Integral and local limit theorems for level crossings of diffusions and the Skorohod problem

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Abstract

Using a new technique, based on the regularization of a càdlàg process via the double Skorohod map, we obtain limit theorems for integrated numbers of level crossings of diffusions. The results are related to the recent results on the limit theorems for the truncated variation. We also extend to diffusions the classical result of Kasahara on the "local" limit theorem for the number of crossings of a Wiener process. We establish the correspondence between the truncated variation and the double Skorohod map. Additionally, we prove some auxiliary formulas for the Skorohod map with time-dependent boundaries.

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1 Introduction

Let $X = (X_t, t \geq 0)$ be a continuous semimartingale adapted to the filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the usual conditions hold. The purpose of this study is to establish a connection between the level crossings of X, the local time of X, quadratic variation $\langle X \rangle$ of X and its *truncated variation*, denoted by TV^c, defined for $c > 0$ and $T > 0$ by the following formula

$$TV^c(X, T) = \sup_n \sup_{0 \leq t_0 < t_1 < \ldots < t_n \leq T} \sum_{i=1}^n \max\left\{ \left| X_{t_i} - X_{t_{i-1}} \right| - c, 0 \right\}. \tag{1.1}$$

The concept of truncated variation of a stochastic process has been recently introduced by Łochowski in [13, 14] and proved relevant for interpreting maximal returns from trading in transaction costs problems. The difference with the total variation is that the truncated variation considers only jumps greater than some constant level c and is always finite for any càdlàg process X.

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In this paper we will present another interpretation of the truncated variation and as a result will obtain an alternative derivation of one of the main findings recently established in [16]: that for a continuous semimartingale X

$$c \cdot TV^c (X, \cdot) \to \langle X \rangle \text{ a.s.} \tag{1.2}$$

as $c \downarrow 0$ and the convergence holds in $C \left([0; +\infty), \mathbb{R} \right)$ equipped with the topology of uniform convergence on compacts.

Remark 1.1. *Throughout this paper we will always apply the convention that for appropriate $d = 1, 2, \ldots$ spaces $C \left([0; +\infty), \mathbb{R}^d \right)$ ($\mathbb{R}^d$-valued continuous functions on $[0; +\infty)$), $D \left([0; +\infty), \mathbb{R}^d \right)$ ($\mathbb{R}^d$-valued càdlàg functions on $[0; +\infty)$) are equipped with the topology of uniform convergence on compacts and spaces $C \left([0; T], \mathbb{R}^d \right)$ ($\mathbb{R}^d$-valued continuous functions on $[0; T]$), $D \left([0; T], \mathbb{R}^d \right)$ ($\mathbb{R}^d$-valued càdlàg functions on $[0; T]$) are equipped with uniform convergence topology.*

We will use classical tools like the relation between the quadratic variation of X and local times of X at different levels (occupation times formula), and then the link between local times of X and the number of interval crossings by X (cf. [12] and references therein). However, our main tool will be a new one (as far as we know) - a direct correspondence between interval crossings by X and *level crossings* by some regularization of X, denoted by $X^{c,x}$, obtained via the double Skorohod map on $[-c/2; c/2]$ (cf. [11], [2]). The direct consequence of this construction is that $X^{c,x}$ has locally finite total variation and the (pathwise) bounds

$$\| X - X^{c,x} \|_\infty \leq c/2, \tag{1.3}$$

$$TV^c (X, T) \leq TV (X^{c,x}, T) \leq TV^c (X, T) + c.$$

Next, we will use a classical result by Banach and Vitali (cf. [3, (3.i)]), stating that the total variation may be obtained by integrating the numbers of levels crossings. In this setting, relation (1.2) corresponds to first order convergence of the integrated number of interval crossings by X to its quadratic variation.

Further, for X being a unique strong solution of the following s.d.e., driven by a standard Brownian motion W,

$$\mathrm{d}X_t = \mu (X_t) \, \mathrm{d}t + \sigma (X_t) \, \mathrm{d}W_t, \ X_0 = x_0,$$

with Lipschitz μ, σ, where $\sigma > 0$, we will use the second order convergence results already obtained in [16] for TV^c to obtain second order convergence of the difference between the integrated number of interval crossings by X and its quadratic variation. More precisely, let $n_c^a(Y, T)$ be the number of times that X crosses the interval $[a; a + c]$ before time T (for the precise definition of $n_c^a(Y, T)$ see Section 3 and Subsection 4.3). We will prove that

$$\frac{1}{c} \left\{ c \int_{\mathbb{R}} n_c^a (X, \cdot) \, \mathrm{d}a - \langle X \rangle \right\} \Rightarrow \frac{1}{\sqrt{3}} B_{\langle X \rangle}, \tag{1.4}$$

where B is another standard Brownian motion, independent from W and the convergence "$\Rightarrow$" is understood as stable (cf. [7, Sect VIII.5]) convergence on $C \left([0; +\infty), \mathbb{R} \right)$ as $c \downarrow 0$ (equipped with the topology defined in Remark 1.1).

Remark 1.2. *Throughout this paper, for $d = 1, 2, \ldots,$ "$\Rightarrow$" will be always understood as the stable (with respect to the σ-field generated by X) convergence as $c \downarrow 0$ on $C \left([0; +\infty), \mathbb{R}^d \right), D \left([0; +\infty), \mathbb{R}^d \right), C \left([0; T], \mathbb{R}^d \right)$ or $D \left([0; T], \mathbb{R}^d \right)$, equipped with the topology defined in Remark 1.1.*

We will call the result given by (1.4) integral limit theorem. The "functional" version of (1.4) is the following. Let $N^a(Y,T)$ be the number of times that Y crosses (from above or from below - for the precise definition of $N^a(Y,T)$ see Section 3 and Subsection 4.4) the level a on the interval $[0;T]$. For any twice differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that f'' is continuous we have

$$\frac{1}{c}\left\{ c\int_{\mathbb{R}} f(a) N^a(X^{c,x},\cdot)\,\mathrm{d}a - \int_0^\cdot f(X_s)\,\mathrm{d}\langle X\rangle_s \right\} \Rightarrow \frac{1}{\sqrt{3}}\int_0^\cdot f(X_s)\,\mathrm{d}B_{\langle X\rangle_s}, \qquad (1.5)$$

where $X^{c,x}$ is the (already mentioned) regularization of X. Moreover, we have a more direct result corresponding to (1.5) which may be expressed in terms of *interval crossings* by X :

$$\frac{1}{c}\left\{ c\int_{\mathbb{R}} f(a) n_c^a(X,\cdot)\,\mathrm{d}a - \int_0^\cdot f(X_s)\,\mathrm{d}\langle X\rangle_s \right\} \Rightarrow \frac{1}{\sqrt{3}}\int_0^\cdot f(X_s)\,\mathrm{d}B_{\langle X\rangle_s}. \qquad (1.6)$$

These results shall be compared with the main result of [18] where it was shown (for a slightly more general family of processes) that for X^ε being a smoothed version of X, i.e.

$$X_t^\varepsilon = \int_{-1}^1 \psi(-u) X_{t+\varepsilon u}\,\mathrm{d}u,$$

where ψ is a smooth (C^∞) kernel with compact support $[-1;1]$, one has

$$\frac{1}{\sqrt{\varepsilon}}\left\{ k_\psi \sqrt{\varepsilon} \int_{\mathbb{R}} f(a) N^a(X^\varepsilon,\cdot)\,\mathrm{d}a - \int_0^\cdot f(X_s)\sigma(X_s)\,\mathrm{d}s \right\}$$
$$\Rightarrow c_\psi \int_0^\cdot f(X_s)\sigma(X_s)\,\mathrm{d}B_s, \qquad (1.7)$$

as $\varepsilon \downarrow 0$. k_ψ and c_ψ are here positive constants depending only on ψ. Notice that the smoothed version X^ε approximates X on average with accuracy $\sqrt{\varepsilon}$, i.e.

$$\mathbb{E}\left\| X - X^\varepsilon \right\|_\infty = O\left(\sqrt{\varepsilon}\right),$$

and in the view of (1.3), (1.5) and (1.7) give the same order of convergence. Other problems of the same type for level crossings by Gaussian processes are an intensive field of study and a good survey of the results obtained so far is [9].

It is worth mentioning that besides the number of interval crossings we consider interval downcrossings and interval upcrossings. For d_c^a, u_c^a being the numbers of relevant interval downcrossings and upcrossings respectively by the process X we establish a joint convergence result for quadruples

$$\left(X, \frac{1}{c}\left\{ c\int_{\mathbb{R}} f(a) u_c^a(X,\cdot)\,\mathrm{d}a - \frac{1}{2}\int_0^\cdot f(X_s)\,\mathrm{d}\langle X\rangle_s \right\}, \right.$$
$$\frac{1}{c}\left\{ c\int_{\mathbb{R}} f(a) d_c^a(X,\cdot)\,\mathrm{d}a - \frac{1}{2}\int_0^\cdot f(X_s)\,\mathrm{d}\langle X\rangle_s \right\},$$
$$\left. \frac{1}{c}\left\{ c\int_{\mathbb{R}} f(a) n_c^a(X,\cdot)\,\mathrm{d}a - \int_0^\cdot f(X_s)\,\mathrm{d}\langle X\rangle_s \right\} \right)$$

and obtain an interesting result (cf. Theorem 4.12 and Theorem 4.11) that e.g.

$$\frac{1}{c}\left\{ c\int_{\mathbb{R}} f(a) u_c^a(X,\cdot)\,\mathrm{d}a - \frac{1}{2}\int_0^\cdot f(X_s)\,\mathrm{d}\langle X\rangle_s \right\}$$
$$\Rightarrow \frac{1}{2\sqrt{3}}\int_0^\cdot f(X_s)\,\mathrm{d}B_{\langle X\rangle_s} + \frac{1}{2}\int_0^\cdot f(X_s)\circ\mathrm{d}X_s,$$

where $\int_0^\cdot f(X_s) \circ \mathrm{d}X_s$ denotes the Stratonovich integral.

With a weaker condition on X- that it is a continuous semimartingale and there exists a probability measure $\mathbb{Q}$ under which X is a local martingale and $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$, we will obtain a "local" counterpart of (1.4), (1.6). Namely, let $n_c(X, T)$ be the number of times that the reflected process $|X|$ crosses down from c to 0 by time T (this is the same as the number of times that X crosses down from c to 0 and crosses up from $-c$ to 0), then

$$\frac{1}{\sqrt{c}}\left\{c \cdot n_c(X, \cdot) - L\right\} \Rightarrow B_L, \tag{1.8}$$

where B is a standard Brownian motion, independent from X, and L is the local time of X at 0. This result is a direct generalisation of the main result of [8], where the same statement was proven for X being a standard Brownian motion. It may be viewed as the "local" counterpart of (1.6) since, by the occupation times formula, $\int_0^T f(X_s) \, \mathrm{d}\langle X\rangle_s = \int_{\mathbb{R}} f(a) L_T^a \mathrm{d}a$. Notice that the integrated process $\int_{\mathbb{R}} f(a) n_c^a(X, \cdot) \, \mathrm{d}a$ reveals much stronger concentration than the process $n_c(X, \cdot)$ (where the multiplication by $\sqrt{c}$ is needed for convergence). Again, the result will identify the limit for the whole quadruple

$$\left(X, \frac{1}{\sqrt{c}}\left\{c \cdot d_c(X, \cdot) - \frac{L}{2}\right\}, \frac{1}{\sqrt{c}}\left\{c \cdot u_c(X, \cdot) - \frac{L}{2}\right\}, L\right).$$

Remark 1.3. *As far as we know, there is no "local" counterpart of (1.7) in the same sense as the generalisation of Kasahara's result, (1.8), is the local counterpart of (1.4), (1.6).*

Let us comment on the organisation of the paper. In the next section we summarize the main results and properties of the truncated variation processes and construct the regularization, $X^{c,x}$, of the process X via the double Skorohod map on $[-c/2; c/2]$. To prove that this regularization satisfies relevant conditions we will need to establish some additional formulas which are (as far as we know) not available in the literature. Thus we will present the solution of the Skorohod problem in a setting suited to our purposes. Next, in Section 3, we establish an important correspondence between the number of interval crossings by the process X and the number of level crossings by the process $X^{c,x}$. Finally, in the last section we prove convergence results.

2 On the truncated variation and the regularization of the process X via Skorohod's map

In this section, first we summarize the main results and properties of the truncated variation processes obtained by Łochowski in [13, 14]. We will assume that $X = (X_t, t \geq 0)$ is a càdlàg process adapted to the filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the usual conditions hold. The truncated variation, given by formula (1.1) is a lower bound for the total variation $TV(Y, T) = TV^0(Y, T)$ of every process Y, uniformly approximating the process X with accuracy $c/2$,

$$\inf_{\|Y - X\|_\infty \leq c/2} TV(Y, T) \geq TV^c(X, T). \tag{2.1}$$

This follows immediately from the fact that $\|X - Y\|_\infty \leq c/2$ implies for any $0 \leq s < t$ the inequality

$$|Y_t - Y_s| \geq \max\left\{|X_t - X_s| - c, 0\right\}.$$

Remark 2.1. *Notice that the truncated variation, unlike the total variation $TV(X, T)$, is always finite. This follows from the fact that every càdlàg function may be uniformly approximated with arbitrary accuracy by step functions, which have finite total variation.*

Together with truncated variation, we consider *upward and downward truncated variations*, defined for $c > 0$ and $T > 0$ by the formulas

$$UTV^c(X, T) = \sup_n \sup_{0 \leq t_0 < t_1 < \ldots < t_n \leq T} \sum_{i=1}^{n} \max\left\{X_{t_i} - X_{t_{i-1}} - c, 0\right\}$$

and

$$DTV^c(X, T) = \sup_n \sup_{0 \leq t_0 < t_1 < \ldots < t_n \leq T} \sum_{i=1}^{n} \max\left\{X_{t_{i-1}} - X_{t_i} - c, 0\right\}$$

respectively. The analogues of (2.1) for UTV and DTV are

$$\inf_{\|Y-X\|_\infty \leq c/2} UTV(Y, T) \geq UTV^c(X, T), \tag{2.2}$$

$$\inf_{\|Y-X\|_\infty \leq c/2} DTV(Y, T) \geq DTV^c(X, T), \tag{2.3}$$

where $UTV = UTV^0$, $DTV = DTV^0$, are called positive and negative total variations respectively (cf. [3, pages 322-323]), and this follows from inequalities: if $\|X - Y\|_\infty \leq c/2$ then for any $0 \leq s < t$, $\max\{Y_t - Y_s, 0\} \geq \max\{X_t - X_s - c, 0\}$ and $\max\{Y_s - Y_t, 0\} \geq \max\{X_s - X_t - c, 0\}$.

Remark 2.2. *We will not need this result in the sequel but it is possible to prove that in fact (cf. [15]):*

$$\inf_{\|Y-X\|_\infty \leq c/2} TV(Y, T) = TV^c(X, T),$$

which means that the lower bound (2.1) is indeed the greatest lower bound. Moreover,

$$TV^c(X, T) = UTV^c(X, T) + DTV^c(X, T)$$

and there exists a càdlàg process X^c with $\|X - X^c\|_\infty \leq c/2$ for which

$$TV^c(X, T) = TV(X^c, T),$$

$$UTV^c(X, T) = UTV(X^c, T), \ DTV^c(X, T) = DTV(X^c, T),$$

but it may be not adapted to $\mathbb{F}$ (see [15, Theorem 4.1 and formula (3.2)]).

