Second quantisation for skew convolution products of measures in Banach spaces[*]

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Abstract

We study measures in Banach space which arise as the skew convolution product of two other measures where the convolution is deformed by a skew map. This is the structure that underlies both the theory of Mehler semigroups and operator self-decomposable measures. We show how that given such a set-up the skew map can be lifted to an operator that acts at the level of function spaces and demonstrate that this is an example of the well known functorial procedure of second quantisation. We give particular emphasis to the case where the product measure is infinitely divisible and study the second quantisation process in some detail using chaos expansions when this is either Gaussian or is generated by a Poisson random measure.

Keywords: Skew convolution product, second quantisation, Ornstein-Uhlenbeck semigroup, Poisson random measure.
AMS MSC 2010: Primary 60B11, Secondary 47D03; 47D07; 60G51; 60H15; 60J35; 60G57.
Submitted to EJP on September 22, 2013, final version accepted on January 11, 2014.

1 Introduction

In recent years there has been considerable interest in skew-convolution semigroups of probability measures in Banach spaces and the so-called Mehler semigroups that they induce on function spaces. These objects arise naturally in the study of infinite dimensional Ornstein-Uhlenbeck processes driven by Banach-space valued Lévy processes. Such processes have attracted much attention as they are the solutions of the simplest non-trivial class of stochastic partial differential equations driven by additive Lévy noise (see [1, 7, 29]). The first systematic study of Mehler semigroups in their own right were [6] and [14] with the former concentrating on Gaussian noise while the latter generalised to the Lévy case. Harnack inequalities were obtained in [31] and the infinitesimal generators were found in [4]. From a different point of view, skew-convolution semigroups also appear naturally in the investigation of continuous state branching processes with immigration [11] and more general affine processes [10].

In this paper we focus on the representation of Mehler semigroups as second quantised operators. Such a result has been known for a long time in the Gaussian case.

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It was first established for Hilbert space valued semigroups in [8] and then extended to Banach spaces in [23]. Once such a representation is known it can be put to good use in proving key properties of the semigroup such as compactness and smoothness [8], symmetry [9], analyticity [15, 22], and in the computation of their L^p spectra [24]. When the semigroups act on Hilbert spaces, the desired second quantisation representation was recently obtained in [28] in the pure jump case using chaotic decomposition techniques from [20], under the assumption that the Ornstein-Uhlenbeck process has an invariant measure. This paper extends that result to the Banach space case and obtains the second quantisation representation without needing to assume the existence of an invariant measure.

In fact, within the main part of our paper we dispense with Mehler semigroups altogether and work with a more general structure which we introduce herein. For this we require that there are measures μ_1 on a Banach space E_1 and μ_2 and ρ on a Banach space E_2 which are related by the identity

$$\mu_2 = T(\mu_1) * \rho,$$

where $T : E_1 \to E_2$ is a Borel mapping and $*$ is the usual convolution of measures. An operator T that has such an induced action is precisely a *skew map* as featured in the abstract of this paper. Note that if $E_1 = E_2 = E$ and $\mu_1 = \mu_2 = \mu$ say, then μ is an *operator self-decomposable* measure and such objects have been intensely studied (see e.g. [18, 19, 34].) The invariant measures arising in [28] are precisely of this form. On the other hand a skew convolution semigroup of measures $(\mu_t, t \geq 0)$ with respect to a C_0-semigroup $(S(t), t \geq 0)$ is characterised by the relations $\mu_{s+t} = S(t)\mu_s * \mu_t$ and these are clearly also examples of our structure. At our more general level, the antecedent of a Mehler semigroup is a bounded linear operator P_T which acts from $L^2(E_2, \mu_2)$ to $L^2(E_1, \mu_1)$. Our main result is then to show that this operator can be seen as a second quantisation of the adjoint $T^* : E_2 \to E_1$ in a natural way in the case where μ_1 and μ_2 are both infinitely divisible and either Gaussian or of pure jump type. With regard to the pure jump case, Surgailis [33] has found necessary and sufficient conditions for a contraction semigroup to second quantise to a Markov semigroup in Poisson Fock space. The emphasis in our paper is different, in that we work at a more general level, but it may be fruitful for future work to explore the confluence of these two streams of thought.

A key part of our approach is the use of a family of vectors that we call *exponential martingale vectors*. We now explain how these arise and contrast them with the more familiar *exponential vectors* (see e.g. [3, 27]). Second quantisation is seen most naturally as a covariant functor Γ within the category whose objects are Hilbert spaces and morphisms are contractions (see e.g. [27]). If H is a Hilbert space and $\Gamma(H)$ is the associated symmetric Fock space, the set of exponential vectors is linearly independent and total in Fock space. If we are given a Gaussian field over H then the exponential vectors correspond to the generating functions of the Hermite polynomials, and from the point of view of stochastic calculus they correspond both to the Doléans-Dade exponentials and to the exponential martingales. When we consider Lévy processes, the latter symmetry is broken. Exponential vectors still correspond to Doléans-Dade exponentials (see [3]) but these are no longer exponential martingales. In this paper, we find that a natural context for defining second quantisation in a non-Gaussian context is to employ vectors that are natural generalisations of exponential martingales, rather than using exponential vectors themselves. Hence we call these exponential martingale vectors. In particular, as we show in Section 2 and the appendix, these are still both total and linearly independent.

Notation. Throughout this article, E is a real Banach space. The space of all bounded

linear operators on E is denoted by $\mathscr{L}(E)$ and the dual of E is denoted by E^*. The action of E^* on E is represented by $x^*(x) = \langle x, x^* \rangle$. Whenever we consider measures on a Banach space E, they are defined on the Borel σ-algebra $\mathscr{B}(E)$. If μ is a Borel measure on E and $T : E \to F$ is a Borel mapping from E into another Banach space F we frequently write $T(\mu)$ to denote the Borel measure $\mu \circ T^{-1}$. The Dirac measure based at $x \in E$ is denoted by δ_x. The Banach space (with respect to the supremum norm) of all bounded Borel measurable functions on E will be denoted $B_{\mathrm{b}}(E; \mathbb{K})$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. If both choices are permitted we simply write $B_{\mathrm{b}}(E)$.

2 Skew convolution of measures and associated skew maps

Let ν be a *finite Radon measure* on a Banach space E, that is, ν is a finite Borel measure on E with the property that for all $\varepsilon > 0$ there exists a compact set K in E such that $\nu(E \setminus K) < \varepsilon$. Recall that if E is separable, then every finite Borel measure is Radon.

