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Hölder continuity property of the densities of SDEs with singular drift coefficients*

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Abstract

We prove that the solution of stochastic differential equations with deterministic diffusion coefficient admits a Hölder continuous density via a condition on the integrability of the Fourier transform of the drift coefficient. In our result, the integrability is an important factor to determine the order of Hölder continuity of the density. Explicit examples and some applications are given.

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1 Introduction

Coefficients of a SDE (stochastic differential equation) play an important role in order to determine the properties of the probability density function of the distribution of the solution of the SDE. For elliptic SDEs if the coefficients are bounded and have bounded derivatives for any order, the solution admits a smooth density (see, e.g., [16]). On the other hand, in the case of non-smooth (especially, discontinuous) coefficients, it is difficult to prove the existence and/or regularity of the density.

In this article, we consider a d-dimensional SDE of the form $dX_t = \sigma(t)dB_t + b(X_t)dt$, where $\{B_t\}_{t\geq 0}$ is a d-dimensional Brownian motion, $b : \mathbf{R}^d \to \mathbf{R}^d$ is a bounded function and $\sigma : [0, +\infty) \to \mathbf{R}^d \times \mathbf{R}^d$. The main purpose of this article is to prove the existence and the pointwise regularity of the density of the SDE with non-smooth drift b. Especially, we are interested in the case when b is discontinuous.

Some related results for this problem have already been obtained. Let us start our discussion with the case $d = 1$. In Section 6.5 of [13], for $\sigma = 1$ and an explicit discontinuous function b, the solution of the above SDE admits a density and also an explicit form of the density is given. As this

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example is quite explicit, one can easily see that this density is not differentiable at the discontinuity point of b.

In [8], the authors proved that the solution to the following one dimensional SDE:

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

admits a density on the set $\{x \in \mathbf{R}; \sigma(x) \neq 0\}$, where σ is α-Hölder continuous with $\alpha > \frac{1}{2}$ and b is at most linear growth. Further improvements have been achieved in [1] weakening the Hölder continuous hypotheses on the coefficients or in [5] and [6] for other type of stochastic equations.

Their approach is attractive due to its simplicity. The key idea is to consider the following random vector $Z_\varepsilon := X_{t_0-\varepsilon} + \sigma(X_{t_0-\varepsilon})(B_{t_0} - B_{t_0-\varepsilon})$ which converges to X_{t_0} as $\varepsilon \to 0$. Then one uses the fact that Z_ε has a smooth density and that Z_ε is close to X_{t_0}. The conditions on the coefficients are used for the latter argument. A careful analysis of their method shows that this argument can not be straightforwardly used to obtain any further pointwise properties of the density (such as the Hölder continuity of the density).

As for the regularity of the density, it is shown in [21] that in the particular case that $\sigma = 1$ and the drift is an indicator function then the density of the solution process exists and is α-Hölder continuous for any $\alpha \in (0, \frac{1}{2})$.

In [4], the author shows that if there exists some ball in $\mathbf{R}^d$ on which both coefficients are smooth and σ is uniformly elliptic, then the density is smooth inside a smaller ball. In the case $d = 1$, [12], shows that if there exists some open interval on which b is Hölder continuous and σ is uniformly elliptic and smooth, then the density is Hölder continuous on the interval. In the present article, the aouthors give a first attempt to overcome the locality of the previous results and establish the regularity of the density across the boundaries of the domain of regularity of the coefficients. In this sense, the present work is related to [1]. In [1], the authors prove the existence of densities for multidimensional SDEs with coefficients that have logarithmic order of Hölder continuity using an interpolation theory approach.

Our result might be regarded as a second alternative stochastic approach to the study of the regularity of fundamental solution to parabolic type PDEs (partial differential equations). It is well known that under suitable regularity conditions, the fundamental solution to a parabolic type PDE is given by the density function of the solution process to the associated SDE. Unfortunately, little is known about how to overcome these requirements.

Leaving completely aside the probabilistic setting, in the theory of parabolic equations, there are many results about the regularity of fundamental solutions to PDEs under weaker conditions on the coefficients, such as Hölder continuity or even bounded measurable. In [9], we can find some classical results on the existence and regularity of fundamental solutions of parabolic equations under global Hölder continuity assumptions on the coefficients of the parabolic equation. Also these equations can be solved in some Sobolev spaces and by using embedding theorems and taking modifications, one can obtain that the solution has Hölder continuous derivatives (see e.g. [7], [14] and [17]). Under our conditions, we do not know any method in order to obtain pointwise regularity properties which may be also lead to a general probabilistic analysis tool in order to analyze pointwise properties of the density of solutions of SDEs with discontinuous drift.

Now, we briefly explain our main result. Assume that the drift coefficient b is bounded and compactly supported. To prove the existence and Hölder continuity property of the density, we rely on Lévy's inversion theorem and a corollary which characterizes the Hölder continuity of the density. Thanks to these results, it is enough to show that the characteristic function of X_{t_0} has some polynomial decay at infinity which in turn, implies the pointwise Hölder continuity of the density. To estimate the decay of the characteristic function of X_{t_0}, we use the integrability of the Fourier transform of b. If the Fourier transform of b belongs to some Sobolev space $H_{\gamma,p}$ with suitable parameters $p > 1$ and $\gamma > 0$, we can show the Hölder continuity of the density up to an order which depends on these two indices and the amount of noise in the model (for an exact statement, see Theorem 3.2).

In general, however, if the support of the function b is not compact the Fourier transform may not exist (in the classical sense) even if b is smooth. In this case, we consider the following truncated approximation of b. For $K > 0$, we define a C_b^∞ function $\varphi_K : \mathbf{R} \to \mathbf{R}$ which satisfies $1_{[-K,K]} \leq \varphi_K \leq 1_{[-(K+1),K+1]}$. Then we can show the Hölder continuity of the density by using the function $b_K : \mathbf{R}^d \to \mathbf{R}^d$ defined by $b_K(x) := (b_1(x)\varphi_K(|x|), \cdots, b_d(x)\varphi_K(|x|))$, instead of b. In any case, the Hölder continuity of the density obtained by our method is almost determined by $\frac{2\gamma}{p}$. For the details, see Section 2 and 3 (in particular, Theorem 3.2).

In our approach, Malliavin calculus plays an important role, but we do not assume any smoothness of the drift term b. So in general, the stochastic process $\{X_t\}_{t\geq 0}$ is not differentiable in the Malliavin sense. To solve this problem, we use Girsanov's theorem in order to reduce our study to the solution of the equation $dX_t = \sigma(t)dW_t$ where $\{W_t\}_{t\geq 0}$ is a new Brownian motion under a new probability measure. Then $\{X_t\}_{t\geq 0}$ has independent increments and is differentiable in the Malliavin calculus sense under this new probability measure.

However this measure change yields an extra exponential type martingale which is not Malliavin differentiable. For this reason, we apply stochastic Taylor expansion to this martingale term and then the characteristic function of X_{t_0} is represented by an infinite series of expectations of multiple Wiener integrals multiplied with an exponential of X_{t_0}. By using integration by parts formula in Malliavin calculus sense, in a time interval where the noise increments are independent of the irregular drift function b, these summands can be represented as Lebesgue integrals whose integrands involve the Fourier transforms of b_K and the transition probability density function (with respect to new probability measure) of $\{X_t\}_{t\geq 0}$. These arguments will be carried out on a fixed short time interval $[t_0 - \tau, t_0]$ and then we will estimate these summands by using the decay of Fourier transform of b_K. Finally, to end the argument, we will choose τ and K as a function of the variable of the characteristic function of X_{t_0}, say θ, and we will obtain the desired result.