In the sequel, for every $\mathcal{F}_0$ measurable random variable $x \in [-c/2; c/2]$ we will construct an adapted process $X^{c,x}$, "$c/2$"-uniform approximation of X with locally finite variation such that its total variation does not exceed the lower bound (2.1) by c. More precisely, it will satisfy the following conditions

(A) $\|X - X^{c,x}\|_\infty \leq c/2$;

(B) $X_0^{c,x} = X_0 - x$;

(C) $X^{c,x}$ is of finite variation with càdlàg paths and is adapted to the filtration $\mathbb{F}$;

(D) for any $s \geq 0$, the jumps (if any) at time s of $X^{c,x}$ and X satisfy

$$\left|\Delta X_s^{c,x}\right| \leq \left|\Delta X_s\right|,$$

where $\Delta X_s^{c,x} = X_s^{c,x} - X_{s-}^{c,x}$ and $\Delta X_s = X_s - X_{s-}$;

(E) for any $T > 0$,

$$TV(X^{c,x}, T) \leq TV^c(X, T) + c;$$

(F) moreover, for any $T > 0$,

$$UTV\left(X^{c,x}, T\right) \leq UTV^c\left(X, T\right) + c,$$

and

$$DTV\left(X^{c,x}, T\right) \leq DTV^c\left(X, T\right) + c.$$

Remark 2.3. *It is easy to see that*

$$TV^c\left(X, T\right) \leq UTV^c\left(X, T\right) + DTV^c\left(X, T\right),$$

which follows directly from the equality

$$\max\left\{|X_t - X_s| - c, 0\right\} = \max\left\{X_t - X_s - c, 0\right\} + \max\left\{X_s - X_t - c, 0\right\}.$$

On the other hand, assuming that the process $X^{c,x}$ is constructed, $UTV(X^{c,x}, \cdot)$ and $DTV(X^{c,x}, \cdot)$ give the Jordan decomposition of $X^{c,x}$ and we have

$$TV(X^{c,x}, T) = UTV(X^{c,x}, T) + DTV(X^{c,x}, T).$$

From this and conditions (E), (2.2) and (2.3) we have

$$
\begin{aligned}
TV^c\left(X, T\right) &\geq TV(X^{c,x}, T) - c \\
&= UTV(X^{c,x}, T) + DTV(X^{c,x}, T) - c \\
&\geq UTV^c\left(X, T\right) + DTV^c\left(X, T\right) - c.
\end{aligned}
$$

To construct the appropriate process $X^{c,x}$ we will need a slight generalisation of the double Skorohod map on the interval $[-c/2; c/2]$ (cf. [11]) as well as some alternative formulas for it. Since the construction of this map for time-dependent boundaries (cf. [2]) is almost the same as for constant boundaries, we will present it in the time-dependent setting.

2.1 The Skorohod problem with time-dependent boundaries and with starting condition

Let $D[0; +\infty)$ denote the set of real-valued càdlàg functions. Let also $BV^+[0; +\infty)$, $BV[0; +\infty)$ denote subspaces of $D[0; +\infty)$ consisting of nondecreasing functions and functions of bounded variation, respectively. We have

Definition 2.4. *Let $\alpha, \beta \in D[0; +\infty)$ and $x \in \mathbb{R}$. A pair of functions $(\phi^x, \eta^x) \in D[0; +\infty) \times BV[0; +\infty)$ is said to be a solution of the Skorohod problem on $[\alpha; \beta]$ with starting condition $\phi^x(0) = x$ for ψ if the following conditions are satisfied:*

(a) *for every $t \geq 0$, $\phi^x(t) = \psi(t) + \eta^x(t) \in [\alpha(t); \beta(t)]$;*

(b) *$\eta^x = \eta_d^x - \eta_u^x$, where $\eta_d^x, \eta_u^x \in BV^+[0; +\infty)$ and the corresponding measures $\mathrm{d}\eta_d^x$, $\mathrm{d}\eta_u^x$ are carried by $\{t \geq 0 : \phi^x(t) = \alpha(t)\}$ and $\{t \geq 0 : \phi^x(t) = \beta(t)\}$ respectively;*

(c) *$\phi^x(0) = x$.*

The usual Skorohod problem is defined with similar conditions (a) and (b), for some càdlàg functions ϕ and η, as the Skorohod problem just defined with starting condition. The difference is such that in the former instead of condition (c) it is assumed $\eta(0-) = 0$, which determines the starting value of the function ϕ, $\phi(0)$, to equal $\max\{\alpha(0), \min\{\psi(0), \beta(0)\}\}$. It is also worth mentioning that the Skorohod problem with

starting condition is the same as what so called the *play operator*, encountered in mathematical models of hysteresis (cf. [4]).

The existence and uniqueness of the solution of the Skorohod problem with time-dependent boundaries and starting condition, for α, β and x such that

$$\varepsilon\left(\alpha,\beta\right) := \inf_{t\geq 0}\left[\beta(t) - \alpha(t)\right] > 0$$

and $x \in [\alpha(0); \beta(0)]$, follows easily from already known results (see the proof of Proposition 2.7). However, in the sequel we will need some formulas for the solution of this problem as well as some additional properties which are (as far as we know) not available in the literature. This is why we will present the solution of the problem in the setting suited to our purposes.

Assume that $\varepsilon\left(\alpha,\beta\right) > 0$ and $x \in [\alpha(0); \beta(0)]$. To solve the Skorohod problem on $[\alpha; \beta]$ with starting condition $\phi^x(0) = x$ let us define two times

$$\begin{aligned} T_u\psi &= \inf\left\{s \geq 0: \psi(s) - \psi(0) + x > \beta(s)\right\}, \\ T_d\psi &= \inf\left\{s \geq 0: \psi(s) - \psi(0) + x < \alpha(s)\right\}. \end{aligned}$$

Assume that $T_d\psi \geq T_u\psi$, i.e. the first instant when the function $\psi - \psi(0) + x$ hits the barrier β appears before the first instant when the function $\psi - \psi(0) + x$ hits the barrier α or both times are infinite (i.e. $\psi(t) - \psi(0) + x \in [\alpha(t); \beta(t)]$ for all $t \geq 0$). The case $T_d\psi < T_u\psi$ is symmetric.

Now we define sequences $(T_{d,k})_{k=-1}^{\infty}$, $(T_{u,k})_{k=0}^{\infty}$ in the following way: $T_{d,-1} = 0$, $T_{u,0} = T_u\psi$ and for $k = 0, 1, 2, ...$

$$T_{d,k} = \begin{cases} \inf\left\{s \geq T_{u,k}: \sup_{t\in[T_{u,k};s]}\left(\psi\left(t\right) - \beta\left(t\right)\right) > \psi\left(s\right) - \alpha\left(s\right)\right\} & \text{if } T_{u,k} < +\infty, \\ +\infty & \text{otherwise,} \end{cases}$$

$$T_{u,k+1} = \begin{cases} \inf\left\{s \geq T_{d,k}: \inf_{t\in[T_{d,k};s]}\left(\psi\left(t\right) - \alpha\left(t\right)\right) < \psi\left(s\right) - \beta\left(s\right)\right\} & \text{if } T_{d,k} < +\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Remark 2.5. *Note that since $\inf_{t\geq 0}\left[\beta(t) - \alpha(t)\right] > 0$ for any $s > 0$ there exists such $K < \infty$ that $T_{u,K} > s$ or $T_{d,K} > s$.*

Now we define the function ψ^x by the formulas

$$\psi^x\left(s\right) = \begin{cases} \psi\left(0\right) - x & \text{if } s \in [T_{d,-1} = 0; T_{u,0}); \\ \sup_{t\in[T_{u,k};s]}\left(\psi\left(t\right) - \beta\left(t\right)\right) & \text{if } s \in [T_{u,k}; T_{d,k}), k = 0, 1, 2, ...; \\ \inf_{t\in[T_{d,k};s]}\left(\psi\left(t\right) - \alpha\left(t\right)\right) & \text{if } s \in [T_{d,k}; T_{u,k+1}), k = 0, 1, 2, ... \end{cases} \tag{2.4}$$

Remark 2.6. *Note that due to Remark 2.5, s belongs to one of the intervals $[0; T_{u,0})$, $[T_{u,k}; T_{d,k})$ or $[T_{d,k}; T_{u,k+1})$ for some $k = 0, 1, 2, ...$ and the function ψ^x is defined for every $s \geq 0$.*

Now we are ready to prove that for $T_d\psi \geq T_u\psi$ the functions $\eta^x := -\psi^x$, $\phi^x := \psi - \psi^x$ solve the Skorohod problem on $[\alpha; \beta]$ with starting condition $\phi^x(0) = x$ for ψ. (The appropriate construction in the case $T_d\psi < T_u\psi$ is symmetric.)

We have

Proposition 2.7. *Let* $\alpha, \beta :\in D[0; +\infty)$ *be such that* $\varepsilon(\alpha, \beta) := \inf_{t \geq 0} [\beta(t) - \alpha(t)] > 0$ *and* $x \in [\alpha(0); \beta(0)]$ *then for every* $\psi \in D[0; +\infty)$ *there exists a unique solution* (ϕ^x, η^x) *of the Skorohod problem on* $[\alpha; \beta]$ *with starting condition* $\phi^x(0) = x$. *Moreover, for* $T_d \psi \geq T_u \psi$ *this solution is given by the functions* $\eta^x := -\psi^x$, $\phi^x := \psi - \psi^x$, *and it has the following property*

$$|\Delta \eta^x(s)| \leq |\Delta \psi(s)| + (\Delta \alpha(s))^+ \, \mathbb{I}_{\{\phi^x = \alpha\}} + (\Delta \beta(s))^- \, \mathbb{I}_{\{\phi^x = \beta\}}, \qquad (2.5)$$

where $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$.

Proof. It is easy to see that the solution of our Skorohod problem with starting condition coincides with the solution of the usual Skorohod problem with time-dependent boundaries for the function $\psi_x = \psi - \psi(0) + x$. Hence the existence and uniqueness follow easily from [2, Theorem 2.6, Proposition 2.3 and Corollary 2.4].

Now, to see that the solution is given by the functions $\eta^x := -\psi^x$, $\phi^x := \psi - \psi^x$ it is enough to check that they satisfy (a)-(b). It is a straightforward task to prove (a). To prove (b) we will show that $\mathrm{d}\eta_u^x$ is carried by $\{t \geq 0 : \phi^x(t) = \beta(t)\}$. In a similar way one also proves that the measure $\mathrm{d}\eta_d^x$ is carried by $\{t \geq 0 : \phi^x(t) = \alpha(t)\}$. By the formula (2.4) the function $\eta^x = -\psi^x$ is nonincreasing on the intervals $[T_{u,k}; T_{d,k})$ and nondecreasing on the intervals $[T_{d,k}; T_{u,k+1})$, $k = 0, 1, 2, \dots$. Thus one may define $\eta_d^x, \eta_u^x \in BV^+[0; +\infty)$ in such a way that $\mathrm{d}\eta^x(s) = \mathrm{d}\eta_d^x(s) = -\mathrm{d} \inf_{t \in [T_{d,k}; s]} (\psi - \alpha)(t)$ and $\mathrm{d}\eta^x(s) = -\mathrm{d}\eta_u^x(s) = -\mathrm{d} \sup_{t \in [T_{u,k}; s]} (\psi - \beta)(t)$ on the intervals $(T_{d,k}; T_{u,k+1})$ and $(T_{u,k}; T_{d,k})$, $k = 0, 1, 2, \dots$, respectively. Now, notice that the only points of increase of the measure $\mathrm{d}\eta_u^x$ from the intervals $(T_{u,k}; T_{d,k})$, $k = 0, 1, 2, \dots$ are the points where the function $\psi - \beta$ attains new suprema on these intervals. But in every such point s we have

$$\psi^x(s) = \sup_{t \in [T_{u,k}; s]} (\psi(t) - \beta(t)) = \psi(s) - \beta(s)$$

and hence $\phi^x(s) = \psi(s) - \psi^x(s) = \beta(s)$. Next, notice that at the point $s = T_{u,0}$ one has $\psi^x(s) = \psi(s) - \beta(s) \geq \psi(0) - x = \psi^x(s-)$, and since for $T_{u,k+1} < +\infty, k = 0, 1, \dots$, one has

$$T_{u,k+1} = \inf \left\{ s \geq T_{d,k} : \psi(s) - \alpha(s) - \inf_{t \in [T_{d,k}; s]} (\psi(t) - \alpha(t)) > \beta(s) - \alpha(s) \right\},$$

then for $s = T_{u,k+1} < +\infty, k = 0, 1, \dots$,

$$\inf_{t \in [T_{d,k}; s]} (\psi(t) - \alpha(t)) = \inf_{t \in [T_{d,k}; s)} (\psi(t) - \alpha(t))$$

and

$$\begin{aligned} \psi^x(s) &= \psi(s) - \beta(s) \geq \inf_{t \in [T_{d,k}; s]} (\psi(t) - \alpha(t)) \\ &= \inf_{t \in [T_{d,k}; s)} (\psi(t) - \alpha(t)) = \psi^x(s-). \end{aligned}$$

Thus, at the points $s = T_{u,k}, k = 0, 1, \dots$ we have $\mathrm{d}\eta_d^x = 0$, $\mathrm{d}\eta_u^x \geq 0$ and $\phi^x(s) = \beta(s)$.

In order to prove inequality (2.5) let us notice that from formula (2.4) it follows that for any $s \notin \{T_{u,k}; T_{d,k}\}, k = 0, 1, \dots$, (2.5) holds, hence let us assume that $s \in \{T_{u,k}; T_{d,k}\}$. We consider three possibilities.

- If $s = T_{u,0}$ then (as already mentioned) $-\Delta \eta^x(s) = \psi^x(s) - \psi^x(s-) \geq 0$ and, by the definition of $T_{u,0}$,

$$-\Delta \eta^x(s) = \psi^x(s) - \psi^x(s-) = \psi(s) - \beta(s) - \psi(0) + x \leq \psi(s) - \psi(s-).$$

- If $s = T_{u,k+1}, k = 0, 1, ...,$ then (as already mentioned) $-\Delta\eta^x = \psi^x(s) - \psi^x(s-) \geq 0$ and, by the definition of $T_{u,k+1}$,

$$-\Delta\eta^x(s) = \psi^x(s) - \psi^x(s-) = \psi(s) - \beta(s) - \inf_{t \in [T_{d,k};s)}(\psi(s) - \alpha(s))$$

$$\leq \psi(s) - \beta(s) - (\psi(s-) - \beta(s-))$$

$$\leq \psi(s) - \psi(s-) + (\beta(s) - \beta(s-))^-.$$

- If $s = T_{d,k}, k = 0, 1, ...,$ then

$$\Delta\eta^x(s) = \psi^x(s-) - \psi^x(s) = \sup_{t \in [T_{u,k};s]}(\psi(t) - \beta(t)) - (\psi(s) - \alpha(s)) \geq 0$$

and, by the definition of $T_{d,k}$,

$$\Delta\eta^x(s) = \psi^x(s-) - \psi^x(s) = \sup_{t \in [T_{u,k};s)}(\psi(t) - \beta(t)) - (\psi(s) - \alpha(s))$$

$$\leq (\psi(s-) - \alpha(s-)) - (\psi(s) - \alpha(s))$$

$$\leq \psi(s-) - \psi(s) + (\alpha(s) - \alpha(s-))^+.$$

$\square$

Remark 2.8. *It is possible to prove that the function ψ^x has the smallest total variation on the intervals $[0;T]$, $T \geq 0$, among all functions $\xi \in \mathcal{D}[0;+\infty)$ such that $\alpha \leq \psi - \xi \leq \beta$, $\xi(0) = \psi(0) - x$. This observation, for constant, symmetric boundaries was proved in [10, Chapt. II, Corollary 1.5] and in full generality in [6, Proposition 6 and Theorem 8], but we will not need this in the sequel.*

2.2 Regularization of the process X via Skorohod's map

Now, for $c > 0$ and $\mathcal{F}_0$-measurable random variable $x \in [-c/2; c/2]$ we define the regularization of X, $X^{c,x}$, satisfying conditions (A)-(F). We have

Proposition 2.9. *Fix $\omega \in \Omega$. For $\alpha \equiv -c/2$, $\beta \equiv c/2$, $x_0 = x(\omega)$ and $\psi = X(\omega)$ we solve the Skorohod problem on $[\alpha;\beta] \equiv [-c/2; c/2]$ with starting condition $\phi^{x_0}(0) = x_0$ and such that (2.5) holds. Let (ϕ^{x_0}, η^{x_0}) be the solution of this problem. Setting*

$$X^{c,x}(\omega) = \psi^{x_0} = -\eta^{x_0} = \psi - \phi^{x_0} \tag{2.6}$$

we obtain a process satisfying conditions (A)-(F).