The *characteristic function* of ν is the mapping $\widehat{\nu} : E^* \to \mathbb{C}$ defined by

$$\widehat{\nu}(x^*) = \int_E \exp(i\langle x, x^* \rangle) \, \nu(dx),$$

for all $x^* \in E^*$. The mapping $\widehat{\nu}$ is continuous with respect to the topology of uniform convergence on compact subsets of E. More generally, for a measurable function $\phi : E \to \mathbb{R}$ we may define

$$\widehat{\nu}(\phi) = \int_E \exp(i\phi(x)) \, \nu(dx).$$

Definition 2.1. *Let μ_1 and μ_2 be Radon probability measures on the Banach spaces E_1 and E_2, respectively, with $\widehat{\mu_2}(x^*) \neq 0$ for all $x^* \in E_2^*$ (e.g. this condition is fulfilled, when μ_2 is infinitely divisible). A Borel mapping $T : E_1 \to E_2$ is called a skew map with respect to the pair (μ_1, μ_2) if there exists a Radon probability measure ρ on E_2 such that*

$$T(\mu_1) * \rho = \mu_2,$$

and we say that μ_2 is the skew-convolution product (with respect to T) of μ_1 and ρ. If T is also a bounded linear operator between E_1 and E_2 we call it a skew operator with respect to (μ_1, μ_2).

Given the pair (μ_1, μ_2), the measure ρ is easily seen to be unique. Indeed, the identity $\widehat{T(\mu_1)}(x^*)\widehat{\rho}(x^*) = \widehat{\mu_2}(x^*) \neq 0$ forces $\widehat{T(\mu_1)}(x^*) \neq 0$, and therefore $\widehat{\rho}$ is uniquely determined by $T(\mu_1)$ and μ_2. We call ρ the *skew convolution factor* associated with T and the pair (μ_1, μ_2).

Proposition 2.2. *Suppose that $T : E_1 \to E_2$ is a skew map with respect to the pair (μ_1, μ_2), where $\widehat{\mu_2}(x^*) \neq 0$ for all $x^* \in E_2^*$. Let ρ be the associated skew convolution factor. For all $1 \leq p < \infty$ the linear mapping $P_T : B_{\mathrm{b}}(E_2) \to B_{\mathrm{b}}(E_1)$ defined by*

$$P_T f(x) := \int_{E_2} f(T(x) + y) \, d\rho(y), \quad x \in E_1,$$

extends uniquely to a linear contraction $P_T : L^p(E_2, \mu_2) \to L^p(E_1, \mu_1)$.

Proof. Fix $1 \leq p < \infty$. By the Hölder inequality, for all $f \in B_{\mathrm{b}}(E_2)$ we have

$$
\begin{aligned}
\|P_T f\|_{L^p(E_1, \mu_1)}^p &= \int_{E_1} \left| \int_{E_2} f(T(x) + y) \, d\rho(y) \right|^p d\mu_1(x) \\
&\leq \int_{E_1} \int_{E_2} |f(T(x) + y)|^p \, d\rho(y) \, d\mu_1(x) \\
&= \int_{E_2} \int_{E_2} |f(y' + y)|^p \, d\rho(y) \, dT(\mu_1)(y') \\
&= \int_{E_2} |f(z)|^p \, d(T(\mu_1) * \rho)(z) \\
&= \int_{E_2} |f(z)|^p \, d\mu_2(z) = \|f\|_{L^p(E_2, \mu_2)}^p,
\end{aligned}
$$

and the required result follows. $\qquad\square$

Example 2.3 (Skew Convolution Semigroups)**.** *Let $(S(t), t \geq 0)$ be a C_0-semigroup on a Banach space E. A skew convolution semigroup is a family $(\mu_t, t \geq 0)$ of Radon probability measures on E for which $\mu_{s+t} = S(t)\mu_s * \mu_t$ for all $s, t \geq 0$. Then $S(t)$ is a skew operator with respect to the pair (μ_s, μ_{s+t}), In this case we write P_t for the linear operator $P_{S(t)}$. Then $(P_t, t \geq 0)$ is a semigroup in that $P_0 = I$ and $P_{s+t} = P_s P_t$ for all $s, t \geq 0$, and is called a Mehler semigroup (see e.g. [4, 6, 10, 11, 14]). Such objects arise naturally in the study of linear stochastic partial differential equations with additive noise of the form:*

$$
dY(t) = AY(t) + dL(t), \tag{2.1}
$$

where A is the infinitesimal generator of $(S(t), t \geq 0)$ and $(L(t), t \geq 0)$ is an E-valued Lévy process. If E is a real Hilbert space then it is well-known (see e.g. [1, 7] and the recent book [29]) that this equation has a unique mild (equivalently weak) solution $(Y(t), t \geq 0)$ which is a Markov process given by the generalised Ornstein-Uhlenbeck process:

$$
Y(t) = S(t)Y(0) + \int_0^t S(t - u) \, dL(u), \tag{2.2}
$$

(where the initial condition $Y(0)$ is assumed to be independent of $(L(t), t \geq 0)$.) Then μ_t is the law of the E-valued random variable $\int_0^t S(t - u) \, dL(u)$ and $(P_t, t \geq 0)$ is the transition semigroup of $(Y(t), t \geq 0)$. On a Banach space we may define the stochastic convolution in (2.2) by using integration by parts as in [19]. Quite general necessary and sufficient conditions for solutions to exist to (2.1) (where the stochastic convolution is defined in the sense of Itô calculus) are given in [30]. If X is a Brownian motion, we refer the reader to [25].

Example 2.4 (Operator Self-Decomposable Measures)**.** *Let μ be a Radon probability measure on E that takes the form*

$$
\mu = T\mu * \rho, \tag{2.3}
$$

where T is a bounded linear operator on E and ρ is another Radon probability measure on E. Then μ is operator self-decomposable (see [34]) and T is a skew operator with respect to the pair (μ, μ). There has been extensive work on such measures in the case where (2.3) holds with $T = S(t)$ for all $t \geq 0$ where $(S(t), t \geq 0)$ is a C_0-semigroup on E (see e.g. [1, 18, 19]). Indeed such measures μ arise as the invariant measures of the Mehler semigroups of Example 2.3 (when these exist - see e.g. [7, 14]) and in the case of (2.2), ρ is the law of $\int_0^\infty S(t - u) \, dL(u)$.

Definition 2.5. *Let μ be a Radon probability measure on E satisfying $\widehat{\mu}(x^*) \neq 0$ for all $x^* \in E^*$. For each Borel function $\phi : E \to \mathbb{R}$ we define the function $K_{\mu,\phi} : E \to \mathbb{C}$ by*

$$K_{\mu,\phi}(x) := \frac{\exp(i\phi(x))}{\widehat{\mu}(\phi)}.$$

We call $K_{\mu,\phi}$ an *exponential martingale vector*.