The rest of the paper is organized as follows. In the following section, we introduce the notation that will be used throughout the article. We state our main result (Theorem 3.2) in Section 3. In Section 4 we will exhibit preparatory lemmas which generalize Lévy's inversion theorem. This lemma implies that the asymptotic behavior of a characteristic function at infinity determines the regularity of the density, hence we may just focus our attention on the asymptotic behavior of characteristic functions (Proposition 4.3). We will prove our main result in Section 5 and give some examples in Section 6. In Section 7, we will give some concluding remarks. Section 8 will be devoted to the proofs of some auxiliary lemmas.

2 Notations and Preliminaries

We introduce the notation that will be used throughout the article.

The symbols $\mathbf{N}$ and $\mathbf{Z}_+$ denote the set of all positive integers and the set of all non-negative integers, respectively. $\lfloor \cdot \rfloor$ denotes the greatest integer function (sometimes also called the floor function).

Vectors will always be interpreted as column vectors unless stated explicitly. $\mathbf{B}(x, r)$ denotes the open ball centered in x and radius $r > 0$. In the particular case that $x = 0$ we use the simplifying notation $\mathbf{B}(r) \equiv \mathbf{B}(0, r)$.

Let $d \in \mathbf{N}$. The transpose of a matrix A is denoted by ${}^tA$ and its inverse is denoted by A^{-1}. The norm of a vector $x \in \mathbf{R}^d$ is denoted by $|x|$.

$1_A(x)$ will denote the indicator function of the set A. $f^{(n)}$ will denote the nth derivative of the function $f : \mathbf{R} \to \mathbf{R}$. $C^\lambda(\mathbf{R}^d;\mathbf{R}^k)$ for $\lambda \in [0,\infty]$ denotes the space of functions from $\mathbf{R}^d$ to $\mathbf{R}^k$ (d, $k \in \mathbf{N}$) which are $\lfloor\lambda\rfloor$-times differentiable and its $\lfloor\lambda\rfloor$-derivative is $\lambda - \lfloor\lambda\rfloor$-Hölder. Sometimes, we simplify this notation to C^λ when d and k are well understood from the context. Similarly, we define the space C_b^λ as the subspace of C^λ of bounded functions with $\lfloor\lambda\rfloor$ bounded derivatives. Also C_c^λ the subspace of C^λ of functions with compact support.

We define the Fourier transform of the function $b = (b_j)_{1\le j\le d} : \mathbf{R}^d \to \mathbf{R}^d$ by

$$\mathscr{F}b(\theta) := (\mathscr{F}b_j(\theta))_{1\le j\le d} := \left(\frac{1}{(2\pi)^d}\int_{\mathbf{R}^d} b_j(x)e^{-i\langle\theta,x\rangle}dx\right)_{1\le j\le d},\ \theta \in \mathbf{R}^d.$$

$\mathscr{S}(\mathbf{R}^d)$ denotes the space of real valued rapidly decreasing functions on $\mathbf{R}^d$. For $\phi \in \mathscr{S}$, $p > 1$ and $\gamma > 0$, we define the Sobolev norm $\|\phi\|_{H^{\gamma,p}}$ as

$$\|\phi\|_{H^{\gamma,p}} := \left[\int_{\mathbf{R}^d}(1+|\xi|^2)^\gamma\,|\mathscr{F}\phi(\xi)\,|^p\,d\xi\right]^{\frac{1}{p}},$$

and the Sobolev space $H^{\gamma,p}$ as the completion of $\mathscr{S}(\mathbf{R}^d)$ with respect to the norm $\|\cdot\|_{H^{\gamma,p}}$.

For a function $f : \mathbf{R}^d \to \mathbf{R}^d$, we define the norm $\|f\|_\infty := \sup_{x\in\mathbf{R}^d}|f(x)|$.

Through this article, we let $\varphi \in C_c^\infty(\mathbf{R};\mathbf{R})$ denote a function which satisfies

$$1_{[-1,1]} \le \varphi \le 1_{[-2,2]}.$$

For $K > 0$, we define the function $\varphi_K : \mathbf{R} \to \mathbf{R}$ by

$$\varphi_K(x) := \varphi(x+1-K)1_{[K,K+1]}(x) + \varphi(x-1+K)1_{[-(K+1),-K]}(x) + 1_{(-K,K)}(x),\ x \in \mathbf{R}.$$

Then for any $K > 0$, $\varphi_K \in C_c^\infty(\mathbf{R};\mathbf{R})$ satisfies

$$1_{[-K,K]} \le \varphi_K \le 1_{[-(K+1),K+1]}.$$

Moreover, for any $n \in \mathbf{N}$, $\|\varphi_K^{(n)}\|_\infty = \|\varphi^{(n)}\|_\infty$.

For $K > 0$ and a function $b = (b_j)_{1\le j\le d} : \mathbf{R}^d \to \mathbf{R}^d$, we define b_K by

$$b_K(x) := (b_{K,j})_{1\le j\le d} := (b_j(x)\varphi_K(|x|))_{1\le j\le d},\ x \in \mathbf{R}^d.$$

Note that the support of b_K is contained in $\mathbf{B}(K+1)$ and $b(x) = b_K(x)$ for $x \in \mathbf{B}(K)$. Moreover, if b is bounded, the Fourier transform of b_K exists for each $K > 0$.

We will also use the integration by parts (or duality formula) of Malliavin Calculus. We refer the reader to Proposition 1.3.11 in [19] for notations and a precise statement. We will also be using two probability measures $\mathbf{P}$ and $\mathbf{Q}$. Their respective expectations will be denoted by $\mathbf{E}$ and $\mathbb{E}$ respectively.

Constants may change values from line to line although in many cases their dependence with respect to the problem parameters is explicitly stated. In particular, all constants may depend upon on σ or b in the sense that they depend on the norms used for these two functions and the constants appearing in the hypothesis which relate to these functions. The constants may also depend on other parameters in the hypothesis such as d, β, γ, p or T, but they are independent of t_0 or n (which will appear as the expansion index for the Girsanov exponentials) unless explicitly stated.

3 Main Result

Now we give our assumptions and main result.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a filtered probability space and $\{B_t\}_{t\geq 0}$ be a d-dimensional $\{\mathcal{F}_t\}_{t\geq 0}$ Brownian motion.

Consider the following SDE;

$$X_t = x_0 + \int_0^t \sigma(s) dB_s + \int_0^t b(X_s) ds, \tag{3.1}$$

where $\sigma = (\sigma_{ij})_{1\leq i,j\leq d} : [0,T] \to \mathbf{R}^d \times \mathbf{R}^d$ and $b = (b_j)_{1\leq j\leq d} : \mathbf{R}^d \to \mathbf{R}^d$ are Borel measurable functions. We assume that these coefficients satisfy the following conditions

(A1). $\|b\|_\infty < \infty$.

(A2). There exist constants $p > 1$ and $\gamma > 0$ such that for any $K > 0$, $b_K \in H^{\gamma,p}$ and it satisfies

$$\|b_K\|_{\gamma,p} := \max_{1\leq j\leq d} \|b_{K,j}\|_{H_{\gamma,p}} \leq g(K) \text{ and } \frac{2\gamma}{p} + 1 > \frac{d}{q},$$

where q is the Hölder conjugate of p and $g(x) := C\left(|x|^m + 1\right)$, $x \in \mathbf{R}$, for some positive constants C and $m \in \mathbf{Z}_+$.