Proof. By Proposition 2.7 we immediately get that $X^{c,x}$ satisfies conditions (A)-(D). To prove that it satisfies conditions (E) and (F) we assume (without loss of generality) that $T_u \leq T_d$ and consider four possibilities.

- $T \in [T_{d,-1} = 0; T_{u,0})$. In this case

$$TV(X^{c,x}, T) = UTV(X^{c,x}, T) = DTV(X^{c,x}, T) = 0.$$

- $T \in [T_{u,0}; T_{d,0})$. In this case

$$UTV(X^{c,x}, T) = \sup_{t \in [T_{u,0};T]} X_t - c/2 - X_0 + x, \ DTV(X^{c,x}, T) = 0,$$

and

$$TV(X^{c,x}, T) = UTV(X^{c,x}, T) + DTV(X^{c,x}, T).$$

Now, by the definition of UTV^c it is not difficult to see that

$$
UTV^c(X,T) \geq \max\left\{\sup_{t\in[T_{u,0};T]} X_t - X_0 - 3c/2 + x, 0\right\}
$$
$$
\geq UTV(X^{c,x},T) - c
$$

and

$$
DTV^c(X,T) \geq 0 = DTV(X^{c,x},T),
$$
$$
TV^c(X,T) \geq TV(X^{c,x},T) - c.
$$

- $T \in [T_{u,k}; T_{d,k})$, for some $k = 1, 2, \ldots$ Denote $M_i = \sup_{t\in[T_{u,i};T_{d,i})} X_t$ and $m_i = \inf_{t\in[T_{d,i};T_{u,i+1})} X_t$, $i = 0, 1, \ldots,$ (times $T_{d,i}$, $T_{u,i}$ are now stopping times, defined for every path separately). Using monotonicity of $X^{c,x}$ on the intervals $[T_{u,i}; T_{d,i}]$ and $[T_{d,i}; T_{u,i+1}]$, and formula (2.4) we calculate

$$
UTV(X^{c,x},T) = (M_0 - c/2 - X_0 + x) + \sum_{i=1}^{k-1}(M_i - m_{i-1} - c)
$$
$$
+ \sup_{t\in[T_{u,k};T]} X_t - m_{k-1} - c
$$

and

$$
DTV(X^{c,x},T) = \sum_{i=0}^{k-1}(M_i - m_i - c),
$$
$$
TV(X^{c,x},T) = UTV(X^{c,x},T) + DTV(X^{c,x},T).
$$

Now it is not difficult to see that

$$
UTV^c(X,T) \geq \max\{M_0 - X_0 - 3c/2 + x, 0\} + \sum_{i=1}^{k-1}(M_i - m_{i-1} - c)
$$
$$
+ \sup_{t\in[T_{u,k};T]} X_t - m_{k-1} - c \geq UTV(X^{c,x},T) - c
$$

and

$$
DTV^c(X,T) \geq \sum_{i=0}^{k-1}(M_i - m_i - c) = DTV(X^{c,x},T),
$$
$$
TV^c(X,T) \geq TV(X^{c,x},T) - c.
$$

- $T \in [T_{d,k}; T_{u,k+1})$, for some $k = 0, 1, 2, \ldots$ The proof follows similarly as in the previous case.

$\square$

3 Interval down- and upcrossings of the process X and level crossings by the regularization $X^{c,x}$

Now for a càdlàg process $X_t, t \geq 0$, (not necessarily starting at 0) and $c > 0$ let us consider the number of *downcrossings of X from above the level c to the level 0 before time T*. We define it in the following way

Definition 3.1. *For $c > 0$ set $\sigma_0^c = 0$ and for $n = 0, 1, \ldots$*

$$
\tau_n^c = \inf\{t > \sigma_n^c : X_t > c\}, \; \sigma_{n+1}^c = \inf\{t > \tau_n^c : X_t \leq 0\}.
$$

The number of downcrossings of X from above the level c to the level 0 before time T *is defined as*

$$
d_c(X,T) = \max\{n : \sigma_n^c \leq T\}.
$$

We will prove that it is almost the same as the *number of crossings the level $c/2$ from above on the interval $[0; T]$ by the regularization* $X^{c,x}$. Here, for a càdlàg process $Y_t, t \geq 0$, we define the number of crossings the level $c/2$ from above on the interval $[0; T]$ in the following way.

Definition 3.2. *For $c > 0$ let $u_0^c = 0$ and for $n = 0, 1, \ldots$ we set*

$$v_n^c = \inf \left\{ t > u_n^c : Y_t > c/2 \right\}, \ u_{n+1}^c = \inf \left\{ t > v_n^c : Y_t \leq c/2 \right\}.$$

We define the number of crossings the level $c/2$ from above on the interval $[0; T]$ by Y as

$$e_c (Y, T) = \max \left\{ n : u_n^c \leq T \right\}.$$

Now we have

Lemma 3.3. *Let $X_t, t \geq 0$, be a càdlàg process adapted to the filtration $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $c > 0$ and a $\mathcal{F}_0$-measurable random variable $x \in [-c/2; c/2]$ consider the regularization $X_t^{c,x}, t \geq 0$, defined in Proposition 2.9. For any $T > 0$ we have*

$$d_c (X, T) \leq e_c (X^{c,x}, T) \leq d_c (X, T) + 1.$$

Moreover, if $x \equiv c/2$ we get exact equality, i.e.

$$d_c (X, T) = e_c (X^{c,x}, T).$$

Proof. We will use the stopping times introduced in Definition 3.1 and Definition 3.2. To prove that $d_c (X, T) \leq e_c (X^{c,x}, T)$ it is enough to notice that for any n such that $\sigma_n^c \leq T$ one easily finds such $v \in \left[\tau_{n-1}^c; \sigma_n^c \right]$ that $X_v > c$. Hence, from $\|X - X^{c,x}\|_\infty \leq c/2$, we get $X_v^{c,x} > c/2$. On the other hand, again by $\|X - X^{c,x}\|_\infty \leq c/2$ and by $X_{\sigma_n^c} \leq 0$, we get $X_{\sigma_n^c}^{c,x} \leq c/2$. Thus, on the interval $\left[\tau_{n-1}^c; \sigma_n^c \right]$ we have at least one crossing the level $c/2$ from above by $X^{c,x}$ and we obtain $d_c (X, T) \leq e_c (X^{c,x}, T)$.

To prove the upper bound we notice that the process $X^{c,x}$ does not change its value as long as $|X_t - X_t^{c,x}| \leq c/2$. More precisely, if $X_t^{c,x} = y$ then $X_s^{c,x} \leq y$ for $t \leq s \leq \inf \{u > t : X_u - y > c/2\}$; similarly, if $X_t^{c,x} = y$ then $X_s^{c,x} \geq y$ for $t \leq s \leq \inf \{u > t : X_u - y < -c/2\}$. By the construction of $X^{c,x}$, for every ω the interval $[0; T]$ may be split into a finite sum of disjoint intervals, such that on each of them $X^{c,x}(\omega)$ is monotonic. Thus $e_c (X^{c,x}, T)$ is a.s. finite. If $e_c (X^{c,x}, T) \in \{0, 1\}$ the inequality $e_c (X^{c,x}, T) \leq d_c (X, T) + 1$ is obvious. Hence assume that $e_c (X^{c,x}, T) \geq 2$ and for some $n = 2, 3, \ldots$ consider such v that $u_{n-1}^c \leq v < u_n^c \leq T$ and $X_v^{c,x} > c/2$. Consider $\overline{X} = \sup_{u_{n-1}^c \leq s \leq u_n^c} X_s$. If $\overline{X} \leq c$, we would have $X_s^{c,x} \leq c/2$ for all $s \in \left[u_{n-1}^c; u_n^c \right]$. Hence, there exists some $w \in \left[u_{n-1}^c; u_n^c \right]$, such that $X_w > c$. Similarly, consider $\underline{X} = \inf_{w \leq s \leq u_n^c} X_s$. If $\underline{X} > 0$, we would have $X_s^{c,x} > c/2$ for all $s \in [w; u_n^c]$. Thus, on the interval $\left[u_{n-1}^c; u_n^c \right]$ we have at least one downcrossing of X from above the level c to the level 0 before time T and we obtain $d_c (X, T) \geq e_c (X^{c,x}, T) - 1$.

To prove the exact equality $d_c (X, T) = e_c (X^{c,x}, T)$ when $x \equiv c/2$ it is enough to see that if $e_c (X^{c,x}, T) \geq 1$ and we consider such v that $0 \leq v < u_1^c \leq T$ and $X_v^{c,x} > c/2$ then $\overline{X} = \sup_{0 \leq s \leq u_1^c} X_s > c$. If $\overline{X} \leq c$, then by condition (B) we would have $X_0^{c,x} \leq c/2$ and thus $X_s^{c,x} \leq c/2$ for all $s \in [0; u_1^c]$. $\qquad\square$

Similarly we consider *upcrossings from below the level $-c$ to the level 0*, $u_c (X, T)$, and *crossings the level $-c/2$ from below*, $g_c (Y, T)$. Note, that their numbers may be easily calculated as numbers of downcrossings or crossings from above, respectively, of the processes $-X, -Y$. Naturally, we have

Lemma 3.4. *Let c, x, X, and $X^{c,x}$ be as in Lemma 3.3. For any $T > 0$ we have*

$$u_c(X, T) \leq g_c(X^{c,x}, T) \leq u_c(X, T) + 1.$$

Moreover, if $x \equiv -c/2$ we get exact equality, i.e.

$$u_c(X, T) = g_c(X^{c,x}, T).$$

4 Limit theorems for truncated variations and interval down- and upcrossings of continuous semimartingales and diffusions

4.1 Strong laws of large numbers for $c \cdot TV^c(X, \cdot)$, $c \cdot UTV^c(X, \cdot)$ and $c \cdot DTV^c(X, \cdot)$

In this subsection we will assume that $X_t, t \geq 0$, is a continuous semimartingale (not necessarily starting from 0). Notice that for $T > 0$ and any $a \in \mathbb{R}$, $u_c(X - a - c, T)$ is equal to the number of times that X upcrosses from below the level a to the level $a + c$ before time T. Assume moreover that the bicontinuous version of the local time L of X exists. By [12, page 18, Theorem II.2.4] we have that for $0 \leq t \leq T$,

$$c \cdot u_c(X - a - c, t) \to \frac{1}{2} L_t^a \text{ a.s.} \tag{4.1}$$

uniformly in t and $a \in \mathbb{R}$ as $c \downarrow 0$.

Remark 4.1. *A small problem we encounter is that the quantity $N^+(0, t, x, x + \varepsilon)$, appearing in [12, Theorem II.2.4] denotes the number of upcrossings (see [12, page 7]) not the number of upcrossings from below, and it may be strictly greater than $u_\varepsilon(X - x - \varepsilon, t)$; but we may always calculate e.g. $N^+(0, t, a - c^2, a + c])$ which is no greater than $u_c(X - a - c, t)$ and for which convergence in [12, Theorem II.2.4] still holds.*

By (4.1), by the continuity of X and by the occupation times formula (cf. [19, Corollary VI.1.6]) we have that

$$\int_{\mathbb{R}} c \cdot u_c(X - a - c, \cdot) \, \mathrm{d}a \to \frac{1}{2} \int_{\mathbb{R}} L^a \, \mathrm{d}a = \frac{1}{2} \langle X \rangle \text{ a.s.} \tag{4.2}$$

as $c \downarrow 0$, in $C([0; T], \mathbb{R})$.

Now let us consider $X^{c, -c/2}$, i.e. regularization of X defined in Proposition 2.9 with $x \equiv -c/2$. Notice that by condition (D), $X^{c, -c/2}$ is also continuous and that for $T > 0$ and any $a \in \mathbb{R}$, $g_c(X^{c, -c/2} - a - c/2, T)$ is equal to the number of times that $X^{c, -c/2}$ crosses the level a from below on the interval $[0; T]$. By the extended version of the Banach-Vitali Indicatrix Theorem (cf. [3, page 328], see also Remark 4.10) for $t > 0$ we have

$$
\begin{aligned}
c \cdot UTV\left(X^{c, -c/2}, t\right) &= \int_{\mathbb{R}} c \cdot g_c\left(X^{c, -c/2} - a - c/2, t\right) \mathrm{d}a \\
&= \int_{\mathbb{R}} c \cdot g_c\left(X^{c, -c/2} - a, t\right) \mathrm{d}a.
\end{aligned}
\tag{4.3}
$$

Now, by Lemma 3.4 we have

$$\int_{\mathbb{R}} c \cdot g_c\left(X^{c, -c/2} - a, t\right) \mathrm{d}a = \int_{\mathbb{R}} c \cdot u_c(X - a, t) \, \mathrm{d}a$$

and from this and (4.2), (4.3) we obtain that

$$c \cdot UTV\left(X^{c, -c/2}, \cdot\right) \to \frac{1}{2} \langle X \rangle \text{ a.s.}$$

as $c \downarrow 0$ in $C([0; T], \mathbb{R})$. Finally, by $0 \leq UTV\left(X^{c, -c/2}, t\right) - UTV^c(X, t) \leq c$, (cf. (2.3) and condition (F)) and the analogous reasoning for crossings from above (analog of (4.2), the Banach-Vitali Indicatrix Theorem and Lemma 3.3), we get

Theorem 4.2. *For a continuous semimartingale X_t, $t \geq 0$, such that the bicontinuous version of its local time exists, and $T > 0$*

$$c \cdot UTV^c(X, \cdot) \to \frac{1}{2} \langle X \rangle \text{ a.s.} \tag{4.4}$$

as $c \downarrow 0$ in $C([0;T], \mathbb{R})$. Similar convergences hold for DTV^c and TV^c, i.e.

$$c \cdot DTV^c(X, \cdot) \to \frac{1}{2} \langle X \rangle \text{ and } c \cdot TV^c(X, \cdot) \to \langle X \rangle \text{ a.s.} \tag{4.5}$$

as $c \downarrow 0$ in $C([0;T], \mathbb{R})$.

Thus, we have obtained an alternative proof of [16, Theorem 1], but using a stronger condition on X - that the bicontinuous version of its local time exists. But we have

Remark 4.3. *A careful examination of the proof of [12, Theorem II.2.4] gives for any $T > 0$ a uniform bound in $t \in [0;T]$ and $a \in \mathbb{R}$ for the difference*

$$\left| c \cdot u_c(X - a - c, t) - \frac{1}{2} L_t^a \right|$$

for any continuous semimartingale X (the second estimate on page 20 in [12]). Thus, applying [12, Theorem III.3.3(a)] and the Lebesgue dominated convergence we get that (4.2) and hence Theorem 4.2 hold for any continuous semimartingale X.