Proposition 2.6. *Let μ_1 and μ_2 be Radon probability measures on E_1 and E_2, respectively, with $\widehat{\mu_2}(x^*) \neq 0$ for all $x^* \in E_2^*$. Let T be a skew map with respect to the pair (μ_1, μ_2). Then for all $x^* \in E_2^*$ we have*

$$P_T K_{\mu_2,x^*} = K_{\mu_1,x^*\circ T}. \tag{2.4}$$

Proof. Let ρ denote the associated skew convolution factor. From the identity

$$\widehat{\mu_2}(x^*) = \widehat{T(\mu_1)}(x^*)\widehat{\rho}(x^*) = \widehat{\mu_1}(x^* \circ T)\widehat{\rho}(x^*)$$

we deduce that for all $x \in E$

$$
\begin{aligned}
P_T K_{\mu_2,x^*}(x) &= \int_{E_2} K_{\mu_2,x^*}(T(x) + y)\, d\rho(y) \\
&= \frac{\exp(i\langle T(x), x^*\rangle)}{\widehat{\mu_2}(x^*)} \int_{E_2} \exp(i\langle y, x^*\rangle)\, d\rho(y) \\
&= \frac{\exp(i\langle T(x), x^*\rangle)}{\widehat{\mu_2}(x^*)}\widehat{\rho}(x^*) = \frac{\exp(i(x^* \circ T)(x))}{\widehat{\mu_1}(x^* \circ T)} = K_{\mu_1,x^*\circ T}(x).
\end{aligned}
$$

$\square$

Fix a Radon probability measure μ on E and let $\mathscr{E}_\mu$ denote the linear span of the set of exponential martingale vectors $\{K_{\mu,x^*}; x^* \in E^*\}$. The proposition implies that, under the stated hypotheses on μ_1, μ_2, and T, the mapping

$$K_{\mu_2,x^*} \mapsto K_{\mu_1,x^*\circ T}$$

has a well-defined linear extension to a contraction from $\mathscr{E}_{\mu_2} \to \mathscr{E}_{\mu_1}$ (this extension being also denoted P_T). Under suitable assumptions on the measures one may show that the functions K_{μ,x^*} are in fact linearly independent. This fact is of some interest by itself but is not needed here; therefore we have included it in an appendix at the end of this paper. Using the injectivity of the Fourier transform, a standard argument shows that $\mathscr{E}_\mu$ is dense in $L^p(E, \mu; \mathbb{C})$ for all $1 \leq p < \infty$ (see e.g. [2, Lemma 5.3.1]), and consequently P_T is the unique such extension.

3 Second quantisation: The Gaussian case

In this section we connect, in the Gaussian setting, the notions of skew operators with second quantisation. The presentation is slightly different from the usual one, in that we introduce a form of the chaos expansion that utilises iterated Malliavin derivatives that was introduced by Stroock [32]. This approach will bring out the analogies between the Gaussian and the Poisson case (which we present in the next section) very elegantly.

We begin by recalling some standard results from the theory of Gaussian measures. Proofs and more details can be found in the monographs [5, 26, 35].

Let μ be a Gaussian measure on the real Banach space E, and let H denote its reproducing kernel Hilbert space, which is defined as follows. The covariance operator Q of μ is given by

$$Qx^* = \int_E \langle x, x^* \rangle x \, \mu(dx), \quad x^* \in E^*.$$

This integral is known to be absolutely convergent in E and defines a bounded operator $Q \in \mathscr{L}(E^*, E)$ which is positive in the sense that $\langle Qx^*, x^* \rangle \geq 0$ for all $x^* \in E^*$ and symmetric in the sense that $\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle$ for all $x^*, y^* \in E^*$. The mapping $(Qx^*, Qy^*) \mapsto \langle Qx^*, y^* \rangle$ defines an inner product on the range of Q. The real Hilbert space H is defined to be the completion of the range of Q with respect to this inner product. The identity mapping $Qx^* \mapsto Qx^*$ extends to a bounded injective operator $j : H \to E$, and we have the factorisation $Q = j \circ j^*$. Here we have identified H and its dual via the Riesz representation theorem.

Each element $h \in H$ of the form $h = j^* x^*$ defines a real-valued function $\phi_h \in L^2(E, \mu)$ by $\phi_h(x) := \langle x, x^* \rangle$, and we have

$$\|\phi_h\|_{L^2(E,\mu)}^2 = \int_E \langle x, x^* \rangle^2 \, d\mu(x) = \|j^* x^*\|_H^2 = \|h\|_H^2.$$

Since j^* has dense range in H, the mapping $h \mapsto \phi_h$ uniquely extends to an isometry from H into $L^2(E, \mu)$.

Suppose now that μ_1 and μ_2 are Gaussian Radon measures on Banach spaces E_1 and E_2, with reproducing kernel Hilbert spaces H_1 and H_2 respectively. In the next two Propositions 3.1 and 3.2 we shall investigate the relationship between linear skew maps from E_1 to E_2 with respect to the pair (μ_1, μ_2) and linear contractions from H_1 to H_2.

We begin by proving that if T is a skew operator with respect to the pair (μ_1, μ_2), then T restricts to a contraction between the reproducing kernel Hilbert spaces. This result and its proof extend a similar result for semigroup operators in [8, 23].

Proposition 3.1. *If T is a bounded linear operator from E_1 to E_2 which is a skew operator with respect to the pair (μ_1, μ_2) of Gaussian measures, then T restricts to a contraction from H_1 to H_2.*

Proof. By assumption we have $T\mu_1 * \rho = \mu_2$ for some Radon probability measure ρ. We claim that ρ is Gaussian. Indeed, using the fact that $T\mu_1$ has mean zero, we have

$$\int_E \langle x, x^* \rangle^2 \, \mu_2(dx) = \int_E \int_E \langle Tx + y, x^* \rangle^2 \, \mu_1(dx) \, \rho(dy)$$
$$= \int_E \langle x, x^* \rangle^2 \, T\mu_1(dx) + \int_E \langle y, x^* \rangle^2 \, \rho(dy).$$

Hence, denoting the covariances of μ_1 and μ_2 by Q_1 and Q_2 (respectively), we see that the operator $R := Q_2 - TQ_1T^*$ is positive and symmetric as an operator from E_2^* to E_2. Since $R \leq Q_2$, a well-known tightness result for Gaussian measures implies that R is the covariance of a Gaussian Radon measure $\tilde{\rho}$ on E_2. The identity $TQ_1T^* + R = Q_2$ implies $T\mu_1 * \tilde{\rho} = \mu_2$. Since μ_2 is a Gaussian measure, its characteristic function vanishes nowhere and hence, by the observation following Definition 2.1, $\rho = \tilde{\rho}$. This proves the claim.