(A3). $\sigma = (\sigma_{ij})_{1\leq i,j\leq d} \in L^2\left([0,T]; \mathbf{R}^d \times \mathbf{R}^d\right)$ and there exists a positive constant c such that $\langle \theta, a(s)\theta\rangle \geq c|\theta|^2$ for all $(s,\theta) \in [0,T] \times \mathbf{R}^d$, where $a := \sigma\, {}^t\sigma$.

(A4). For some $t_0 \in (0,T]$, there exist constants $c \in (0,+\infty)$, $\beta \in (0,1]$ and $\delta \in (0,t_0)$ such that for any $s \in [t_0 - \delta, t_0]$,

$$\int_s^{t_0} \langle a(u)\theta, \theta\rangle du \geq c|\theta|^2 (t_0 - s)^\beta. \tag{3.2}$$

(A5). There exists a unique weak solution X to the SDE (3.1).

Remark 3.1. *For sufficient conditions for existence and uniqueness for the equation* (3.1)*, we refer the reader to the traditional results in Section 1.2 in [3]. For recent results, we refer to [2], [11], [15] or [22] between others.*

Our main result is the following.

Theorem 3.2. *Fix $t_0 \in (0,T]$ as in hypothesis* (A4)*. Assume that* (A1)-(A5) *hold. Then X_{t_0} admits a density in the class C^λ for any $\lambda \in (0, \frac{2\gamma}{p} + \frac{2}{\beta} - 1 - d)$.*

Remark 3.3.
1. *If $\lfloor \frac{2\gamma}{p} + \frac{2}{\beta} - 1 - d \rfloor = 0$ in the above theorem, then the density of X_{t_0} is λ-Hölder continuous.*
2. *Note that the parameters γ and p measure the regularity of the drift coefficient b. Furthermore, if* (A3) *holds then* (A4) *also holds for all $t \in (0,T]$ with $\beta = 1$. If $\sigma(s)$ is close to $+\infty$ at $s = t_0$ then we may have $\beta < 1$. In Section 6.3, we will give an example of σ for which $\beta < 1$. The interest in this case stems from the fact that there is a widespread belief that "more noise implies more smoothness of the density". In our case, the amount of noise is measured by the parameter β. The smaller the value of β, the more noise we have in the model and the more regularity the density of X_{t_0} will have.*
3. *The fact that m in* (A2) *does not play any role in the final result will be clearly understood from the proof. In fact, the Gaussian tails of the Wiener process make the effect of m negligible at the end.*

4 Preparatory Lemmas

To show Theorem 3.2, we use the well known relation between the integrability of characteristic functions and the regularity of densities.

Theorem 4.1. (*Lévy's inversion theorem*) *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be any probability space,* X *be a* $\mathbf{R}^d$*-valued random vector defined on that space and* $\phi(\theta) = \mathbf{E}[e^{i\langle\theta, X\rangle}]$ *be its characteristic function. If* $\phi \in L^1(\mathbf{R}^d)$*, then* f_X*, the density function of the law of* X*, exists and is continuous. Moreover, for any* $x \in \mathbf{R}^d$*, we have*

$$f_X(x) = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{i\langle\theta,x\rangle} \phi(\theta) d\theta.$$

The following corollary of Theorem 4.1 gives us a more precise criterion for the Hölder continuity of the density. For the proof of this corollary in one dimension, see [12]. The multidimensional version can be proved in the same way.

Corollary 4.2. *Let* X *be a random vector under the same setting as in Theorem 4.1 and* ϕ *be its characteristic function. Assume that there exist a constant* $\lambda > 0$ *such that*

$$\int_{\mathbf{R}^d} |\phi(\theta)||\theta|^\lambda d\theta < +\infty.$$

Then the density function of the law of X *exists and it belongs to the set* C^λ*.*

Given the above result, we will concentrate on obtaining the asymptotic behavior of the characteristic function at infinity. Under our assumptions, we obtain the following result.

Proposition 4.3. *Fix* $t_0 \in (0, T]$ *as in hypothesis* (A4) *and assume that* (A1)-(A5) *hold. Then for any* $\lambda < \frac{2\gamma}{p} + \frac{2}{\beta} - 1$*, there exists a constant* $C > 0$ *such that*

$$\left| \mathbf{E}\left[e^{i\langle\theta, X_{t_0}\rangle} \right] \right| \leq C(1 + |\theta|)^{-\lambda}.$$

Proof of Theorem 3.2: From Corollary 4.2 and Proposition 4.3, it is easy to see that Theorem 3.2 holds. Therefore, in the following section, we will give the proof of Proposition 4.3.

5 Proof of Proposition 4.3

We recall the reader that we are assuming hypotheses (A1)-(A5) throughout this section.

5.1 Measure change

Fix $t_0 \in (0, T]$ as in hypothesis (A4) and $\delta \in (0, t_0)$ which satisfy (3.2). Let us fix some $\tau \in (t_0 - \delta, t_0)$. Before estimating the characteristic function of X_{t_0}, we apply Girsanov's theorem. Define the function h as

$$h(s, x) := {}^t\sigma(s)^{-1} \cdot b(x),\ s \in [\tau, t_0],\ x \in \mathbf{R}^d.$$

Remark 5.1. *Note that the assumptions* (A1) *and* (A3) *imply that* h *is bounded.*

Define a new probability measure $\mathbf{Q}$ as

$$\left.\frac{d\,\mathbf{Q}}{d\mathbf{P}}\right|_{\mathscr{F}_u} = \exp\left(-\int_\tau^u h(s, X_s) dB_s - \frac{1}{2}\int_\tau^u |h(s, X_s)|^2 ds \right),\ u \in [\tau, t_0].$$

Then by Girsanov's theorem,

$$W_u := B_u - B_\tau + \int_\tau^u h(s, X_s)ds,\ u \in [\tau, t_0],$$

is a Brownian motion under the new measure $\mathbf{Q}$.

Define the following processes for $u \in [\tau, t_0]$,

$$\begin{aligned} Y_u &:= X_u - X_\tau = \int_\tau^u \sigma(s)dW_s, \\ Z_u(z) &:= \exp\left(\int_\tau^u h(s, Y_s + z)dW_s - \frac{1}{2}\int_\tau^u |h(s, Y_s + z)|^2 ds\right). \end{aligned} \tag{5.1}$$

First we state some basic properties of Y. Their proof is straightforward.

Lemma 5.2. *The process $\{Y_s\}_{\tau \le s \le t_0}$ is a Gaussian process with independent increments. Moreover, for any $s, u \in [\tau, t_0]$ with $s < u$, $Y_u - Y_s$ is a centered Gaussian random vector with covariance matrix given by $\int_u^s a(v)dv$. Therefore the characteristic function of the increments of Y is real valued and*

$$\mathbf{Q}\left(Y_u \in dx | Y_s = y\right) = p_{s,u}(x - y)dx,$$

where $p_{s,u}$ is the density function of the law of $Y_u - Y_s$.

Next, we explain the role of Z in the calculation of the characteristic function of X_{t_0}. Recall that we denote by $\mathbf{E}$ and $\mathbb{E}$ the expectations under $\mathbf{P}$ and $\mathbf{Q}$ respectively.

