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Maximum principle for quasilinear stochastic PDEs with obstacle

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Abstract

We prove a maximum principle for local solutions of quasilinear stochastic PDEs with obstacle (in short OSPDE). The proofs are based on a version of Itô's formula and estimates for the positive part of a local solution which is non-positive on the lateral boundary. Our method is based on a version of Moser's iteration scheme developed first by Aronson and Serrin [2] in the context of non-linear parabolic PDEs and recently adapted in the context of quasilinear SPDEs in [5, 7].

Keywords: Stochastic PDEs; Obstacle problems; Itô's formula; L^p-estimate; Local solution; Comparison theorem; Maximum principle; Moser's iteration.
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1 Introduction

In this paper, we consider an obstacle problem for the following parabolic Stochastic PDE (SPDE in short)

$$\begin{cases} du_t(x) = \partial_i \left(a_{i,j}(x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x))\right)dt + f(t,x,u_t(x),\nabla u_t(x))dt \\ \qquad\qquad + \displaystyle\sum_{j=1}^{+\infty} h_j(t,x,u_t(x),\nabla u_t(x))dB_t^j + \nu(t,dx), \\ u_t \geq S_t\,, \\ u_0 = \xi\,. \end{cases} \tag{1.1}$$

Here, S is a given obstacle, a is a matrix defining a symmetric operator on an open bounded domain $\mathcal{O}$, f, g, h are random coefficients.
In a recent work [9] we have proved existence and uniqueness of the solution of equation (1.1) under standard Lipschitz hypotheses and L^2-type integrability conditions on the coefficients. Let us recall that the solution is a couple (u,ν), where u is a process with values in the first order Sobolev space and ν is a random regular measure forcing

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u to stay above S and satisfying a minimal Skohorod condition.
In order to give a rigorous meaning to the notion of solution, inspired by the works of M. Pierre in the deterministic case (see [18, 19]), we introduce the notion of parabolic capacity. The key point is that in [9], we construct a solution which admits a quasi-continuous version hence defined outside a polar set and that regular measures which in general are not absolutely continuous w.r.t. the Lebesgue measure, do not charge polar sets.
There is a huge literature on parabolic SPDEs without obstacle. The study of the L^p-norms w.r.t. the randomness of the space-time uniform norm on the trajectories of a stochastic PDE was started by N. V. Krylov in [13], for a more complete overview of existing works on this subject see [7, 8] and the references therein. Concerning the obstacle problem, there are two approaches, a probabilistic one (see [15, 12]) based on the Feynman-Kac's formula via the backward doubly stochastic differential equations and the analytical one (see [10, 17, 22]) based on the Green function.

To our knowledge, up to now there is no maximum principle result for quasilinear SPDEs with obstacle and even very few results in the deterministic case. The aim of this paper is to obtain, under suitable integrability conditions on the coefficients, L^p-estimates for the uniform norm (in time and space) of the solution, a maximum principle for local solutions of equation (1.1) and comparison theorems similar to those obtained in the without obstacle case in [5, 7]. This yields for example the following result:

Theorem 1.1. *Let $(M_t)_{t\geq 0}$ be an Itô process satisfying some integrability conditions, $p \geq 2$ and u be a local weak solution of the obstacle problem (1.1). Assume that $\partial\mathcal{O}$ is Lipschitz and $u \leq M$ on $\partial\mathcal{O}$, then for all $t \in [0,T]$:*

$$E\left\|(u-M)^+\right\|^p_{\infty,\infty;t} \leq k\,(p,t)\,\mathcal{C}(S,f,g,h,M)$$

where $\mathcal{C}(S,f,g,h,M)$ depends only on the barrier S, the initial condition ξ, the coefficients f,g,h, the boundary condition M and k is a function which only depends on p and t, $\|\cdot\|_{\infty,\infty;t}$ is the uniform norm on $[0,t]\times\mathcal{O}$.

Let us point out that, due to the presence of the barrier random field S, the study of the maximum principle for the weak solution of the obstacle problem is not an obvious extension of the previous works [5, 7, 6] on quasilinear SPDEs (without obstacle). First of all, in order to get such a result, we define the notion of local solutions to the obstacle problem (1.1) and so introduce what we call *local regular measures*. Then, the main difficulty consists in implementing a stochastic version of Moser's iteration scheme in our case. This Moser iteration is based on a version of Itô's formula and estimates for the positive part of a local solution for the obstacle problem (1.1), involving local time terms coming from the reflection on the barrier. Finally, another difficulty comes from the fact that we do not make any regularity assumption on the barrier S.

The paper is organized as follows: in section 2 we introduce notations and hypotheses. In section 3, we establish the L^p-estimate for uniform norm of the solution with null Dirichlet boundary condition. Section 4 is devoted to the main result: the maximum principle for local solutions whose proof is based on an Itô formula satisfied by the positive part of any local solution with lateral boundary condition, M. The last section is an Appendix in which we give the proofs of several lemmas.

2 Preliminaries

2.1 $L^{p,q}-$space

Let $\mathcal{O} \subset \mathbb{R}^d$ be an open bounded domain and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$, it is an Hilbert space equipped with the usual scalar product and norm as follows

$$(u,v) = \int_{\mathcal{O}} u(x)v(x)dx, \qquad \| u \| = (\int_{\mathcal{O}} u^2(x)dx)^{1/2}.$$

In general, we shall extend the notation

$$(u,v) = \int_{\mathcal{O}} u(x)v(x)dx,$$

where $u,\ v$ are measurable functions defined on $\mathcal{O}$ such that $uv \in L^1(\mathcal{O})$.
The first order Sobolev space of functions vanishing at the boundary will be denoted by $H_0^1(\mathcal{O})$, its natural scalar product and norm are

$$(u,v)_{H_0^1(\mathcal{O})} = (u,v) + \int_{\mathcal{O}} \sum_{i=1}^{d} \left(\partial_i u\left(x\right)\right)\left(\partial_i v\left(x\right)\right)dx,\ \|u\|_{H_0^1(\mathcal{O})} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

As usual we shall denote $H^{-1}(\mathcal{O})$ its dual space.
We shall denote by $H^1_{loc}(\mathcal{O})$ the space of functions which are locally square integrable in $\mathcal{O}$ and which admit first order derivatives that are also locally square integrable.
For each $t > 0$ and for all real numbers $p,\ q \geq 1$, we denote by $L^{p,q}([0,t] \times \mathcal{O})$ the space of (classes of) measurable functions $u : [0,t] \times \mathcal{O} \longrightarrow \mathbb{R}$ such that

$$\|u\|_{p,q;\,t} := \left(\int_0^t \left(\int_{\mathcal{O}} |u(s,x)|^p\, dx\right)^{q/p} ds\right)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the use of the essential sup norm.
Now we introduce some other spaces of functions and discuss a certain duality between them. Like in [5] and [7], for self-containeness, we recall the following definitions:
Let (p_1,q_1), $(p_2,q_2) \in [1,\infty]^2$ be fixed and set

$$I = I\left(p_1,q_1,p_2,q_2\right) := \Big\{(p,q) \in [1,\infty]^2 \,/\, \exists\, \rho \in [0,1]\, s.t.$$

$$\frac{1}{p} = \rho\frac{1}{p_1} + (1-\rho)\frac{1}{p_2}, \frac{1}{q} = \rho\frac{1}{q_1} + (1-\rho)\frac{1}{q_2}\Big\}.$$

This means that the set of inverse pairs $\left(\frac{1}{p},\frac{1}{q}\right)$, (p,q) belonging to I, is a segment contained in the square $[0,1]^2$, with the extremities $\left(\frac{1}{p_1},\frac{1}{q_1}\right)$ and $\left(\frac{1}{p_2},\frac{1}{q_2}\right)$.
We introduce:

$$L_{I;t} = \bigcap_{(p,q)\in I} L^{p,q}\left([0,t] \times \mathcal{O}\right).$$

We know that this space coincides with the intersection of the extreme spaces,

$$L_{I;t} = L^{p_1,q_1}\left([0,t] \times \mathcal{O}\right) \cap L^{p_2,q_2}\left([0,t] \times \mathcal{O}\right)$$

and that it is a Banach space with the following norm

$$\|u\|_{I;t} := \|u\|_{p_1,q_1;t} \vee \|u\|_{p_2,q_2;t}\,.$$

The other space of interest is the algebraic sum

$$L^{I;t} := \sum_{(p,q)\in I} L^{p,q}\left([0,t]\times\mathcal{O}\right),$$

which represents the vector space generated by the same family of spaces. This is a normed vector space with the norm

$$\|u\|^{I;t} := \inf\left\{\sum_{i=1}^{n}\|u_i\|_{\tilde{p}_i,\tilde{q}_i;\,t}\ /\ u=\sum_{i=1}^{n}u_i, u_i\in L^{\tilde{p}_i,\tilde{q}_i}\left([0,t]\times\mathcal{O}\right),\,(\tilde{p}_i,\tilde{q}_i)\in I,\ i=1,...n;\ n\in\mathbb{N}^*\right\}.$$

Clearly one has $L^{I;t}\subset L^{1,1}\left([0,t]\times\mathcal{O}\right)$ and $\|u\|_{1,1;t}\le c\,\|u\|^{I;t}$, for each $u\in L^{I;t}$, with a certain constant $c>0$.
We also remark that if $(p,q)\in I$, then the conjugate pair (p',q'), with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1$, belongs to another set, I', of the same type. This set may be described by

$$I'=I'\left(p_1,q_1,p_2,q_2\right):=\left\{(p',q')\,/\,\exists\,(p,q)\in I\ s.t.\ \frac{1}{p}+\frac{1}{p'}=\frac{1}{q}+\frac{1}{q'}=1\right\}$$

and it is not difficult to check that $I'\left(p_1,q_1,p_2,q_2\right)=I\left(p_1',q_1',p_2',q_2'\right)$, where p_1',q_1',p_2' and q_2' are defined by $\frac{1}{p_1}+\frac{1}{p_1'}=\frac{1}{q_1}+\frac{1}{q_1'}=\frac{1}{p_2}+\frac{1}{p_2'}=\frac{1}{q_2}+\frac{1}{q_2'}=1$.
Moreover, by Hölder's inequality, it follows that one has

$$\int_0^t\int_{\mathcal{O}}u\left(s,x\right)v\left(s,x\right)dxds\le\|u\|_{I;t}\,\|v\|^{I';t},\tag{2.1}$$

for any $u\in L_{I;t}$ and $v\in L^{I';t}$. This inequality shows that the scalar product of $L^2\left([0,t]\times\mathcal{O}\right)$ extends to a duality relation for the spaces $L_{I;t}$ and $L^{I';t}$.
Now let us recall that the Sobolev inequality states that

$$\|u\|_{2^*}\le c_S\,\|\nabla u\|_2\,,\tag{2.2}$$

for each $u\in H_0^1\left(\mathcal{O}\right)$, where $c_S>0$ is a constant that depends on the dimension and $2^*=\frac{2d}{d-2}$ if $d>2$, while 2^* may be any number in $]2,\infty[$ if $d=2$ and $2^*=\infty$ if $d=1$. Therefore one has

$$\|u\|_{2^*,2;t}\le c_S\,\|\nabla u\|_{2,2;t}\,,$$

for each $t\ge0$ and each $u\in L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathcal{O}\right)\right)$. If $u\in L^\infty_{loc}\left(\mathbb{R}_+;L^2\left(\mathcal{O}\right)\right)\bigcap L^2_{loc}\left(\mathbb{R}_+;H_0^1\left(\mathcal{O}\right)\right)$, one has

$$\|u\|_{2,\infty;t}\vee\|u\|_{2^*,2;t}\le c_1\left(\|u\|^2_{2,\infty;t}+\|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}},$$

with $c_1=c_S\vee1$.
Let $\theta\in[0,1[$ be some fixed parameter. For $d\ge3$ and we set:

$$\Gamma_\theta=\left\{(p,q)\in[1,\infty]^2\,,\ \frac{d}{2p}+\frac{1}{q}=\frac{d}{2}+\theta\right\},$$

$$\Gamma_\theta^*=\left\{(p,q)\in[1,\infty]^2\,/\ \frac{d}{2p}+\frac{1}{q}=1-\theta\right\}.$$

On the space $L_{\theta,t}:=L_{\Gamma_\theta;t}$, we define

$$\|u\|_{\theta;\,t}:=\|u\|_{\frac{d}{d-2(1-\theta)},1;\,t}\vee\|u\|_{1,\frac{1}{\theta};\,t}=\sup_{(p,q)\in\Gamma_\theta}\|u\|_{p,q;\,t}\ ,$$

$$L^*_{\theta;t}=\sum_{(p,q)\in\Gamma_\theta^*}L^{p,q}\left([0,t]\times\mathcal{O}\right)\ ,$$

$$\|u\|_{\theta;t}^{*} := \inf\left\{\sum_{i=1}^{n}\|u_i\|_{p_i,q_i;\,t} \ / \ u=\sum_{i=1}^{n}u_i, u_i\in L^{p_i,q_i}\left([0,t]\times\mathcal{O}\right),\right.$$
$$\left.(p_i,q_i)\in\Gamma_\theta^*,\ i=1,...n;\ n\in\mathbf{N}^*\right\}.$$

We remark that

$$\Gamma_\theta^* = I\left(\infty,\frac{1}{1-\theta},\frac{d}{2\left(1-\theta\right)},\infty\right)$$

and that the norm $\|u\|_{\theta;t}^{*}$ coincides with $\|u\|^{\Gamma_\theta^*;t} = \|u\|^{I\left(\infty,\frac{1}{1-\theta},\frac{d}{2(1-\theta)},\infty\right);t}$.
If $d=1,2$ we put

$$\Gamma_\theta = \left\{(p,q)\in[1,\infty]^2 \ / \ \frac{2^*}{2^*-2}\frac{1}{p}+\frac{1}{q}=\frac{2^*}{2^*-2}+\theta\right\},$$
$$\Gamma_\theta^* = \left\{(p,q)\in[1,\infty]^2 \ / \ \frac{2^*}{2^*-2}\frac{1}{p}+\frac{1}{q}=1-\theta\right\}$$

with the convention $\frac{2^*}{2^*-2}=1$ if $d=1$ and similar definitions hold.
Moreover we have the following duality relation:

$$\int_0^t\int_{\mathcal{O}}u\left(s,x\right)v\left(s,x\right)dxds \le \|u\|_{\theta;t}\|v\|_{\theta;t}^{*}\,, \tag{2.3}$$

for any $u\in L_{\theta;t}$ and $v\in L_{\theta;t}^{*}$ and the following inequality:

$$\|u\|_{\theta;t}\le c_1\left(\|u\|_{2,\infty;t}^2+\|\nabla u\|_{2,2;t}^2\right)^{1/2}. \tag{2.4}$$

2.2 Hypotheses

We consider a sequence $((B^i(t))_{t\ge0})_{i\in\mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},P)$ satisfying the usual conditions.
Let A be a symmetric second order differential operator defined on the open bounded subset $\mathcal{O}\subset\mathbb{R}^d$, with domain $\mathcal{D}(A)$, given by

$$A := -L = -\sum_{i,j=1}^{d}\partial_i(a_{i,j}\partial_j).$$

We assume that $a=(a_{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2\le\sum_{i,j=1}^{d}a_{i,j}(x)\xi^i\xi^j\le\Lambda|\xi|^2,\ \forall x\in\mathcal{O},\ \xi\in\mathbb{R}^d,$$

where λ and Λ are positive constants. The energy associated with the matrix a will be denoted by

$$\mathcal{E}\left(w,v\right)=\sum_{i,j=1}^{d}\int_{\mathcal{O}}a_{i,j}(x)\partial_i w(x)\partial_j v(x)\,dx. \tag{2.5}$$

It's defined for functions $w,\,v\in H_0^1(\mathcal{O})$, or for $w\in H_{loc}^1(\mathcal{O})$ and $v\in H_0^1(\mathcal{O})$ with compact support.
We assume that we have predictable random functions

$$f:\mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R},$$
$$g=(g_1,...,g_d):\mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d,$$
$$h=(h_1,...,h_i,...):\mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^{\mathbb{N}^*}.$$

We define

$$f(\cdot,\cdot,\cdot,0,0):=f^0,\ g(\cdot,\cdot,\cdot,0,0):=g^0=(g_1^0,...,g_d^0) \text{ and } h(\cdot,\cdot,\cdot,0,0):=h^0=(h_1^0,...,h_i^0,...).$$

In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,\omega,x,y,z)|^2=\sum_{i=1}^{+\infty}|h_i(t,\omega,x,y,z)|^2.$$

Remark 2.1. *Let us note that this general setting of SPDE* (1.1) *we consider, encompasses the case of an SPDE driven by a space-time noise, colored in space and white in time as in [21] for example (see also Example 1 in [9]).*

Assumption (H): There exist non-negative constants $C,\ \alpha,\ \beta$ such that for almost all ω, the following inequalities hold for all $(t,x,y,z)\in\mathbb{R}_+\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d$:

1. $|f(t,\omega,x,y,z)-f(t,\omega,x,y',z')|\le C(|y-y'|+|z-z'|)$,
2. $|g(t,\omega,x,y,z)-g(t,\omega,x,y',z')|\le C|y-y'|+\alpha|z-z'|$,
3. $|h(t,\omega,x,y,z)-h(t,\omega,x,y',z')|\le C|y-y'|+\beta|z-z'|$,
4. the contraction property: $2\alpha+\beta^2<2\lambda$.