Corollary 4.4. *Let X, Y be two continuous semimartingales. For $T > 0$*

$$c \cdot \{TV^c(X + Y, \cdot) - TV^c(X - Y, \cdot)\} \to 4 \langle X, Y \rangle \text{ a.s.} \tag{4.6}$$

as $c \downarrow 0$ in $C([0;T], \mathbb{R})$.

4.2 The local limit theorem - generalisation of Kasahara's result on CLT for number of interval crossings

Let $T > 0$ be given and fixed. In this subsection we will work with a continuous semimartingale X satisfying the following conditions.

(i) There exists a probability measure $\mathbb{Q}$, under which X is a local martingale;

(ii) the measure $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$.

The Girsanov theorem for unbounded drifts (cf. [5, Theorem 1]) provides examples of processes satisfying (i)-(ii). Namely, consider the following s.d.e. driven by a standard Brownian motion W,

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \ X_0 = x_0, \tag{4.7}$$

where $x_0 \in \mathbb{R}$, $\mu : [0; +\infty) \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0; +\infty) \times \mathbb{R} \to \mathbb{R}$ are measurable and locally Lipschitz with respect to x. Moreover, assume that (4.7) has a strong solution and σ is separated from 0, i.e. there exists $\varepsilon > 0$ such that $\sigma \geq \varepsilon$. By [5, Theorem 1], the strong solution of (4.7) satisfies (i)-(ii).

Now we will prove a generalisation of the main result of [8], namely we have the following.

Theorem 4.5. *Fix $T > 0$. Let X be as above and $d_c(X, t)$ and $u_c(X, t)$, $t \in [0;T]$, be numbers of times that the process X downcrosses from above and upcrosses from*

below the intervals $[0; c]$ and $[-c; 0]$ till time t respectively. Moreover, let L be the local time of X at 0. We have that

$$\frac{1}{\sqrt{c}} \left(c \cdot d_c \left(X, \cdot \right) - \frac{L}{2}, c \cdot u_c \left(X, \cdot \right) - \frac{L}{2} \right) \Rightarrow \left(B^1_{L/2}, B^2_{L/2} \right), \tag{4.8}$$

where B^1, B^2, are independent standard Brownian motions, which are also independent from X.

Remark 4.6. *In view of Theorem 4.5, it seems that in [19, Chapt. XIII, Exercise (2.13)] γ_l should be replaced by $\gamma_{l/2}$.*

The immediate consequence of the obtained result is the following generalisation of the main result of [8].

Corollary 4.7. *Let $n_c \left(X, t \right) = d_c \left(X, t \right) + u_c \left(X, t \right)$ be the number of downcrossings the interval $[0; c]$ by the process $|X|$ till time t. Then*

$$\frac{1}{\sqrt{c}} \left(c \cdot n_c \left(X, \cdot \right) - L \right) \Rightarrow B_L,$$

where B is a standard Brownian motion, independent from X.

Proof. (Of Theorem 4.5) Let us recall the definition of the sequences of stopping times corresponding to downcrossings from above the interval $[0; c]$, σ_n^c, τ_n^c, $n = 0, 1, \dots$. Similarly, let us define the sequence of stopping times corresponding to upcrossings from below of the interval $[-c; 0] : \tilde{\sigma}_0^c = 0$ and for $n = 0, 1, \dots$,

$$\tilde{\tau}_n^c = \inf \left\{ t > \tilde{\sigma}_n^c : X_t < -c \right\}, \tilde{\sigma}_{n+1}^c = \inf \left\{ t > \tilde{\tau}_n^c : X_t \geq 0 \right\}.$$

Now the beginning of the proof goes along the same lines as the proof of [19, Proposition VI.1.10]. For simplicity, we will write only $\sigma_n, \tau_n, \tilde{\sigma}_n, \tilde{\tau}_n, \dots$ instead of $\sigma_n^c, \tau_n^c, \tilde{\sigma}_n^c, \tilde{\tau}_n^c$. By Tanaka's formula, for $n = 1, 2, \dots$

$$X^+_{\tau_n \wedge t} - X^+_{\sigma_n \wedge t} = \int_{(\sigma_n; \tau_n]} \mathbb{I}_{(0; c]} \left(X_s \right) \mathrm{d} X_s + \frac{1}{2} \left(L_{\tau_n \wedge t} - L_{\sigma_n \wedge t} \right)$$

and because X does not vanish on $[\tau_n; \sigma_{n+1})$, $L_{\sigma_{n+1} \wedge t} = L_{\tau_n \wedge t}$. As a result

$$\int_0^t \eta_s^c \mathrm{d} X_s = c \cdot d_c \left(X, t \right) - \frac{1}{2} L_t + r_1 \left(c, t \right)$$

and similarly

$$\int_0^t \tilde{\eta}_s^c \mathrm{d} X_s = c \cdot u_c \left(X, t \right) - \frac{1}{2} L_t + r_2 \left(c, t \right),$$

where η and $\tilde{\eta}$ are two predictable processes

$$\eta_s^c := \sum_{n=1}^{\infty} \mathbb{I}_{(\sigma_n; \tau_n]} \left(s \right) \mathbb{I}_{(0; c]} \left(X_s \right), \quad \tilde{\eta}_s^c := - \sum_{n=1}^{\infty} \mathbb{I}_{(\tilde{\sigma}_n; \tilde{\tau}_n]} \left(s \right) \mathbb{I}_{[-c; 0)} \left(X_s \right) \tag{4.9}$$

and random variables $r_1(c, t)$, $r_2(c, t)$ belong to the interval $[0; c]$.

We define

$$K_t^c = \frac{1}{\sqrt{c}} \int_0^t \eta_s^c \mathrm{d} X_s, \quad \tilde{K}_t^c = \frac{1}{\sqrt{c}} \int_0^t \tilde{\eta}_s^c \mathrm{d} X_s.$$

With this notation we have

$$\frac{1}{\sqrt{c}} \left(c \cdot d_c \left(X, t \right) - \frac{1}{2} L_t \right) = K_t^c - \frac{r_1 \left(c, t \right)}{\sqrt{c}} \tag{4.10}$$

and

$$\frac{1}{\sqrt{c}}\left(c \cdot u_c\left(X,t\right) - \frac{1}{2}L_t\right) = \tilde{K}_t^c - \frac{r_2\left(c,t\right)}{\sqrt{c}}.$$

By the usual localization argument, we may assume that there exists some $M > 0$ such that $\mathbb{Q}$ a.s. $\sup_{t\in[0;T]}|X_t| < M$.

For $0 \leq s < t$, by the Burkholder inequality, the definition of K^c and occupation times formula, for some constant A_1

$$\mathbb{E}_\mathbb{Q}\left(K_t^c - K_s^c\right)^2 \leq A_1 \cdot \mathbb{E}_\mathbb{Q}\left(\langle K^c\rangle_t - \langle K^c\rangle_s\right)$$

$$= A_1 \cdot \mathbb{E}_\mathbb{Q}\left(\frac{1}{c}\int_s^t (\eta_u^c)^2\, \mathrm{d}\langle X\rangle_u\right)$$

$$\leq A_1 \cdot \mathbb{E}_\mathbb{Q}\left(\frac{1}{c}\int_s^t \mathbb{I}_{(0;c]}\left(X\right)\mathrm{d}\langle X\rangle_u\right)$$

$$= A_1 \cdot \mathbb{E}_\mathbb{Q}\left(\frac{1}{c}\int_0^c \left(L_t^x(X) - L_s^x(X)\right)\mathrm{d}x\right)$$

$$\leq A_1 \cdot \mathbb{E}_\mathbb{Q}\sup_{x\in\mathbb{R}}\left(L_t^x(X) - L_s^x(X)\right). \tag{4.11}$$

Now, since X is a local martingale under $\mathbb{Q}$, by the Barlow-Yor inequality [19, Theorem XI.2.4], for some universal constant A_2 we have

$$\mathbb{E}_\mathbb{Q}\sup_{x\in\mathbb{R}}\left(L_t^x(X) - L_s^x(X)\right) \leq A_2 \cdot \mathbb{E}_\mathbb{Q}\sup_{u\in[s;t]}\left(X_u - X_s\right). \tag{4.12}$$

Now, combining (4.11) and (4.12), for $A = A_1 \cdot A_2$ we get

$$\mathbb{E}_\mathbb{Q}\left(K_t^c - K_s^c\right)^2 \leq A \cdot \mathbb{E}_\mathbb{Q}\sup_{u\in[s;t]}\left(X_u - X_s\right). \tag{4.13}$$

Similarly,

$$\mathbb{E}_\mathbb{Q}\left(\tilde{K}_t^c - \tilde{K}_s^c\right)^2 \leq A \cdot \mathbb{E}_\mathbb{Q}\sup_{u\in[s;t]}\left(X_u - X_s\right). \tag{4.14}$$

By Tanaka's formula applied to the function $x \mapsto \left(x^+\right)^2$,

$$c^2 = \left(X_{\tau_n}^+\right)^2 - \left(X_{\sigma_n}^+\right)^2 = 2\int_{\sigma_n}^{\tau_n} X_s^+\mathrm{d}X_s + \int_{\sigma_n}^{\tau_n}\mathbb{I}_{(0;+\infty)}\left(X_s\right)\mathrm{d}\langle X\rangle_s$$

$$= 2\int_{\sigma_n}^{\sigma_{n+1}} X_s\eta_s^c\mathrm{d}X_s + \int_{\sigma_n}^{\sigma_{n+1}}(\eta_s^c)^2\,\mathrm{d}\langle X\rangle_s.$$

Hence, for $t \geq 0$,

$$\left|c^2 d_c\left(X,t\right) - 2\int_0^t X_s\eta_s^c\mathrm{d}X_s - \int_0^t (\eta_s^c)^2\,\mathrm{d}\langle X\rangle_s\right| \leq c^2. \tag{4.15}$$

Now, using (4.10), (4.15) and several times estimate $(a+b)^2 \leq 2a^2 + 2b^2$ we get

$$\mathbb{E}_\mathbb{Q}\left(\frac{1}{2}L_t - \langle K^c\rangle_t\right)^2 = \mathbb{E}_\mathbb{Q}\left(\frac{1}{2}L_t - \frac{1}{c}\int_0^t (\eta_s^c)^2\,\mathrm{d}\langle X\rangle_s\right)^2$$

$$\leq 8c^2 + 4c \cdot \mathbb{E}_\mathbb{Q}\left(K_t^c\right)^2 + 16\mathbb{E}_\mathbb{Q}\left(\frac{1}{c}\int_0^t X_s\eta_s^c\mathrm{d}X_s\right)^2. \tag{4.16}$$

Using the Burkholder inequality and $|X_s\eta_s^c| \leq c\,|\eta_s^c|$ we estimate the last two terms in (4.16) similarly as $\mathbb{E}_\mathbb{Q}\left(K_t^c - K_s^c\right)^2$ in (4.11), and by (4.12) we get that for some universal constant D,

$$\mathbb{E}_\mathbb{Q}\left(\frac{1}{2}L_t - \langle K^c\rangle_t\right)^2 \leq D\left(c^2 + c \cdot \mathbb{E}_\mathbb{Q}\sup_{s\in[0;t]}|X_s|\right). \tag{4.17}$$

Similarly,

$$\mathbb{E}_{\mathbb{Q}} \left(\frac{1}{2} L_t - \left\langle \tilde{K}^c \right\rangle_t \right)^2 \leq D \left(c^2 + c \cdot \mathbb{E}_{\mathbb{Q}} \sup_{s \in [0;t]} |X_s| \right). \qquad (4.18)$$

Since $dK^c = (\sqrt{c})^{-1} \mathbb{I}_{(0;c]}(X) dX$ on intervals $(\sigma_n; \tau_n)$, $n = 1, 2, \ldots$, and $dK^c = 0$ otherwise (except maybe $\tau_n, \sigma_n, n = 0, 1, \ldots$), we have $\mathbb{Q}$ a.s. $\langle K^c, X \rangle_t \leq \sqrt{c} \langle K^c \rangle_t$ and using (4.11), (4.12) we estimate

$$\mathbb{E}_{\mathbb{Q}} \langle K^c, X \rangle_t \leq \sqrt{c} \cdot \mathbb{E}_{\mathbb{Q}} \langle K^c \rangle_t \leq 2A\sqrt{c} \cdot \mathbb{E}_{\mathbb{Q}} \sup_{s \in [0;t]} |X_s|. \qquad (4.19)$$

Similarly,

$$\mathbb{E}_{\mathbb{Q}} \left\langle \tilde{K}^c, X \right\rangle_t \leq \sqrt{c} \cdot \mathbb{E}_{\mathbb{Q}} \langle \tilde{K}^c \rangle_t \leq 2A\sqrt{c} \cdot \mathbb{E}_{\mathbb{Q}} \sup_{s \in [0;t]} |X_s|. \qquad (4.20)$$

Finally, let us notice that by the definition of continuous processes K^c and $\tilde{K}^c$, they have disjoint intervals where they are non-constant, hence

$$\left\langle K^c, \tilde{K}^c \right\rangle \equiv 0. \qquad (4.21)$$

Now we are ready to prove the convergence result. Following [8], let $\mathbb{Q}^c$ denote the probability measure on $C\left([0;T], \mathbb{R}^4\right)$ induced by $\left(X, K^c, \tilde{K}^c, L \right)$ under the measure $\mathbb{Q}$. By (4.13)-(4.14) and Chebyschev's inequality, with the aid of Aldous' criterion we have that the family of measures $\mathbb{Q}^c$, $c \in (0;1)$, is weakly relatively compact (cf. [7, Theorem VI.4.5]) and notice that by the Lebesgue dominated convergence theorem and continuity of X we have

$$\lim_{\theta \downarrow 0} \mathbb{E}_{\mathbb{Q}} \sup_{0 \leq s \leq u \leq s + \theta \leq T} (X_u - X_s) = 0).$$

Let $\mathbb{Q}^*$ be any limit of $\mathbb{Q}^c$, $c \in (0;1)$, and let $\left(x, k, \tilde{k}, l \right)$ denote the coordinate process in $\left(C\left([0;T], \mathbb{R}^4\right), \mathbb{Q}^*, \mathcal{G}_t, t \geq 0 \right)$ where $\mathcal{G}_t, t \geq 0$, is the natural increasing family of $\sigma-$fields. Now x, k and $\tilde{k}$ are continuous $\mathbb{Q}^*$ martingales with respect to $\mathcal{G}_t, t \geq 0$ (recall that by localization $\sup_{t \in [0;T]} |X_t| < M$, $\mathbb{Q}$ a.s., thus the martingale property follows from the weak convergence). By (4.17)-(4.21) we get that for all $t \in [0;T]$

$$\langle k \rangle_t = \left\langle \tilde{k} \right\rangle_t = l_t/2, \langle x, k \rangle_t = \left\langle x, \tilde{k} \right\rangle_t = \left\langle k, \tilde{k} \right\rangle_t = 0 \quad \mathbb{Q}^* \text{ a.s.}$$

Therefore, by the Knight representation theorem for continuous local martingales we have that for a two-dimensional standard Brownian motion $\left(B^1, B^2 \right)$, independent from X,

$$\left(x, k, \tilde{k}, l \right) =^d \left(X, B^1_{L/2}, B^2_{L/2}, L \right),$$

where "$=^d$" denotes the equality in distributions (in $C\left([0;T], \mathbb{R}^4\right)$). Notice that the assumption in the Knight therorem that $\mathbb{Q}$ a.s. $\langle K^c \rangle \wedge \langle \tilde{K}^c \rangle \to +\infty$ may be omitted, since we consider the Brownian motion $\left(B^1, B^2 \right)$ on the interval $[0; L_T/2]$ only (see remark below [19, Theorem V.1.9]). Since $\mathbb{Q}^*$ is unique we get the desired weak convergence.