Lemma 5.3. *For any $\theta \in \mathbf{R}^d$ and $\tau \in (t_0 - \delta, t_0]$ we have*

$$\mathbf{E}\left[e^{i\langle\theta, X_{t_0}\rangle}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+z\rangle} Z_{t_0}(z)\right]\Big|_{z=X_\tau}\right].$$

Proof. We have

$$\begin{aligned} \mathbf{E}\left[e^{i\langle\theta, X_{t_0}\rangle}\right] &= \mathbf{E}\left[e^{i\langle\theta, X_{t_0}-X_\tau+X_\tau\rangle}\right] \\ &= \mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+X_\tau\rangle} \exp\left(\int_\tau^{t_0} h(s, X_s)dB_s + \frac{1}{2}\int_\tau^{t_0} |h(s, X_s)|^2 ds\right)\right] \\ &= \mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+X_\tau\rangle} \exp\left(\int_\tau^{t_0} h(s, Y_s + X_\tau)dW_s - \frac{1}{2}\int_\tau^{t_0} |h(s, Y_s + X_\tau)|^2 ds\right)\right]. \end{aligned}$$

Taking conditional expectations with respect to X_τ, we have

$$\begin{aligned} &\mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+X_\tau\rangle} \exp\left(\int_\tau^{t_0} h(s, Y_s + X_\tau)dW_s - \frac{1}{2}\int_\tau^{t_0} |h(s, Y_s + X_\tau)|^2 ds\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+z\rangle} Z_{t_0}(z)\right]\Big|_{z=X_\tau}\right]. \end{aligned}$$

□

From Lemma 5.3, one sees that in order to prove Proposition 4.3 it is enough to find an upper bound estimate for

$$\left|\mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+z\rangle} Z_{t_0}(z)\right]\right|; \quad z, \theta \in \mathbf{R}^d. \tag{5.2}$$

Itô's formula implies that Z satisfies the following linear SDE;

$$Z_u(z) = 1 + \int_\tau^u Z_s(z) dM_s,$$

where

$$M_u := \int_\tau^u h(s, Y_s + z) dW_s,\ u \in [\tau, t_0]. \tag{5.3}$$

Let $M_u^{(0)} := 1$ and define recursively for $n \in \mathbf{N}$,

$$M_u^{(n)} := \int_\tau^u M_s^{(n-1)} dM_s,\ u \in [\tau, t_0]. \tag{5.4}$$

Then for any $N \in \mathbf{Z}_+$, we have

$$\mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+z\rangle} Z_{t_0}(z)\right] = \sum_{n=0}^N I_n(t_0, \theta, z) + R_N(t_0, \theta, z), \tag{5.5}$$

where for $n, N \in \mathbf{Z}_+$

$$I_n(u, \theta, z) := \mathbb{E}\left[e^{i\langle\theta, Y_u+z\rangle} M_u^{(n)}\right],\ u \in [\tau, t_0], \tag{5.6}$$

and

$$R_N(t_0, \theta, z) := \mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+z\rangle} \int_\tau^{t_0} \cdots \int_\tau^{s_{N-1}} Z_{s_N}(z) dM_{s_N} \cdots dM_{s_1}\right]. \tag{5.7}$$

Therefore continuing with the reasoning in (5.2), we need to obtain upper bound estimates for $|I_n|$ and $|R_N|$. These estimates are obtained in Propositions 5.4 and 5.6.

5.2 Estimates for I_n and R_N

Recall that when we say that a constant depends on b or σ we mean that they depend on $\|b\|_\infty$ and the constants appearing in hypothesis (A2) or the ellipticity coefficient c appearing in hypothesis (A3) and (A4) respectively.

5.2.1 Estimate for R_N

Proposition 5.4. *Assume that* (A1) *and* (A3) *hold and* t_0 *satisfies* (A4). *Then there exists positive constant* C_N *which depends only on* T, N, b *and* σ *such that for any* θ, $z \in \mathbf{R}^d$

$$|R_N(t_0, \theta, z)| \le C_N (t_0 - \tau)^{\frac{N}{2}}.$$

Proof. The L^2-isometry of the stochastic integral applied to (5.7) yields

$$|R_N(t_0, \theta, z)| \le \|h\|_\infty^N \left\{\int_\tau^{t_0} \cdots \int_\tau^{s_{N-1}} \mathbb{E}\left[Z_{s_N}(z)^2\right] ds_N \cdots ds_1\right\}^{\frac{1}{2}}. \tag{5.8}$$

Since for any $u \in [\tau, t_0]$ and $z \in \mathbf{R}^d$,

$$\mathbb{E}[Z_u(z)^2] \le e^{(u-\tau)\|h\|_\infty^2}$$

holds, then we have

$$\int_{\tau}^{t_0}\cdots\int_{\tau}^{s_{N-1}}\mathbb{E}[Z_{s_N}(z)^2]ds_N\dots ds_1 \le \int_{\tau}^{t_0}\cdots\int_{\tau}^{s_{N-1}} e^{(s_N-\tau)\|h\|_\infty^2}ds_N\dots ds_1. \tag{5.9}$$

We remark that the right hand side of (5.9) is the N-th remainder term of the Maclaurin expansion of the function $e^{(u-\tau)\|h\|_\infty^2}$, hence

$$\int_{\tau}^{t_0}\cdots\int_{\tau}^{s_{N-1}}\mathbb{E}\left[Z_{s_N}(z)^2\right]ds_N\dots ds_1 \le \frac{(t_0-\tau)^N}{N!\|h\|_\infty^{2N}}e^{(t_0-\tau)\|h\|_\infty^2}. \tag{5.10}$$

Substitute (5.10) in (5.8) and define $C_N := \frac{e^{T\|h\|_\infty^2}}{\sqrt{N!}}$. From here, the result follows. □

5.2.2 Estimates for the summands I_n

Now we turn to the estimate of I_n defined in (5.6). We remark here, that I_0 is essentially treated differently from the other summands $(I_n)_{n\ge1}$ because that term does not depend upon the drift coefficient. In fact, for any $u\in[\tau,t_0]$, we can calculate I_0 explicitly as follows;

$$I_0(u,\theta,z)=\mathbb{E}\left[e^{i\langle\theta,Y_u+z\rangle}\right]=e^{i\langle\theta,z\rangle}\exp\left(-\int_{\tau}^{u}\langle\theta,a(s)\theta\rangle ds\right).$$

By the assumptions (A3) and (A4), we see that there exists positive a constant c which depends only on σ such that for any θ, $z\in\mathbf{R}^d$,

$$|I_0(u,\theta,z)|\le\begin{cases} e^{-c(t_0-\tau)^\beta|\theta|^2},\ u=t_0,\\ e^{-c(u-\tau)|\theta|^2},\ u\in[\tau,t_0).\end{cases}$$

These estimates are enough for our purposes, but calculations in the proof become very complicated if we use this exact functional form. Therefore, we use the following rough but manageable estimate which follows from the basic inequality $x^ae^{-x}\le a^ae^{-a}$ for $x>0$ and $a>0$.

Lemma 5.5. *Let $\beta\in(0,1]$ and $\rho,\nu,T>0$. Then there exists a positive constant C which depends on β, ρ, ν and T such that for any $x>0$, $r\ge\beta\nu$, $x\in\mathbf{R}$ and $s\in(0,T]$,*

$$e^{-\rho s^\beta x^2}\le\frac{C}{s^r(1+x^2)^\nu}.$$

From this lemma, we have

$$|I_0(u,\theta,z)|\le\begin{cases}\frac{C}{(t_0-\tau)^{r_1}(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac12}},\ u=t_0,\ r_1\ge\frac{\beta\gamma}{p}+1-\frac{\beta}{2},\\ \frac{C}{(u-\tau)^{r_2}(1+|\theta|^2)^{\frac{\gamma}{p}+\frac12}},\ u\in[\tau,t_0),\ r_2\ge\frac{\gamma}{p}+\frac12,\end{cases} \tag{5.11}$$

where C is a positive constant which depends on σ, γ, p, β and T. The reason why there is a difference in the cases $u=t_0$ and $u<t_0$ is due to the fact that in general there is more noise at t_0 than otherwise. In fact, if $\beta=1$, then both estimates are equal.