Moreover we introduce some integrability conditions on the coefficients $f^0,\ g^0,\ h^0$ and the initial data ξ. Along this article, we fix a terminal time $T>0$.

Assumption (HI2)

$$E\left(\|\xi\|_2^2+\left\|f^0\right\|_{2,2;T}^2+\left\||g^0|\right\|_{2,2;T}^2+\left\||h^0|\right\|_{2,2;T}^2\right)<\infty.$$

Assumption (HIL)

$$E\int_K|\xi(x)|^2dx+E\int_0^T\int_K\left(|f_s^0(x)|^2+|g_s^0(x)|^2+|h_s^0(x)|^2\right)dxds<\infty,$$

for any compact set $K\subset\mathcal{O}$.

2.3 Weak solutions

We now introduce $\mathcal{H}_T$, the space of $H_0^1(\mathcal{O})$-valued predictable processes $(u_t)_{t\in[0,T]}$ such that

$$\left(E\sup_{0\le s\le T}\|u_s\|_2^2+E\int_0^T\mathcal{E}\left(u_s\right)ds\right)^{1/2}\ <\ \infty\ .$$

We define $\mathcal{H}_{loc}=\mathcal{H}_{loc}(\mathcal{O})$ to be the set of $H^1_{loc}(\mathcal{O})$-valued predictable processes defined on $[0,T]$ such that for any compact subset K in $\mathcal{O}$:

$$\left(E\sup_{0\le s\le T}\int_K u_s(x)^2\,dx+E\int_0^T\int_K|\nabla u_s(x)|^2\,dxds\right)^{1/2}\ <\ \infty.$$

The space of test functions is the algebraic tensor product $\mathcal{D}=\mathcal{C}_c^\infty(\mathbb{R}^+)\otimes\mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}^+)$ denotes the space of all real infinite differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Now we recall the definition of the regular measure which has been defined in [9].
$\mathcal{K}$ denotes $L^\infty([0,T];L^2(\mathcal{O}))\cap L^2([0,T];H_0^1(\mathcal{O}))$ equipped with the norm:

$$\begin{aligned}\| v \|_{\mathcal{K}}^2 &= \| v \|^2_{L^\infty([0,T];L^2(\mathcal{O}))} + \| v \|^2_{L^2([0,T];H_0^1(\mathcal{O}))} \\ &= \sup_{t\in[0,T[} \| v_t \|^2 + \int_0^T \left(\| v_t \|^2 + \mathcal{E}(v_t)\right) dt.\end{aligned}$$

$\mathcal{C}$ denotes the space of continuous functions with compact support in $[0,T[\times\mathcal{O}$ and finally:

$$\mathcal{W} = \{\varphi \in L^2([0,T];H_0^1(\mathcal{O}));\ \frac{\partial \varphi}{\partial t} \in L^2([0,T];H^{-1}(\mathcal{O}))\},$$

endowed with the norm$\| \varphi \|^2_{\mathcal{W}} = \| \varphi \|^2_{L^2([0,T];H_0^1(\mathcal{O}))} + \| \dfrac{\partial \varphi}{\partial t} \|^2_{L^2([0,T];H^{-1}(\mathcal{O}))}$.
It is known (see [14]) that $\mathcal{W}$ is continuously embedded in $C([0,T];L^2(\mathcal{O}))$, the set of $L^2(\mathcal{O})$-valued continuous functions on $[0,T]$. So without ambiguity, we will also consider $\mathcal{W}_T = \{\varphi\in\mathcal{W}; \varphi(T)=0\}$, $\mathcal{W}^+ = \{\varphi\in\mathcal{W}; \varphi\geq 0\}$, $\mathcal{W}_T^+ = \mathcal{W}_T\cap\mathcal{W}^+$.

Definition 2.2. *An element $v\in\mathcal{K}$ is said to be a* **parabolic potential** *if it satisfies:*

$$\forall \varphi\in\mathcal{W}_T^+,\ \int_0^T -(\frac{\partial \varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt \geq 0.$$

We denote by $\mathcal{P}$ the set of all parabolic potentials.

The next representation property is crucial:

Proposition 2.3. *(Proposition 1.1 in [19]) Let $v\in\mathcal{P}$, then there exists a unique positive Radon measure on $[0,T[\times\mathcal{O}$, denoted by ν^v, such that:*

$$\forall \varphi\in\mathcal{W}_T\cap\mathcal{C},\ \int_0^T (-\frac{\partial \varphi_t}{\partial t}, v_t)dt + \int_0^T \mathcal{E}(\varphi_t, v_t)dt = \int_0^T\int_{\mathcal{O}} \varphi(t,x)d\nu^v.$$

Moreover, v admits a right-continuous (resp. left-continuous) version $\hat{v}$ (resp. $\bar{v}$) : $[0,T]\mapsto L^2(\mathcal{O})$.
Such a Radon measure, ν^v is called **a regular measure** *and we write:*

$$\nu^v = \frac{\partial v}{\partial t} + Av.$$

Definition 2.4. *Let $K\subset[0,T[\times\mathcal{O}$ be compact, $v\in\mathcal{P}$ is said to be $\nu-$superior than 1 on K, if there exists a sequence $v_n\in\mathcal{P}$ with $v_n\geq 1$ $a.e.$ on a neighborhood of K converging to v in $L^2([0,T];H_0^1(\mathcal{O}))$.*

We denote:

$$\mathcal{S}_K = \{v\in\mathcal{P};\ v \text{ is } \nu - superior \text{ to } 1 \text{ on } K\}.$$

Proposition 2.5. *(Proposition 2.1 in [19]) Let $K\subset[0,T[\times\mathcal{O}$ compact, then $\mathcal{S}_K$ admits a smallest $v_K\in\mathcal{P}$ and the measure ν_K^v whose support is in K satisfies*

$$\int_0^T\int_{\mathcal{O}} d\nu_K^v = \inf_{v\in\mathcal{P}}\{\int_0^T\int_{\mathcal{O}} d\nu^v;\ v\in\mathcal{S}_K\}.$$

Definition 2.6. *(Parabolic Capacity)*

- *Let $K\subset[0,T[\times\mathcal{O}$ be compact, we define $cap(K) = \int_0^T\int_{\mathcal{O}} d\nu_K^v$;*
- *let $O\subset[0,T[\times\mathcal{O}$ be open, we define $cap(O) = \sup\{cap(K);\ K\subset O\ compact\}$;*

- *for any borelian $E \subset [0,T[\times\mathcal{O}$, we define $cap(E) = \inf\{cap(O);\ O \supset E\ open\}$.*

Definition 2.7. *A property is said to hold quasi-everywhere (in short q.e.) if it holds outside a set of null capacity.*

Definition 2.8. *(Quasi-continuous)*
A function $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is called quasi-continuous, if there exists a decreasing sequence of open subsets O_n of $[0,T[\times\mathcal{O}$ with:

1. *for all n, the restriction of u to the complement of O_n is continuous;*
2. *$\lim_{n\to+\infty} cap\ (O_n) = 0$.*

We say that u admits a quasi-continuous version, if there exists $\tilde{u}$ quasi-continuous such that $\tilde{u} = u\ a.e.$

The next proposition, whose proof may be found in [18] or [19] shall play an important role in the sequel:

Proposition 2.9. *Let $K \subset \mathcal{O}$ a compact set, then $\forall t \in [0,T[$,*

$$cap(\{t\} \times K) = \lambda_d(K),$$

where λ_d is the Lebesgue measure on $\mathcal{O}$.
As a consequence, if $u : [0,T[\times\mathcal{O} \to \mathbb{R}$ is a map defined quasi-everywhere then it defines uniquely a map from $[0,T[$ into $L^2(\mathcal{O})$. In other words, for any $t \in [0,T[$, u_t is defined without any ambiguity as an element in $L^2(\mathcal{O})$. Moreover, if $u \in \mathcal{P}$, it admits version $\bar{u}$ which is left continuous on $[0,T]$ with values in $L^2(\mathcal{O})$ so that $u_T = \bar{u}_{T^-}$ is also defined without ambiguity.

Remark 2.10. *The previous proposition applies if for example u is quasi-continuous.*

To establish a maximum principle for local solutions we need to define the notion of *local regular measures*:

Definition 2.11. *We say that a Radon measure ν on $[0,T[\times\mathcal{O}$ is a local regular measure if for any non-negative ϕ in $\mathcal{C}_c^\infty(\mathcal{O})$, $\phi\nu$ is a regular measure.*

Proposition 2.12. *Local regular measures do not charge polar sets (i.e. sets of capacity 0).*

Proof. Let A be a polar set and consider a sequence (ϕ_n) in $\mathcal{C}_c^\infty(\mathcal{O})$, $0 \le \phi_n \le 1$, converging to 1 everywhere on $\mathcal{O}$. By Fatou's lemma,

$$0 \le \int_{[0,T[\times\mathcal{O}} \mathbb{I}_A d\nu(x,t) \le \liminf_{n\to\infty} \int_{[0,T[\times\mathcal{O}} \mathbb{I}_A \phi_n d\nu(x,t) = 0.$$

□

We end this part by a convergence lemma which plays an important role in our approach (Lemma 3.8 in [19]):

Lemma 2.13. *If $v^n \in \mathcal{P}$ is a bounded sequence in $\mathcal{K}$ and converges weakly to v in $L^2([0,T];H_0^1(\mathcal{O}))$; if u is a quasi-continuous function and $|u|$ is bounded by a element in $\mathcal{P}$. Then*

$$\lim_{n\to+\infty} \int_0^T \int_{\mathcal{O}} u d\nu^{v^n} = \int_0^T \int_{\mathcal{O}} u d\nu^v.$$

We now give the assumptions on the obstacle that we shall need in the different cases that we shall consider.

Assumption (O): The obstacle $S:[0,T]\times\Omega\times\mathcal{O}\to\mathbb{R}$ is an adapted random field almost surely quasi-continuous, in the sense that for P-almost all $\omega\in\Omega$, the map $(t,x)\to S_t(\omega,x)$ is quasi-continuous. Moreover, $S_0\le\xi$ P-almost surely and S is controlled by the solution of an SPDE, i.e. $\forall t\in[0,T]$,

$$S_t\le S'_t,\quad dP\otimes dt\otimes dx-a.e. \tag{2.6}$$

where S' is the solution of the linear SPDE

$$\left\{\begin{array}{rcl} dS'_t &=& LS'_t dt+f'_t dt+\sum_{i=1}^d\partial_i g'_{i,t}dt+\sum_{j=1}^{+\infty}h'_{j,t}dB^j_t\\ S'(0)&=&S'_0,\end{array}\right. \tag{2.7}$$

with null boundary Dirichlet conditions.

Assumption (OL): The obstacle $S:[0,T]\times\Omega\times\mathcal{O}\to\mathbb{R}$ is an adapted random field, almost surely quasi-continuous, such that $S_0\le\xi$ P-almost surely and S is controlled by a **local** solution of an SPDE, i.e. $\forall t\in[0,T]$,

$$S_t\le S'_t,\quad dP\otimes dt\otimes dx-a.e.$$

where S' is a **local** solution of the linear SPDE

$$\left\{\begin{array}{rcl} dS'_t &=& LS'_t dt+f'_t dt+\sum_{i=1}^d\partial_i g'_{i,t}dt+\sum_{j=1}^{+\infty}h'_{j,t}dB^j_t\\ S'(0)&=&S'_0.\end{array}\right.$$

Remark 2.14. *For the definition of local solution of SPDEs, one can see, for example, [7].*

Assumption (HO2)

$$E\left(\|\xi\|_2^2+\|f'\|^2_{2,2;T}+\||g'|\|^2_{2,2;T}+\||h'|\|^2_{2,2;T}\right)<\infty.$$

Assumption (HOL)

$$E\int_K|S'_0|^2dx+E\int_0^T\int_K\left(|f'_t(x)|^2+|g'_t(x)|^2+|h'_t(x)|^2\right)dxdt<\infty$$

for any compact set $K\subset\mathcal{O}$.

Remark 2.15. *It is well-known that under* **(HO2)** *S' belongs to $\mathcal{H}_T$, is unique and satisfies the following estimate:*

$$E\sup_{t\in[0,T]}\|S'_t\|^2+E\int_0^T\mathcal{E}(S'_t)dt\le CE\left[\|S'_0\|^2+\int_0^T(\|f'_t\|^2+\||g'_t|\|^2+\||h'_t|\|^2)dt\right], \tag{2.8}$$

see for example Theorem 8 in [4]. Moreover, as a consequence of Theorem 3 in [9], we know that S' admits a quasi-continuous version.

Definition 2.16. *A pair (u,ν) is said to be a solution of problem (1.1) if*

1. *$u\in\mathcal{H}_T$, $u(t,x)\ge S(t,x)$, $dP\otimes dt\otimes dx-a.e.$ and $u_0(x)=\xi$, $dP\otimes dx-a.e.$;*

2. *ν is a random regular measure defined on $[0,T[\times\mathcal{O}$;*
3. *the following relation holds almost surely, for all $t\in[0,T]$ and all $\varphi\in\mathcal{D}$,*

$$\begin{aligned}(u_t,\varphi_t)=&(\xi,\varphi_0)+\int_0^t(u_s,\partial_s\varphi_s)ds-\int_0^t\mathcal{E}(u_s,\varphi_s)ds\\&-\sum_{i=1}^d\int_0^t(g_s^i(u_s,\nabla u_s),\partial_i\varphi_s)ds+\int_0^t(f_s(u_s,\nabla u_s),\varphi_s)ds\\&+\sum_{j=1}^{+\infty}\int_0^t(h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds);\end{aligned}\tag{2.9}$$

4. *u admits a quasi-continuous version, $\tilde{u}$, and we have*

$$\int_0^T\int_{\mathcal{O}}(\tilde{u}(s,x)-S(s,x))\nu(dx,ds)=0,\quad P-a.s.$$

We denote by $\mathcal{R}(\xi,f,g,h,S)$ the solution of the obstacle problem when it exists and is unique.