Recall that $Q_1 = j_1 \circ j_1^*$, where $j_1 : H_1 \hookrightarrow E$ is the canonical inclusion mapping, and likewise we have $Q_2 = j_2 \circ j_2^*$ and $R = j_R \circ j_R^*$. For all $x^* \in E^*$ we have

$$\|j_1^* T^* x^*\|_{H_1}^2 = \langle TQ_1T^* x^*, x^* \rangle \tag{3.1}$$
$$= \langle Q_2 x^*, x^* \rangle - \langle R^* x^*, x^* \rangle \leq \langle Q_2^* x^*, x^* \rangle = \|j_2^* x^*\|_{H_2}^2.$$

Hence,

$$|\langle Q_1 T^* x^*, y^* \rangle| = |[j_1^* T^* x^*, j_1^* y^*]_{H_1}| \leq \|j_2^* x^*\|_{H_2} \|j_1^* y^*\|_{H_1}. \tag{3.2}$$

Define a linear functional ψ_{y^*} on the range of j_2^* by

$$\psi_{y^*}(j_2^* x^*) := \langle Q_1 T^* x^*, y^* \rangle.$$

If $j_2^* x^* = 0$, then $j_1^* T^* x^* = 0$ by (3.1), so ψ_{y^*} is well-defined. By (3.2), ψ_{y^*} extends to a bounded linear functional on H_2 of norm $\leq \|j_1^* y^*\|_{H_1}$. Identifying ψ_{y^*} with an element of H_2, for all $x^* \in E^*$ we have

$$\langle j_2 \psi_{y^*}, x^* \rangle = [j_2^* x^*, \psi_{y^*}]_{H_2} = \langle Q_1 T^* x^*, y^* \rangle = \langle T Q_1^* y^*, x^* \rangle.$$

Hence, $T Q_1 y^* = j_2 \psi_{y^*}$ and $\|T Q_1 y^*\|_{H_2} \leq \|j_1^* y^*\|_{H_1}$. Writing $Q_1 = j_1 j_1^*$ we see that the restriction of $T|_{H_1}$ to H_1 maps $j_1^* y^*$ to the element $j_2 \psi_{y^*}$ of H_1, and that $T|_H$ is contractive on the dense range of j_1^* in H_1. This gives the result. $\qquad\square$

In the converse direction we have the following result.

Proposition 3.2. *Suppose $T : H_1 \to H_2$ is a linear contraction. Then T admits a linear Borel measurable extension $\bar{T} : E_1 \to E_2$ with the following properties:*

1. *the image measure $\bar{T}(\mu_1)$ is a Gaussian Radon measure;*
2. *there exists a Gaussian Radon measure ρ on E_2 such that $\bar{T}(\mu_1) * \rho = \mu_2$.*

In particular, $\bar{T}$ is a linear skew map for the pair (μ_1, μ_2).

Proof. The following facts follows from the general theory of Gaussian measures (see, e.g., [5, 13]):

1. the mapping $T : H_1 \to H_2$ admits an extension to a linear Borel mapping $\bar{T} : E_1 \to E_2$;
2. the operator $Q = j_2 T T^* j_2^*$ is the covariance of a Gaussian measure μ on E_2;
3. μ coincides with the image measure $\bar{T}(\mu_1)$.

In terms of the covariance operators Q and Q_2 of μ and μ_2 we have

$$\langle Q x^*, x^* \rangle = \|T^* j_2^* x^*\|_{H_1}^2 \leq \|j_2^* x^*\|_{H_2}^2 = \langle Q_2 x^*, x^* \rangle.$$

Hence the positive symmetric operator $R := Q_2 - Q$ is the covariance of a Gaussian measure ρ for which we have $\bar{T}(\mu_1) * \rho = \mu * \rho = \mu_2$. $\qquad\square$

Our next objective is to relate the abstract second quantisation procedure of the previous section to the Wiener-Itô decompositions of $L^2(E_1, \mu_2)$ and $L^2(E_2, \mu_2)$.

Following the presentation in [26], for each $n \geq 1$ we define $\mathscr{H}_n$ to be the closed linear subspace of $L^2(E, \mu)$ spanned by the functions $H_n(\phi_h)$, where $h \in H$ has norm one and H_n is the n-th Hermite polynomial given by the generating function expansion

$$\exp\left(tx - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x).$$

The Wiener-Itô decomposition theorem asserts that we have an orthogonal direct sum decomposition

$$L^2(E, \mu) = \bigoplus_{n \geq 0} \mathscr{H}_n.$$

Let S_n be the permutation group on n elements. The range of the symmetrising projection $\Sigma_n : H^{\otimes n} \to H^{\otimes n}$ defined by

$$\Sigma_n(h_1 \otimes \ldots \otimes h_n) := \sum_{\sigma \in S_n} (h_{\sigma(1)} \otimes \ldots \otimes h_{\sigma(n)})$$

is denoted by $H^{\circledS n}$ and is called the *n-fold symmetric tensor product* of H. Let $(h_n)_{n \geq 1}$ be an orthonormal basis of H (the Hilbert space H, being a reproducing kernel Hilbert space of a Gaussian Radon measure, is separable (see e.g. [5])).

Consider the n-fold stochastic integral $I_n : H^{\circledS n} \to \mathscr{H}_n$, defined by

$$I_n\big(\Sigma_n(h_{j_1}^{\otimes k_1} \otimes \ldots \otimes h_{j_m}^{\otimes k_m})\big) := \prod_{l=1}^{m} H_{k_l}(\phi_{h_{j_l}})$$

with $j_1 < \cdots < j_m$ and $k_1 + \cdots + k_m = n$. Then $\frac{1}{\sqrt{n!}}I_n$ sets up an isometric isomorphism $H^{\circledS n} \simeq \mathscr{H}_n$. Stated differently, the mapping $I = \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}}I_n$ defines an isometric isomorphism

$$L^2(E, \mu) \simeq \Gamma(H),$$

where

$$\Gamma(H) := \bigoplus_{n=0}^{\infty} H^{\circledS n}$$

with norm $\|(h_n)_{n=0}^{\infty}\|_{\Gamma(H)}^2 = \sum_{n=0}^{\infty} \|h_n\|_{H^{\circledS n}}^2$ is the *symmetric Fock space* over H.

For a function $f : E \to \mathbb{R}$ of the form

$$f = g(\phi_{h_1}, \ldots, \phi_{h_n})$$

with $h_1, \ldots, h_n$ orthonormal in H and $g : \mathbb{R}^n \to \mathbb{C}$ of class C^1, we define the *Malliavin derivative in the direction of H* as the function $Df : E \to H$ given by

$$Df = \sum_{j=1}^{n} \partial_j g(\phi_{h_1}, \ldots, \phi_{h_n}) \otimes h_j.$$

As is well known (see e.g. [26]), for all $1 \leq p < \infty$ the linear operator D is closable and densely defined from $L^p(E, \mu)$ to $L^p(E, \mu; H)$. From now on we will denote its closure by D as well, and denote the domain of its closure by $W^{1,p}(E, \mu)$. The higher order derivatives $D^k f : E \to H^{\otimes k}$ are defined recursively by $D^k f := D(D^{k-1}f)$. These operators are closable as well and the domains of their closures will be denoted by $W^{k,p}(E, \mu)$. We define the spaces $W^{\infty,p}(E, \mu) := \bigcap_{k \in \mathbb{N}} W^{k,p}(E, \mu)$.

The next proposition is due to Stroock [32] in the context of an abstract Wiener space. We give a different proof for Gaussian measures on Banach spaces. We write $\mathbb{E}_\mu f = \int_E f \, d\mu$.