The stable convergence follows e.g. from the fact that the inequalities (4.17)-(4.21), (4.13)-(4.14), have their counterparts when we restrict to any subset $F \in \mathcal{F}$ with $\mathbb{Q}(F) > 0$, for example

$$\mathbb{E}_{\mathbb{Q}} \left[\left(\frac{1}{2} L_t - \langle K^c \rangle_t \right)^2 | F \right] \leq \frac{D \left(c^2 + c \cdot \mathbb{E}_{\mathbb{Q}} \sup_{s \in [0;t]} |X_s| \right)}{\mathbb{Q}(F)},$$

and one may again apply the Knight theorem to obtain the desired weak convergence on F. Now the stable convergence follows from [7, Sect. VIII, Proposition 5.33,(iv)].

To obtain the stable convergence under the measure $\mathbb{P}$ one may notice that for any $\varepsilon \in (0; 1)$ there exists $F_\varepsilon \in \mathcal{F}$ with $\mathbb{P}(F_\varepsilon) > 1 - \varepsilon$ such that for some $M_\varepsilon < +\infty$, $d\mathbb{P}/d\mathbb{Q} < M_\varepsilon$ and $\mathbb{P}$ a.s. $\sup_{0 \leq t \leq T} |X_s| \leq M_\varepsilon$ on F_ε. Now notice that the inequalities (4.17)-(4.21), (4.13)-(4.14) have their counterparts under measure $\mathbb{P}$ when we restrict to the subset F_ε, for example

$$
\begin{aligned}
\mathbb{E}_{\mathbb{P}} \left[\left(\frac{1}{2} L_t - \langle K^c \rangle_t \right)^2 ; F_\varepsilon \right] &\leq M_\varepsilon \cdot \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{1}{2} L_t - \langle K^c \rangle_t \right)^2 ; F_\varepsilon \right] \\
&\leq M_\varepsilon \cdot D \left(c^2 + c \cdot \mathbb{E}_{\mathbb{Q}} \sup_{s \in [0;t]} |X_s| \right).
\end{aligned}
$$

$\square$

4.3 Integral limit theorem - CLT for integrated number of interval crossings

In this subsection we will assume that X is the unique strong solution of the following s.d.e., driven by a standard Brownian motion W,

$$
dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \; X_0 = x_0, \tag{4.22}
$$

with Lipschitz μ, σ and $\sigma > 0$. For X satisfying (4.22) we have a more precise result than Theorem 4.2, namely (cf. [16, Theorem 5])

$$
\begin{aligned}
&\left(X, UTV^c(X, \cdot) - \frac{\langle X \rangle}{2c}, DTV^c(X, \cdot) - \frac{\langle X \rangle}{2c}, TV^c(X, \cdot) - \frac{\langle X \rangle}{c} \right) \\
&\Rightarrow \left(X, \frac{1}{2} \left(\frac{1}{\sqrt{3}} B_{\langle X \rangle} + X - x_0 \right), \frac{1}{2} \left(\frac{1}{\sqrt{3}} B_{\langle X \rangle} - X + x_0 \right), \frac{1}{\sqrt{3}} B_{\langle X \rangle} \right),
\end{aligned} \tag{4.23}
$$

where B is another standard Brownian motion, independent from W.

Remark 4.8. *The just cited Theorem 5 from [16] describes convergence of a different vector than the vector appearing in (4.23) and only for diffusions starting from 0, but (4.23) follows simply from the Mapping Theorem (cf. [1, Sect. 2]) and from the fact that we may set an arbitrary starting value for X, x_0, and then consider the diffusion $X - x_0$, which has the same values of UTV^c, DTV^c, TV^c and $\langle \cdot \rangle$ as X. The stable convergence follows from [16, Remark 6], see also [7, Chapt. VIII, Proposition 5.33(ii),(iii)].*

From (4.23) we shall obtain a convergence result concerning the integrated number of down(up-)crossings.

For a càdlàg $Y_t, t \geq 0$, let us denote

$$
d_c^a(Y, t) = d_c(Y - a, t), \; u_c^a(Y, t) = u_c(Y - a - c, t)
$$

and

$$
n_c^a(Y, t) = d_c^a(Y, t) + u_c^a(Y, t),
$$

i.e. $d_c^a(Y, t)$ is the number of downcrossings by X the interval $[a; a+c]$ from above before time t, $u_c^a(Y, t)$ is the number of upcrossings by X the interval $[a; a+c]$ from below before time t and $n_c^a(Y, t)$ is the number of crossings by X the interval $[a; a + c]$ from below or above before time t.

We have

Theorem 4.9. *Fix $T > 0$. For X satisfying (4.22) we have*

$$\left(X, \frac{1}{c} \left\{ c \int_{\mathbb{R}} u_c^a\left(X, \cdot\right) \mathrm{d}a - \frac{\langle X \rangle}{2} \right\}, \frac{1}{c} \left\{ c \int_{\mathbb{R}} d_c^a\left(X, \cdot\right) \mathrm{d}a - \frac{\langle X \rangle}{2} \right\}, \right.$$
$$\left. \frac{1}{c} \left\{ c \int_{\mathbb{R}} n_c^a\left(X, \cdot\right) \mathrm{d}a - \langle X \rangle \right\} \right)$$
$$\Rightarrow \left(X, \frac{1}{2} \left(\frac{1}{\sqrt{3}} B_{\langle X \rangle} + X - x_0 \right), \frac{1}{2} \left(\frac{1}{\sqrt{3}} B_{\langle X \rangle} - X + x_0 \right), \frac{1}{\sqrt{3}} B_{\langle X \rangle} \right), \qquad (4.24)$$

where B is another standard Brownian motion, independent from W.

Proof. By Lemma 3.3 and the extended Banach-Vitali Indicatrix Theorem ([3, page 328]) we have

$$\int_{\mathbb{R}} d_c^a\left(X, t\right) \mathrm{d}a = \int_{\mathbb{R}} e_c\left(X^{c,c/2} - a, t\right) \mathrm{d}a$$
$$= DTV\left(X^{c,c/2}, t\right) = DTV^c\left(X, t\right) + r_d(c, t), \qquad (4.25)$$

where by (2.3) and condition (F), the random variable $r_d(c, t)$ defined by equality (4.25) belongs to $[0; c]$. Similarly,

$$\int_{\mathbb{R}} u_c^a\left(X, t\right) \mathrm{d}a = \int_{\mathbb{R}} g_c\left(X^{c,-c/2} - a - c, t\right) \mathrm{d}a$$
$$= UTV\left(X^{c,-c/2}, t\right) = UTV^c\left(X, t\right) + r_u(c, t), \qquad (4.26)$$

where $r_u(c, t) \in [0; c]$. Finally, by the just obtained equalities (4.25), (4.26) and by Remark 2.3,

$$\int_{\mathbb{R}} n_c^a\left(X, t\right) \mathrm{d}a = UTV^c\left(X, t\right) + DTV^c\left(X, t\right) + r_u(c, t) + r_d(c, t)$$
$$= TV^c\left(X, t\right) + r_n(c, t), \qquad (4.27)$$

where $r_n(c, t) \in [0; 3c]$.

Now, by (4.26), (4.25), (4.27) and (4.23) we get (4.24). $\qquad \square$

It is interesting to compare the formulas just obtained for the integrated number of crossings with that obtained in the Subsection 4.2 generalisation of Kasahara's result. By the occupation times formula

$$\frac{1}{c} \left\{ c \int_{\mathbb{R}} d_c^a\left(X, \cdot\right) \mathrm{d}a - \frac{\langle X \rangle}{2} \right\} = \frac{1}{c} \int_{\mathbb{R}} \left\{ c \cdot d_c\left(X - a, \cdot\right) - \frac{L^a}{2} \right\} \mathrm{d}a$$
$$\Rightarrow \frac{1}{2\sqrt{3}} B_{\langle X \rangle} - \frac{1}{2}\left(X - x_0\right).$$

Thus we get that the integrated with respect to a process $d_c^a\left(X, \cdot\right) - L^a/(2c)$ reveals much stronger concentration than the same process for a given a. Moreover, for numbers of (up-)crossings we get

$$\frac{1}{c} \int_{\mathbb{R}} \left\{ c \cdot u_c^a\left(X, \cdot\right) - \frac{L^a}{2} \right\} \mathrm{d}a \Rightarrow \frac{1}{2\sqrt{3}} B_{\langle X \rangle} + \frac{1}{2}\left(X - x_0\right),$$
$$\frac{1}{c} \int_{\mathbb{R}} \left\{ c \cdot n_c^a\left(X, \cdot\right) - L^a \right\} \mathrm{d}a \Rightarrow \frac{1}{\sqrt{3}} B_{\langle X \rangle}.$$

4.4 Functional integral limit theorems

In this subsection we assume that X is the same process as in Subsection 4.3. Let us fix $T > 0$ and let $f \in C^2$, where C^2 is the class of functions with continuous second derivative. The aim of this subsection is to obtain a "functional" limit theorem of the form

$$\frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, n_c^a(X, \cdot)\, \mathrm{d}a - \int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\} \Rightarrow \frac{1}{\sqrt{3}}\int_0^{\cdot} f(X_s)\, dB_{\langle X\rangle_s}. \qquad (4.28)$$

This is a considerable generalisation of the theorem obtained in the previous section and this result can not be obtained (as far as we know) by a straightforward application of [16, Theorem 5], like in the previous subsection. The convergence result (4.28) and its relevant version for the quadruple

$$\mathbf{Q}_1^c = \left(X, \frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, u_c^a(X, \cdot)\, \mathrm{d}a - \frac{1}{2}\int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\},\right.$$
$$\frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, d_c^a(X, \cdot)\, \mathrm{d}a - \frac{1}{2}\int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\},$$
$$\left.\frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, n_c^a(X, \cdot)\, \mathrm{d}a - \int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\}\right)$$

will be obtained via establishing the convergence result for the quadruple of integrals involving level crossings of the regularised processes $X^{c,x}$, as in Proposition 2.9, where for each $c > 0$, $x = x(c)$ is a $\mathcal{F}_0$-measurable random variable such that $|x| \leq c/2$.

$$\mathbf{Q}_2^c = \left(X, \frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, U^a(X^{c,x}, \cdot)\, \mathrm{d}a - \frac{1}{2}\int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\},\right.$$
$$\frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, D^a(X^{c,x}, \cdot)\, \mathrm{d}a - \frac{1}{2}\int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\},$$
$$\left.\frac{1}{c}\left\{c\int_{\mathbb{R}} f(a)\, N^a(X^{c,x}, \cdot)\, \mathrm{d}a - \int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X\rangle_s\right\}\right). \qquad (4.29)$$

Numbers $U^a(X^{c,x}, t)$, $D^a(X^{c,x}, t)$ and $N^a(X^{c,x}, t)$ in (4.29) denote the number of level crossings and for any càdlàg process $Y_t, t \geq 0$, are defined as

$$U^a(Y, t) = g_c(Y - a - c/2, t),\ D^a(Y, t) = e_c(Y - a + c/2, t),$$
$$N^a(Y, t) = U^a(Y, t) + D^a(Y, t),$$

i.e. $U^a(Y, t)$ is the number of times that the process Y crosses the level a from below and $D^a(Y, t)$ is the number of times that the process Y crosses the level a from above the interval $[0; t]$.

The main tool we will use will be the "functional" version of the Banach-Vitali Indicatrix Theorem, from which follows that for any continuous function $f : \mathbb{R} \to \mathbb{R}$ and $t > 0$ we have

$$\int_0^t f(X_s^{c,x})\, |\mathrm{d}X_s^{c,x}| := \int_0^t f(X_s^{c,x})\, \mathrm{d}TV(X^{c,x}, s)$$
$$= \int_{\mathbb{R}} f(a)\, N^a(X^{c,x}, t)\, \mathrm{d}a. \qquad (4.30)$$

Further, we also have two other equalities analogous to (4.30):

$$\int_{\mathbb{R}} f(a)\, U^a(X^{c,x}, t)\, \mathrm{d}a = \int_0^t f(X_s^{c,x})\, \mathrm{d}UTV(X^{c,x}, s), \qquad (4.31)$$

$$\int_{\mathbb{R}} f(a)\, D^a(X^{c,x}, t)\, \mathrm{d}a = \int_0^t f(X_s^{c,x})\, \mathrm{d}DTV(X^{c,x}, s).$$

Remark 4.10. *Since these results are not easily found in the literature, for the reader's convenience we present a simple proof of (4.31) and (4.30).*

Proof. Let us notice that for every $\omega \in \Omega$ the continuous function $\psi := X^{c,x}(\omega)$ is monotonic on disjoint intervals $[T_{d,k}; T_{u,k+1})$, $[T_{u,k+1}; T_{d,k+1})$, $k = -1, 0, 1, \ldots$. On every interval $[T_{u,k}; T_{d,k})$, $k = 0, 1, \ldots$, ψ is no-decreasing and

$$U^a\left(\psi, T_{d,k} \wedge t\right) - U^a\left(\psi, T_{u,k} \wedge t\right) = \begin{cases} 1 & \text{if } a \in \left(\psi\left(T_{u,k} \wedge t\right); \psi\left(T_{d,k} \wedge t\right)\right], \\ 0 & \text{otherwise.} \end{cases}$$

From this we have

$$\int_{\mathbb{R}} f\left(a\right) U^a\left(\psi, T_{d,k} \wedge t\right) \mathrm{d}a - \int_{\mathbb{R}} f\left(a\right) U^a\left(\psi, T_{u,k} \wedge t\right) \mathrm{d}a$$
$$= \int_{\psi\left(T_{u,k} \wedge t\right)}^{\psi\left(T_{d,k} \wedge t\right)} f\left(a\right) \mathrm{d}a = \int_{T_{u,k} \wedge t}^{T_{d,k} \wedge t} f\left(\psi\left(s\right)\right) \mathrm{d}UTV\left(\psi, s\right). \tag{4.32}$$

Moreover,

$$\int_{\mathbb{R}} f\left(a\right) U^a\left(\psi, T_{u,k+1} \wedge t\right) \mathrm{d}a - \int_{\mathbb{R}} f\left(a\right) U^a\left(\psi, T_{d,k} \wedge t\right) \mathrm{d}a$$
$$= 0 = \int_{T_{d,k} \wedge t}^{T_{u,k+1} \wedge t} f\left(\psi\left(s\right)\right) \mathrm{d}UTV\left(\psi, s\right). \tag{4.33}$$

By (4.32) and (4.33) we get (4.31). Similar arguments work for DTV. Finally, (4.30) follows from the fact that $N^a = U^a + D^a$ and $\mathrm{d}TV = \mathrm{d}UTV + \mathrm{d}DTV$. $\qquad\square$

Now let $X^{c,x}, c > 0$, be a sequence of regularizations of X as in Proposition 2.9 with $x = x(c)$ such that $|x| \le c/2$. We have

Theorem 4.11. *For any $f \in C^2$ the following convergence holds*

$$\mathbf{Q}_2^c \Rightarrow \left(X, \frac{1}{2\sqrt{3}} \int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X \rangle_s} + \frac{1}{2} \int_0^{\cdot} f\left(X_s\right) \circ \mathrm{d}X_s, \right.$$
$$\frac{1}{2\sqrt{3}} \int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X \rangle_s} - \frac{1}{2} \int_0^{\cdot} f\left(X_s\right) \circ \mathrm{d}X_s$$
$$\left. \frac{1}{\sqrt{3}} \int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X \rangle_s} \right),$$

where $\int_0^{\cdot} f\left(X_s\right) \circ \mathrm{d}X_s$ denotes the Stratonovich integral.