Definition 2.17. *A pair (u,ν) is said to be a local solution of problem (1.1) if*

1. *$u\in\mathcal{H}_{loc}$, $u(t,x)\geq S(t,x),\ dP\otimes dt\otimes dx-a.e.$ and $u_0(x)=\xi,\ dP\otimes dx-a.e.$;*
2. *ν is a local random regular measure defined on $[0,T[\times\mathcal{O}$;*
3. *the following relation holds almost surely, for all $t\in[0,T]$ and all $\varphi\in\mathcal{D}$,*

$$\begin{aligned}(u_t,\varphi_t)=&(\xi,\varphi_0)+\int_0^t(u_s,\partial_s\varphi_s)ds-\int_0^t\mathcal{E}(u_s,\varphi_s)ds\\&-\sum_{i=1}^d\int_0^t(g_s^i(u_s,\nabla u_s),\partial_i\varphi_s)ds+\int_0^t(f_s(u_s,\nabla u_s),\varphi_s)ds\\&+\sum_{j=1}^{+\infty}\int_0^t(h_s^j(u_s,\nabla u_s),\varphi_s)dB_s^j+\int_0^t\int_{\mathcal{O}}\varphi_s(x)\nu(dx,ds);\end{aligned}\tag{2.10}$$

4. *u admits a quasi-continuous version, $\tilde{u}$, and we have*

$$\int_0^T\int_{\mathcal{O}}(\tilde{u}(s,x)-S(s,x))\nu(dx,ds)=0,\quad P-a.s.$$

We denote by $\mathcal{R}_{loc}(\xi,f,g,h,S)$ the set of all the local solutions (u,ν).
Finally, in the sequel, we introduce some constants ϵ, $\delta>0$, we shall denote by C_ϵ, C_δ some constants depending only on ϵ, δ, typically those appearing in the kind of inequality

$$|ab|\leq\epsilon a^2+C_\epsilon b^2.\tag{2.11}$$

3 L^p-estimate for the uniform norm of solutions with null Dirichlet boundary condition

In this section, we want to study, for some $p\geq 2$, the L^p- estimate for the uniform norm of the solution of (1.1). To get such estimate, we need stronger integrability conditions on the coefficients and the initial condition. To this end, we consider the following assumptions: for $\theta\in[0,1[$ and $p\geq 2$:

Assumption (HI2p)

$$E\left(\|\xi\|_\infty^p + \left\|f^0\right\|_{2,2;T}^2 + \left\||g^0|\right\|_{2,2;T}^2 + \left\||h^0|\right\|_{2,2;T}^2\right) < \infty.$$

Assumption (HO∞p)

$$S_0' \in L^\infty(\Omega \times \mathcal{O}) \text{ and } E\left((\|f'\|_{\infty,\infty;T})^p + (\left\||g'|^2\right\|_{\infty,\infty;T})^{p/2} + (\left\||h'|^2\right\|_{\infty,\infty;T})^{p/2}\right) < \infty.$$

To get the estimates that we need, we apply Itô's formula to $u - S'$, in order to take advantage of the fact that $S - S'$ is non-positive and that as u is the solution of (1.1) and S' satisfies (2.7), $u - S'$ satisfies

$$\begin{cases} d(u_t - S_t') = \partial_i(a_{i,j}(x)\partial_j(u_t(x) - S_t'(x)))dt + (f(t,x,u_t(x),\nabla u_t(x)) - f'(t,x))dt \\ \quad + \partial_i(g_i(t,x,u_t(x),\nabla u_t(x)) - g_i'(t,x))dt + (h_j(t,x,u_t(x),\nabla u_t(x)) - h_j'(t,x))dB_t^j \\ \quad + \nu(x,dt), \\ (u - S')_0 = \xi - S_0', \\ u - S' \geq S - S'. \end{cases} \tag{3.1}$$

that is why we introduce the following functions:

$$\bar{f}(t,\omega,x,y,z) = f(t,\omega,x,y+S_t',z+\nabla S_t') - f'(t,\omega,x),$$

$$\bar{g}(t,\omega,x,y,z) = g(t,\omega,x,y+S_t',z+\nabla S_t') - g'(t,\omega,x),$$

$$\bar{h}(t,\omega,x,y,z) = h(t,\omega,x,y+S_t',z+\nabla S_t') - h'(t,\omega,x).$$

Let us remark that the Skohorod condition for $u - S'$ is satisfied since

$$\int_0^T \int_{\mathcal{O}} (u_s(x) - S_s'(x)) - (S_s(x) - S_s'(x))\nu(ds,dx) = \int_0^T \int_{\mathcal{O}} (u_s(x) - S_s(x))\nu(ds,dx) = 0.$$

It is obvious that $\bar{f}$, $\bar{g}$ and $\bar{h}$ satisfy the Lipschitz conditions with the same Lipschitz coefficients as f, g and h and $\|\xi - S_0'\|_\infty \in L^p(\Omega, P)$. Nevertheless, we need a supplementary hypothesis:

Assumption (HDθp)

$$E((\left\|\bar{f}^0\right\|_{\theta;T}^*)^p + (\left\||\bar{g}^0|^2\right\|_{\theta;T}^*)^{p/2} + (\left\||\bar{h}^0|^2\right\|_{\theta;T}^*)^{p/2}) < \infty.$$

This assumption is fulfilled in the following case:

Example 3.1. *If $\|\nabla S'\|_{\theta;T}^*$, $\left\|f^0\right\|_{\theta;T}^*$, $\left\|g^0\right\|_{\theta;T}^*$ and $\left\|h^0\right\|_{\theta;T}^*$ belong to $L^p(\Omega, P)$, and assumptions* **(H)** *and* **(HO∞p)** *hold, then:*

$\bar{f}$ satisfies the integrability condition:

$$\begin{aligned} \left\|\bar{f}^0\right\|_{\theta;T}^* &= \|f(S',\nabla S') - f'\|_{\theta;T}^* \leq \|f(S',\nabla S')\|_{\theta;T}^* + \|f'\|_{\theta;T}^* \\ &\leq \left\|f^0\right\|_{\theta;T}^* + C\|S'\|_{\theta;T}^* + C\|\nabla S'\|_{\theta;T}^* + \|f'\|_{\infty,\infty;T}. \end{aligned}$$

And the same for $\bar{g}$ and $\bar{h}$, which proves that **(HDθp)** *holds.*

We now give the main result of this section, which is a version of the maximum principle in the case of a solution vanishing on the boundary of $\mathcal{O}$:

Theorem 3.2. *Suppose that assumptions* **(H)**, **(O)**, **(HI2p)**, **(HO∞p)** *and* **(HDθp)** *hold, for some $\theta \in [0,1[$ and $p \geq 2$ and that the constants of Lipschitz conditions satisfy*

$$\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda.$$

Let $(u,\ \nu)$ be the solution of OSPDE (1.1) with null boundary condition, then for all $t \in [0,T]$,

$$E\,\|u\|_{\infty,\infty;t}^p \leq c(p)k(t)E\big(\,\|\xi\|_\infty^p + \|S_0'\|_\infty^p + \|f'\|_{\theta;t}^{*p} + \big\||g'|^2\big\|_{\theta;t}^{*p/2} + \big\||h'|^2\big\|_{\theta;t}^{*p/2}$$
$$+ \big\|\bar{f}^0\big\|_{\theta;t}^{*p} + \big\||\bar{g}^0|^2\big\|_{\theta;t}^{*p/2} + \big\||\bar{h}^0|^2\big\|_{\theta;t}^{*p/2}\,\big),$$

where $c(p)$ is a constant which depends on p and $k(t)$ is a constant which depends on the structure constants and $t \in [0,T]$.

Remark 3.3. *The relations $\|f'\|_{\theta;t}^{*p} \leq (\|f'\|_{\infty,\infty;t})^p$, $\big\||g'|^2\big\|_{\theta;t}^{*p/2} \leq (\big\||g'|^2\big\|_{\infty,\infty;t})^{p/2}$ and $\big\||h'|^2\big\|_{\theta;t}^{*p/2} \leq (\big\||h'|^2\big\|_{\infty,\infty;t})^{p/2}$ and assumption* **(HO∞p))** *yield*

$$E\left(\|f'\|_{\theta;t}^{*p} + \big\||g'|^2\big\|_{\theta;t}^{*p/2} + \big\||h'|^2\big\|_{\theta;t}^{*p/2}\right) < +\infty.$$

As the proof of this theorem is quite long, we split it into several steps.

3.1 The case where ξ, $\bar{f}^0$, $\bar{g}^0$ and $\bar{h}^0$ are uniformly bounded

In this subsection, we assume that the hypotheses **(H)**, **(O)**, **(HI2p)**, **(HO∞p)** hold and we add the following stronger ones:

$$\xi \in L^\infty(\Omega \times \mathcal{O}),$$

and

$$\bar{f}^0,\ \bar{g}^0,\ \bar{h}^0 \in L^\infty(\mathbb{R}_+ \times \Omega \times \mathcal{O}).$$

Then it is obviously that $\xi - S_0' \in L^\infty(\Omega \times \mathcal{O})$.
Under these hypotheses, we know that OSPDE (1.1) admits a unique weak solution $(u,\nu) = \mathcal{R}(\xi, f, g, h, S)$ (see [9]) and that $(u - S', \nu) = \mathcal{R}(\xi - S_0', \bar{f}, \bar{g}, \bar{h}, S - S')$. We start by proving the following L^l-estimate:

Lemma 3.4. *The solution u of problem (1.1) belongs to $\cap_{l\geq 2} L^l([0,T]\times\mathcal{O}\times\Omega)$. Moreover there exist constants $c,\ c' > 0$ which only depend on $C,\ \alpha,\ \beta$ and on the quantity*

$$K = \|\xi - S_0'\|_{L^\infty(\Omega\times\mathcal{O})} \vee \big\|\bar{f}^0\big\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})} \vee \big\|\bar{g}^0\big\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})} \vee \big\|\bar{h}^0\big\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}$$

such that, for all real $l \geq 2$,

$$E\int_{\mathcal{O}} |u_t(x) - S_t'(x)|^l dx \leq cK^2 l(l-1)e^{cl(l-1)t}\,, \tag{3.2}$$

$$E\int_0^t\int_{\mathcal{O}} |u_s(x) - S_s'(x)|^{l-2} |\nabla(u_s(x) - S_s'(x))|^2 dxds \leq c'K^2 e^{cl(l-1)t}\,, \tag{3.3}$$

and

$$E\int_0^t\int_{\mathcal{O}} |u_s(x) - S_s'(x)|^{l-1} \nu(dxds) < +\infty\,. \tag{3.4}$$

Proof. Notice first that if $(u - S', \nu) = \mathcal{R}(\xi - S'_0, \bar f, \bar g, \bar h, S - S')$, then

$$\bar f\left(u - S', \nabla(u - S')\right), \bar g_i\left(u - S', \nabla(u - S')\right), \bar h_i\left(u - S', \nabla(u - S')\right) \in L^2\left([0,T]; L^2\left(\Omega \times \mathcal{O}\right)\right)$$

and consequently we can apply Itô's formula to $(u - S', \nu)$ (see Theorem 5 in [9]).
We fix a real $l \geq 2$, $T > 0$ and introduce the sequence $(\varphi_n)_{n\in\mathbb{N}^*}$ of functions such that for all $n \in \mathbb{N}^*$:

$$\forall x \in \mathbb{R},\, \varphi_n(x) = \begin{cases} \mid x \mid^l & \text{if } \mid x \mid \leq n \\ n^{l-2}\left[\frac{l(l-1)}{2}(|x| - n)^2 + l\, n(|x| - n) + n^2\right] & \text{if } \mid x \mid > n \end{cases}$$

One can easily verify that for fixed n, φ_n is twice differentiable with bounded second derivative, $\varphi''_n(x) \geq 0$, and as $n \to \infty$ one has $\varphi_n(x) \longrightarrow |x|^l$, $\varphi'_n(x) \longrightarrow l\ \mathrm{sgn}(x)|x|^{l-1}$, $\varphi''_n(x) \longrightarrow l(l-1)|x|^{l-2}$. Moreover, the following relations hold, for all $x \in \mathbb{R}$ and $n \geq l$:

1. $\mid x\varphi'_n(x) \mid \leq l\varphi_n(x)$,
2. $\mid \varphi'_n(x) \mid \leq \mid x\varphi''_n(x) \mid$,
3. $\mid x^2\varphi''_n(x) \mid \leq l(l-1)\varphi_n(x)$,
4. $|\varphi'_n(x)| \leq l(\varphi_n(x) + 1)$,
5. $|\varphi''_n(x)| \leq l(l-1)(\varphi_n(x) + 1)$.

Applying Itô's formula to $\varphi_n(u - S')$, we have P-a.s. for all $t \in [0,T]$,

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi_n(u_t(x) - S'_t(x))\, dx \;+\; \int_0^t \mathcal{E}\left(\varphi'_n(u_s - S'_s),\, u_s - S'_s\right) ds = \int_{\mathcal{O}} \varphi_n(\xi(x) - S'_0(x))\, dx \\
&+ \int_0^t \int_{\mathcal{O}} \varphi'_n(u_s(x) - S'_s(x)) \bar f(s, x, u_s - S'_s, \nabla(u_s - S'_s))\, dxds \\
&- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi''_n(u_s(x) - S'_s(x)) \partial_i(u_s(x) - S'_s(x))\, \bar g_i(s, x, u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\
&+ \sum_{j=1}^\infty \int_0^t \int_{\mathcal{O}} \varphi'_n(u_s(x) - S'_s(x))\, \bar h_j(s, x, u_s - S'_s, \nabla(u_s - S'_s))\, dx dB^j_s \\
&+ \frac{1}{2} \sum_{j=1}^\infty \int_0^t \int_{\mathcal{O}} \varphi''_n(u_s(x) - S'_s(x)) \bar h_j^2(s, x, u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\
&+ \int_0^t \int_{\mathcal{O}} \varphi'_n(u_s(x) - S'_s(x)) \nu(dxds)\,.
\end{aligned} \tag{3.5}$$

Since the support of ν is $\{u = S\}$, the last term is equal to

$$\int_0^t \int_{\mathcal{O}} \varphi'_n(S_s(x) - S'_s(x)) \nu(dxds)$$

and it is non-positive, thanks to Lemma III.1 p.210 in [19] which ensures that P-almost surely, $S \leq S'$, $\nu(dx, dt)$-a.e. ,

$$\int_0^t \int_{\mathcal{O}} \varphi'_n(S_s(x) - S'_s(x)) \mathbb{I}_{\{|S-S'| \leq n\}} \nu(dx, ds) = l \int_0^t \int_{\mathcal{O}} \mathrm{sgn}(S - S')\, |S_s(x) - S'_s(x)|^{l-1}\, \nu(dxds) \leq 0$$

and

$$\begin{aligned}
\int_0^t \int_{\mathcal{O}} \varphi'_n(S_s(x) \quad &- \quad S'_s(x)) \mathbb{I}_{\{|S-S'| > n\}} \nu(dxds) \\
&= \quad \int_0^t \int_{\mathcal{O}} n^{l-2}[l(l-1)(|S - S'| - n)\mathrm{sgn}(S - S') + nl\, \mathrm{sgn}(S - S')] \nu(dxds) \leq 0.
\end{aligned}$$

By the uniform ellipticity of the operator A we get

$$\mathcal{E}\big(\varphi_n'(u_s - S_s'),\, u_s - S_s'\big) \,\geq\, \lambda \int_{\mathcal{O}} \varphi_n''(u_s - S_s')|\nabla(u_s - S_s')|^2\, dx.$$

Let $\epsilon > 0$ be fixed. Using the Lipschitz condition on $\bar{f}$ and the properties of the functions $(\varphi_n)_n$ we get

$$\begin{aligned}
&|\varphi_n'(u_s - S_s')|\,|\bar{f}(s,x,u_s - S_s', \nabla(u_s - S_s'))| \\
&\leq\, |\varphi_n'(u_s - S_s')|\,\big(|\bar{f}^0(s,x)| + C\,(|u_s - S_s'| + |\nabla(u_s - S_s')|)\,\big) \\
&\leq\, |\varphi_n'(u_s - S_s')||\bar{f}^0(s,x)| + |u_s - S_s'||\varphi_n''(u_s - S_s')|\,(C|u_s - S_s'| + C|\nabla(u_s - S_s')|)\,) \\
&\leq l(\varphi_n(u_s - S_s') + 1)\,|\bar{f}^0(s,x)| + C|u_s - S_s'|^2|\varphi_n''(u_s - S_s')| + C|u_s - S_s'||\nabla(u_s - S_s')||\varphi_n''(u_s - S_s')| \\
&\leq l(\varphi_n(u_s - S_s') + 1)\,|\bar{f}^0(s,x)| + (C + c_\epsilon)\,|u_s - S_s'|^2\varphi_n''(u_s - S_s') + \epsilon\varphi_n''(u_s - S_s')|\nabla(u_s - S_s')|^2.
\end{aligned}$$

Now using Cauchy-Schwarz's inequality and the Lipschitz condition on $\bar{g}$ we get

$$\begin{aligned}
&\sum_{i=1}^{d} \varphi_n''(u_s - S_s')\partial_i(u_s - S_s')\,\bar{g}_i(s,x,u_s - S_s', \nabla(u_s - S_s')) \\
&\leq\, \varphi_n''(u_s - S_s')\,|\nabla(u_s - S_s')|\,\big(|\bar{g}^0(s,x)| + C|u_s - S_s'| + \alpha|\nabla(u_s - S_s')|\,\big) \\
&\leq\, \epsilon\,\varphi_n''(u_s - S_s')|\nabla(u_s - S_s')|^2 + 2c_\epsilon\varphi_n''(u_s - S_s')\,\big(K^2 + C^2|u_s - S_s'|^2\,\big) + \alpha\,\varphi_n''(u_s - S_s')|\nabla(u_s - S_s')|^2 \\
&\leq l(l-1)c_\epsilon K^2 + 2c_\epsilon(K^2 + C^2)l(l-1)|\varphi_n(u_s - S_s')| + (\alpha + \epsilon)\,\varphi_n''(u_s - S_s')|\nabla(u_s - S_s')|^2.
\end{aligned}$$