Proposition 3.3. *The space $W^{\infty,2}(E, \mu)$ is dense in $L^2(E, \mu)$ and for all $f \in W^{\infty,2}(E, \mu)$ we have*

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E}_\mu D^n f).$$

Proof. For each $h \in H$, the function $e_h : E \to \mathbb{R}$ is defined by $e_h := \exp(\phi_h - \frac{1}{2}\|h\|_H^2)$. It is well known that the linear span of $\{e_h, h \in E\}$ is dense in $L^2(E, \mu)$ (see e.g. [26] for a proof). Since

$$D^n e_h = e_h \otimes (\underbrace{h \otimes \ldots \otimes h}_{n \text{ times}})$$

for all $n \in \mathbb{N}$, the first assertion follows. We clearly have

$$\mathbb{E}_\mu D^n e_h = \mathbb{E}_\mu e_h \otimes (h \otimes \ldots \otimes h) = h \otimes \ldots \otimes h.$$

Applying the n-fold stochastic integral and using the generating function identity for the Hermite polynomials we obtain

$$\sum_{n=0}^\infty \frac{1}{n!} I_n(\mathbb{E}_\mu D^n e_h) = \sum_{n=0}^\infty \frac{1}{n!} I_n(\underbrace{h \otimes \ldots \otimes h}_{n \text{ times}})$$
$$= \sum_{n=0}^\infty \frac{1}{n!} H_n(\phi_h^n) = \exp\left(\phi_h - \frac{1}{2}\|h\|_H^2\right) = e_h,$$

and the required result follows by density. $\qquad\square$

Let us now return to the setting where μ_1 and μ_2 are Gaussian measures on E_1 and E_2, having reproducing kernel Hilbert spaces H_1 and H_2, respectively. In order to avoid unnecessary notational complexity we will use the notation D for Malliavin derivatives acting on both $L^2(E_1, \mu_1)$ and $L^2(E_2, \mu_2)$, and define

$$D_h f := [Df, h].$$

Lemma 3.4. *Let $T : H_1 \to H_2$ be a linear contraction. Then for all $f \in W^{n,2}(E_2, \mu_2)$ and $h_1, \ldots h_n \in H_1$,*

$$\mathbb{E}_{\mu_1} D_{h_1, \ldots, h_n}^n P_T f = \mathbb{E}_{\mu_2} D_{Th_1, \ldots, Th_n}^n f.$$

Proof. Let us check this first for $n = 1$. By an easy computation (see [22]), for $f \in W^{1,2}(E_2, \mu_2)$ we have $P_T f \in W^{1,2}(E_2, \mu_2)$ and

$$DP_T f = (P_T \otimes T^*)Df.$$

Consequently,

$$\mathbb{E}_{\mu_1} D_h P_T f = \mathbb{E}_{\mu_1}[(P_T \otimes T^*)Df, h]$$
$$= \int_E \int_E [Df(Tx + y), Th]\, d\rho(y)\, d\mu_1(x) \tag{3.3}$$
$$= \int_E [Df(z), Th]\, d\mu_2(z) = \mathbb{E}_{\mu_2} D_{Th} f.$$

Here, ρ is the measure constructed in Proposition 3.2. The higher order case is proved along similar lines. $\qquad\square$

In terms of the global derivative, the computation (3.3) shows that $\mathbb{E}_{\mu_1} DP_T f = T^* \mathbb{E}_{\mu_2} Df$ and more generally we have

$$\mathbb{E}_{\mu_1} D^n P_T f = (T^*)^{\circledS n} \mathbb{E}_{\mu_2} D^n f. \tag{3.4}$$

Applying I_n to both sides of (3.4) and using Proposition 3.3 together with the density of $W^{\infty,2}(E, \mu)$ in $L^2(E, \mu)$ we have proved:

Theorem 3.5. *The following diagram commutes:*

$$
\begin{array}{ccc}
L^2(E_2, \mu_2) & \xrightarrow{\ \ P_T\ \ } & L^2(E_1, \mu_1) \\[4pt]
{\scriptstyle\bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}} I_n}\Big\uparrow & & \Big\uparrow{\scriptstyle\bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}} I_n} \\[4pt]
\Gamma(H_2) & \xrightarrow{\ \bigoplus_{n=0}^\infty (T^*)^{\circledS n}\ } & \Gamma(H_1)
\end{array}
$$

The operator $\Gamma(T^*) := \bigoplus_{n=0}^{\infty}(T^*)^{\circledS n}$ is usually called the *symmetric second quantisation* of the operator T^*.

Remark 3.6. *The operator* $\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} I_n$ *is inverse to* $\bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_\mu D^n$ *by Proposition 3.3, so we may rewrite the commutative diagram in the following equivalent form:*

$$
\begin{array}{ccc}
L^2(E_2, \mu_2) & \xrightarrow{\quad P_T \quad} & L^2(E_1, \mu_1) \\
{\scriptstyle \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_{\mu_2} D^n} \Big\downarrow & & \Big\downarrow {\scriptstyle \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \mathbb{E}_{\mu_1} D^n} \\
\Gamma(H_2) & \xrightarrow{\bigoplus_{n=0}^{\infty}(T^*)^{\circledS n}} & \Gamma(H_1)
\end{array}
$$

This diagram should be compared with the one in the next section.

Let us finally return to the setting of the previous section and derive the identity (2.4) by the methods of the present section. Fix $x^* \in E_2^*$ and let $h := j_2^* x^*$. Then $K_{\mu_2, x^*} = \exp(i\phi_h - \|h\|_{H_2}^2)$ and therefore by Lemma 3.4 (which we apply to the real and imaginary parts of K_{μ_2, x^*}), for all $g \in H_1$ we have

$$
\mathbb{E}_{\mu_1} D_g P_T K_{\mu_2, x^*} = \mathbb{E}_{\mu_2} D_{Tg} K_{\mu_2 . x^*} = i[h, Tg] \mathbb{E}_{\mu_2} K_{\mu_2, x^*} = i[T^* h, g],
$$

so $\mathbb{E}_{\mu_1} D P_T K_{\mu_2, x^*} = i T^* h$. Likewise we have

$$
\mathbb{E}_{\mu_1} D^n P_T K_{\mu_2, x^*} = i^n \otimes (\underbrace{T^* h \otimes \cdots \otimes T^* h}_{n \text{ times}}).
$$

Applying the n-fold stochastic integral, using Proposition 3.3 (again considering real and imaginary parts separately), and using the (analytic extension of the) generating function identity for the Hermite polynomials, we obtain

$$
\begin{aligned}
P_T K_{\mu_2, x^*} &= \sum_{n=0}^{\infty} \frac{1}{n!} I_n (\mathbb{E}_\mu D^n P_T K_{\mu_2, x^*}) \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} I_n (\underbrace{T^* h \otimes \ldots \otimes T^* h}_{n \text{ times}}) \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} H_n(\phi_{T^* h}^n) \\
&= \exp\left(i\phi_{T^* h} - \frac{1}{2}\|T^* h\|_H^2\right) = K_{\mu_1, T^* x^*},
\end{aligned}
$$

where the last identity used that $T \circ j_2 = j_2 \circ T$ implies $T^* h = T^* j_2^* x^* = j_2^* T^* x^*$.