The analog of Theorem 4.11 for $\mathbf{Q}_1^c$ is

Theorem 4.12. *The same convergence as for the quadruple $\mathbf{Q}_2^c$ holds for the quadruple $\mathbf{Q}_1^c$.*

First, we will prove the following two-dimensional version of Theorem 4.11.

Theorem 4.13. *For any $f \in C^2$ the following convergence holds*

$$\left(X, \frac{1}{c} \left\{ c \int_{\mathbb{R}} f\left(a\right) N^a\left(X^{c,x}, \cdot\right) \mathrm{d}a - \int_0^{\cdot} f\left(X_s\right) \mathrm{d}\langle X \rangle_s \right\} \right)$$
$$\Rightarrow \left(X, \frac{1}{\sqrt{3}} \int_0^{\cdot} f\left(X_s\right) dB_{\langle X \rangle_s} \right). \tag{4.34}$$

Proof. (of Theorem 4.13.) We start with the following properties of the Skorohod map, which are simply the translation of condition (b) in the definition of the Skorohod problem in Subsection 2.1:

$$\text{if } \mathrm{d}UTV\left(X^{c,x}, \cdot\right) > 0 \text{ then } X^{c,x} - X = -c/2 \text{ and } \mathrm{d}DTV\left(X^{c,x}, \cdot\right) = 0, \tag{4.35}$$

and

$$\text{if } \mathrm{d}DTV\left(X^{c,x}, \cdot\right) > 0 \text{ then } X^{c,x} - X = c/2 \text{ and } \mathrm{d}UTV\left(X^{c,x}, \cdot\right) = 0. \tag{4.36}$$

Moreover,

$$\mathrm{d}TV\left(X^{c,x}, \cdot\right) = \mathrm{d}UTV\left(X^{c,x}, \cdot\right) + \mathrm{d}DTV\left(X^{c,x}, \cdot\right) \tag{4.37}$$

and

$$\mathrm{d}X^{c,x} = \mathrm{d}UTV\left(X^{c,x}, \cdot\right) - \mathrm{d}DTV\left(X^{c,x}, \cdot\right). \tag{4.38}$$

By the usual localization argument we may assume that

$$\sup_{t\in[0;T]} |X_t| \le M; \tag{4.39}$$

notice that it implies that

$$\langle X \rangle_T = \int_0^T \sigma^2\left(X_s\right) \mathrm{d}s \le T \sup_{x\in[-M;M]} \sigma^2\left(x\right). \tag{4.40}$$

The total variation of the drift part of X is globally bounded:

$$TV\left(\int_0^\cdot \mu\left(X_s\right)\mathrm{d}s, T\right) \le \int_0^T |\mu\left(X_s\right)|\, ds \le T \sup_{x\in[-M;M]} |\mu\left(x\right)|, \tag{4.41}$$

and, since f'' is continuous,

$$R := \sup_{x\in[-M-1;M+1]} |f''(x)| < +\infty. \tag{4.42}$$

Now let us consider the difference

$$\int_{\mathbb{R}} f\left(a\right) N^a\left(X^{c,x}, t\right) \mathrm{d}a - \frac{1}{c}\int_0^t f\left(X_s\right) \mathrm{d}\left\langle X\right\rangle_s.$$

By (4.30) it may be split into two parts:

$$S_1^c\left(t\right) \quad =: \quad \int_0^t f\left(X_s^{c,x}\right) \mathrm{d}TV\left(X^{c,x}, s\right) - \int_0^t f\left(X_s\right) \mathrm{d}TV\left(X^{c,x}, s\right)$$

and

$$S_2^c\left(t\right) = \int_0^t f\left(X_s\right) \mathrm{d}TV\left(X^{c,x}, s\right) - \frac{1}{c}\int_0^t f\left(X_s\right) \mathrm{d}\left\langle X\right\rangle_s.$$

Step 1. **Proof that $S_1^c \to 0$ a.s. as $c \downarrow 0$ in** $C\left([0;T], \mathbb{R}\right)$. By Taylor's formula, for any real x and δ,

$$f\left(x+\delta\right) - f\left(x\right) = f'\left(x\right)\delta + r\left(x,\delta\right)\delta^2, \tag{4.43}$$

where the remainder r, defined for $\delta \neq 0$ by the equality (4.43), may be represented as

$$r(x,\delta) = \frac{1}{2}f''(\theta) \tag{4.44}$$

for some $\theta \in (x; x+\delta)$. For $\delta = 0$ we may simply set $r(x,0) = \frac{1}{2}f''(x)$.

Now we fix positive integer K and split the interval $[-M; M]$ into finite sum of intervals

$$I_i = \left[-M + 2M \frac{i-1}{K}; -M + 2M \frac{i}{K} \right], \quad i = 1, 2, ..., K.$$

Let us consider the stopping times: $v_0 = 0$ and

$$v_k = \inf \left\{ t > v_{k-1} : X_t \in I_i \text{ for some } i = 1, 2, ..., K \text{ such that } X_{v_{k-1}} \notin I_i \right\}. \qquad (4.45)$$

To simplify the notation we will omit x in the superscript $X^{c,x}$. By (4.43) we get

$$
\begin{aligned}
S_1^c(t) &= \int_0^t \left(f(X_s^c) - f(X_s) \right) \mathrm{d}TV(X^c, s) \\
&= \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} f'(X_s)(X_s^c - X_s) \mathrm{d}TV(X^c, s) \\
&\quad + \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} r(X_s, X_s^c - X_s)(X_s^c - X_s)^2 \mathrm{d}TV(X^c, s), \\
&= \sum_{k=0}^{\infty} f'(X_{v_k}) \int_{v_k \wedge t}^{v_{k+1} \wedge t} (X_s^c - X_s) \mathrm{d}TV(X^c, s) \qquad (4.46) \\
&\quad + \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \{ f'(X_s) - f'(X_{v_k}) \}(X_s^c - X_s) \mathrm{d}TV(X^c, s) \\
&\quad + \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} r(X_s, X_s^c - X_s)(X_s^c - X_s)^2 \mathrm{d}TV(X^c, s).
\end{aligned}
$$

To bound the first term in (4.46), notice that by (4.35)-(4.38)

$$
\begin{aligned}
&\int_{v_k \wedge t}^{v_{k+1} \wedge t} (X_s^c - X_s) \mathrm{d}TV(X^c, s) \\
&= \frac{c}{2} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \mathrm{d}DTV(X^c, s) - \frac{c}{2} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \mathrm{d}UTV(X^c, s) \\
&= \frac{c}{2} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \mathrm{d} \left(DTV(X^c, s) - UTV(X^c, s) \right) \\
&= -\frac{c}{2} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \mathrm{d}X_s^c = -\frac{c}{2} \left(X_{v_{k+1} \wedge t}^c - X_{v_k \wedge t}^c \right). \qquad (4.47)
\end{aligned}
$$

A small difficulty may arise from the fact that the number of summands in (4.46) may be arbitrary large, but since X is a.s. continuous, we may always restrict to a subset of $\Theta \subset \Omega$ with probability arbitrary close to 1 and such that for $X \in \Theta$ the number of summands in (4.46) is bounded by some fixed integer. Let us also notice that the absolute value of $X_{v_{k+1} \wedge T}^c - X_{v_k \wedge T}^c$ is bounded by $(2M/K + c)$ and the absolute value of $f'(X_{v_k})$ is bounded by $\sup_{x \in [-M; M]} |f'(x)|$. By these observations and (4.47)

$$
\begin{aligned}
&\sum_{k=0}^{\infty} f'(X_{v_k}) \int_{v_k \wedge t}^{v_{k+1} \wedge t} (X_s^c - X_s) \mathrm{d}TV(X^c, s) \\
&= -\frac{c}{2} \sum_{k=0}^{\infty} f'(X_{v_k}) \left(X_{v_{k+1} \wedge t}^c - X_{v_k \wedge t}^c \right) \to 0 \text{ a.s.} \qquad (4.48)
\end{aligned}
$$

as $c \downarrow 0$ in $C([0; T], \mathbb{R})$.

To bound the second term in (4.46) notice that by the mean value theorem, for $v_k \leq s \leq v_{k+1} \wedge T$

$$f'(X_s) - f'(X_{v_k}) = (X_s - X_{v_k}) f''(Z),$$

where Z is a random number belonging to the interval $[X_s \wedge X_{v_k}; X_s \vee X_{v_k}]$. Hence, by (4.39), (4.42) and the definition of stopping times v_k, $k = 0, 1, \ldots$, for $v_k \leq s \leq v_{k+1} \wedge T$ we get $|f'(X_s) - f'(X_{v_k})| \leq 2MR/K$. From this observation, the inequality $|X_s^c - X_s| \leq c/2$ and condition (E) we have

$$\sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} |f'(X_s) - f'(X_{v_k})| \, |X_s^c - X_s| \, \mathrm{d}TV(X^c, s)$$

$$\leq \frac{2M \cdot R}{K} \frac{c}{2} \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \mathrm{d}TV(X^c, s)$$

$$= \frac{M \cdot R}{K} c \cdot TV(X^c, t)$$

$$\leq \frac{M \cdot R}{K} c \cdot (TV^c(X, t) + c) \to \frac{M \cdot R}{K} \langle X \rangle_t \quad \text{a.s. as} \quad c \downarrow 0. \tag{4.49}$$

Thus, for any $\varepsilon > 0$, by the choice of sufficiently large K, the absolute value of the second term in (4.46) may be bounded by ε as $c \downarrow 0$.

To prove the convergence of the third term in (4.46), notice that for $v_k \leq s \leq v_{k+1} \wedge T$ and $c \leq 2$

$$|r(X_s, X_s^c - X_s)| \leq \frac{R}{2}.$$

This follows from (4.44) and (4.42). Now we estimate

$$\sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \left| r(X_s, X_s^c - X_s)(X_s^c - X_s)^2 \right| \mathrm{d}TV(X^c, s)$$

$$\leq \frac{R}{2} \frac{c^2}{4} \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \mathrm{d}TV(X^c, s) = \frac{R}{8} c^2 TV(X^c, t)$$

$$\leq \frac{R}{8} c^2 TV^c(X, t) + \frac{R}{8} c^3. \tag{4.50}$$

Hence, by (4.48), (4.49) and (4.50) we get $S_1^c \to 0$ a.s. as $c \downarrow 0$ in $C([0; T], \mathbb{R})$.

Step 2. Convergence in probability of S_2^c. By integration by parts formula (recall that $TV(X^c, [0; t]) - \langle X \rangle_t / c$ is a process with locally finite total variation), we have

$$S_2^c(t) = \int_0^t f(X_s) \, \mathrm{d}\left\{ TV(X^c, s) - \frac{\langle X \rangle_s}{c} \right\} \tag{4.51}$$

$$= f(X_t) \left\{ TV(X^c, t) - \frac{\langle X \rangle_t}{c} \right\} - \int_0^t \left\{ TV(X^c, s) - \frac{\langle X \rangle_s}{c} \right\} \mathrm{d}f(X_s).$$

By (4.35)-(4.38) we have

$$TV(X^c, t) = \frac{2}{c} \int_0^t (X_s - X_s^c) \, \mathrm{d}X_s^c \tag{4.52}$$

and, since X^c is continuous and has locally finite total variation,

$$\langle X \rangle_t = \langle X - X^c \rangle_t = (X_t - X_t^c)^2 - (X_0 - X_0^c)^2 - 2 \int_0^t (X_s - X_s^c) \, \mathrm{d}(X_s - X_s^c). \tag{4.53}$$

Now we have

$$
\begin{aligned}
TV\left(X^c, t\right) - \frac{\langle X \rangle_t}{c} &= \frac{2}{c} \int_0^t \left(X_s - X_s^c\right) \mathrm{d} X_s^c + \frac{2}{c} \int_0^t \left(X_s - X_s^c\right) \mathrm{d} \left(X_s - X_s^c\right) - r_t \\
&= \frac{2}{c} \int_0^t \left(X_s - X_s^c\right) \mathrm{d} X_s - r_t,
\end{aligned}
\tag{4.54}
$$

where

$$
r_t = \frac{1}{c} \left\{ \left(X_t - X_t^c\right)^2 - \left(X_0 - X_0^c\right)^2 \right\} \in \left[-\frac{c}{4}; \frac{c}{4} \right].
$$

Thus, in view of (4.51) (and (4.39) - (4.41)), it is enough to consider

$$
S_3^c\left(t\right) := f\left(X_t\right) \left\{ \frac{2}{c} \int_0^t \left(X_s - X_s^c\right) \mathrm{d} X_s \right\} - \int_0^t \left\{ \frac{2}{c} \int_0^s \left(X_u - X_u^c\right) \mathrm{d} X_u \right\} \mathrm{d} f\left(X_s\right).
$$

Again, by integration by parts

$$
S_3^c\left(t\right) = \int_0^t f\left(X_s\right) \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d} X_s + \left\langle \int_0^{\cdot} \mathrm{d} f\left(X_s\right), \int_0^{\cdot} \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d} X_s \right\rangle_t.
$$

By Itô's formula, $\mathrm{d} f\left(X_s\right) = \frac{1}{2} f''\left(X_s\right) \mathrm{d} \langle X \rangle_s + f'\left(X_s\right) \mathrm{d} X_s$, and

$$
\left\langle \int_0^{\cdot} \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d} X_s, \int_0^{\cdot} \mathrm{d} f\left(X_s\right) \right\rangle_t = \int_0^t \frac{2}{c} \left(X_s - X_s^c\right) f'\left(X_s\right) \mathrm{d} \langle X \rangle_s.
$$

Finally,

$$
S_3^c\left(t\right) = \int_0^t f\left(X_s\right) \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d} X_s + \int_0^t \frac{2}{c} \left(X_s - X_s^c\right) f'\left(X_s\right) \mathrm{d} \langle X \rangle_s.
\tag{4.55}
$$

We prove that the second term in (4.55), i.e.

$$
\int_0^t \frac{2}{c} \left(X_s - X_s^c\right) f'\left(X_s\right) \mathrm{d} \langle X \rangle_s
$$

vanishes as $c \to 0$.

First, using the stopping times υ_k, $k = 0, 1, \ldots$, defined in Step 1, we decompose

$$
\begin{aligned}
&\int_0^t \frac{2}{c} \left(X_s - X_s^c\right) f'\left(X_s\right) \mathrm{d} \langle X \rangle_s \\
&= \sum_{k=0}^{\infty} \int_{\upsilon_k \wedge t}^{\upsilon_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c\right) f'\left(X_s\right) \mathrm{d} \langle X \rangle_s \\
&= \sum_{k=0}^{\infty} f'\left(X_{\upsilon_k \wedge t}\right) \int_{\upsilon_k \wedge t}^{\upsilon_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d} \langle X \rangle_s \\
&\quad + \sum_{k=0}^{\infty} \int_{\upsilon_k \wedge t}^{\upsilon_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c\right) \left\{ f'\left(X_s\right) - f'\left(X_{\upsilon_k \wedge t}\right) \right\} \mathrm{d} \langle X \rangle_s.
\end{aligned}
\tag{4.56}
$$

To bound the second term in (4.56) we use the mean value theorem and similarly as in Step 1, for $\upsilon_k \leq s < \upsilon_{k+1} \wedge t$ obtain

$$
\left| f'\left(X_s\right) - f'\left(X_{\upsilon_k \wedge t}\right) \right| \leq 2 \frac{M \cdot R}{K}.
$$

Hence

$$\left| \sum_{k=0}^{\infty} \int_{v_k \wedge t}^{v_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c \right) \left\{ f'\left(X_s \right) - f'\left(X_{v_k \wedge t} \right) \right\} \mathrm{d} \left\langle X \right\rangle_s \right|$$

$$\leq \int_0^t \frac{2}{c} \left| X_s - X_s^c \right| 2 \frac{M \cdot R}{K} \mathrm{d} \left\langle X \right\rangle_s$$

$$\leq 2 \frac{M \cdot R}{K} \left\langle X \right\rangle_t . \tag{4.57}$$

Now we will prove that for every $k = 0, 1, \ldots,$

$$\int_{v_k \wedge t}^{v_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c \right) \mathrm{d} \left\langle X \right\rangle_s \to 0 \text{ a.s.}$$

as $c \downarrow 0$. First, let us introduce the "natural clock" for the process X, i.e. we define

$$\gamma(t) := \inf \left\{ s : \left\langle X \right\rangle_s > t \right\}.$$

By this time-change we obtain the process $\tilde{X}_t := X_{\gamma(t)}$ satisfying the following s.d.e. (cf. [17, Theorem 8.5.7]):

$$\mathrm{d} \tilde{X}_t = \frac{\mu\left(\tilde{X}_t \right)}{\sigma^2\left(\tilde{X}_t \right)} \mathrm{d}t + \mathrm{d} \tilde{W}_t, \tag{4.58}$$

where $\tilde{W}$ is another Brownian motion.