For $n\in\mathbf{N}$, define $K\equiv K(z,\theta):=|z|+\sqrt{4\gamma p^{-1}d\sum_{i,j=1}^d\|\sigma_{ij}\|_{L^2[0,T]}^2\log(1+|\theta|^2)}$ and the function $J_n:\mathbf{R}^d\times\mathbf{R}^d\to(0,+\infty)$ as follows;

$$J_n(z,\theta):=g(K(z,\theta))^n,$$

where g is the function defined in (A2). Using this function, we have the following estimate for I_n.

Proposition 5.6. *Let $n \in \mathbf{Z}_+$ and $r \geq \max\{\frac{\beta\gamma}{p} + 1 - \frac{\beta}{2}, \frac{\gamma}{p} + \frac{1}{2}\}$. Then there exists a positive constant $C_{n,r}$ which depends only on $p, \gamma, d, T, \sigma, b, g, n, r$ and β such that for any θ, $z \in \mathbf{R}^d$,*

$$|I_n(u, \theta, z)| \leq \begin{cases} \frac{C_{n,r} J_n(z,\theta)}{(t_0-\tau)^r (1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}}, & u = t_0, \\ \frac{C_{n,r} J_n(z,\theta)}{(u-\tau)^r (1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{2}}}, & u \in [\tau, t_0). \end{cases}$$

To prove Proposition 5.6, we need several lemmas. Lemma 5.7 immediately follows from the the repeated application of the L^2-isometry of stochastic integrals to (5.6) together with (5.4) and (5.3). We prove Lemmas 5.8, 5.9 and 5.10 in the Appendix.

Lemma 5.7. *Let $n \in \mathbf{N}$ and $u \in [\tau, t_0]$. Then for any θ, $z \in \mathbf{R}^d$, we have*

$$|I_n(u, \theta, z)| \leq \left\| M_u^{(n)} \right\|_{L^2(\Omega, \mathbf{Q})} \leq \frac{\|h\|_\infty^n T^{\frac{n}{2}}}{\sqrt{n!}}.$$

Lemma 5.8. *Let $n \in \mathbf{N}$ and $u \in [\tau, t_0]$. Then for any θ, $z \in \mathbf{R}^d$, we have*

$$I_n(u, \theta, z) = (2\pi)^d \int_\tau^u \mathscr{F} p_{s,u}(\theta) \mathbb{E}\left[i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_s + z\rangle} M_s^{(n-1)} \right] ds.$$

Lemma 5.9. *Let ν and μ be positive constants which satisfy $\nu + \mu > \frac{d}{2}$. Assume that $n \in \mathbf{Z}_+$, $C_0 > 0$ and g is the function defined in* (A2)*. Then we have*

$$\int_{\mathbf{R}^d} \frac{1}{(1 + |\eta - \theta|^2)^\nu} \frac{g\left(|z| + \sqrt{C_0 \log(1 + |\eta|^2)} \right)^n}{(1 + |\eta|^2)^{\nu+\mu}} d\eta \leq \frac{C_n g(|z|)^n}{(1 + |\theta|^2)^\nu},$$

where C_n is a positive constant which depends only on d, ν, C_0, g, n and μ.

Lemma 5.10. *Let $n \in \mathbf{N}$ and $s \in [\tau, t_0]$. Then for any bounded and compactly supported function $\vartheta : \mathbf{R}^d \to \mathbf{R}^d$, $\theta, z \in \mathbf{R}^d$ we have*

$$\mathbb{E}\left[i\langle \theta, \vartheta(Y_s + z)\rangle e^{i\langle \theta, Y_s + z\rangle} M_s^{(n-1)} \right] = \int_{\mathbf{R}^d} i\langle \theta, \mathscr{F}\vartheta(\eta - \theta)\rangle I_{n-1}(s, \eta, z) d\eta.$$

Proof of Proposition 5.6. The proof is done by an induction argument on I_n. Due to (5.11), the statement is true for $n = 0$. Assume that $n \in \mathbf{N}$ and the statement is true for $n - 1$. We prove now that the inequality holds for $u = t_0$, that is,

$$|I_n(t_0, \theta, z)| \leq \frac{C_{n,r} J_n(z, \theta)}{(t_0 - \tau)^r (1 + |\theta|^2)^{\frac{\gamma}{p} + \frac{1}{\beta} - \frac{1}{2}}}$$

holds. The other case, $u < t_0$, will follow similarly.

By Lemma 5.8, we have

$$\begin{aligned} I_n(t_0, \theta, z) &= (2\pi)^d \int_\tau^{\frac{t_0+\tau}{2}} \mathscr{F} p_{s,t_0}(\theta) \mathbb{E}\left[i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_s + z\rangle} M_s^{(n-1)} \right] ds \\ &\quad + (2\pi)^d \int_{\frac{t_0+\tau}{2}}^{t_0} \mathscr{F} p_{s,t_0}(\theta) \mathbb{E}\left[i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_s + z\rangle} M_s^{(n-1)} \right] ds. \end{aligned}$$

Recall that according to Lemma 5.2, $\{Y_s\}_{\tau\leq s\leq t_0}$ has independent increments, therefore for any $s\in[\tau,\frac{t_0+\tau}{2}]$ we have

$$\mathscr{F}p_{s,t_0}(\theta)=\mathscr{F}p_{s,\frac{t_0+\tau}{2}}(\theta)\mathscr{F}p_{\frac{t_0+\tau}{2},t_0}(\theta).$$

Note that from Lemma 5.2 and (A4) we have that $0<\mathcal{F}p_{\frac{t_0+\tau}{2},t_0}(\theta)\leq e^{-\frac{c(t_0-\tau)^\beta}{2^\beta}|\theta|^2}$. Hence, from Lemma 5.8, we have

$$\begin{aligned}
&|I_n(t_0,\theta,z)|\\
&\leq\left|(2\pi)^d\mathscr{F}p_{\frac{t_0+\tau}{2},t_0}(\theta)\int_\tau^{\frac{t_0+\tau}{2}}\mathscr{F}p_{s,\frac{t_0+\tau}{2}}(\theta)\mathbb{E}\left[i\langle\theta,b(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|\\
&+\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|\\
&\leq e^{-\frac{c(t_0-\tau)^\beta}{2^\beta}|\theta|^2}\left|I_n\left(\frac{t_0+\tau}{2},\theta,z\right)\right|+\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|.
\end{aligned}\tag{5.12}$$

Apply Lemmas 5.5 and 5.7, to obtain that

$$e^{-\frac{c(t_0-\tau)^\beta}{2^\beta}|\theta|^2}\left|I_n\left(\frac{t_0+\tau}{2},\theta,z\right)\right|\leq\frac{\hat{C}_n}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}},\tag{5.13}$$

where $\hat{C}_n$ is some positive constant which depends only on p,γ,T,σ,b,n and β.