4 Second quantisation: the Poisson random measure case

We proceed with a similar result in the case where μ_1 and μ_2 are infinitely divisible measures of pure jump type. For this we need do delve a bit deeper into the structure of such measures and develop their connection with Poisson random measures.

Let $(Y, \mathscr{Y}, \nu)$ be a σ-finite measure space and let $\mathbb{N}(Y)$ denote the set of all $\mathbb{N}$-valued measures on Y. We endow this space with the σ-algebra $\sigma(\mathscr{Y})$ generated by $\mathscr{Y}$, that is, the smallest σ-algebra which renders the mappings $\xi \mapsto \xi(B)$ measurable for all $B \in \mathscr{Y}$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and Π be a Poisson random measure having intensity measure ν. We denote by $\mathbb{P}_\Pi$ the distribution of Π, that is, $\mathbb{P}_\Pi$ is the probability measure on $(\mathbb{N}(Y), \sigma(\mathscr{Y}))$ given by

$$
\mathbb{P}_\Pi(B) = \mathbb{P}(\Pi \in B)
$$

for measurable $B \subseteq \mathbb{N}(Y)$.

Following Last and Penrose [20], for a measurable function $f : \mathbb{N}(Y) \to \mathbb{R}$ and $y \in Y$ we define the measurable function $D_y f : \mathbb{N}(Y) \to \mathbb{R}$ by

$$D_y f(\eta) := f(\eta + \delta_y) - f(\eta).$$

The function $D^n_{y_1,\ldots,y_n} f : \mathbb{N}(Y) \to \mathbb{R}$ is defined recursively by

$$D^n_{y_1,\ldots,y_n} f = D_{y_n} D^{n-1}_{y_1,\ldots,y_{n-1}} f,$$

for $y_1, \ldots, y_n \in Y$. This function is symmetric, i.e. it is invariant under any permutation of the variables.

We have a canonical isometry $L^2_{\circledS}(Y^n) = (L^2(Y))^{\circledS n}$, where the former denotes the closed subspace of $L^2(Y^n)$ comprised of all symmetric functions. We set

$$\Gamma(L^2_{\circledS}(Y)) = \bigoplus_{n=0}^{\infty} L^2_{\circledS}(Y^n)$$

with norm $\|(f_n)_{n=0}^{\infty}\|^2_H = \sum_{n=0}^{\infty} \|f_n\|^2_{L^2_{\circledS}(Y^n)}$; for $n = 0$ it is understood that $(L^2(Y))^{\circledS 0} = L^2_{\circledS}(Y^0) := \mathbb{R}$. By $I^n : L^2_{\circledS}(Y^n) \to L^2(\Omega)$ we denote the n-fold stochastic integral associated with Π as defined in [20]. We note that part (3) of the Last-Penrose theorem is essentially a Stroock formula for Poisson measures (cf. Proposition 3.3) and that a version of this result for a class of real-valued Lévy processes may be found in [12].

Theorem 4.1 (Last-Penrose [20]). *1. For all $n \in \mathbb{N}$, $y_1, \ldots, y_n \in Y$, and $f \in L^2(\mathbb{P}_\Pi)$ we have $\tau^n f \in L^2_{\circledS}(Y^n)$, where*

$$\tau^n f(y_1, \ldots, y_n) := \mathbb{E} D^n_{y_1,\ldots,y_n} f(\Pi).$$

2. The mapping $\tau := \bigoplus_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \tau^n$ is a surjective isometry from $L^2(\mathbb{P}_\Pi)$ onto $\Gamma(L^2_{\circledS}(Y))$.

3. For all $f \in L^2(P_\Pi)$ we have

$$f(\Pi) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E} D^n f(\Pi)).$$

From this point on, we shall consider the special case $Y = E$, where E is a separable real Banach space. We use the shorthand notation

$$\bar{\Pi}(dx) = \mathbf{1}_{\{0 < \|x\| \le 1\}} \widehat{\Pi}(dx) + \mathbf{1}_{\{\|x\| > 1\}} \Pi(dx),$$

where $\widehat{\Pi}$ is the compensated Poisson random measure,

$$\widehat{\Pi}(B) = \Pi(B) - \nu(B),$$

and ν is now assumed to be a Lévy measure on E (see e.g. [17, 21] for the definition). We will need to use the Lévy-Itô decomposition for Banach space-valued Lévy processes, as established in [30], and the next lemma is key in that regard.

Lemma 4.2. *The function $x \mapsto x$ is Pettis integrable with respect to $\bar{\Pi}$.*

Proof. Let N be a Poisson random measure on $[0, \infty) \times E$ with intensity measure $dt \times \nu$. By a theorem of Riedle and Van Gaans [30], $x \mapsto x$ is Pettis integrable with respect to $\bar{N}$. It follows that $x \mapsto x$ is Pettis integrable with respect to $\bar{M}$, where M is the Poisson random measure on E given by $M(B) = N([0, 1] \times B)$. Since M and Π are identically distributed (both being Poisson random measures with intensity measure ν), this proves the lemma. $\square$

We will be interested in Borel probability measures μ on E which arise as the distribution of E-valued random variables X of the form

$$X = \xi + \int_E x \, \bar{\Pi}(dx) \tag{4.1}$$

where Π is a Poisson random measure on E whose intensity measure ν is a Lévy measure and $\xi \in E$ is a given vector. The interest of such random variables comes from the Lévy-Itô decomposition for E-valued Lévy processes, which asserts that if $(L(t))_{t \geq 0}$ is a Lévy process without Gaussian part, then $L(1)$ is precisely of this form (see [30]). Note that μ is a Radon measure (since every Borel measure on a separable Banach space is Radon) and infinitely divisible. In particular, its characteristic function vanishes nowhere.

It will be convenient to define, for $f \in L^2(E, \mu)$,

$$\tilde{D}_y f(x) := f(x + y) - f(x) \tag{4.2}$$

The higher order derivatives are defined recursively by $\tilde{D}^n_{y_1,\ldots,y_n} = \tilde{D}_{y_n} \tilde{D}^{n-1}_{y_1,\ldots,y_{n-1}}$.

Suppose now that μ_1 and μ_2 are two measures of the above form, associated with random variables $X_1 : \Omega \to E_1$ and $X_2 : \Omega \to E_2$ which are given in terms of the vectors $\xi_1 \in E_1$ and $\xi_2 \in E_2$ and Poisson random measures Π_1 and Π_2 as in (4.1). Consider a linear skew map $T : E_1 \to E_2$ with respect to the pair (μ_1, μ_2), so that

$$T\mu_1 * \rho = \mu_2$$

for some unique Borel probability measure ρ.

