqrt{1-\rho^2})]d\gamma_n(y)\frac{\rho}{\sqrt{1-\rho^2}} \\
&= -\int f(x\rho + y\sqrt{1-\rho^2})\frac{\partial}{\partial y_i}[d\gamma_n(y)]\frac{\rho}{\sqrt{1-\rho^2}} \\
&= \frac{\rho}{\sqrt{1-\rho^2}}\int y_i f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y).
\end{aligned} \tag{9.1}$$

$$\begin{aligned}
\frac{\partial^2}{\partial x_i^2}T_\rho f(x) &\overset{(9.1)}{=} \frac{\rho}{\sqrt{1-\rho^2}}\int \frac{\partial}{\partial x_i}[f(x\rho + y\sqrt{1-\rho^2})]y_i d\gamma_n(y) \\
&= \frac{\rho}{\sqrt{1-\rho^2}}\int \frac{\partial f(x\rho + y\sqrt{1-\rho^2})}{\partial z_i} y_i d\gamma_n(y)\rho \\
&= \frac{\rho^2}{1-\rho^2}\int \frac{\partial}{\partial y_i}[f(x\rho + y\sqrt{1-\rho^2})]y_i d\gamma_n(y) \\
&= -\frac{\rho^2}{1-\rho^2}\int f(x\rho + y\sqrt{1-\rho^2})\frac{\partial}{\partial y_i}[y_i d\gamma_n(y)] \\
&= -\frac{\rho^2}{1-\rho^2}\int f(x\rho + y\sqrt{1-\rho^2})(-y_i^2+1)d\gamma_n(y) \\
&= \frac{\rho^2}{1-\rho^2}\int (y_i^2-1)f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y).
\end{aligned} \tag{9.2}$$

$$\begin{aligned}
&\frac{d}{d\rho}T_\rho f(x) = \frac{d}{d\rho}\int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y) \\
&= \int_{\mathbb{R}^n}\sum_{i=1}^n \frac{\partial f(x\rho + y\sqrt{1-\rho^2})}{\partial z_i}\left(x_i - y_i\frac{\rho}{\sqrt{1-\rho^2}}\right)d\gamma_n(y) \\
&= \int_{\mathbb{R}^n}\sum_{i=1}^n \frac{\partial}{\partial y_i}[f(x\rho + y\sqrt{1-\rho^2})]\left(x_i - y_i\frac{\rho}{\sqrt{1-\rho^2}}\right)\frac{d\gamma_n(y)}{\sqrt{1-\rho^2}} \\
&= -\int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2})\sum_{i=1}^n \frac{\partial}{\partial y_i}\left[\left(x_i - y_i\frac{\rho}{\sqrt{1-\rho^2}}\right)\frac{d\gamma_n(y)}{\sqrt{1-\rho^2}}\right] \\
&= -\int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2})\sum_{i=1}^n \left[\left(x_i - y_i\frac{\rho}{\sqrt{1-\rho^2}}\right)(-y_i) - \frac{\rho}{\sqrt{1-\rho^2}}\right]\frac{d\gamma_n(y)}{\sqrt{1-\rho^2}} \\
&= -\int_{\mathbb{R}^n} f(x\rho + y\sqrt{1-\rho^2})\sum_{i=1}^n \left[(y_i^2-1)\frac{\rho}{\sqrt{1-\rho^2}} - x_i y_i\right]\frac{d\gamma_n(y)}{\sqrt{1-\rho^2}} \\
&= \frac{1}{\rho}\Bigg[\frac{\rho}{\sqrt{1-\rho^2}}\left\langle x, \int_{\mathbb{R}^n} y f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\right\rangle \\
&\qquad + \frac{\rho^2}{1-\rho^2}\int_{\mathbb{R}^n}\left(\sum_{i=1}^n (1-y_i^2)\right)f(x\rho + y\sqrt{1-\rho^2})d\gamma_n(y)\Bigg] \\
&\overset{(9.1)\wedge(9.2)}{=} \frac{1}{\rho}\left(\langle x, \nabla T_\rho f(x)\rangle - \Delta T_\rho f(x)\right).
\end{aligned}$$