In the same way as before

$$\begin{aligned}
&\sum_{j=1}^{\infty} \varphi_n''(u_s - S_s')\bar{h}_j^2(s, u_s - S_s', \nabla(u_s - S_s')) \\
&\leq\, \varphi_n''(u_s - S_s')\,\big(\,c_\epsilon'(|\bar{h}^0(s,x)| + C|u_s - S_s'|)^2 \,+\, (1+\epsilon)\beta^2\,|\nabla(u_s - S_s')|^2\,\big) \\
&\leq \varphi_n''(u_s - S_s')\,\big(2c_\epsilon'K^2 \,+\, 2c_\epsilon'C^2|u_s - S_s'|^2 +\, (1+\epsilon)\beta^2\,|\nabla(u_s - S_s')|^2\,\big) \\
&\leq\, 2c_\epsilon' l(l-1)K^2 + 2c_\epsilon'(K^2 + C^2)l(l-1)\varphi_n(u_s - S_s') + (1+\epsilon)\,\beta^2\,\varphi_n''(u_s - S_s')|\nabla(u_s - S_s')|^2.
\end{aligned}$$

Thus taking the expectation, we deduce

$$\begin{aligned}
&E\int_{\mathcal{O}} \varphi_n(u_t(x) - S_t'(x))\,dx + (\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha + 2\epsilon)\,)\,E\int_0^t\int_{\mathcal{O}} \varphi_n''(u_s - S_s')\,|\nabla(u_s - S_s')|^2\,dx\,ds \\
&\leq\, l(l-1)c_\epsilon''K^2 \,+\, c_\epsilon'' l(l-1)\big(K^2 + C^2 + C + c_\epsilon\big)E\int_0^t\int_{\mathcal{O}} \varphi_n(u_s(x) - S_s'(x))\,dx\,ds.
\end{aligned} \tag{3.6}$$

On account of the contraction condition, one can choose $\epsilon > 0$ small enough such that

$$\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha + 2\epsilon) > 0$$

and then

$$E\int_{\mathcal{O}} \varphi_n(u_t(x) - S_t'(x))\,dx \leq\, cK^2l(l-1) \,+\, cl(l-1)E\int_0^t\int_{\mathcal{O}} \varphi_n(u_s(x) - S_s'(x))\,dx\,ds\,.$$

We obtain by Gronwall's Lemma, that

$$E\int_{\mathcal{O}} \varphi_n(u_t(x) - S_t'(x))\,dx \leq\, c\,K^2l(l-1)\,\exp\big(c\,l(l-1)\,t\big) \tag{3.7}$$

and so it is now easy from (3.6) to get

$$E\int_0^t\int_{\mathcal{O}}\varphi_n''(u_s(x)-S_s'(x))\,|\nabla(u_s-S_s')|^2\,dx\,ds\;\le\;c'\,K^2 l\,(l-1)\,\exp\big(cl(l-1)\,t\big). \tag{3.8}$$

Finally, letting $n\to\infty$ by Fatou's lemma we deduce (3.2) and (3.3).
Then with (3.5), we know that

$$-\int_0^t\int_{\mathcal{O}}\varphi_n'(u_s-S_s')\nu(dxds)=-\int_0^t\int_{\mathcal{O}}\varphi_n'(S_s-S_s')\nu(dxds)\le C.$$

This yields (3.4) by Fatou's lemma. □

With the help of Lemma 3.4, we are able to prove the following Itô formula:

Proposition 3.5. *Assume the hypotheses of the previous lemma. Let (u,ν) be the solution of problem (1.1). Then for $l\geq 2$, we get the following Itô's formula, P-almost surely, for all $t\in[0,T]$:*

$$\begin{aligned}&\int_{\mathcal{O}}|u_t(x)-S_t'(x)|^l\,dx+\int_0^t\mathcal{E}\left(l\,(u_s-S_s')^{l-1}\mathbf{sgn}(u_s-S_s'),\,u_s-S_s'\right)ds=\int_{\mathcal{O}}|\xi(x)-S_0'(x)|^l\,dx\\&+l\int_0^t\int_{\mathcal{O}}\mathbf{sgn}(u_s-S_s')\,|u_s(x)-S_s'(x)|^{l-1}\,\bar f(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dxds\\&-l(l-1)\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}|u_s(x)-S_s'(x)|^{l-2}\,\partial_i(u_s(x)-S_s'(x))\,\bar g_i(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dx\,ds\\&+l\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\mathbf{sgn}(u_s-S_s')\;|u_s(x)-S_s'(x)|^{l-1}\,\bar h_j(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dxdB_s^j\\&+\frac{l(l-1)}{2}\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}|u_s(x)-S_s'(x)|^{l-2}\,\bar h_j^2(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dx\,ds\\&+l\int_0^t\int_{\mathcal{O}}\mathbf{sgn}(u_s-S_s')\,|u_s(x)-S_s'(x)|^{l-1}\,\nu(dx\,ds)\,.\end{aligned} \tag{3.9}$$

Proof. From Itô's formula (see Theorem 5 in [9]), with the same notations as in the previous lemma, we have P-almost surely, and for all $t\in[0,T]$ and all $n\in\mathbb{N}^*$,

$$\begin{aligned}&\int_{\mathcal{O}}\varphi_n(u_t(x)-S_t'(x))\,dx\;+\;\int_0^t\mathcal{E}\big(\varphi_n'(u_s-S_s'),\,u_s-S_s'\big)\,ds=\int_{\mathcal{O}}\varphi_n(\xi(x)-S_0'(x))\,dx\\&+\int_0^t\int_{\mathcal{O}}\varphi_n'(u_s(x)-S_s'(x))\bar f(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dxds\\&-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi_n''(u_s(x)-S_s'(x))\partial_i(u_s(x)-S_s'(x))\,\bar g_i(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dx\,ds\\&+\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\varphi_n'(u_s(x)-S_s'(x))\,\bar h_j(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dxdB_s^j\\&+\frac12\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\varphi_n''(u_s(x)-S_s'(x))\bar h_j^2(s,x,u_s-S_s',\nabla(u_s-S_s'))\,dx\,ds\\&+\int_0^t\int_{\mathcal{O}}\varphi_n'(u_s(x)-S_s'(x))\nu(dxds)\,.\end{aligned}$$

Therefore, passing to the limit as $n \to \infty$, all the terms converge thanks to Lemma 3.4 and the dominated convergence theorem. The last term converges to

$$l \int_0^t \int_{\mathcal{O}} \operatorname{sgn}(u_s - S'_s) \left| u_s(x) - S'_s(x) \right|^{l-1} \nu(dx\, ds) \, , \quad a.s.$$

which is equal to

$$\begin{aligned} & l \int_0^t \int_{\mathcal{O}} \operatorname{sgn}(u_s - S'_s) \left| u_s(x) - S'_s(x) \right|^{l-1} \nu(dxds) \\ = \quad & l \int_0^t \int_{\mathcal{O}} \operatorname{sgn}(S_s - S'_s) \left| S_s(x) - S'_s(x) \right|^{l-1} \nu(dxds) \\ = \quad & -l \int_0^t \int_{\mathcal{O}} \left| S_s(x) - S'_s(x) \right|^{l-1} \nu(dxds) \leq 0 \end{aligned}$$

since ν acts on $\{u = S\}$ and $S \leq S'$ ν-a.e. by Lemma III.1 in [19]. □

From now on, we assume the following stronger hypothesis:

$$\alpha + \frac{1}{2}\beta^2 + 72\beta^2 < \lambda. \tag{3.10}$$

At this stage, the idea is to adapt the Moser iteration technics to our setting. To this end, in order to control uniformly the L^l-norms and make l tend to $+\infty$, we introduce for each $l \geq 2$, the processes v and v' given by

$$\begin{aligned} v_t :&= \sup_{s \leq t} \left(\int_{\mathcal{O}} \left| u_s - S'_s \right|^l dx + \gamma l \,(l-1) \int_0^s \int_{\mathcal{O}} \left| u_r - S'_r \right|^{l-2} \left| \nabla (u_r - S'_r) \right|^2 \, dx\, dr \right) , \\ v'_t :&= \int_{\mathcal{O}} \left| \xi - S'_0 \right|^l \, dx + l^2 c_1 \left\| |u - S'|^l \right\|_{1,1;t} + l \left\| \bar{f}^0 \right\|^*_{\theta,t} \left\| |u - S'|^{l-1} \right\|_{\theta;t} \\ & \quad + l^2 \left(c_2 \left\| |\bar{g}^0|^2 \right\|^*_{\theta;t} + c_3 \left\| |\bar{h}^0|^2 \right\|^*_{\theta;t} \right) \left\| |u - S'|^{l-2} \right\|_{\theta;t} , \end{aligned}$$

where the constants are given by

$$\begin{aligned} \gamma &= \lambda - \alpha - \frac{\epsilon l}{l-1} - \frac{1+\epsilon}{2}\beta^2 \, , \\ c_1 &= \frac{C}{2} \left(1 + \frac{C}{4\epsilon} \right) + \frac{3 + 2\epsilon}{2\epsilon} C^2 + 3 \frac{1+\epsilon}{\epsilon^2} C^2 \, , \\ c_2 &= \frac{1}{2\epsilon} \quad \text{and} \quad c_3 = \frac{(3+\epsilon)\,(1+\epsilon)}{\epsilon} \, , \end{aligned} \tag{3.11}$$

where ϵ is chosen small enough in order to have $\gamma > 0$.
The main difficulty in the stochastic case is to control the martingale part. We start by estimating the bracket of the local martingale in (3.9)

$$M_t := l \sum_{j=1}^{\infty} \int_0^t \int_{\mathcal{O}} \operatorname{sgn}(u_s - S'_s) \, \left| u_s(x) - S'_s(x) \right|^{l-1} \bar{h}_j(s, x, u_s - S'_s, \nabla (u_s - S'_s)) \, dx dB^j_s$$

Lemma 3.6. *For arbitrary $\varepsilon > 0$, small enough, one has*

$$\begin{aligned} \langle M \rangle_t^{\frac{1}{2}} \leq \varepsilon v_t + \frac{l^2}{2\varepsilon} & \left(\frac{1+\varepsilon}{\varepsilon} \left\| |\bar{h}^0|^2 \right\|^*_{\theta;t} \left\| |u - S'|^{l-2} \right\|_{\theta;t} + \frac{1+\varepsilon}{\varepsilon} C^2 \left\| |u - S'|^l \right\|_{1,1;t} \right) \\ & + \sqrt{1+\varepsilon} \sqrt{\frac{l}{l-1}} \frac{\beta}{\sqrt{\gamma}} v_t \, . \end{aligned} \tag{3.12}$$

The proof is the same as Lemma 12 in [5] replacing u by $u - S'$ and also h by $\bar{h}$.
In what follows we will use the notion of domination, which is essential to handle the martingale part. We recall the definition from Revuz and Yor [20].

Definition 3.7. *A non-negative, adapted right continuous process X is dominated by an increasing process A, if*

$$E\left[X_\rho\right] \leq E\left[A_\rho\right]$$

for any bounded stopping time, ρ.

One important result related to this notion is the following domination inequality (see Proposition IV.4.7 in Revuz-Yor, p. 163), for any $k \in]0, 1[$,

$$E\big[(X_\infty^*)^k\big] \leq C_k\, E\big[(A_\infty)^k\big] \tag{3.13}$$

where C_k is a positive constant and $X_t^* := \sup_{s\leq t}|X_s|$.

We will also use the fact that if A, A' are increasing processes, then the domination of a process X by A is equivalent to the domination of $X + A'$ by $A + A'$.

Lemma 3.8. *The Process τv is dominated by the process v' where*

$$\tau = 1 - 6\epsilon - 6\sqrt{1+\epsilon}\sqrt{\frac{l}{l-1}}\frac{\beta}{\sqrt{\gamma}}.$$

In other words, we have

$$\begin{aligned} &\tau\, E \sup_{0\leq s\leq t}\left(\int_{\mathcal{O}} |u_s - S'_s|^l\, dx + \gamma l\,(l-1)\int_0^s\int_{\mathcal{O}} |u_r - S'_r|^{l-2}\,|\nabla(u_r - S'_r)|^2\, dxdr\right) \\ &\leq E\int_{\mathcal{O}} |\xi - S'_0|^l\, dx + l^2 c_1 E\,\left\| |u - S'|^l \right\|_{1,1;t} + lE\,\|\bar{f}^0\|^*_{\theta,t}\left\||u - S'|^{l-1}\right\|_{\theta;t} \\ &\quad + l^2 E\,\left(c_2\,\||\bar{g}^0|^2\|^*_{\theta;t} + c_3\,\||\bar{h}^0|^2\|^*_{\theta;t}\right)\left\||u - S'|^{l-2}\right\|_{\theta;t}, \end{aligned} \tag{3.14}$$

where $\gamma,\ c_1,\ c_2$ and c_3 are the constants given above.

Proof. Starting from the relation (3.9):

$$\begin{aligned} &\int_{\mathcal{O}} |u_t(x) - S'_t(x)|^l\, dx + \int_0^t \mathcal{E}\left(l\,(u_s - S'_s)^{l-1}\mathrm{sgn}(u_s - S'_s),\, u_s - S'_s\right) ds = \int_{\mathcal{O}} |\xi(x) - S'_0(x)|^l\, dx \\ &+ l\int_0^t\int_{\mathcal{O}} \mathrm{sgn}(u_s - S'_s)\,|u_s(x) - S'_s(x)|^{l-1}\,\bar{f}(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dxds \\ &- l(l-1)\sum_{i=1}^d\int_0^t\int_{\mathcal{O}} |u_s(x) - S'_s(x)|^{l-2}\,\partial_i(u_s(x) - S'_s(x))\,\bar{g}_i(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\ &+ l\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}} \mathrm{sgn}(u_s - S'_s)\ |u_s(x) - S'_s(x)|^{l-1}\,\bar{h}_j(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dxdB^j_s \\ &+ \frac{l(l-1)}{2}\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}} |u_s(x) - S'_s(x)|^{l-2}\,\bar{h}^2_j(s,x,u_s - S'_s, \nabla(u_s - S'_s))\, dx\, ds \\ &+ l\int_0^t\int_{\mathcal{O}} \mathrm{sgn}(u_s - S'_s)\,|u_s(x) - S'_s(x)|^{l-1}\,\nu(dx\, ds)\,, \quad a.s. \end{aligned}$$

The last term is negative: from the condition of minimality, we have the following relation,

$$\begin{aligned} &\int_0^t\int_{\mathcal{O}} \mathrm{sgn}(u_s - S'_s)\,|u_s(x) - S'_s(x)|^{l-1}\,\nu(dxds) \\ =\ &\int_0^t\int_{\mathcal{O}} \mathrm{sgn}(S_s - S'_s)\,|S_s(x) - S'_s(x)|^{l-1}\,\nu(dxds) \leq 0. \end{aligned}$$

Then we can do the same calculus as in the proof of Lemma 14 in [5], replacing u by $u - S'$ and f, g, h by $\bar{f}$, $\bar{g}$, $\bar{h}$ respectively. □

The proofs of the next 3 lemmas are similar to the proofs of Lemmas 15, 16 and 17 in [5], just replacing u by $u - S'$ and replacing f, g and h by $\bar{f}$, $\bar{g}$ and $\bar{h}$ respectively.

Lemma 3.9. *The process v satisfies the estimate*

$$v_t \geq \delta \left\| |u - S'|^l \right\|_{0;t}$$

with $\delta = 1 \wedge \left(2c_S^{-1}\gamma\right)$, where c_S is the constant in the Sobolev inequality (2.2).

Lemma 3.10. *The process*

$$w_t := \left[\left\| |u - S'|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \|\xi - S'_0\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right]$$

is dominated by the process

$$w'_t := 6k(t)\, l^2 \left[\left\| |u - S'|^l \right\|_{\theta;t} \vee \|\xi - S'_0\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right],$$

where $\sigma = \frac{d+2\theta}{d}$ and $k : \mathbb{R}_+ \to \mathbb{R}_+$ is a function independent of l, depending only on the structure constants.