We have the following analogue of Lemma 3.4:

Lemma 4.3. *For all $f \in L^2(E_2, \mu_2)$ and $y_1, \ldots, y_n \in E_1$,*

$$\mathbb{E}_{\mu_1} \tilde{D}^n_{y_1,\ldots,y_n} P_T f = \mathbb{E}_{\mu_2} \tilde{D}^n_{Ty_1,\ldots,Ty_n} f. \tag{4.3}$$

Proof. Suppose the random variable $R : \tilde{\Omega} \to E_2$, defined on an independent probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$, has distribution ρ. Then using the fact that $TX_1 + R$ and X_2 are identically distributed,

$$\mathbb{E}_{\mu_1} P_T f = \mathbb{E} \tilde{\mathbb{E}} f(TX_1 + R) = \mathbb{E} f(X_2) = \mathbb{E}_{\mu_2} f$$

and

$$\begin{aligned}
\mathbb{E}_{\mu_1} \tilde{D}_y P_T f &= \mathbb{E}_{\mu_1} P_T f(\cdot + y) - \mathbb{E}_{\mu_1} P_T f(\cdot) \\
&= \mathbb{E} \tilde{\mathbb{E}} f(Ty + TX_1 + R) - f(TX_1 + R) \\
&= \mathbb{E} f(Ty + X_2) - f(X_2) \\
&= \mathbb{E}_{\mu_2} f(\cdot + Ty) - \mathbb{E}_{\mu_2} f(\cdot) \\
&= \mathbb{E}_{\mu_2} \tilde{D}_{Ty} f.
\end{aligned}$$

For the higher derivatives we use a straightforward inductive argument. $\square$

Below we will think of the left and right hand side of (4.3) as symmetric functions on E^n. As such, the identity will be written as

$$\mathbb{E}_{\mu_1} \tilde{D}^n P_T f = \mathbb{E}_{\mu_1} \tilde{D}^n f \circ T^{\circledS n},$$

where

$$(g \circ T^{\circledS n})(y_1, \ldots, y_n) := g(Ty_1, \ldots, Ty_n).$$

Define, for $k = 1, 2$, the operators $j_k : L^2(E_k, \mu_k) \to L^2(\mathbb{P}_{\Pi_k})$ by

$$j_k f(\eta) = f\Big(\xi + \int_E x\, \bar{\eta}(dx)\Big), \quad \eta \in \mathbb{N}(E).$$

The rigorous interpretation of this identity is provided by noting that

$$\|j_k f\|^2_{L^2(\mathbb{P}_\Pi)} = \mathbb{E}\Big| f\Big(\xi + \int_E x\, \bar{\Pi}(dx)\Big)\Big|^2 = \|f\|^2_{L^2(E, \mu_k)},$$

which means that $j_k f(\eta)$ is well-defined for $\mathbb{P}_\Pi$-almost all η and that j_k establishes an isometry from $L^2(E_k, \mu_k)$ into $L^2(\mathbb{P}_{\Pi_k})$. Note that

$$j_k f(\Pi_k) = f(X_k) \tag{4.4}$$

and

$$j_k \circ \tilde{D} = D \circ j_k,$$

and therefore, for all $g \in L^2(E_k, \mu_k)$,

$$(\tau_k^n \circ j_k)g = \mathbb{E}D^n j_k g(\Pi_k) = \mathbb{E}j_k \tilde{D}^n g(\Pi_k) = \mathbb{E}\tilde{D}^n g(X_k) = \mathbb{E}_{\mu_k} \tilde{D}^n g.$$

Using this identity in combination with Lemma 4.3, for all $f \in L^2(E_2, \mu_2)$ we obtain

$$(\tau_1^n \circ j_1)P_T f = \mathbb{E}_{\mu_1} \tilde{D}^n P_T f = \mathbb{E}_{\mu_1} \tilde{D}^n f \circ T^{\circledS n} = (\tau_1^n \circ j_1)f \circ T^{\circledS n}.$$

When combined with the contractivity of P_T and the surjectivity of τ (see Theorem 4.1), this identity implies that the mapping $f \mapsto f \circ T^{\circledS n}$ is a is a linear contraction from $L^2_{\circledS}(E_2^n, \nu_2^n)$ to $L^2_{\circledS}(E_1^n, \nu_1^n)$.

In summary we have proved the following theorem.

Theorem 4.4. *Let $T : E_1 \to E_2$ be a linear Borel mapping. Under the above assumptions, the mapping $(T^*)^{\circledS n} : f \mapsto f \circ T^{\circledS n}$ is a linear contraction from $L^2_{\circledS}(E_2^n, \nu_2^n)$ to $L^2_{\circledS}(E_1^n, \nu_1^n)$, and the following diagram commutes:*

$$
\begin{array}{ccc}
L^2(E_2, \mu_2) & \xrightarrow{P_T} & L^2(E_1, \mu_1) \\
{\scriptstyle\bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}\mathbb{E}_{\mu_2}\tilde{D}^n}\Big\downarrow & & \Big\downarrow{\scriptstyle\bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}\mathbb{E}_{\mu_1}\tilde{D}^n} \\
\Gamma(L^2(E_2, \nu_2)) & \xrightarrow{\bigoplus_{n=0}^\infty (T^*)^{\circledS n}} & \Gamma(L^2(E_1, \nu_1))
\end{array}
$$

To make the connection with the commuting diagram in the Gaussian case, which features the n-fold stochastic integrals rather than n-fold derivatives, we note that by Theorem 4.1 the following diagram commutes as well for $k = 1, 2$:

$$
\begin{array}{ccc}
L^2(E, \mu_k) & \xrightarrow{f \mapsto f(X_k)} & L^2(\Omega) \\
{\scriptstyle\bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}\mathbb{E}_{\mu_k}\tilde{D}^n}\Big\downarrow & & \Big\uparrow{\scriptstyle\bigoplus_{n=0}^\infty \frac{1}{\sqrt{n!}}I_n} \\
\Gamma(L^2(E_k, \nu_k)) & \xrightarrow{=} & \Gamma(L^2(E_k, \nu_k))
\end{array}
$$

Theorem 4.4 is a generalisation of the result obtained by Peszat [28] in the case where $\mu_1 = \mu_2$ is an invariant measure associated with a Mehler semigroup on a Hilbert space $E_1 = E_2$.

As we did in the previous section, we wish to make the link with the results on skew operators. In principle we could repeat the Gaussian computation at the end of Section 3, but this requires the evaluation of a rather intractable Poisson stochastic integral. There is, however, a simpler argument.