Remark 4.14. *Due to localisation this s.d.e. might be satisfied only on the time interval* $[0; \tilde{\tau}_M]$, *where* $\tilde{\tau}_M := U \wedge \inf \left\{ t : \left| \tilde{X}_t \right| \geq M \right\}$ *and*

$$U := T \sup_{x \in [-M; M]} \sigma^2(x),$$

but this is sufficient for our purposes. (By (4.40), U is sufficiently large for the inequality $\left\langle X \right\rangle_T \leq U$ *to hold on* $\sup_{0 \leq t \leq T} \left| X_t \right| < M$.)

Defining $\tilde{X}_t^c := X_{\gamma(t)}^c$ and stopping times $\tilde{v}_k$, $k = 0, 1, \ldots$ analogous to the times v_k, $k = 0, 1, \ldots$, i.e. $\tilde{v}_k = 0$ and for $k = 1, 2, \ldots,$

$$\tilde{v}_k = \inf \left\{ t > \tilde{v}_{k-1} : \tilde{X}_t \in I_i \text{ for some } i = 1, 2, \ldots, K \text{ such that } \tilde{X}_{\tilde{v}_{k-1}} \notin I_i \right\},$$

we obtain the equality

$$\int_{v_k \wedge t}^{v_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c \right) \mathrm{d} \left\langle X \right\rangle_s \quad = \quad \int_{\tilde{v}_k \wedge \left\langle X \right\rangle_t}^{\tilde{v}_{k+1} \wedge \left\langle X \right\rangle_t} \frac{2}{c} \left(\tilde{X}_s - \tilde{X}_s^c \right) \mathrm{d}s. \tag{4.59}$$

Moreover, we may assume that the process $\tilde{X}_t - x_0$ is a standard Brownian motion on the set $\sup_{0 \leq t \leq T} \left| \tilde{X}_t \right| < M$ under some measure $\mathbb{Q}$, such that the measure $\mathbb{P}$ is absolute continuous with respect to $\mathbb{Q}$.

More precisely, take a standard Brownian motion B on some filtered probability space $\left(\tilde{\Omega}, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, \tilde{\mathbb{Q}} \right)$ with probability measure $\tilde{\mathbb{Q}}$, such that usual conditions hold, define $v(M) = \inf \left\{ t \geq 0 : \left| B_t + x_0 \right| \geq M \right\}$ and the measure $\tilde{\mathbb{P}}$, absolutely continuous with respect to $\tilde{\mathbb{Q}}$, with the Radon-Nikodym derivative

$$\frac{\mathrm{d} \tilde{\mathbb{P}}}{\mathrm{d} \tilde{\mathbb{Q}}} = \exp \left(\int_0^{U \wedge v(M)} \frac{\mu\left(B_t \right)}{\sigma^2\left(B_t \right)} \mathrm{d}B_t - \frac{1}{2} \int_0^{U \wedge v(M)} \frac{\mu^2\left(B_t \right)}{\sigma^4\left(B_t \right)} \mathrm{d}t \right).$$

By Girsanov's theorem we obtain that

$$B - \int_0^\cdot \frac{\mu(B_s)}{\sigma^2(B_s)} \mathrm{d}s$$

is a standard Brownian motion under $\tilde{\mathbb{P}}$ on the interval $[0; U \wedge v(M)]$. More precisely, by appropriate enlargement of filtration (cf. [19, Theorem V.1.7]) we may assume that there exists a Brownian motion $\tilde{B}$ under $\tilde{\mathbb{P}}$ such that B satisfies the equation

$$\mathrm{d}B_t = \frac{\mu(B_t)}{\sigma^2(B_t)} \mathrm{d}t + \mathrm{d}\tilde{B}_t, \tag{4.60}$$

on the interval $[0; U \wedge v(M)]$. It is easy to see that for the solutions of (4.60) uniqueness in law holds on the interval $[0; U \wedge v(M)]$. Thus, by (4.58) and (4.60)

$$\left\{ \left(\tilde{X}_t - x_0, 0 \le t \le U \wedge \tilde{\tau}_M \right) \text{ under } \mathbb{P} \right\} =^d \left\{ (B_t, 0 \le t \le U \wedge v(M)) \text{ under } \tilde{\mathbb{P}} \right\}. \tag{4.61}$$

Now we will prove that for $k = 0, 1, \ldots,$

$$\int_{w_k}^{w_{k+1}} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s \to^{\tilde{\mathbb{Q}}} 0 \tag{4.62}$$

as $c \downarrow 0$, where $\to^{\tilde{\mathbb{Q}}}$ denotes convergence in probability with respect to the measure $\tilde{\mathbb{Q}}$, $B^c = B^{c,x}$ (with $x = x(c) \in [-c/2; c/2]$ being a $\mathcal{G}_0$-measurable random variable) is the regularization of B, defined similarly as the regularization $X^{c,x}$ in Proposition 2.9; $w_0 = 0$ and for $k = 1, 2, \ldots,$

$$w_k = \inf \left\{ t > v_{k-1} : B_t + x_0 \in I_i \text{ for some } i = 1, \ldots, K \text{ such that } B_{w_{k-1}} + x_0 \notin I_i \right\}.$$

For $\psi = B$ define stopping times $U_{u,k}^c$ and $U_{d,k}^c$ analogously as stopping times $T_{u,k}$ and $T_{d,k}$ in Subsection 2.1, with $\alpha \equiv -c/2$, $\beta \equiv c/2$. It is easy to see that by symmetry of standard Brownian, for $l = 1, 2, \ldots,$ under the measure $\tilde{\mathbb{Q}}$,

$$\int_{[U_{u,l}^c; U_{d,l}^c]} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s =^d - \int_{[U_{d,l}^c; U_{u,l+1}^c]} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s. \tag{4.63}$$

Moreover, by scaling properties of standard Brownian motion, for $l = 1, 2, \ldots,$ under the measure $\tilde{\mathbb{Q}}$,

$$\int_{[U_{u,l}^c; U_{d,l}^c]} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s =^d c^2 \int_{[U_{u,l}^1; U_{d,l}^1]} 2 (B_s - B_s^1) \, \mathrm{d}s, \tag{4.64}$$

$$[U_{u,l}^c; U_{d,l}^c] =^d c^2 [U_{u,l}^1; U_{d,l}^1] \tag{4.65}$$

and similar equalities hold for intervals $\left[U_{d,l}^c; U_{u,l+1}^c \right]$. The integral

$$\int_{w_k}^{v_{w+1}} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s,$$

$k = 0, 1, \ldots,$ consists, except maybe two marginal integrals $\int_{w_k}^{U_{u,l_0^c+1}^c}$ and $\int_{U_{u,l_1^c}^c}^{w_{k+1}}$, where l_0^c and l_1^c are such that $w_k \in \left[U_{u,l_0^c}^c; U_{u,l_0^c+1}^c \right]$, $w_{k+1} \in \left[U_{u,l_1^c}^c; U_{u,l_1^c+1}^c \right]$, of the sum of the integrals of the form

$$\int_{\left[U_{u,l}^c; U_{d,l}^c \right]} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s + \int_{\left[U_{d,l}^c; U_{u,l+1}^c \right]} \frac{2}{c} (B_s - B_s^c) \, \mathrm{d}s.$$

Define $U_t^{c,k}$, $t \geq 0$, in the following way: $U_t^{c,k} = 0$ for $t \leq U_{u,l_0^c+1}^c$ and for $t > U_{u,l_0^c+1}^c$

$$U_t^{c,k} = \int_{\left[U_{u,l_0^c+1}^c; U_{u,l_1^c(t)}^c\right]} \frac{2}{c} \left(B_s - B_s^c\right) \mathrm{d}s,$$

where $l_1^c(t) := \sup\left\{l : U_{u,l}^c \leq t\right\} \wedge (l_1^c + 1)$. By (4.63) and renewal structure of $B_s - B_s^c$, $U^{c,k}$ is a martingale under $\tilde{\mathbb{Q}}$ (for $l = 1, 2, \ldots$ consecutive integrals

$$\int_{\left[U_{u,l}^c; U_{u,l+1}^c\right]} \frac{2}{c} \left(B_s - B_s^c\right) \mathrm{d}s,$$

are independent). By (4.65), similar equalities for intervals $\left[U_{d,l}^c; U_{u,l+1}^c\right]$, independence of consecutive intervals $\left[U_{u,l}^c; U_{u,l+1}^c\right]$, $l = 1, 2, \ldots$, and Wald's identity we have

$$\lim_{c \downarrow 0} c^2 \mathbb{E}_{\tilde{\mathbb{Q}}} \left(l_1^c + 1\right) = \frac{\mathbb{E}_{\tilde{\mathbb{Q}}} w_{k+1}}{\mathbb{E}_{\tilde{\mathbb{Q}}} \left(U_{u,2}^1 - U_{u,1}^1\right)} \tag{4.66}$$

(notice that for fixed $k = 0, 1, \ldots$, $\mathbb{E} w_{k+1} < +\infty$). Now, denoting by $\left[U^{c,k}\right]$ the quadratic variation of $U^{c,k}$, by (4.66) and (4.64) we have that

$$\mathbb{E}_{\tilde{\mathbb{Q}}} \left[U^{c,k}\right]_\infty = \mathbb{E}_{\tilde{\mathbb{Q}}} \left(l_1^c - l_0^c\right) \mathbb{E}_{\tilde{\mathbb{Q}}} \left(\int_{\left[U_{u,1}^c; U_{u,2}^c\right]} \frac{2}{c} \left(B_s - B_s^c\right) \mathrm{d}s\right)^2$$

is of order c^2. Now, to get (4.62) it is enough to notice that $U_\infty^{c,k}$ differs from the integral $\int_{w_k}^{v_{w+1}} \frac{2}{c} \left(B_s - B_s^c\right) \mathrm{d}s$, $k = 0, 1, \ldots$, by two marginal integrals $\int_{w_k}^{U_{u,l_0^c+1}^c}$ and $\int_{w_{k+1}}^{U_{u,l_1^c+1}^c}$ (which vanish as $c \downarrow 0$) and apply Burkholder's inequality.

The same holds for B under the (absolutely continuous with respect to $\tilde{\mathbb{Q}}$) measure $\tilde{\mathbb{P}}$. Thus, by this and by (easy to prove) equality $\tilde{X}_s - x_0 - \left(\tilde{X}_s - x_0\right)^c = \tilde{X}_s - \tilde{X}_s^c$, where $\left(\tilde{X}_s - x_0\right)^c = \left(\tilde{X}_s - x_0\right)^{c,x}$ is the regularization of $\tilde{X}_s - x_0$ with the same x as the regularization $X^c = X^{c,x}$ of X, for $k = 0, 1, \ldots$

$$\int_{\tilde{v}_k}^{\tilde{v}_{k+1}} \frac{2}{c} \left(\tilde{X}_s - \tilde{X}_s^c\right) \mathrm{d}s \to^{\mathbb{P}} 0.$$

Hence, by (4.59) we have

$$\sum_{k=0}^{\infty} f'\left(X_{v_k \wedge t}\right) \int_{v_k \wedge t}^{v_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d}\langle X\rangle_s \to^{\mathbb{P}} 0. \tag{4.67}$$

Thus, by (4.57) and (4.67), since M and K may be arbitrary large (first we choose M then K), we get that

$$\int_0^t \frac{2}{c} \left(X_s - X_s^c\right) f'\left(X_s\right) \mathrm{d}\langle X\rangle_s \to^{\mathbb{P}} 0.$$

Now we are left with the term

$$S_4^c(T) := \int_0^T f\left(X_s\right) \frac{2}{c} \left(X_s - X_s^c\right) \mathrm{d}X_s \tag{4.68}$$

and we know that $S_2^c - S_4^c \to^{\mathbb{P}} 0$.

Step 3. Approximation of S_4^c with step processes and establishing its weak convergence. Let us recall the stopping times defined in (4.45) and define the càdlàg process

$$\tilde{X}_t := \sum_{k=0}^{\infty} X_{v_k} \mathbb{I}_{[v_k; v_{k+1})}(t).$$

Notice that $\tilde{X}$ is uniformly close to X, more precisely

$$\left\| \tilde{X} - X \right\|_{\infty} \leq \frac{2M}{K}, \tag{4.69}$$

where K is the same number which was used in the definition of the intervals I_i, $i = 1, 2, ..., K$. From (4.69), (4.39) and the mean value theorem it follows that

$$\left\| f\left(\tilde{X}\right) - f(X) \right\|_{\infty} \leq \frac{2M}{K} \sup_{x \in [-M;M]} |f'(x)|. \tag{4.70}$$

Let S_5^c be defined as

$$\begin{aligned}
S_5^c(t) &:= \int_0^t f\left(\tilde{X}_s\right) \frac{2}{c} \left(X_s - X_s^c\right) dX_s \\
&= \sum_{k=0}^{\infty} f\left(X_{v_k \wedge t}\right) \int_{v_k \wedge t}^{v_{k+1} \wedge t} \frac{2}{c} \left(X_s - X_s^c\right) dX_s.
\end{aligned}$$

By (4.70), $\left\| \frac{2}{c}\left(X - X^c\right) \right\|_{\infty} \leq 1$, (4.39)-(4.41), and the Burkholder-Davis-Gundy inequality it follows

$$\mathbb{E} \sup_{t \in [0;T]} \left| S_5^c(t) - S_4^c(t) \right|^2$$

$$= \mathbb{E} \sup_{t \in [0;T]} \left| \int_0^t \left(f\left(\tilde{X}_s\right) - f\left(X_s\right) \right) \frac{2}{c} \left(X_s - X_s^c\right) dX_s \right|^2$$

$$\leq C_2 \mathbb{E} \int_0^T \left(f\left(\tilde{X}_s\right) - f\left(X_s\right) \right)^2 \left(\frac{2}{c} \left(X_s - X_s^c\right) \right)^2 d\langle X \rangle_s$$

$$+ C_2 \mathbb{E} \left(\int_0^T \left| f\left(\tilde{X}_s\right) - f\left(X_s\right) \right| \left| \frac{2}{c} \left(X_s - X_s^c\right) \right| dV_s \right)^2$$

$$\leq C_2 \left(\frac{2M}{K} \right)^2 \sup_{x \in [-M;M]} \left(f'(x) \right)^2 \left(\mathbb{E}\langle X \rangle_T + T^2 \sup_{x \in [-M;M]} \left(\mu(x) \right)^2 \right), \tag{4.71}$$

where C_2 is a universal constant and V denotes finite variation part of X.