Lemma 3.11. *There exists a function $k_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our problem and such that the following estimate holds*

$$Ev_t \leq k_1(l,t)\, E\left(\int_{\mathcal{O}} |\xi - S'_0|^l \, dx + \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} + \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right).$$

We are now able to proof the annunciated result.

Proof of Theorem 3.2: the bounded case

Set $l = p\sigma^n$, with some $n \in \mathbb{N}^*$. By Lemma 3.10 and the domination inequality (3.13) we deduce, for $n \geq 1$,

$$E\left(\left\| |u - S'|^{\sigma l} \right\|_{\theta;t}^{\frac{1}{\sigma}} \vee \|\xi - S'_0\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right)^{\frac{1}{\sigma^n}}$$
$$\leq C_{\sigma^{-n}} \left(6k(t)\, l^2\right)^{\frac{1}{\sigma^n}} E\left(\left\| |u - S'|^l \right\|_{\theta;t} \vee \|\xi - S'_0\|_\infty^l \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*l} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right)^{\frac{1}{\sigma^n}},$$

where $C_{\sigma^{-n}}$ is the constant in the domination inequality.
The rest of the proof is similar to the first step of the proof of Theorem 11 in [5], p. 458-459, and based on a stochastic version of Moser iteration ([2]). One just has to apply the same technics with the following sequence:

$$a_n := \left\| |u - S'|^{p\sigma^n} \right\|_{\theta;t}^{\frac{1}{\sigma^n}} \vee \|\xi - S'_0\|_\infty^p \vee \left\| \bar{f}^0 \right\|_{\theta;t}^{*p} \vee \left\| |\bar{g}^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \vee \left\| |\bar{h}^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}}.$$

□

3.2 Proof of Theorem 3.2 in the general case

We now assume that **(H)**, **(O)**, **(HI2p)**, **(HO∞p)** and **(HDθp)** hold. We are going to prove Theorem 3.2 in the general case by using an approximation argument. The same type of approximation have been used in [7], p. 460-461, nevertheless in the obstacle problem, we have to deal additionally with the approximation of the regular random measure and the convergence of such sequence is not obvious in general. For this, for all $n \in \mathbb{N}^*$, $1 \le i \le d, 1 \le j \le \infty$ and all (t, w, x, y, z) in $\mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d$, we set

$$\begin{aligned} \bar{f}_n(t,w,x,y,z) &= \bar{f}(t,w,x,y,z) - \bar{f}^0(t,w,x) + \bar{f}^0(t,w,x)\cdot \mathbf{1}_{\{|\bar{f}^0(t,w,x)|\le n\}} \\ \bar{g}_{i,n}(t,w,x,y,z) &= \bar{g}_i(t,w,x,y,z) - \bar{g}_i^0(t,w,x) + \bar{g}_i^0(t,w,x)\cdot \mathbf{1}_{\{|\bar{g}_i^0(t,w,x)|\le n\}} \\ \bar{h}_{j,n}(t,w,x,y,z) &= \bar{h}_j(t,w,x,y,z) - \bar{h}_j^0(t,w,x) + \bar{h}_j^0(t,w,x)\cdot \mathbf{1}_{\{|\bar{h}_j^0(t,w,x)|\le n\}} \\ \xi_n(w,x) &= \xi(w,x)\cdot \mathbf{1}_{\{|\xi(\omega,x)|\le n\}} \end{aligned} \quad (3.15)$$

One can check that for all n, $\bar{f}_n$, $\bar{g}_n$, $\bar{h}_n$ and $\xi^n - S_0'$ satisfy all the assumptions of the Step 1 of the proof, and that Lipschitz constants do not depend on n. And the obstacle $S - S'$ is controlled by 0, which obviously satisfies **(HO2)**. For each $n \in \mathbb{N}^*$, we put $(\bar{u}^n, \nu^n) = \mathcal{R}(\xi^n - S_0', \bar{f}^n, \bar{g}^n, \bar{h}^n, S - S')$ and we know that $\bar{u}^n$ satisfies the estimate of Step 1. We are now going to prove that $(\bar{u}^n,\, \nu^n)$ converges to $(\bar{u}, \nu) = \mathcal{R}(\xi - S_0', \bar{f}, \bar{g}, \bar{h}, S - S')$. Let us fix $n \le m$ in $\mathbb{N}^*$ and put $\bar{u}^{n,m} := \bar{u}^n - \bar{u}^m$ and $\nu^{n,m} := \nu^n - \nu^m$ We first note that $\bar{u}^{n,m}$ satisfies the equation

$$\begin{aligned} d\bar{u}_t^{n,m}(x) + A\bar{u}_t^{n,m}(x)\,dt = {} & \bar{f}_{n,m}\left(t,x,\bar{u}_t^{n,m}(x), \nabla \bar{u}_t^{n,m}(x)\right)dt \\ & - \sum_{i=1}^d \partial_i \bar{g}_{i,n,m}\left(t,x,\bar{u}_t^{n,m}(x), \nabla \bar{u}_t^{n,m}(x)\right)dt \\ & + \sum_{j=1}^\infty \bar{h}_{j,n,m}\left(t,x,\bar{u}_t^{n,m}(x), \nabla \bar{u}_t^{n,m}(x)\right)dB_t^j + \nu^{n,m}(x,\,dt) \end{aligned}$$

where

$$\begin{aligned} \bar{f}_{n,m}(t,w,x,y,z) = {} & \bar{f}\left(t,w,x,y+\bar{u}_t^m(x), z + \nabla \bar{u}_t^m(x)\right) - \bar{f}\left(t,w,x,\bar{u}_t^m(x), \nabla \bar{u}_t^m(x)\right) \\ & + \bar{f}_n^0(t,w,x) - \bar{f}_m^0(t,w,x) \end{aligned}$$

and $\bar{g}_{i,n,m}, \bar{h}_{j,n,m}$ have similar expressions. Clearly one has

$$\bar{f}_{n,m}(t,w,x,0,0) = \bar{f}_n^0(t,w,x) - \bar{f}_m^0(t,w,x) := \bar{f}_{n,m}^0(t,w,x)$$

and some similar relations for $\bar{g}_{i,n,m}(t,w,x,0,0)$ and $\bar{h}_{j,n,m}(t,w,x,0,0)$. On the other hand, one can easily verify that

$$\begin{aligned} E\|\xi_n - \xi\|_\infty^p \longrightarrow 0, \qquad & E\left\|\bar{f}_n^0 - \bar{f}^0\right\|_{\theta;T}^{*p} \longrightarrow 0, \\ E\left\|\bar{g}_n^0 - \bar{g}^0\right\|_{\theta;T}^{*p} \longrightarrow 0, \qquad & E\left\|\bar{h}_n^0 - \bar{h}^0\right\|_{\theta;T}^{*p} \longrightarrow 0. \end{aligned}$$

By Lemma 5.4 with $l = 2$ (see Appendix) we deduce that

$$E\|\bar{u}^n - \bar{u}^m\|_T^2 \longrightarrow 0, \quad as\ n,\, m \to \infty. \quad (3.16)$$

Therefore, $(\bar{u}^n)$ has a limit $\bar{u}$ in $\mathcal{H}_T$.

We now study the convergence of (ν^n). Denote by v^n the parabolic potential associated to ν^n, and $z^n = \bar{u}^n - v^n$, so z^n satisfies the following SPDE

$$\begin{aligned} dz_t^n(x) + Az_t^n(x)dt = {} & \bar{f}_n(t,x,\bar{u}_t^n(x), \nabla \bar{u}_t^n(x))dt - \sum_{i=1}^d \partial_i \bar{g}_{i,n}(t,x,\bar{u}_t^n(x), \nabla \bar{u}_t^n(x))dt \\ & + \sum_{j=1}^\infty \bar{h}_{j,n}(t,x,\bar{u}_t^n(x), \nabla \bar{u}_t^n(x))\,dB_t^j. \end{aligned}$$

We define $z^{1,n}$ to be the solution of the following SPDE with initial value $\xi^n - S'_0$ and zero boundary condition:

$$dz_t^{1,n}(x) + Az_t^{1,n}(x)dt = (\bar{f}(t,x,\bar{u}_t^n(x),\nabla\bar{u}_t^n(x)) - \bar{f}^0(t,x))dt - \sum_{i=1}^d \partial_i(\bar{g}_i(t,x,\bar{u}_t^n(x),\nabla\bar{u}_t^n(x)) - \bar{g}^0(t,x))dt + \sum_{j=1}^\infty (\bar{h}_j(t,x,\bar{u}_t^n(x),\nabla\bar{u}_t^n(x)) - \bar{h}^0(t,x))\, dB_t^j.$$

This is a linear SPDE in $z^{1,n}$, its solution uniquely exists and belongs to $\mathcal{H}_T$. Applying Itô's formula to $(z^{1,n})^2$ and doing a classical calculation, we get:

$$E\left\|z^{1,n} - z^{1,m}\right\|_T^2 \le CE\big(\,\|\xi^n - \xi^m\|_2^2 + \|\bar{u}^n - \bar{u}^m\|_T^2\,\big) \to 0, \quad as\ n,\ m \to \infty.$$

Then, we define $z^{2,n}$ to be the solution of the following SPDE with initial value 0 and zero boundary condition:

$$dz_t^{2,n}(x) + Az_t^{2,n}(x)dt = \bar{f}_n^0(t,x)dt - \sum_{i=1}^d \partial_i \bar{g}_{i,n}^0(t,x))dt + \sum_{j=1}^\infty \bar{h}_{j,n}^0(x)\, dB_t^j.$$

This is still a linear SPDE in $z^{2,n}$, its solution uniquely exists and from the proof of Theorem 11 in [5], we know that

$$E\left\|z^{2,n} - z^{2,m}\right\|_T^2 \le CE\left(\left\|\bar{f}_{n,m}^0\right\|_{\theta;T}^{*2} + \left\|\left|\bar{g}_{n,m}^0\right|^2\right\|_{\theta;T}^* + \left\|\left|\bar{h}_{n,m}^0\right|^2\right\|_{\theta;T}^*\right) \to 0, \quad as\ n,\ m \to \infty.$$

This yields since $z^n = z^{1,n} + z^{2,n}$:

$$E\left\|z^n - z^m\right\|_T^2 \longrightarrow 0, \quad as \quad n, m \to \infty.$$

Hence, using (3.16) and the fact that $\bar{u}^n = z^n + v^n$, we get:

$$E\left\|v^n - v^m\right\|_T^2 \longrightarrow 0, \quad as\ n, m \to \infty.$$

Therefore, (v^n) has a limit v in $\mathcal{H}_T$. So, by extracting a subsequence, we can assume that (v^n) converges to v in $\mathcal{K}$ almost-surely. Then, it's clear that $v \in \mathcal{P}$, and we denote by ν the random regular measure associated to the potential v. Moreover, we have P-a.s., $\forall \varphi \in \mathcal{W}_t^+$:

$$\begin{aligned}\int_0^t\int_{\mathcal{O}}\varphi(x,s)\nu(dxds) &= \lim_{n\to\infty}\int_0^t\int_{\mathcal{O}}\varphi(x,s)\nu^n(dxds)\\ &= \lim_{n\to\infty}\int_0^t -(v_s^n,\frac{\partial\varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s^n,\varphi_s)ds\\ &= \int_0^t -(v_s,\frac{\partial\varphi_s}{\partial s})ds + \int_0^t \mathcal{E}(v_s,\varphi_s)ds.\end{aligned}$$

As a consequence of Lemma 5.3 in the Appendix, we know that

$$E\left\|\bar{u}^n - \bar{u}^m\right\|_{\infty,\infty;T}^p \longrightarrow 0.$$

Therefore, we can apply Proposition 3.5 to $\bar{u}^n$ and pass to the limit and so we obtain that this proposition remains valid in this case. Then, one can end the proof by repeating the first part of Step 1 starting from Proposition 3.5.
We conclude thanks to the uniqueness of the solution of the obstacle problem ensuring that $\bar{u}$ is equal to $u - S'$.

□

4 Maximum Principle for local solutions

We now introduce the lateral condition on the boundary that we consider:

Definition 4.1. *If u belongs to $\mathcal{H}_{loc}$, we say that u is non-positive on the boundary of $\mathcal{O}$ if u^+ belongs to $\mathcal{H}_T$ and we denote it simply: $u \leq 0$ on $\partial\mathcal{O}$. More generally, if M is a random field defined on $[0,T] \times \mathcal{O}$, we note $u \leq M$ on $\partial\mathcal{O}$ if $u - M \leq 0$ on $\partial\mathcal{O}$.*

4.1 Itô's formula for the positive part of a local solution

The following proposition represents a key technical result which leads to a generalization of the estimates of the positive part of a local solution. Let $(u,\nu) \in \mathcal{R}_{loc}(\xi, f, g, h, S)$, denote by u^+ its positive part. For this we need the following notations:

$$f^{u,0} = 1_{\{u>0\}} f^0,\ g^{u,0} = 1_{\{u>0\}} g^0,\ h^{u,0} = 1_{\{u>0\}} h^0,$$
$$f^{u,0+} = 1_{\{u>0\}} \left(f^0 \vee 0 \right),\ \xi^+ = \xi \vee 0. \tag{4.1}$$

Proposition 4.2. *Assume that $\partial\mathcal{O}$ is Lipschitz and that u^+ belongs to $\mathcal{H}_T$, i.e. u is non-positive on the boundary of $\mathcal{O}$ and that the data satisfy the following integrability conditions*

$$E\left\|\xi^+\right\|_2^2 < \infty,\ E\left(\left\|f^{u,0}\right\|_{\theta;t}^*\right)^2 < \infty,\ E\left\|g^{u,0}\right\|_{2,2;t}^2 < \infty,\ E\left\|h^{u,0}\right\|_{2,2;t}^2 < \infty,$$

for each $t \geq 0$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$, which admits a bounded second order derivative and such that $\varphi(0) = \varphi'(0) = 0$. Then the following relation holds, a.s., for each $t \in [0,T]$,

$$\int_{\mathcal{O}} \varphi(u_t^+(x))dx + \int_0^t \mathcal{E}(\varphi'(u_s^+), u_s^+)ds = \int_{\mathcal{O}} \varphi(\xi^+(x))dx + \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x)) f_s(x) dxds$$
$$- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi''(u_s^+(x)) \partial_i u_s^+(x) g_s^i(x) dxds + \frac{1}{2} \int_0^t \int_{\mathcal{O}} \varphi''(u_s^+(x)) 1\!\!1_{\{u_s>0\}} |h_s(x)|^2 dxds$$
$$+ \sum_{i=1}^\infty \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x)) h_s^j(x) dx dB_s^j + \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x)) \nu(dxds). \tag{4.2}$$

Proof. We consider $\phi \in \mathcal{C}_c^\infty(\mathcal{O}),\ 0 \leq \phi \leq 1$, and put

$$\forall t \in [0,T], \quad w_t = \phi u_t.$$

A direct calculation yields the following relation:

$$dw_t = Lw_t dt + \bar{f}_t dt + \sum_{i=1}^d \partial_i \widetilde{g_{i,t}} dt + \sum_{j=1}^\infty \widetilde{h_{j,t}} dB_t^j + \phi\nu(x, dt)$$

where

$$\bar{f}_t = \phi f_t - \sum a_{i,j} (\partial_i \phi)(\partial_j u_t) - \sum (\partial_i \phi) g_{i,t}\,,$$

$$\widetilde{g_{i,t}} = \phi g_{i,t} - u_t \sum a_{i,j} \partial_j \phi\,, \quad \widetilde{h_{j,t}} = \phi h_{j,t}\,.$$

Now we prove that $\phi\nu$ is a regular measure:
We know that:

$$\forall \varphi \in \mathcal{W}_T^+, \quad \int (-\frac{\partial \varphi_s}{\partial s}, v_s) ds + \int \mathcal{E}(\varphi_s, v_s) ds = \int \int \varphi(s,x) d\nu. \tag{4.3}$$

We replace φ by $\phi\varphi$ in (4.3), where ϕ is the same as before, and we obtain the following relation:

$$\int(-\frac{\partial\phi\varphi_s}{\partial s}, v_s)ds + \int \mathcal{E}(\phi\varphi_s, v_s)ds = \int\int \phi\varphi(s,x)d\nu.$$