We start with some preparations. If X and μ are as in (4.1), then $K_{\mu,x^*} = \exp(i\langle x, x^*\rangle - \zeta(x^*))$, where $\widehat{\mu}(x^*) = \exp(\zeta(x^*))$ is the characteristic function of μ (with ζ the *Lévy symbol* of μ; see [2, page 31]). Then, for all $y \in E$ and for all $x^* \in E^*$,

$$
\begin{aligned}
\mathbb{E}_\mu \tilde{D}_y K_{\mu,x^*} &= \mathbb{E}_\mu K_{\mu,x^*}(\cdot + y) - \mathbb{E}_\mu K_{\mu,x^*}(\cdot) \\
&= \exp(-\zeta(x^*))\mathbb{E}_\mu \exp(i\langle\cdot, x^*\rangle)[\exp(i\langle y, x^*\rangle) - 1] = \exp(i\langle y, x^*\rangle) - 1.
\end{aligned}
$$

Likewise, for $y_1, \ldots, y_n \in E$,

$$
\mathbb{E}_\mu \tilde{D}^n_{y_1,\ldots,y_n} K_{\mu,x^*} = \prod_{j=1}^n [\exp(i\langle y_j, x^*\rangle) - 1]. \tag{4.5}
$$

Now suppose that $T : E_1 \to E_2$ is a skew operator with respect to (μ_1, μ_2), where the measures μ_k are the distributions of random variables X_k as in (4.1) for $k = 1, 2$. Then, by (4.4), the Last-Penrose theorem, Lemma 4.3 and (4.5), for all $x^* \in E_2^*$ we have

$$
\begin{aligned}
P_T K_{\mu_2,x^*}(X_1) = j_1 P_T K_{\mu_2,x^*}(\Pi_1) &= \sum_{n=0}^\infty \frac{1}{n!} I_n(\mathbb{E}D^n j_1 P_T K_{\mu_2,x^*}(\Pi_1)) \\
&= \sum_{n=0}^\infty \frac{1}{n!} I_n(\mathbb{E}_{\mu_1} \tilde{D}^n P_T K_{\mu_2,x^*}) \\
&= \sum_{n=0}^\infty \frac{1}{n!} I_n(\mathbb{E}_{\mu_2} \tilde{D}^n_T. K_{\mu_2,x^*}) \\
&= \sum_{n=0}^\infty \frac{1}{n!} I_n\Big(\prod_{j=1}^n [\exp(i\langle T\cdot_j, x^*\rangle) - 1] \Big)
\end{aligned}
$$

and, by duality and then repeating the same computation backwards,

$$
\begin{aligned}
&= \sum_{n=0}^\infty \frac{1}{n!} I_n\Big(\prod_{j=1}^n [\exp(i\langle\cdot_j, T^*x^*\rangle) - 1] \Big) \\
&= \sum_{n=0}^\infty \frac{1}{n!} I_n(\mathbb{E}_{\mu_1} \tilde{D}^n K_{\mu_1,T^*x^*}) \\
&= \sum_{n=0}^\infty \frac{1}{n!} I_n(\mathbb{E}D^n j_1 K_{\mu_1,T^*x^*}(\Pi_1)) \\
&= j_1 K_{\mu_1,T^*x^*}(\Pi_1) = K_{\mu_1,T^*x^*}(X_1).
\end{aligned}
$$

It follows that $P_T K_{\mu_2,x^*} = K_{\mu_1,x^*\circ T}$, in agreement with (2.4).

Remark 4.5. *The results of Sections 3 and 4 suggest the problem of extending the theory to that case where μ_1 and μ_2 are arbitrary infinitely divisible measures. We conjecture that Theorems 3.5 and 4.4 extend to this more general framework.*

A Linear independence of the functions K_{μ,x^*}

The *support* of a Radon measure μ on E is the complement of the union of all open μ-null sets in E. We denote the support of μ by $\mathrm{supp}(\mu)$ and its closed linear span by E_μ. We say that μ has *linear support* if $\mathrm{supp}(\mu) = E_\mu$. The proof of the next result uses a variant of a standard technique of reduction to a system of linear equations that can be found in [16, pp. 20-21] or [27, pp. 126-7].

Proposition A.1. *Suppose that μ has linear support and let $F \subseteq E^*$ be such that its points are separated by E_μ. Then the family $\{x \mapsto \exp(i\langle x, x^* \rangle); x^* \in F\}$ is linearly independent in $L^2(E, \mu)$.*

Proof. Let $x_1^*, \ldots x_N^* \in F$ be distinct linear functionals and let $c_1, \ldots, c_N \in \mathbb{R}$ be such that $\sum_{n=1}^N c_n \exp(i\langle \cdot, x_n^* \rangle) = 0$ in $L^2(E, \mu)$. In particular, the set G of all $x \in E_\mu$ such that $\sum_{n=1}^N c_n \exp(i\langle x, x_n^* \rangle) = 0$ has full measure. By the assumption on μ, every open set V which intersects E_μ has positive μ-measure and therefore intersects G in a set of positive μ-measure. It follows that G is dense in E_μ. But then, by continuity, we find that $G = E_\mu$, that is, $\sum_{n=1}^N c_n \exp(i\langle x, x_n^* \rangle) = 0$ for all $x \in E_\mu$. Hence $\sum_{n=1}^N c_n \exp(it\langle x, x_n^* \rangle) = 0$ for all $t \in \mathbb{R}$ and $x \in E_\mu$. For $r = 0, 1, 2, \ldots, N$ differentiate r times with respect to t (where $1 \le r \le n-1$) and then put $t = 0$. This yields $\sum_{n=1}^N c_n \langle x, x_n^* \rangle^r = 0$ for all $x \in E_\mu$. We thus obtain a system of N linear equations in $c_1, \ldots c_N$ and it has a non-zero solution if and only if

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \langle x, x_1^* \rangle & \langle x, x_2^* \rangle & \cdots & \langle x, x_N^* \rangle \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \langle x, x_1^* \rangle^{N-1} & \langle x, x_2^* \rangle^{N-1} & \cdots & \langle x, x_N^* \rangle^{N-1} \end{vmatrix} = 0.$$

The left hand side of this equation is a Vandermonde determinant and so the equation simplifies to

$$\prod_{1 \le m < n \le N} (\langle x, x_m^* \rangle - \langle x, x_n^* \rangle) = 0.$$

Hence for each $x \in E$ there exist $1 \le m, m \le N$ such that $\langle x, x_m^* - x_n^* \rangle = 0$. The choice of m and n here depends on x. We now prove that in fact they are independent of the choice of vector $x \in E_\mu$. To this end for each $1 \le m, n \le N$ define $F_{mn} := \{x \in E_\mu : \langle x, x_m^* - x_n^* \rangle = 0\}$. Then F_{mn} is closed and $\bigcup_{m,n=1}^N F_{mn} = E_\mu$. By the Baire category theorem, at least one pair (m, n) must be such that F_{mn} has non-empty interior O_{mn} in E_μ. Let (m_0, n_0) be such a pair and fix $x_0 \in O_{m_0 n_0}$. Then by linearity $\langle x - x_0, x_{m_0}^* - x_{n_0}^* \rangle = 0$ for all $x \in O_{m_0 n_0}$. In other words $\langle y, x_{m_0}^* - x_{n_0}^* \rangle = 0$ for all $y \in O_{m_0 n_0} - \{x_0\}$. But $O_{m_0 n_0} - \{x_0\}$ contains an open neighbourhood of 0 in E_μ and hence by linearity again, $\langle x, x_{m_0}^* - x_{n_0}^* \rangle = 0$ for all $x \in E_\mu$, contradicting our assumptions. So we must have $c_1 = \cdots = c_N = 0$. $\qquad \square$

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