The second term of the right hand side of (5.12) can be estimated as follows

$$\begin{aligned}
&\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|\\
&\leq\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b_K(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|\\
&+\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b(Y_s+z)-b_K(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|.
\end{aligned}\tag{5.14}$$

Now we turn to the estimate of the first term on the right hand side of the inequality (5.14). From Lemma 5.10 with $\vartheta=b_K$, we have

$$\begin{aligned}
&\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b_K(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|\\
&=\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\int_{\mathbf{R}^d}i\langle\theta,\mathscr{F}b_K(\eta-\theta)\rangle I_{n-1}(s,\eta,z)d\eta ds\right|\\
&\leq(2\pi)^d|\theta|\int_{\frac{t_0+\tau}{2}}^{t_0}e^{-c(t_0-s)^\beta|\theta|^2}\int_{\mathbf{R}^d}|\mathscr{F}b_K(\eta-\theta)||I_{n-1}(s,\eta,z)|d\eta ds.
\end{aligned}\tag{5.15}$$

From the assumption (A2) and Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$, we have that for any $s\in[\frac{t_0+\tau}{2},t_0]$,

$$\begin{aligned}&\int_{\mathbf{R}^d}|\mathscr{F}b_K(\eta-\theta)||I_{n-1}(s,\eta,z)|d\eta\\&=\int_{\mathbf{R}^d}(1+|\eta-\theta|^2)^{\frac{\gamma}{p}}|\mathscr{F}b_K(\eta-\theta)|(1+|\eta-\theta|^2)^{-\frac{\gamma}{p}}|I_{n-1}(s,\eta,z)|d\eta\\&\le\|b_K\|_{p,\gamma}\left\{\int_{\mathbf{R}^d}\frac{1}{(1+|\eta-\theta|^2)^{\frac{\gamma q}{p}}}|I_{n-1}(s,\eta,z)|^q d\eta\right\}^{\frac{1}{q}}.\end{aligned}$$

Now, the inductive hypothesis and Lemma 5.9 with $\nu+\mu:=(\frac{\gamma}{p}+\frac{1}{2})q>\frac{d}{2}$ imply that for any $s\in[\frac{t_0+\tau}{2},t_0]$,

$$\begin{aligned}&\|b_K\|_{p,\gamma}\left\{\int_{\mathbf{R}^d}\frac{1}{(1+|\eta-\theta|^2)^{\frac{\gamma q}{p}}}|I_{n-1}(s,\eta,z)|^q d\eta\right\}^{\frac{1}{q}}\\&\le\frac{C_{n-1,r}\|b_K\|_{p,\gamma}}{(s-\tau)^r}\left\{\int_{\mathbf{R}^d}\frac{1}{(1+|\eta-\theta|^2)^{\frac{\gamma q}{p}}}\frac{J_{n-1}(z,\eta)}{(1+|\eta|^2)^{(\frac{\gamma}{p}+\frac{1}{2})q}}d\eta\right\}^{\frac{1}{q}}\\&\le\frac{\overline{C}_{n,r}\|b_K\|_{p,\gamma}g(|z|)^{n-1}}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}}}\end{aligned}\tag{5.16}$$

holds, where $\overline{C}_{n,r}$ is some positive constant which depends only on $p,\gamma,d,T,\sigma,g,n,r$ and β. Moreover, from the assumption (A2) we see that $\|b_K\|_{p,\gamma}\le g(K)$. Since the function g is monotone increasing on $[0,+\infty)$ and the inequality $K\ge|z|$ holds, then we have

$$\|b_K\|_{p,\gamma}g(|z|)^{n-1}\le g(K)^n=J_n(z,\theta).\tag{5.17}$$

Now, the inequalities (5.15)−(5.17) yield that

$$\begin{aligned}&(2\pi)^d|\theta|\int_{\frac{t_0+\tau}{2}}^{t_0}e^{-c(t_0-s)^\beta|\theta|^2}\int_{\mathbf{R}^d}|\mathscr{F}b_K(\eta-\theta)||I_{n-1}(s,\eta,z)|d\eta ds\\&\le\frac{\overline{C}_{n,r}J_n(z,\theta)}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}}}|\theta|\int_{\frac{t_0+\tau}{2}}^{t_0}e^{-c(t_0-s)^\beta|\theta|^2}ds.\end{aligned}$$

Lemma 8.1 (introduced and proved in Section 8.4) and the assumption (A2) imply that

$$\frac{\overline{C}_{n,r}J_n(z,\theta)}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}}}|\theta|\int_{\frac{t_0+\tau}{2}}^{t_0}e^{-c(t_0-s)^\beta|\theta|^2}ds\le\frac{\overline{C}_{n,r}J_n(z,\theta)}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}}\tag{5.18}$$

holds with some positive constant $\overline{C}_{n,r}$ which depends only on $p,\gamma,d,T,\sigma,g,n,r$ and β. This finishes our estimation of the first term on the right hand side of (5.14).

For the second term in the right hand side of (5.14), applying Hölder's inequality and Jensen's inequality we have

$$\begin{aligned}&\left|(2\pi)^d\int_{\frac{t_0+\tau}{2}}^{t_0}\mathscr{F}p_{s,t_0}(\theta)\mathbb{E}\left[i\langle\theta,b(Y_s+z)-b_K(Y_s+z)\rangle e^{i\langle\theta,Y_s+z\rangle}M_s^{(n-1)}\right]ds\right|\\&\le(2\pi)^d|\theta|\int_{\frac{t_0+\tau}{2}}^{t_0}e^{-c(t_0-s)^\beta|\theta|^2}\|b(Y_s+z)-b_K(Y_s+z)\|_{L^2(\Omega,\mathbf{Q})}\left\|M_s^{(n-1)}\right\|_{L^2(\Omega,\mathbf{Q})}ds.\end{aligned}\tag{5.19}$$

By Lemma 5.7, we have that

$$\left\| M_s^{(n-1)} \right\|_{L^2(\Omega,\mathbf{Q})} \leq \frac{\|h\|_\infty^{n-1} T^{\frac{n-1}{2}}}{\sqrt{(n-1)!}}. \tag{5.20}$$

Now, we estimate the first $L^2(\Omega, \mathbf{Q})$-norm term in (5.19). By the definition of b_K, we have

$$|b(x) - b_K(x)| \leq \|b\|_\infty 1_{(K,\infty)}(|x|)$$

for any $x \in \mathbf{R}^d$. Therefore, for any $K > 0$, the following inequality holds.

$$\|b(Y_s + z) - b_K(Y_s + z)\|_{L^2(\Omega,\mathbf{Q})} \leq \|b\|_\infty \sqrt{\mathbf{Q}\left(|Y_s + z| \geq K\right)}. \tag{5.21}$$

To estimate this tail probability, we use a classical result for tail probabilities of Gaussian random vectors. In fact, since Y_s is a centered Gaussian random vector with covariance matrix

$$A_{\tau,s} := \int_\tau^s a(u)du,$$

and according to Proposition 6.8 in [20], we have that

$$\mathbf{Q}\left(|Y_s + z| \geq K\right) \leq \mathbf{Q}\left(|Y_s| \geq K - |z|\right) \leq 2d \exp\left(- \frac{(K - |z|)^2}{2d \sum_{i,j=1}^d \|\sigma_{ij}\|^2_{L^2[\tau,s]}} \right).$$

Therefore as $\|\sigma_{ij}\|^2_{L^2[\tau,s]} \leq \|\sigma_{ij}\|^2_{L^2[0,T]}$ and the definition of K, we obtain that

$$\mathbf{Q}\left(|Y_s + z| \geq K\right) \leq 2d \left(1 + |\theta|^2\right)^{-\frac{2\gamma}{p}} \tag{5.22}$$

for any $s \in [\frac{\tau+t_0}{2}, t_0]$.