Note that ϕ does not depend on t and by a similar calculation as before, we get

$$\int(-\frac{\partial\varphi_s}{\partial s}, \phi v_s)ds + \int \mathcal{E}(\varphi_s, \phi v_s)ds + \int(K_s,\varphi_s)ds - \int(k_s, \nabla\varphi_s)ds = \int\int \varphi(s,x)d\phi\nu$$

where

$$K_t = \sum a_{i,j}(\partial_i\phi)(\partial_j v_t), \quad k_t = v_t\sum a_{i,j}\partial_j\phi.$$

We denote by $\bar{z}$ the solution of the following PDE with Dirichlet boundary condition and the initial value 0:

$$d\bar{z}_t + A\bar{z}_t dt = K_t dt + div k_t dt.$$

If we set $\bar{v} = \phi v + \bar{z}$, then $\bar{v}$ satisfies the following relation:

$$\int_0^t(-\frac{\partial\varphi_s}{\partial s}, \bar{v}_s)ds + \int_0^t \mathcal{E}(\varphi_s, \bar{v}_s)ds = \int_0^t\int_{\mathcal{O}} \varphi(x,s)d\phi\nu.$$

It is easy to verify that $\bar{v}\in\mathcal{P}$. Thus $\phi\nu$ is a regular measure associated to $\bar{v}$.
Hence, we deduce that $(\phi u, \phi\nu)$ satisfies an OSPDE with $\phi\xi$ as initial data and zero Dirichlet boundary conditions.
Now, we approximate the function $\psi:\ y\in\mathbb{R}\to\varphi(y^+)$ by a sequence (ψ_n) of regular functions. Let ζ be a $\mathcal{C}^\infty$ increasing function such that

$$\forall y\in]-\infty,1],\ \zeta(y)=0 \text{ and } \forall y\in[2,+\infty[,\ \zeta(y)=1.$$

We set for all n:

$$\forall y\in\mathbb{R},\ \ \psi_n(y)=\varphi(y)\zeta(ny).$$

It is easy to verify that (ψ_n) converges uniformly to the function ψ, (ψ_n') converges everywhere to the function $(y\to\varphi'(y^+))$ and (ψ_n'') converges everywhere to the function $(y\to \mathbb{1}_{\{y>0\}}\varphi''(y^+))$. Moreover we have the estimates:

$$\forall y\in\mathbb{R}^+,\ n\in\mathbb{N}^*,\ \ 0\le\psi_n(y)\le\psi(y),\ \ 0\le\psi_n'(y)\le Cy,\ \ |\psi_n''(y)|\le C, \tag{4.4}$$

where C is a constant. Thanks to Itô's formula for the solution of OSPDE (1.1) (see Theorem 5 in [9]), we have almost surely, for $t\in[0,T]$,

$$\begin{aligned}&\int_{\mathcal{O}}\psi_n(w_t(x))dx + \int_0^t\mathcal{E}(\psi_n'(w_s),w_s)ds = \int_{\mathcal{O}}\psi_n(\phi(x)\xi(x))dx + \int_0^t\int_{\mathcal{O}}\psi_n'(w_s(x))\bar{f}_s(x)dxds\\ &-\sum\int_0^t\int_{\mathcal{O}}\psi_n''(w_s(x))\partial_i w_s(x)\widetilde{g_{i,s}}(x)dxds + \sum\int_0^t\int_{\mathcal{O}}\psi_n'(w_s(x))\widetilde{h_{j,s}}(x)dxdB_s^j\\ &+\frac{1}{2}\int_0^t\int_{\mathcal{O}}\psi_n''(w_s(x))|\widetilde{h_{j,s}}(x)|^2dxds + \int_0^t\int_{\mathcal{O}}\psi_n'(w_s(x))d\phi\nu(x,s).\end{aligned}$$

Making n tends to $+\infty$ and using the fact that $\mathbb{1}_{\{w_s>0\}}\partial_i w_s = \partial_i w_s^+$, we get by the dominated convergence theorem:

$$\begin{aligned}&\int_{\mathcal{O}}\varphi(w_t^+(x))dx + \int_0^t\mathcal{E}(\varphi'(w_s^+),w_s^+)ds = \int_{\mathcal{O}}\varphi(\phi(x)\xi^+(x))dx + \int_0^t\int_{\mathcal{O}}\varphi'(w_s^+(x))\bar{f}_s(x)dxds\\ &-\sum\int_0^t\int_{\mathcal{O}}\varphi''(w_s^+(x))\partial_i w_s^+(x)\widetilde{g_{i,s}}(x)dxds + \sum\int_0^t\int_{\mathcal{O}}\varphi'(w_s^+(x))\widetilde{h_{j,s}}(x)dxdB_s^j\\ &+\frac{1}{2}\int_0^t\int_{\mathcal{O}}\varphi''(w_s^+(x))\mathbb{1}_{\{w_s>0\}}|\widetilde{h_{j,s}}(x)|^2dxds + \int_0^t\int_{\mathcal{O}}\phi\varphi'(w_s^+(x))d\nu(x,s),\quad a.s.\end{aligned}$$

Then we consider a sequence (ϕ_n) in $\mathcal{C}_c^\infty(\mathcal{O})$, $0 \le \phi_n \le 1$, converging to 1 everywhere on $\mathcal{O}$ and such that for any $y \in H_0^1(\mathcal{O})$ the sequence $(\phi_n y)$ tends to y in $H_0^1(\mathcal{O})$ and

$$\sup_n \|\phi_n y\|_{H_0^1(\mathcal{O})} \le C \|y\|_{H_0^1(\mathcal{O})},$$

where C is a constant which does not depend on y. Such a sequence (ϕ_n) exists because $\partial\mathcal{O}$ is assumed to be Lipschitz (see Lemma 19 in [8]).
One has to remark that if $i \in \{1, ...d\}$ and $y \in H_0^1(\mathcal{O})$, then $(y\partial_i\phi_n)$ tends to 0 in $L^2(\mathcal{O})$.
Now, we set $w_n = \phi_n u$ and

$$\widetilde{f_t^n} = \phi_n f_t - \sum a_{i,j}(\partial_i\phi_n)(\partial_j u_t) - \sum(\partial_i\phi_n)g_{i,t}$$

$$\widetilde{g_{i,t}^n} = \phi_n g_{i,t} - u_t \sum a_{i,j}\partial_j\phi_n, \quad \widetilde{h_{j,t}^n} = \phi_n h_{j,t}$$

Applying the above Itô formula to $\varphi(w_n^+)$, we get

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi(w_{n,t}^+(x))dx + \int_0^t \mathcal{E}(\varphi'(w_{n,s}^+), w_{n,s}^+)ds = \int_{\mathcal{O}} \varphi(\phi_n(x)\xi^+(x))dx + \int_0^t\int_{\mathcal{O}} \varphi'(w_{n,s}^+(x))\bar{f}_s(x)dxds \\
&- \sum \int_0^t\int_{\mathcal{O}} \varphi''(w_{n,s}^+(x))\partial_i w_{n,s}^+(x)\widetilde{g_{i,s}}(x)dxds + \sum\int_0^t\int_{\mathcal{O}} \varphi'(w_{n,s}^+(x))\widetilde{h_{j,s}}(x)dxdB_s^j \\
&+ \frac{1}{2}\int_0^t\int_{\mathcal{O}} \varphi''(w_{n,s}^+(x))\mathbb{1}_{\{w_{n,s}>0\}}|\widetilde{h_{j,s}}(x)|^2dxds + \int_0^t\int_{\mathcal{O}} \phi_n\varphi'(w_{n,s}^+(x))d\nu(x,s), \quad a.s.
\end{aligned} \tag{4.5}$$

We have

$$\begin{aligned}
\varphi'(w_{n,s}^+)\bar{f}_s^n \quad &- \quad \sum \varphi''(w_{s,n}^+)\partial_i w_{n,s}^+\widetilde{g_{i,s}^n} = \varphi'(w_{n,s}^+)\phi_n f_s - \sum a_{i,j}\varphi'(w_{n,s}^+)\partial_j\phi_n\partial_i u_s^+ \\
&+ \quad \sum a_{i,j}\varphi''(w_{n,s}^+)u_s^+\partial_i w_{n,s}^+\partial_j\phi_n - \sum(\varphi'(w_{n,s}^+))g_{i,s}\partial_i\phi_n + \varphi''(w_{n,s}^+)\phi_n g_{i,s}\partial_i w_{n,s}^+.
\end{aligned}$$

Remarking that for all $s \in (0,T]$, $(\phi_n\varphi'(w_{n,s}^+))$ (resp. $(\partial_i\phi_n\varphi'(w_{n,s}^+))$) tends to $\varphi'(u_s^+)$ (resp. 0) in $H_0^1(\mathcal{O})$ (resp. $L^2(\mathcal{O})$) we get by the dominated convergence theorem the convergence of all the terms in equality (4.5) excepted the one involving the measure ν. For this last term, we know that w_n is quasi-continuous and from (4.4) and (4.5) it is easy to verify

$$\sup_n \int_0^t\int_{\mathcal{O}} \phi_n\varphi'(w_{n,s}^+(x))d\nu(x,s) \le C.$$

Then, by Fatou's lemma, we have

$$\int_0^t\int_{\mathcal{O}} \varphi'(u_s^+)\nu(dxds) = \liminf_{n\to\infty}\int_0^t\int_{\mathcal{O}} \phi_n\varphi'(w_{n,s}^+(x))d\nu(x,s) < +\infty, \quad a.s.$$

Hence, the convergence of the last term comes from the dominated convergence theorem. □

4.2 The comparison theorem for local solutions

Firstly, we prove an Itô formula for the difference of local solutions of two OSPDE, $(u^1,\nu^1) \in \mathcal{R}_{loc}(\xi^1, f^1, g, h, S^1)$ and $(u^2,\nu^2) \in \mathcal{R}_{loc}(\xi^2, f^2, g, h, S^2)$, where (ξ^i, f^i, g, h, S^i) satisfy assumptions **(H)**, **(HIL)**, **(OL)** and **(HOL)**. We denote by $\hat{u} = u^1 - u^2$, $\hat{\nu} = \nu^1 - \nu^2$, $\hat{\xi} = \xi^1 - \xi^2$, and

$$\begin{aligned}
\hat{f}(t,\omega,x,y,z) &= f^1\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - f^2\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right), \\
\hat{g}(t,\omega,x,y,z) &= g\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - g\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right), \\
\hat{h}(t,\omega,x,y,z) &= h\left(t,\omega,x,y+u_t^2(x), z+\nabla u_t^2(x)\right) - h\left(t,\omega,x,u_t^2(x),\nabla u_t^2(x)\right).
\end{aligned}$$

Proposition 4.3. *Assume that $\partial\mathcal{O}$ is Lipschitz and that $\hat{u}^+$ belongs to $\mathcal{H}_T$. Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a function of class $\mathcal{C}^2$, which admits a bounded second order derivative and such that $\varphi(0)=\varphi'(0)=0$. Then the following relation holds for each $t\in[0,T]$,*

$$\int_{\mathcal{O}}\varphi(\hat{u}_t^+(x))dx+\int_0^t\mathcal{E}(\varphi'(\hat{u}_s^+),\hat{u}_s^+)ds=\int_{\mathcal{O}}\varphi(\hat{\xi}^+(x))dx+\int_0^t\int_{\mathcal{O}}\varphi'(\hat{u}_s^+(x))\hat{f}_s(x)dxds$$
$$-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi''(\hat{u}_s^+(x))\partial_i\hat{u}_s^+(x)\hat{g}_s^i(x)dxds+\frac{1}{2}\int_0^t\int_{\mathcal{O}}\varphi''(\hat{u}_s^+(x))\mathbb{1}_{\{\hat{u}_s>0\}}|\hat{h}_s(x)|^2dxds$$
$$+\sum_{i=1}^{\infty}\int_0^t\int_{\mathcal{O}}\varphi'(\hat{u}_s^+(x))\hat{h}_s^j(x)dxdB_s^j+\int_0^t\int_{\mathcal{O}}\varphi'(\hat{u}_s^+(x))\hat{\nu}(dxds)\qquad a.s. \tag{4.6}$$

Proof. We consider $\phi\in\mathcal{C}_c^\infty(\mathcal{O}),\ 0\le\phi\le1$, and put

$$\forall t\in[0,T],\quad \hat{w}_t=\phi\hat{u}_t.$$

From the proof of Proposition 4.2, we know that $(\phi u^1,\phi\nu^1)$ and $(\phi u^2,\phi\nu^2)$ are the solutions of problem (1.1) with null Dirichlet boundary conditions. We have the Itô formula for $\hat{w}$, see Theorem 6 in [9]. Then we do the same approximations as in the proof of Proposition 4.2, we can get the desired formula. □

We have the following comparison theorem:

Theorem 4.4. *Assume that (ξ^i,f^i,g,h,S^i), $i=1,2$, satisfy assumptions* **(H)**, **(HIL)**, **(OL)** *and* **(HOL)**. *Let $(u^i,\nu^i)\in\mathcal{R}_{loc}\left(\xi^i,f^i,g,h,S^i\right),i=1,2$ and suppose that the process $\left(u^1-u^2\right)^+$ belongs to $\mathcal{H}_T$ and that one has*

$$E\left(\left\|f^1\left(.,.,u^2,\nabla u^2\right)-f^2\left(.,.,u^2,\nabla u^2\right)\right\|_{\theta;t}^*\right)^2<\infty,\quad \textit{for all}\quad t\in[0,T].$$

If $\xi^1\le\xi^2$ a.s., $f^1\left(t,\omega,u^2,\nabla u^2\right)\le f^2\left(t,\omega,u^2,\nabla u^2\right)$, $dt\otimes dx\otimes dP$-a.e. and $S^1\le S^2$, $dt\otimes dx\otimes dP$-a.s., then one has $u^1(t,x)\le u^2(t,x)$, $dt\otimes dx\otimes dP$-a.e.

Proof. Applying Itô's formula (4.6) to $(\hat{u}^+)^2$, we have $\forall t\in[0,T]$,

$$\int_{\mathcal{O}}(\hat{u}_t^+(x))^2dx+2\int_0^t\mathcal{E}((\hat{u}_s^+))ds=\int_{\mathcal{O}}(\hat{\xi}^+(x))^2dx+2\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{f}_s(x,\hat{u}_s(x),\nabla\hat{u}_s(x))dxds$$
$$-2\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\partial_i\hat{u}_s^+(x)\hat{g}_s^i(x,\hat{u}_s(x),\nabla\hat{u}_s(x))dxds+\int_0^t\int_{\mathcal{O}}\mathbb{1}_{\{\hat{u}_s>0\}}|\hat{h}_s(x,\hat{u}_s(x),\nabla\hat{u}_s(x))|^2dxds$$
$$+2\sum_{i=1}^{\infty}\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{h}_s^j(x,\hat{u}_s(x),\nabla\hat{u}_s(x))dxdB_s^j+2\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{\nu}(dxds),\qquad a.s. \tag{4.7}$$

Remarking the following relation

$$\int_0^t\int_{\mathcal{O}}\hat{u}_s^+(x)\hat{\nu}(dxds)=\int_0^t\int_{\mathcal{O}}(S^1-u^2)^+\nu^1(dxds)-\int_0^t\int_{\mathcal{O}}(u^1-S^2)^+\nu^2(dxds)\le0$$

The Lipschitz conditions in $\hat{g}$ and $\hat{h}$ and Cauchy-Schwarz's inequality lead the following relations: for δ, $\epsilon>0$, we have

$$\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\partial_i\hat{u}_s^+(x)\hat{g}_s^i(x,\hat{u}_s(x),\nabla\hat{u}_s(x))dxds\le(\alpha+\epsilon)\left\|\nabla\hat{u}^+\right\|_{2,2;t}^2+c_\epsilon\left\|\hat{u}^+\right\|_{2,2;t}^2+c_\epsilon\left\|\hat{g}^{\hat{u},0}\right\|_{2,2;t}^2,$$

and

$$\int_0^t \left\| \mathbb{I}_{\{\hat{u}_s>0\}} \hat{h}_s(\hat{u}_s, \nabla \hat{u}_s) \right\|^2 ds \le (\beta^2+\epsilon) \left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 + c_\epsilon \left\|\hat{u}^+\right\|_{2,2;t}^2 + c_\epsilon \left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2 .$$

Moreover, the Lipschitz condition in $\hat{f}$, the duality relation between elements in $L_{\theta;t}$ and $L^*_{\theta;t}$ (2.3) and Young's inequality (2.11) yield the following relation:

$$\int_0^t \int_{\mathcal{O}} \hat{u}_s^+(x) \hat{f}_s\left(\hat{u}_s(x), \hat{u}_s(x)\right) ds \le \epsilon \left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 + c_\epsilon \left\|\hat{u}^+\right\|_{2,2;t}^2 + \delta \left\|\hat{u}^+\right\|_{\theta;t}^2 + c_\delta \left(\left\| \hat{f}^{\hat{u},0+} \right\|_{\theta;t}^* \right)^2 .$$