By (4.54)

$$TV\left(X^c, \cdot\right) - \frac{\langle X \rangle}{c} - \int_0^{\cdot} \frac{2}{c} \left(X_s - X_s^c\right) dX_s \to^{\mathbb{P}} 0 \text{ as } c \downarrow 0. \tag{4.72}$$

By (4.72) we also have the same convergence for the differences,

$$\int_0^{\cdot} \mathbb{I}_{[v_k \wedge T; v_{k+1} \wedge T]}(s) \, d\left\{ TV\left(X^c, s\right) - \frac{\langle X \rangle_s}{c} \right\}$$

$$- \int_0^{\cdot} \mathbb{I}_{[v_k \wedge T; v_{k+1} \wedge T]}(s) \frac{2}{c} \left(X_s - X_s^c\right) dX_s \to^{\mathbb{P}} 0 \text{ as } c \downarrow 0. \tag{4.73}$$

Since $\tilde{X}_s$ is a step process,

$$\int_0^T f\left(\tilde{X}_s\right) \mathrm{d}\left\{TV\left(X^c, s\right) - \frac{\langle X\rangle_s}{c}\right\}$$
$$= \sum_{k=0}^{\infty} f\left(X_{v_k \wedge T}\right) \int_{v_k \wedge T}^{v_{k+1} \wedge T} \mathrm{d}\left\{TV\left(X^c, s\right) - \frac{\langle X\rangle_s}{c}\right\}, \qquad (4.74)$$

and hence, by (4.73) we also have

$$\int_0^{\cdot} f\left(\tilde{X}_s\right) \mathrm{d}\left\{TV\left(X^c, s\right) - \frac{\langle X\rangle_s}{c}\right\} - \int_0^{\cdot} f\left(\tilde{X}_s\right) \frac{2}{c}\left(X_s - X_s^c\right) \mathrm{d}X_s$$
$$= \int_0^{\cdot} f\left(\tilde{X}_s\right) \mathrm{d}\left\{TV\left(X^c, s\right) - \frac{\langle X\rangle_s}{c}\right\} - S_5^c \to^{\mathbb{P}} 0 \text{ as } c \downarrow 0. \qquad (4.75)$$

Let π be the Prohorov metric (cf. [1, Sect. 6]) defined on the space of probability laws on $C\left([0; T], \mathbb{R}\right)$. Reasoning similarly as above (but instead of convergence in probability considering the Prohorov metric), by the weak convergence result for $TV^c\left(X, \cdot\right) - \langle X\rangle/c$ (i.e. [16, Theorem 5]) and the fact that $TV^c\left(X, \cdot\right)$ and $TV\left(X^c, \cdot\right)$ differ by no more than c, we get

$$\pi\left(\mathcal{L}\left(\int_0^{\cdot} f\left(\tilde{X}_s\right) \mathrm{d}\left\{TV\left(X^c, s\right) - \frac{\langle X\rangle_s}{c}\right\}\right), \mathcal{L}\left(\frac{1}{\sqrt{3}}\int_0^{\cdot} f\left(\tilde{X}_s\right) \mathrm{d}B_{\langle X\rangle_s}\right)\right) \to 0 \qquad (4.76)$$

as $c \downarrow 0$.

Finally, notice that by (4.69)

$$\pi\left(\mathcal{L}\left(\int_0^{\cdot} f\left(\tilde{X}_s\right) \mathrm{d}B_{\langle X\rangle_s}\right), \mathcal{L}\left(\int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X\rangle_s}\right)\right) =: \delta\left(K\right) \to 0 \qquad (4.77)$$

as $K \uparrow +\infty$. Now, from (4.71), (4.75), (4.76) and (4.77) we have that

$$\limsup_{c \downarrow 0} \pi\left(\mathcal{L}\left(S_4^c\right), \mathcal{L}\left(\frac{1}{\sqrt{3}}\int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X\rangle_s}\right)\right) \le \delta\left(K\right).$$

Since K may be chosen arbitrary large and $\delta\left(K\right) \to 0$ as $K \uparrow +\infty$, we get

$$\limsup_{c \downarrow 0} \pi\left(\mathcal{L}\left(S_4^c\right), \mathcal{L}\left(\frac{1}{\sqrt{3}}\int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X\rangle_s}\right)\right) = 0.$$

Step 4. Convergence. By Steps 1-3 and the fact that the convergence in the defined Prohorov metric is equivalent to weak convergence in $C\left([0; T], \mathbb{R}\right)$, c.f. [1, Theorem 6.8], we obtain the weak convergence of

$$\int_{\mathbb{R}} f\left(a\right) N^a\left(X^c, \cdot\right) \mathrm{d}a - \frac{1}{c}\int_0^{\cdot} f\left(X_s\right) \mathrm{d}\langle X\rangle_s$$

to

$$\frac{1}{\sqrt{3}}\int_0^{\cdot} f\left(X_s\right) \mathrm{d}B_{\langle X\rangle_s}.$$

To finish the proof let us notice that instead of considering the processes S_i^c, $i = 1, \ldots, 5$ alone, we could consider pairs $\left(X, S_i^c\right)$. By [16, Theorem 5 and Remark 6] we have stable convergence of the pair $\left(X, TV^c(X, \cdot) - \langle X\rangle/c\right)$ and thus we obtain the desired stable convergence with respect to the σ-field generated by X (as described in Remark 1.2). $\qquad \square$

To prove Theorem 4.11 we will need

Lemma 4.15. *For any $f \in C^2$ we have*

$$\int_0^{\cdot} f(X_s^c) \, dX_s^c \to^{\mathbb{P}} \int_0^{\cdot} f(X_s) \circ dX_s \text{ as } c \downarrow 0, \tag{4.78}$$

where $\int_0^{\cdot} f(X_s) \circ dX_s$ denotes the Stratonovich integral and "$\to^{\mathbb{P}}$" denotes convergence in probability in $C([0;T], \mathbb{R})$.

Proof. Let us decompose

$$\int_0^{\cdot} f(X_s^c) \, dX_s^c = \int_0^{\cdot} \{f(X_s^c) - f(X_s)\} \, dX_s^c + \int_0^{\cdot} f(X_s) \, dX_s^c. \tag{4.79}$$

Using integration by parts we get that

$$\int_0^{\cdot} f(X_s) \, dX_s^c = X^c f(X) - X_0^c f(X_0) - \int_0^{\cdot} X_s^c df(X_s). \tag{4.80}$$

Now, using $\|X - X^c\|_\infty \le c/2$,

$$\int_0^{\cdot} X_s^c df(X_s) \to^{\mathbb{P}} \int_0^{\cdot} X_s df(X_s) \tag{4.81}$$

as $c \downarrow 0$. From (4.80), (4.81) and again by integration by parts and Itô's formula $df(X_s) = \frac{1}{2} f''(X_s) \, d\langle X \rangle_s + f'(X_s) \, dX_s$ we get

$$\int_0^{\cdot} f(X_s) \, dX_s^c \to^{\mathbb{P}} \int_0^{\cdot} f(X_s) \, dX_s + \left\langle \int_0^{\cdot} df(X_s), X \right\rangle$$

$$= \int_0^{\cdot} f(X_s) \, dX_s + \int_0^{\cdot} f'(X_s) \, d\langle X \rangle_s. \tag{4.82}$$

Further, using the fact that $\|X - X^c\|_\infty \le c/2$, the Taylor expansion and similar reasoning as in Step 1 of the proof of Theorem 4.13 we get that

$$\int_0^{\cdot} \{f(X_s^c) - f(X_s)\} \, dX_s^c - \int_0^{\cdot} f'(X_s)(X_s^c - X_s) \, dX_s^c \to 0 \text{ a.s.} \tag{4.83}$$

as $c \downarrow 0$. Finally, from (4.79), (4.83) and (4.82) we get that

$$\int_0^{\cdot} f(X_s^c) \, dX_s^c - \int_0^{\cdot} f'(X_s)(X_s^c - X_s) \, dX_s^c - \int_0^{\cdot} f'(X_s) \, d\langle X \rangle_s \to^{\mathbb{P}} \int_0^{\cdot} f(X_s) \, dX_s. \tag{4.84}$$

Now, using $\frac{2}{c}(X_s^c - X_s) \, dX_s^c = -dTV(X^c, s)$ and again integration by parts we get

$$\int_0^{\cdot} f'(X_s)(X_s^c - X_s) \, dX_s^c + \int_0^{\cdot} f'(X_s) \, d\langle X \rangle_s$$

$$= -\frac{1}{2} \int_0^{\cdot} f'(X_s) \, d\{cTV(X^c, s) - \langle X \rangle_s\} + \frac{1}{2} \int_0^{\cdot} f'(X_s) \, d\langle X \rangle_s$$

$$= -\frac{1}{2} f'(X) \{cTV(X^c, \cdot) - \langle X \rangle\} + \frac{1}{2} \int_0^{\cdot} \{cTV(X^c, s) - \langle X \rangle_s\} \, df'(X_s)$$

$$+ \frac{1}{2} \int_0^{\cdot} f'(X_s) \, d\langle X \rangle_s \tag{4.85}$$

which, in view of (4.54) converges in probability as $c \downarrow 0$ to the same limit as

$$
\begin{aligned}
- f'(X) &\int_0^{\cdot} (X_s - X_s^c)\, \mathrm{d}X_s + \int_0^{\cdot} \int_0^t (X_s - X_s^c)\, \mathrm{d}X_s \mathrm{d}f'(X_t) \\
&+ \frac{1}{2} \int_0^{\cdot} f'(X_s)\, \mathrm{d}\langle X \rangle_s \\
= - &\int_0^{\cdot} f'(X_s)(X_s - X_s^c)\, \mathrm{d}X_s - \int_0^{\cdot} (X_s - X_s^c) f''(X_s)\, \mathrm{d}\langle X \rangle_s \\
&+ \frac{1}{2} \int_0^{\cdot} f'(X_s)\, \mathrm{d}\langle X \rangle_s \to^{\mathbb{P}} \frac{1}{2} \int_0^{\cdot} f'(X_s)\, \mathrm{d}\langle X \rangle_s .
\end{aligned} \tag{4.86}
$$

Thus, by (4.84)-(4.86) we have

$$
\int_0^{\cdot} f(X_s^c)\, \mathrm{d}X_s^c \to^{\mathbb{P}} \int_0^{\cdot} f(X_s)\, \mathrm{d}X_s + \frac{1}{2} \int_0^{\cdot} f'(X_s)\, \mathrm{d}\langle X \rangle_s ,
$$

which is the Stratonovich integral and we get (4.78). $\qquad\square$

Now we proceed to the proof of Theorem 4.11.

Proof. By (4.31),

$$
\int_{\mathbb{R}} f(a) U^a(X^c, t)\, \mathrm{d}a = \int_0^t f(X_s^c)\, \mathrm{d}UTV(X^c, s)
$$

and

$$
\int_{\mathbb{R}} f(a) D^a(X^c, t)\, \mathrm{d}a = \int_0^t f(X_s^c)\, \mathrm{d}DTV(X^c, s) .
$$

To finalize the proof we need to notice that

$$
UTV(X^c, \cdot) = \frac{1}{2} \left\{ X^c + TV(X^c, \cdot) \right\},
$$

$$
DTV(X^c, \cdot) = \frac{1}{2} \left\{ -X^c + TV(X^c, \cdot) \right\} .
$$

From this we get

$$
\begin{aligned}
\int_{\mathbb{R}} f(a) U^a(X^c, t)\, \mathrm{d}a &= \int_0^{\cdot} f(X_s^c)\, \mathrm{d}UTV(X^c, s) \\
&= \frac{1}{2} \int_0^{\cdot} f(X_s^c)\, \mathrm{d}\left\{ X_s^c + TV(X^c, s) \right\} \\
&= \frac{1}{2} \int_0^{\cdot} f(X_s^c)\, \mathrm{d}X_s^c + \frac{1}{2} \int_0^{\cdot} f(X_s^c)\, \mathrm{d}TV(X^c, s)
\end{aligned} \tag{4.87}
$$

and

$$
\int_{\mathbb{R}} f(a) D^a(X^c, t)\, \mathrm{d}a = -\frac{1}{2} \int_0^{\cdot} f(X_s^c)\, \mathrm{d}X_s^c + \frac{1}{2} \int_0^{\cdot} f(X_s^c)\, \mathrm{d}TV(X^c, s) . \tag{4.88}
$$

By Theorem 4.13 we have stable convergence of the pair

$$
\left(X, \int_{\mathbb{R}} f(a) N^a(X^c, \cdot)\, \mathrm{d}a - \frac{1}{c} \int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X \rangle_s \right)
$$

which (recall (4.30)) is equal to

$$
\left(X, \int_0^{\cdot} f(X_s^c)\, \mathrm{d}TV(X^c, s) - \frac{1}{c} \int_0^{\cdot} f(X_s)\, \mathrm{d}\langle X \rangle_s \right) .
$$

By this, (4.87), (4.88) and by Lemma 4.15 we have the thesis. $\qquad\square$

We finish with the proof of Theorem 4.12.

Proof. By (4.87), Lemma 4.15, (4.30) and Step 1 of the proof of Theorem 4.13, for any sequences of $\mathcal{F}_0$-measurable r.v.s $x_1 = x_1(c)$, $x_2 = x_2(c)$ such that $x_1, x_2 \in [-c/2; c/2]$ we have the convergence

$$\int_{\mathbb{R}} f(a) U^a (X^{c,x_1}, \cdot) \, \mathrm{d}a - \int_{\mathbb{R}} f(a) U^a (X^{c,x_2}, \cdot) \, \mathrm{d}a$$
$$- \frac{1}{2} \int_0^{\cdot} f(X_s) \, \mathrm{d} \{ TV(X^{c,x_1}, s) - TV(X^{c,x_2}, s) \} \to^{\mathbb{P}} 0$$

in $C([0;T], \mathbb{R})$ as $c \downarrow 0$, and a similar assertion holds for crossings from above. But by (2.1) and condition (E), $|TV(X^{c,x_1}, s) - TV(X^{c,x_2}, s)| \le c$, and integration by parts yields

$$\int_0^{\cdot} f(X_s) \, \mathrm{d} \{ TV(X^{c,x_1}, s) - TV(X^{c,x_2}, s) \} \to^{\mathbb{P}} 0$$

in $C([0;T], \mathbb{R})$ as $c \downarrow 0$. Thus the triple

$$\mathbf{T} = \left(X, \int_{\mathbb{R}} f(a) U^a (X^{c,x_1}, t) \, \mathrm{d}a - \frac{1}{2c} \int_0^t f(X_s) \, \mathrm{d} \langle X \rangle_s, \right.$$
$$\left. \int_{\mathbb{R}} f(a) D^a (X^{c,x_2}, t) \, \mathrm{d}a - \frac{1}{2c} \int_0^t f(X_s) \, \mathrm{d} \langle X \rangle_s \right)$$

converges stably (as $c \downarrow 0$) to the same limit for any sequences x_1 and x_2 as above. Now, it is enough to notice that by Lemma 3.4 we have

$$\int_{\mathbb{R}} f(a) u_c^a (X, t) \, \mathrm{d}a = \int_{\mathbb{R}} f(a) U^a \left(X^{c,-c/2}, t \right) \mathrm{d}a$$

and similarly

$$\int_{\mathbb{R}} f(a) d_c^a (X, t) \, \mathrm{d}a = \int_{\mathbb{R}} f(a) D^a \left(X^{c,c/2}, t \right) \mathrm{d}a.$$

The convergence of the quadruple $\mathbf{Q}_1$ holds by the Mapping Theorem, since

$$n_c^a (X, t) = u_c^a (X, t) + d_c^a (X, t).$$

$\square$

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