Now, from (5.21) and (5.22), we have that

$$\|b(Y_s + z) - b_K(Y_s + z)\|_{L^2(\Omega,\mathbf{Q})} \leq \|b\|_\infty \sqrt{\mathbf{Q}\left(|Y_s + z| \geq K\right)} \leq C_{b,d} \left(1 + |\theta|^2\right)^{-\frac{\gamma}{p}}, \tag{5.23}$$

where $C_{b,d} := \|b\|_\infty \sqrt{2d}$. Therefore, from the inequalities (5.20) and (5.23) and the range of the integral below, there exists some positive constant $\tilde{C}_n$ which depends only on d, T, σ, b and n such that

$$\begin{aligned} &(2\pi)^d \int_{\frac{t_0+\tau}{2}}^{t_0} e^{-c(t_0-s)^\beta |\theta|^2} |\theta| \|b(Y_s + z) - b_K(Y_s + z)\|_{L^2(\Omega,\mathbf{Q})} \|M_s^{(n-1)}\|_{L^2(\Omega,\mathbf{Q})} ds \qquad (5.24) \\ &\leq \tilde{C}_n \left(1 + |\theta|^2\right)^{-\frac{\gamma}{p}} |\theta| \int_{\frac{t_0+\tau}{2}}^{t_0} e^{-c(t_0-s)^\beta |\theta|^2} ds. \end{aligned}$$

Hence, from (5.24) and (8.4), there exists some constant $\tilde{C}_n$ which depends only on d, T, σ, b, n and β such that

$$\begin{aligned} &(2\pi)^d \int_{\frac{t_0+\tau}{2}}^{t_0} e^{-c(t_0-s)^\beta |\theta|^2} |\theta| \|b(Y_s + z) - b_K(Y_s + z)\|_{L^2(\Omega,\mathbf{Q})} \|M_s^{(n-1)}\|_{L^2(\Omega,\mathbf{Q})} ds \\ &\leq \tilde{C}_n \left(1 + |\theta|^2\right)^{-\frac{\gamma}{p} - \frac{1}{\beta} + \frac{1}{2}}. \qquad (5.25) \end{aligned}$$

Now from the inequalities (5.13), (5.25) and (5.18), we see that

$$|I_n(t_0, \theta, z)| \leq \frac{C_{n,r} J_n(z, \theta)}{(t_0 - \tau)^r (1 + |\theta|^2)^{\frac{\gamma}{p} + \frac{1}{\beta} - \frac{1}{2}}}$$

holds with some positive constant $C_{n,r}$ which depends only on $p, \gamma, d, T, \sigma, b, g, n, r$ and β.

On the other hand, the estimate for $|I_n(u, \theta, z)|$ with $u \in [\tau, t_0)$ can be obtained using the same argument as above with $\beta = 1$ and u instead of t_0. □

Remark 5.11. *Inspecting the proof, one realizes that given the estimate in* (5.15)*, one can not improve the present estimate in terms of the power of* $1 + |\theta|^2$*.*

5.3 Proof of Proposition 4.3

Now we prove Proposition 4.3. Let $\tau \in [t_0 - \delta, t_0)$. From Lemma 5.3 and (5.5), we see that

$$\left|\mathbf{E}\left[e^{i\langle\theta, X_{t_0}\rangle}\right]\right| = \left|\mathbb{E}\left[\mathbb{E}\left[e^{i\langle\theta, Y_{t_0}+z\rangle} Z_{t_0}(z)\right]\Big|_{z=X_\tau}\right]\right| \le \sum_{n=0}^{N} \mathbb{E}\left[|I_n(t_0,\theta,X_\tau)|\right] + \sup_{z\in\mathbf{R}^d} |R_N(t_0,\theta,z)|$$

holds for any $N \in \mathbf{Z}_+$. From (5.11) and Propositions 5.6 and 5.4 we see that

$$\sum_{n=0}^{N} \mathbb{E}\left[|I_n(t_0,\theta,X_\tau)|\right] + \sup_{z\in\mathbf{R}^d} |R_N(t_0,\theta,z)| \le \sum_{n=0}^{N} \frac{C_{n,r}\mathbb{E}\left[J_n(X_\tau,\theta)\right]}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}} + C_N(t_0-\tau)^{\frac{N}{2}}.$$

By the definition of J_n, the hypotheses (A1) and (A3) yield that there exists some positive constant C_n which depends only on $p,\gamma,d,T,\sigma,b,n,g$ and β such that $\mathbb{E}[|X_\tau|^n] \le C_n$ and

$$\mathbb{E}\left[J_n(X_\tau,\theta)\right] \le C_n\left(1+\sqrt{\log(1+|\theta|^2)}\right)^{nm}.$$

So that we have

$$\sum_{n=0}^{N} \frac{C_{n,r}\mathbb{E}\left[J_n(X_\tau,\theta)\right]}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}} + C_N(t_0-\tau)^{\frac{N}{2}} \le \sum_{n=0}^{N} \frac{C_{n,r}\left(1+\sqrt{\log(1+|\theta|^2)}\right)^{nm}}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}} + C_N(t_0-\tau)^{\frac{N}{2}}.$$

Remark here, that this inequality holds for any $\tau \in [t_0-\delta, t_0)$ and the constants $C_{n,r}$ are independent of τ, θ and X_τ.

Let $\lambda \in (0, \frac{2\gamma}{p} + \frac{2}{\beta} - 1)$. Take a positive number ε such that

$$\varepsilon < \frac{1}{r}\left(\frac{\gamma}{p} + \frac{1}{\beta} - \frac{1}{2} - \frac{\lambda}{2}\right)$$

and $N \in \mathbf{N}$ big enough such that $N > \frac{\lambda}{\varepsilon}$. Then for large enough $|\theta|$, we can take $\tau = t_0 - (1+|\theta|^2)^{-\varepsilon}$ and hence we have

$$\sum_{n=0}^{N} \frac{C_{n,r}(1+\log(1+|\theta|^2))^{\frac{nm}{2}}}{(t_0-\tau)^r(1+|\theta|^2)^{\frac{\gamma}{p}+\frac{1}{\beta}-\frac{1}{2}}} + C_N(t_0-\tau)^{\frac{N}{2}} \le C(1+|\theta|^2)^{-\frac{\lambda}{2}},$$

where C is some positive constant which depends only on $p,\gamma,d,T,\sigma,b,g,N,\varepsilon,r$ and β. Moreover, since any characteristic function is bounded by one, the above estimate holds for any θ. This completes the proof of Proposition 4.3. □

6 Examples and Applications

In this section, we will discuss some applications and an explicit example of non-differentiable densities. We will also give an example of irregular drift coefficient b which satisfies our hypothesis and introduce an example of diffusion coefficient σ which shows the meaning of β which appeared in Theorem 3.2 in the case $\beta < 1$.

6.1 Relation between the Hölder Continuity of the Density and the Fourier Transform of the Drift Coefficient

For this subsection, consider the following condition (A2′);

(A2′). Suppose that $b : \mathbf{R}^d \to \mathbf{R}^d$ satisfies the following inequality for some $\alpha > 0$ and any $K > 0$;

$$|\mathscr{F} b_K(\theta)| \le \frac{g(K)}{(1+|\theta|)^{d-1+\alpha}},$$

where g is the function defined in (A2).

Assume that $\gamma > 0$ and $p > 1$ satisfy the inequalities $(d-1+\alpha)p - 2\gamma > d$ and $(d-1)p - 2\gamma < d$ which correspond respectively to the integrability condition in the norm $H_{\gamma,p}$ and the condition on (γ, p) appearing in hypothesis (A2). Then (A2′) implies (A2).