Since $\mathcal{E}(\hat{u}^+) \ge \lambda \left\|\nabla \hat{u}^+\right\|_2^2$, we deduce from (4.7) that for all $t \in [0,T]$, almost surely,

$$\begin{aligned} \left\|\hat{u}_t^+\right\|_2^2 &+ 2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\epsilon\right) \left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 \le \left\|\hat{\xi}^+\right\|_2^2 + \delta \left\|\hat{u}^+\right\|_{\theta;t}^2 + 2c_\delta \left(\left\| \hat{f}^{\hat{u},0+} \right\|_{\theta;t}^* \right)^2 \\ &+ 2c_\epsilon \left\|\hat{g}^{\hat{u},0}\right\|_{2,2;t}^2 + c_\epsilon \left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2 + 5c_\epsilon \left\|\hat{u}^+\right\|_{2,2;t}^2 + 2M_t, \end{aligned} \tag{4.8}$$

where $M_t := \sum_{j=1}^\infty \int_0^t \left(\hat{u}_s^+, \hat{h}_s^j\left(\hat{u}_s, \nabla \hat{u}_s\right)\right) dB_s^j$ represents the martingale part. Further, using a stopping procedure while taking the expectation, the martingale part vanishes, so that

$$\begin{aligned} &E\left\|\hat{u}_t^+\right\|_2^2 + 2\left(\lambda-\alpha-\frac{\beta^2}{2}-\frac{5}{2}\epsilon\right) E\left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 \le E\left\|\hat{\xi}^+\right\|_2^2 + \delta E \left\|\hat{u}^+\right\|_{\theta;t}^2 \\ &+2c_\delta E\left(\left\| \hat{f}^{\hat{u},0+} \right\|_{\theta;t}^* \right)^2 + 2c_\epsilon E \left\|\hat{g}^{\hat{u},0}\right\|_{2,2;t}^2 + c_\epsilon E \left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2 + 5c_\epsilon \int_0^t E\left\|\hat{u}_s^+\right\|_2^2 ds. \end{aligned}$$

Then we choose $\epsilon = \frac{1}{5}\left(\lambda-\alpha-\frac{\beta^2}{2}\right)$, set $\gamma = \lambda-\alpha-\frac{\beta^2}{2}$ and apply Gronwall's lemma obtaining

$$E\left\|\hat{u}_t^+\right\|_2^2 + \gamma E\left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 \le \left(\delta E\left\|\hat{u}^+\right\|_{\theta;t}^2 + E\left[F\left(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t\right)\right]\right) e^{5c_\epsilon t}, \tag{4.9}$$

with $F(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t) = \left(\|\hat{\xi}^+\|^2 + 2c_\delta\left(\|\hat{f}^{\hat{u},0+}\|_{\theta;t}^*\right)^2 + 2c_\epsilon \|\hat{g}^{\hat{u},0}\|_{2,2;t}^2 + c_\epsilon \|\hat{h}^{\hat{u},0}\|_{2,2;t}^2\right)$. As a consequence one gets

$$E\left\|\hat{u}^+\right\|_{2,2;t}^2 \le \frac{1}{5c_\epsilon}\left(\delta E\left\|\hat{u}^+\right\|_{\theta;t}^2 + E\left[F\left(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t\right)\right]\right)\left(e^{5c_\epsilon t}-1\right). \tag{4.10}$$

Now we return to the inequality (4.8) and take the supremum in time, getting

$$\left\|\hat{u}^+\right\|_{2,\infty;t}^2 \le \delta\left\|\hat{u}^+\right\|_{\theta;t}^2 + F\left(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t\right) + 5c_\epsilon \left\|\hat{u}^+\right\|_{2,2;t}^2 + 2\sup_{s\le t} M_s \tag{4.11}$$

We would like to take the expectation in this relation and for that reason we need to estimate the bracket of the martingale part,

$$\langle M \rangle_t^{\frac{1}{2}} \le \left\|\hat{u}^+\right\|_{2,\infty;t} \left\|\hat{h}(\hat{u}, \nabla \hat{u})\right\|_{2,2;t} \le \eta \left\|\hat{u}^+\right\|_{2,\infty;t}^2 + c_\eta \left(\left\|\hat{u}^+\right\|_{2,2;t}^2 + \left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 + \left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2 \right)$$

with η another small parameter to be properly chosen. Using this estimate and the inequality of Burkholder-Davis-Gundy we deduce from the inequality (4.11):

$$\begin{aligned} (1-2C_{BDG}\eta) E\left\|\hat{u}^+\right\|_{2,\infty;t}^2 &\le \delta E\left\|\hat{u}^+\right\|_{\theta;t}^2 + E\left[F\left(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t\right)\right] \\ + (5c_\varepsilon + 2C_{BDG}c_\eta) E\left\|\hat{u}^+\right\|_{2,2;t}^2 &+ 2C_{BDG}c_\eta E\left\|\nabla \hat{u}^+\right\|_{2,2;t}^2 + 2C_{BDG}c_\eta E\left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2 \end{aligned}$$

where C_{BDG} is the constant corresponding to the Burkholder-Davis-Gundy inequality. Further we choose the parameter $\eta = \frac{1}{4C_{BDG}}$ and combine this estimate with (4.9) and (4.10) to deduce an estimate of the form:

$$E\left(\left\|\hat{u}^+\right\|_{2,\infty;t}^2 + \left\|\nabla\hat{u}^+\right\|_{2,2;t}^2\right) \leq \delta c_2\left(t\right) E\left\|\hat{u}^+\right\|_{\theta;t}^2 + c_3(\delta,t) E\left[R\left(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t\right)\right]$$

where $R\left(\delta, \hat{\xi}^+, \hat{f}^{\hat{u},0+}, \hat{g}^{\hat{u},0}, \hat{h}^{\hat{u},0}, t\right) = \left(\left\|\hat{\xi}^+\right\|^2 + \left(\left\|\hat{f}^{\hat{u},0+}\right\|_{\theta;t}^*\right)^2 + \left\|\hat{g}^{\hat{u},0}\right\|_{2,2;t}^2 + \left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2\right)$ and $c_3(\delta,t)$ is a constant that depends on δ and t, while $c_2\left(t\right)$ is independent of δ. Dominating the term $E\left\|\hat{u}^+\right\|_{\theta;t}^2$ by using the estimate (2.4) and then choosing $\delta = \frac{1}{2c_1^2 c_2(t)}$, we get the following estimate:

$$E\left(\left\|\hat{u}^+\right\|_{2,\infty;t}^2 + \left\|\nabla\hat{u}^+\right\|_{2,2;t}^2\right) \leq k(t) E\left(\left\|\hat{\xi}^+\right\|_2^2 + \left(\left\|\hat{f}^{\hat{u},0+}\right\|_{\theta;t}^*\right)^2 + \left\|\hat{g}^{\hat{u},0}\right\|_{2,2;t}^2 + \left\|\hat{h}^{\hat{u},0}\right\|_{2,2;t}^2\right).$$

This implies the desired result since $\hat{\xi} \leq 0$, $\hat{f}^0 \leq 0$ and $\hat{g}^0 = \hat{h}^0 = 0$. □

4.3 Maximum principle

We first consider the case of a solution u such that $u \leq 0$ on $\partial\mathcal{O}$.

Theorem 4.5. *Suppose that Assumptions* **(H)**, **(OL)**, **(HIL)**, **(HOL)**,**(HI2p)**, **(HO∞p)** *and* **(HDθp)** *hold for some $\theta \in [0,1[$, $p \geq 2$ and that the constants of the Lipschitz conditions satisfy*

$$\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda.$$

Let $(u,\nu) \in \mathcal{R}_{loc}\left(\xi, f, g, h, S\right)$ be such that $u^+ \in \mathcal{H}_T$. Then one has

$$\begin{aligned} E\left\|u^+\right\|_{\infty,\infty;t}^p &\leq k(t)c(p)E\big(\left\|\xi^+ - S_0'\right\|_\infty^p + (\left\|\bar{f}^{0,+}\right\|_{\theta;t}^*)^p + (\left\||\bar{g}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||\bar{h}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} \\ &+ \left\|(S_0')^+\right\|_\infty^p + (\left\|f^{',+}\right\|_{\theta;t}^*)^p + (\left\||g'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||h'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}}\big) \end{aligned}$$

where $k\left(t\right)$ is constant that depends on the structure constants and $t \in [0,T]$.

Proof. Set $(y,\nu') = \mathcal{R}(\xi^+, \check{f}, g, h, S)$ the solution with zero Dirichlet boundary conditions, where the function $\check{f}$ is defined by $\check{f} = f + f^{0,-}$, with $f^{0,-} = 0 \vee (-f^0)$. The assumption on the Lipschitz constants ensures the application of Section 3, which gives the following estimate :

$$E\left\|y - S'\right\|_{\infty,\infty;t}^p \leq k(t)E(\left\|\xi^+ - S_0'\right\|_\infty^p + (\left\|\bar{f}^{0,+}\right\|_{\theta;t}^*)^p + (\left\||\bar{g}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||\bar{h}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}}),$$

where $\bar{f}^{0,+} = \check{f}^0 - f' = f^{0,+} - f'$. On the boundary, $y = 0$ and $u \leq 0$, hence, $u - y \leq 0$ on the boundary, i.e. $(u-y)^+ \in \mathcal{H}_T$. Moreover, the other conditions of Theorem 4.4 are satisfied so that we can apply it and deduce that $u - S' \leq y - S'$. This implies that $(u - S')^+ \leq (y - S')^+$ and the above estimate of $y - S'$ leads to the following estimate:

$$E\left\|(u - S')^+\right\|_{\infty,\infty;t}^p \leq k(t)E(\left\|\xi^+ - S_0'\right\|_\infty^p + (\left\|\bar{f}^{0,+}\right\|_{\theta;t}^*)^p + (\left\||\bar{g}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||\bar{h}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}}).$$

with the estimate of S'

$$E\left\|(S')^+\right\|_{\infty,\infty;t}^p \leq k(t)E(\left\|(S_0')^+\right\|_\infty^p + (\left\|f^{',+}\right\|_{\theta;t}^*)^p + (\left\||g'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||h'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}}).$$

Therefore,

$$\begin{aligned} E\left\|u^+\right\|_{\infty,\infty;t}^p &\leq k(t)c(p)E\big(\left\|\xi^+ - S_0'\right\|_\infty^p + (\left\|\bar{f}^{0,+}\right\|_{\theta;t}^*)^p + (\left\||\bar{g}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||\bar{h}^0|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} \\ &+ \left\|(S_0')^+\right\|_\infty^p + (\left\|f^{',+}\right\|_{\theta;t}^*)^p + (\left\||g'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}} + (\left\||h'|^2\right\|_{\theta;t}^*)^{\frac{p}{2}}\big). \end{aligned}$$

□

Let us generalize the previous result by considering a real Itô process of the form

$$M_t = m + \int_0^t b_s ds + \sum_{j=1}^{+\infty} \int_0^t \sigma_{j,s} dB_s^j \, ,$$

where m is a random variable and $b = (b_t)_{t\geq 0}$, $\sigma = (\sigma_{1,t}, ..., \sigma_{n,t}, ...)_{t\geq 0}$ are adapted processes.

Theorem 4.6. *Suppose that Assumptions* **(H)**, **(OL)**, **(HIL)**, **(HOL)**,**(HI2p)**, **(HO∞p)** *and* **(HDθp)** *hold for some $\theta \in [0,1[$, $p \geq 2$ and that the constants of the Lipschitz conditions satisfy*

$$\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda.$$

Assume also that m and the processes b and σ satisfy the following integrability conditions

$$E\,|m|^p < \infty, \; E\left(\int_0^t |b_s|^{\frac{1}{1-\theta}} ds\right)^{p(1-\theta)} < \infty, \; E\left(\int_0^t |\sigma_s|^{\frac{2}{1-\theta}} ds\right)^{\frac{p(1-\theta)}{2}} < \infty,$$

for each $t \in [0,T]$. Let $(u,\nu) \in \mathcal{R}_{loc}(\xi, f, g, h, S)$ be such that $(u-M)^+$ belongs to $\mathcal{H}_T$. Then one has

$$\begin{aligned} E\left\|(u-M)^+\right\|_{\infty,\infty;t}^p \quad \leq \quad & c(p)k(t)E\big[\left\|(\xi-m)^+ - (S_0'-m)\right\|_\infty^p + \left(\left\|\bar{f}^{0,+}\right\|_{\theta;t}^*\right)^p \\ + \quad & \left(\left\||\bar{g}^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||\bar{h}^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left\|(S_0'-m)^+\right\|_\infty^p \qquad (4.12) \\ + \quad & \left(\left\|(f'-b)^+\right\|_{\theta;t}^*\right)^p + \left(\left\||g'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||h'-\sigma|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} \big] \end{aligned}$$

where $k(t)$ is the constant from the preceding corollary. The right hand side of this estimate is dominated by the following quantity which is expressed directly in terms of the characteristics of the process M,

$$\begin{aligned} c(p)k(t)E\big[&\left\|(\xi-m)^+ - (S_0'-m)\right\|_\infty^p + \left(\left\|\bar{f}^{0,+}\right\|_{\theta;t}^*\right)^p + \left(\left\||\bar{g}^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||\bar{h}^0|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} \\ &+ \left\|(S_0'-m)^+\right\|_\infty^p + \left(\left\|f^{',+}\right\|_{\theta;t}^*\right)^p + \left(\left\||g'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\||h'|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} \\ &+ \left(\int_0^t |b_s|^{\frac{1}{1-\theta}} ds\right)^{p(1-\theta)} + \left(\int_0^t |\sigma_s|^{\frac{2}{1-\theta}} ds\right)^{\frac{p(1-\theta)}{2}} \big]. \end{aligned}$$

Proof. One immediately observes that $u - M$ belongs to $\mathcal{R}_{loc}\big(\xi - m, \check{f}, \check{g}, \check{h}, S - M\big)$, where

$$\check{f}(t,\omega,x,y,z) = f(t,\omega,x,y+M_t(\omega), z + \nabla M_t(\omega)) - b_t(\omega),$$

$$\check{g}(t,\omega,x,y,z) = g(t,\omega,x,y+M_t(\omega), z + \nabla M_t(\omega)),$$

$$\check{h}(t,\omega,x,y,z) = h(t,\omega,x,y+M_t(\omega), z + \nabla M_t(\omega)) - \sigma_t(\omega).$$

In order to apply the preceding theorem we only have to estimate the zero terms of the following functions:

$$\bar{\check{f}}(t,\omega,x,y,z) = \check{f}(t,\omega,x,y+S'-M, z+\nabla(S'-M)) - f'(t,\omega,x) + b_t(\omega),$$

$$\bar{\check{g}}(t,\omega,x,y,z) = \check{g}(t,\omega,x,y+S'-M, z+\nabla(S'-M)) - g'(t,\omega,x),$$

$$\bar{\check{h}}(t,\omega,x,y,z)=\check{h}(t,\omega,x,y+S'-M,z+\nabla(S'-M))-h'(t,\omega,x)+\sigma_t(\omega).$$

So we have:

$$\bar{\check{f}}_t^0=\check{f}_t(S'-M,\nabla(S'-M))-f'_t+b_t=f_t(S',\nabla S')-f'_t=\bar{f}^0\,,$$
$$\bar{\check{g}}_t^0=\check{g}_t(S'-M,\nabla(S'-M))-g'_t=g_t(S',\nabla S')-g'_t=\bar{g}^0\,,$$
$$\bar{\check{h}}_t^0=\check{h}_t(S'-M,\nabla(S'-M))-h'_t+\sigma_t=h_t(S',\nabla S')-h'_t=\bar{h}^0\,.$$

Therefore, applying the preceding theorem to $u-M$, we obtain (4.12).
On the other hand, one has the following estimates:

$$\left\|(f'-b)^+\right\|^*_{\theta;t}\leq c\left[\left\|f^{',+}\right\|^*_{\theta;t}+\left(\int_0^t|b_s|^{\frac{1}{1-\theta}}\,ds\right)^{1-\theta}\right],$$

$$\left\||h'-\sigma|^2\right\|^*_{\theta;t})^{\frac{p}{2}}\leq c\left[\left(\left\||h'|^2\right\|^*_{\theta;t}\right)^{\frac{p}{2}}+\left(\int_0^t|\sigma_s|^{\frac{2}{1-\theta}}\,ds\right)^{1-\theta}\right].$$

This allows us to conclude the proof. □

5 Appendix

In this section, we prove some technical lemmas that we need in the Step 2 of the proof of Theorem 3.2. For simplicity, we put, for fixed $n\leq m$, $\hat{u}:=\bar{u}^n-\bar{u}^m$, $\hat{\xi}:=\xi^n-\xi^m$, $\hat{f}(t,\omega,x,y,z):=\bar{f}_{n,m}(t,\omega,x,y,z)$ and similar for $\hat{g}$ and $\hat{h}$. Note that in this step, the initial values $\hat{\xi}$ and $\hat{f}^0$, $\hat{g}^0$, $\hat{h}^0$ are assumed to be uniformly bounded.
The next Lemma ensures the L^l-integrability of $\hat{u}$ with respect to both $dt\otimes dx\otimes dP$ and $(\nu^n+\nu^m)\otimes dP$ and will allow us to pass to the limit and therefore get an Ito formula for $\hat{u}^l$.

Lemma 5.1. *Denote*

$$K=\left\|\hat{\xi}\right\|_{L^\infty(\Omega\times\mathcal{O})}\vee\left\|\hat{f}^0\right\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}\vee\left\|\hat{g}^0\right\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}\vee\left\|\hat{h}^0\right\|_{L^\infty(\mathbb{R}_+\times\Omega\times\mathcal{O})}.$$

Then there exist constants c, $c'>0$ which only depend on K, C, α, β such that, for all real $l\geq 2$, one has

$$E\int_{\mathcal{O}}|\hat{u}_t(x)|^l dx\leq cK^2 l(l-1)e^{cl(l-1)t},\tag{5.1}$$

$$E\int_0^t\int_{\mathcal{O}}|\hat{u}_s(x)|^{l-2}|\nabla\hat{u}_s(x)|^2dxds\leq c'K^2e^{cl(l-1)t},\tag{5.2}$$

and

$$E\int_0^t\int_{\mathcal{O}}|\hat{u}_s(x)|^{l-1}(\nu^n+\nu^m)(dxds)<+\infty.\tag{5.3}$$

Proof. Beginning from the Itô formula for the difference of solutions of two obstacle problems which has been proved in [9]: we take the same φ_n as in the proof of Lemma 3.4,

$$\begin{aligned}&\int_{\mathcal{O}}\varphi_n(\hat{u}_t(x))\,dx\;+\;\int_0^t\mathcal{E}\big(\varphi'_n(\hat{u}_s),\,\hat{u}_s\big)\,ds=\int_{\mathcal{O}}\varphi_n(\hat{\xi}(x))dx+\int_0^t\int_{\mathcal{O}}\varphi'_n(\hat{u}_s(x))\,\hat{f}(s,x)\,dxds\\&-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi''_n(\hat{u}_s(x))\partial_i(\hat{u}_s(x)))\,\hat{g}_i(s,x)\,dx\,ds+\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\varphi'_n(\hat{u}_s(x))\,\hat{h}_j(s,x)\,dxdB^j_s\\&+\frac{1}{2}\sum_{j=1}^\infty\int_0^t\int_{\mathcal{O}}\varphi''_n(\hat{u}_s(x))\,\hat{h}^2_j(s,x)\,dx\,ds+\int_0^t\int_{\mathcal{O}}\varphi'_n(\hat{u}_s(x))\,(\nu^n-\nu^m)(dx\,ds)\,,\quad a.s.\end{aligned}\tag{5.4}$$

The support of ν^n is $\{\bar{u}^n = S\}$ and the support of ν^m is $\{\bar{u}^m = S\}$, so the last term is equal to

$$\int_0^t \int_{\mathcal{O}} \varphi_n'(S_s(x) - \bar{u}_s^m(x))\, \nu^n(dx\, ds) - \int_0^t \int_{\mathcal{O}} \varphi_n'(\bar{u}_s^n(x) - S_s(x))\, \nu^m(dx\, ds)$$

and the fact that $\varphi_n'(x) \leq 0$, when $x \leq 0$ and $\varphi_n'(x) \geq 0$, when $x \geq 0$, ensure that the last term is always negative.
By the uniform ellipticity of the operator A, we get

$$\mathcal{E}\big(\varphi_n'(\hat{u}_s),\, \hat{u}_s\big) \;\geq\; \lambda \int_{\mathcal{O}} \varphi_n''(\hat{u}_s)|\nabla \hat{u}_s|^2\, dx.$$

Let $\epsilon > 0$ be fixed. Using the Lipschitz condition on $\hat{f}$ and the properties of the functions $(\varphi_n)_n$ we get

$$|\varphi_n'(\hat{u}_s)|\,|\hat{f}(s,x)| \leq l(\varphi_n(\hat{u}_s) + 1)\,|\hat{f}^0| + (C + c_\epsilon)\,|\hat{u}_s|^2\varphi_n''(\hat{u}_s) + \,\epsilon\varphi_n''(\hat{u}_s)|\nabla(\hat{u}_s)|^2.$$

Now using Cauchy-Schwarz's inequality and the Lipschitz condition on $\hat{g}$ we get

$$\sum_{i=1}^d \varphi_n''(\hat{u}_s)\partial_i(\hat{u}_s)\,\hat{g}(s,x) \leq l(l-1)c_\epsilon K^2 + 2c_\epsilon(K^2 + C^2)l(l-1)|\varphi_n(\hat{u}_s)| + (\alpha + \epsilon)\,\varphi_n''(\hat{u}_s)|\nabla(\hat{u}_s)|^2.$$

In the same way as before

$$\sum_{j=1}^\infty \varphi_n''(\hat{u}_s)\,\hat{h}(s,x) \leq\; 2c_\epsilon' l(l-1)K^2 + 2c_\epsilon'(K^2 + C^2)l(l-1)\varphi_n(\hat{u}_s) + (1+\epsilon)\,\beta^2\,\varphi_n''(\hat{u}_s)|\nabla(\hat{u}_s)|^2.$$

Thus taking the expectation, we deduce

$$\begin{aligned} &E\int_{\mathcal{O}} \varphi_n(\hat{u}_t(x))\,dx + (\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha + 2\epsilon)\,)\,E\int_0^t\int_{\mathcal{O}} \varphi_n''(\hat{u}_s)\,|\nabla(\hat{u}_s)|^2\,dx\,ds \\ &\leq\; l(l-1)c_\epsilon'' K^2 \;+\; c_\epsilon'' l(l-1)\big(K^2 + C^2 + C + c_\epsilon\big)E\int_0^t\int_{\mathcal{O}} \varphi_n(\hat{u}_s(x))\,dx\,ds. \end{aligned}$$

On account of the contraction condition, one can choose $\epsilon > 0$ small enough such that

$$\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha + 2\epsilon) > 0$$

and then

$$E\int_{\mathcal{O}} \varphi_n(\hat{u}_t(x))\,dx \leq\; cK^2 l(l-1) \;+\; cl(l-1)E\int_0^t\int_{\mathcal{O}} \varphi_n(\hat{u}_s(x))\,dx\,ds\,.$$

We obtain by Gronwall's Lemma, that

$$E\int_{\mathcal{O}} \varphi_n(\hat{u}_t(x))\,dx \leq\; c\,K^2 l(l-1)\,\exp\big(c\,l(l-1)\,t\big),$$

and so it is easy to get

$$E\int_0^t\int_{\mathcal{O}} \varphi_n''(\hat{u}_s(x))\,|\nabla\hat{u}_s|^2\,dx\,ds \;\leq\; c'\,K^2 l\,(l-1)\,\exp\big(cl(l-1)\,t\big).$$

Then, letting $n \to \infty$, by Fatou's lemma we get (5.1) and (5.2).
From (5.4), we know that

$$-\int_0^t\int_{\mathcal{O}} \varphi_n'(\hat{u}_s(x))(\nu^n - \nu^m)(dxds) \leq C.$$

Moreover,

$$\begin{aligned}&-\int_0^t\int_{\mathcal{O}}\varphi_n'(\hat{u}_s(x))(\nu^n-\nu^m)(dxds)\\ =\;&-\int_0^t\int_{\mathcal{O}}\varphi_n'(S_s(x)-\bar{u}_s^m(x))\,\nu^n(dx\,ds)+\int_0^t\int_{\mathcal{O}}\varphi_n'(\bar{u}_s^n(x)-S_s(x))\,\nu^m(dx\,ds)\\ =\;&\int_0^t\int_{\mathcal{O}}\varphi_n'(\bar{u}_s^m(x)-S_s(x))\,\nu^n(dx\,ds)+\int_0^t\int_{\mathcal{O}}\varphi_n'(\bar{u}_s^n(x)-S_s(x))\,\nu^m(dx\,ds)\end{aligned}$$

By Fatou's lemma, we obtain

$$\int_0^t\int_{\mathcal{O}}|\bar{u}_s^m(x)-S_s(x)|^{l-1}\nu^n(dxds)+\int_0^t\int_{\mathcal{O}}|\bar{u}^n(x)-S_s(x)|^{l-1}\nu^m(dxds)<+\infty,\ a.s.$$

which implies (5.3). □

Lemma 5.2. *One has the following formula for* $\hat{u}$*:* $\forall t\geq 0$*, almost surely,*

$$\begin{aligned}&\int_{\mathcal{O}}|\hat{u}_t(x)|^l\,dx+\int_0^t\mathcal{E}\left(l\,(\hat{u}_s)^{l-1}\mathbf{sgn}(\hat{u}_s),\,\hat{u}_s\right)ds=\int_{\mathcal{O}}\left|\hat{\xi}(x)\right|^l dx\\ &+l\int_0^t\int_{\mathcal{O}}\mathbf{sgn}(\hat{u}_s)\,|\hat{u}_s(x)|^{l-1}\,\hat{f}(s,x)\,dxds-l(l-1)\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}|\hat{u}_s(x)|^{l-2}\,\partial_i(\hat{u}_s(x))\,\hat{g}_i(s,x)\,dx\,ds\\ &+l\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}}\mathbf{sgn}(\hat{u}_s)\,|\hat{u}_s(x)|^{l-1}\,\hat{h}_j(s,x)\,dxdB_s^j+\frac{l(l-1)}{2}\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}}|\hat{u}_s(x)|^{l-2}\,\hat{h}_j^2(s,x,)\,dx\,ds\\ &+l\int_0^t\int_{\mathcal{O}}\mathbf{sgn}(\hat{u}_s)\,|\hat{u}_s(x)|^{l-1}\,(\nu^1-\nu^2)(dx\,ds)\,.\end{aligned}\tag{5.5}$$

Proof. From the Itô formula for the difference of two solutions (see Theorem 6 in [9]), we have P-almost surely for all $t\in[0,T]$ and $n\in\mathbb{N}^*$:

$$\begin{aligned}&\int_{\mathcal{O}}\varphi_n(\hat{u}_t(x))\,dx\;+\;\int_0^t\mathcal{E}\big(\varphi_n'(\hat{u}_s),\,\hat{u}_s\big)\,ds=\int_{\mathcal{O}}\varphi_n(\hat{\xi}(x))dx\\ &+\int_0^t\int_{\mathcal{O}}\varphi_n'(\hat{u}_s(x))\,\hat{f}(s,x)\,dxds-\sum_{i=1}^d\int_0^t\int_{\mathcal{O}}\varphi_n''(\hat{u}_s(x))\partial_i\hat{u}_s(x)\,\hat{g}_i(s,x)\,dx\,ds\\ &+\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}}\varphi_n'(\hat{u}_s(x))\,\hat{h}_j(s,x)\,dxdB_s^j+\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t\int_{\mathcal{O}}\varphi_n''(\hat{u}_s(x))\,\hat{h}_j^2(s,x)\,dx\,ds\\ &+\int_0^t\int_{\mathcal{O}}\varphi_n'(\hat{u}_s(x))(\nu^1-\nu^2)(dxds)\,.\end{aligned}$$

Then, passing to the limit as $n\to\infty$, the convergences come from the dominated convergence theorem and the previous lemma. □

Similar as before, we define the processes $\hat{v}$ and $\hat{v}'$ by

$$\begin{aligned}\hat{v}_t:&=\sup_{s\leq t}\left(\int_{\mathcal{O}}|\hat{u}_s|^l\,dx+\gamma l\,(l-1)\int_0^s\int_{\mathcal{O}}|\hat{u}_r|^{l-2}\,|\nabla\hat{u}_r|^2\,dx\,dr\right)\\ \hat{v}_t':&=\int_{\mathcal{O}}\left|\hat{\xi}\right|^l dx+l^2c_1\left\||\hat{u}|^l\right\|_{1,1;t}+l\left\|\hat{f}^0\right\|_{\theta,t}^*\left\||\hat{u}|^{l-1}\right\|_{\theta;t}\\ &\quad+l^2\left(c_2\left\||\hat{g}^0|^2\right\|_{\theta;t}^*+c_3\left\||\hat{h}^0|^2\right\|_{\theta;t}^*\right)\left\||\hat{u}|^{l-2}\right\|_{\theta;t},\end{aligned}$$

where above and below γ, c_1, c_2 and c_3 are the constants given by relations (3.11).
We remark first that the last term in (5.5) is non-positive, indeed:

$$\begin{aligned}
&\int_0^t \int_{\mathcal{O}} \operatorname{sgn}(\hat{u}_s) \left|S_s - u_s^2(x)\right|^{l-1} (\nu^1 - \nu^2)(dx\, ds) \\
= \quad &\int_0^t \int_{\mathcal{O}} \operatorname{sgn}(S_s - u_s^2) \left|S_s - u_s^2(x)\right|^{l-1} \nu^1(dx\, ds) \\
&- \int_0^t \int_{\mathcal{O}} \operatorname{sgn}(u_s^1 - S_s) \left|u_s^1(x) - S_s(x)\right|^{l-1} \nu^2(dx\, ds) \le 0.
\end{aligned}$$

Then applying the same proof as the one of Lemma 3.8, we obtain:

$$\begin{aligned}
&\tau E \sup_{0\le s\le t} \left(\int_{\mathcal{O}} |\hat{u}_s|^l \, dx + \gamma l\,(l-1) \int_0^s \int_{\mathcal{O}} |\hat{u}_r|^{l-2} \left|\nabla \hat{u}_r\right|^2 dx dr \right) \\
&\le E \int_{\mathcal{O}} \left|\hat{\xi}\right|^l dx + l^2 c_1 E \left\| |\hat{u}|^l \right\|_{1,1;t} + lE \left\| \hat{f}^0 \right\|_{\theta,t}^* \left\| |\hat{u}|^{l-1} \right\|_{\theta;t} \\
&\quad + l^2 E \left(c_2 \left\| |\hat{g}^0|^2 \right\|_{\theta;t}^* + c_3 \left\| |\hat{h}^0|^2 \right\|_{\theta;t}^* \right) \left\| |\hat{u}|^{l-2} \right\|_{\theta;t}.
\end{aligned}$$

and this yields that the process $\tau\hat{v}$ is dominated by $\hat{v}'$.
Starting from here, we can repeat line by line the proofs of Lemmas 15-17 in [5] and apply the Moser iteration as at the end of Subsection 3.1 to obtain the desired estimations, namely:

Lemma 5.3. *There exists a function $k_2 : \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our problem and such that the following estimate holds*

$$E\|\hat{u}\|_{\infty,\infty:t}^p \le k_2(t)\, E \left(\left\|\hat{\xi}\right\|^p + \left\|\hat{f}^0\right\|_{\theta;t}^{*p} + \left\| |\hat{g}^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} + \left\| |\hat{h}^0|^2 \right\|_{\theta;t}^{*\frac{p}{2}} \right).$$

Lemma 5.4. *There exists a function $k_1 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which involves only the structure constants of our problem and such that the following estimate holds*

$$E\hat{v}_t \le k_1(l,t) E \left(\int_{\mathcal{O}} |\hat{\xi}|^l dx + \left\|\hat{f}^0\right\|_{\theta;t}^{*l} + \left\| \left|\hat{g}^0\right|^2 \right\|_{\theta;t}^{*\frac{l}{2}} + \left\| \left|\hat{h}^0\right|^2 \right\|_{\theta;t}^{*\frac{l}{2}} \right).$$

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