A basic analysis of these two inequalities in the coordinates (p, γ) shows that

$$\bigcup_{(p,\gamma)\in\Gamma} \left(0, \frac{2\gamma}{p} + \frac{2}{\beta} - 1 - d\right) \supset \left(0, \alpha + \frac{2}{\beta} - 2\right),$$

where Γ is the subset of $\mathbf{R}^2$ defined by

$$\Gamma := \{(p,\gamma) \in (1,+\infty) \times (0,+\infty); \text{(A2) holds with } p \text{ and } \gamma\}.$$

Since $\beta \in (0,1]$ and α is positive, the above interval is not empty. Hence, Theorem 3.2 gives us the following result.

Corollary 6.1. *Fix $t_0 \in (0,T]$ as in* (A4)*. Assume that* (A1), (A3), (A4), (A5) *and* (A2′) *hold. Then X_{t_0} admits a C^λ density, where $\lambda \in (0, \alpha + \frac{2}{\beta} - 2)$.*

6.1.1 Example: Indicator Function of the Unit Ball

Now we introduce a concrete example of an irregular drift coefficient b which satisfies the assumption (A2′). Define the function $b = (b_j)_{1\le j\le d}$ as

$$b_j(x) = 1_{\mathbf{B}(1)}(x).$$

Then its Fourier transform $\mathscr{F} b$ can be calculated explicitly as follows;

$$\mathscr{F} b(\theta) = (\mathscr{F} b_j(\theta))_{1\le j\le d} = \left(\frac{J_{\frac{d}{2}}(2\pi|\theta|)}{|\theta|^{\frac{d}{2}}}\right)_{1\le j\le d},$$

where $J_{\frac{d}{2}}$ is the Bessel function of order $\frac{d}{2}$. It is known that for each $d \ge 1$, the asymptotic behavior of $J_{\frac{d}{2}}(|\theta|)$ at $|\theta| = 0$ is $\mathcal{O}(|\theta|^{\frac{d}{2}})$ and at $+\infty$ is $\mathcal{O}\left(|\theta|^{-\frac{1}{2}}\right)$ (see Appendix B in [10]). Here $\mathcal{O}$ denotes Landau symbol.

Hence there exists some positive constant C such that

$$|\mathscr{F} b(\theta)| \le \frac{C}{(1+|\theta|)^{\frac{d+1}{2}}}. \tag{6.1}$$

This estimate implies that (A2′) holds for $\alpha = \frac{3-d}{2}$. In order to have that $\alpha > 0$ we need to impose that $d \le 2$. Then, the solution of (3.1) has a continuous density and for any $\lambda \in (0, \frac{3-d}{2})$, it is λ-Hölder continuous. A slight generalization of the above argument gives the following result.

Corollary 6.2. *Let X denote the unique weak solution to the following SDE*

$$X_t = x + \int_0^t \sigma(s) dB_s + \int_0^t \sum_{i=1}^d a_i e_i 1_{\mathbf{B}(x_i, r_i)}(X_s) ds,\ t \in [0, T].$$

Here σ is a Borel measurable function which satisfies the assumptions (A3) *and* (A4) *with $\beta = 1$, $a_i \in \mathbf{R}$, $x_i \in \mathbf{R}^d$, $r_i > 0$, $i = 1, ..., d$ and $\{e_i; i = 1, ..., d\}$ denotes the canonical base of $\mathbf{R}^d$.*

Further assume that $d \leq 2$. Then X_t has a density which belongs to the space C^λ for $\lambda \in (0, \frac{3-d}{2})$ for any $t \in (0, T]$.

Remark 6.3. *1. Corollary* 3.3 *in [18] shows us that we can find many more irregular drift functions which satisfy the inequality* (6.1).

2. It is also possible to treat the case of the indicator function of a cube. The techniques are similar although the arguments have to be redone even from the statement of Theorem 3.2 as the calculations have to be carried out coordinate-wise. Therefore we will treat this case in another article.

3. We remark that the case $d = 1$, $b(x) = 1_{(-\infty,0)}(x) - 1_{(0,\infty)}(x)$, $\sigma \equiv 1$ has been treated in Section 6.5 of [13]. The density of X_t has an explicit form due to the particular form of b. One can therefore compute the derivative of the density and prove that it is discontinuous at 0. This result is included in Corollary 6.1 for $\alpha = 1$ and $\beta = 1$.

6.2 An SDE with Non-deterministic Diffusion Coefficient

Let $d = 1$ and $\{X_t\}_{0 \leq t \leq T}$ be the solution of the following SDE;

$$X_t = \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds,\ t \in [0, T], \tag{6.2}$$

where $\sigma \in C_b^2(\mathbf{R}; \mathbf{R})$ is uniformly elliptic, $b(x) = 1_{(a,b)}(x)$, $x \in \mathbf{R}$ and $-\infty \leq a < b \leq \infty$. Note that for (6.2), the hypothesis (A5) holds (see [3]).

Since we have only treated the deterministic diffusion coefficient case, Theorem 3.2 can not be applied directly. However, using the Lamperti transform, we may obtain the Hölder continuity result for the density of X_t.

Let

$$F(x) := \int_0^x \frac{dy}{\sigma(y)},\ x \in \mathbf{R}.$$

Since $\sigma \in C_b^2(\mathbf{R}; \mathbf{R})$ is uniformly elliptic, F is invertible and belongs to $C_b^2(\mathbf{R}; \mathbf{R})$. We denote by F^{-1} the inverse function of F which is a twice differentiable function with bounded derivatives.

Define $Y_t := F(X_t)$ for $t \in [0, T]$. Then using Itô's formula, Y_t can be represented as

$$\begin{aligned} Y_t &= B_t + \int_0^t \left(\frac{b}{\sigma} - \frac{\sigma'}{2} \right) (X_s) ds \\ &= B_t + \int_0^t \left(\frac{b}{\sigma} - \frac{\sigma'}{2} \right) \left(F^{-1}(Y_s) \right) ds, \end{aligned}$$

where σ' denotes the derivative of σ.

For this SDE, (A3) and (A4) clearly hold with $\beta = 1$ and since $\sigma \in C_b^2$ is uniformly elliptic. Furthermore (A1) also holds clearly. Moreover, one can show that there exists some positive constant C such that

$$\left| \mathscr{F}\left(\left(\frac{b}{\sigma} - \frac{\sigma'}{2} \right) \circ \left(F^{-1} \right) \varphi_K \right) \right| \leq \frac{CK}{1+|\theta|}.$$

In fact, the above result is obtained by considering separately the Fourier transforms of $\frac{b}{\sigma} \circ F^{-1}\varphi_K$ and $\frac{\sigma'}{2} \circ F^{-1}\varphi_K$. For the second, one applies integration by parts once and then the following two facts: $\sigma \in C_b^2$ and that φ_K is supported in $[-K-1, K+1]$. For the first, one considers two subcases: in the case that $|a|, |b| < \infty$ and then $K = \max\{|a|, |b|\}$ will suffice together with an integration by parts. Otherwise, if $|a|$ or $|b|$ are ∞ then the calculation is similar as the first case (applying integration by parts once).

Therefore, (A2′) in Section 6.1 holds with $\alpha = 1$. Hence, we see that Y_t has a continuous density q_t for each $t \in (0, T]$. Moreover, q_t is λ-Hölder continuous for any $\lambda \in (0, 1)$. For $t \in (0, T]$, define

$$p_t(x) := \frac{q_t(F(x))}{\sigma(x)}.$$

Then by the change of variables theorem, it is easy to see that p_t is the density function of X_t and has the same Hölder continuity as that of q_t. Summarizing the above, we have the following result

Corollary 6.4. *Consider the SDE* (6.2)*. If $\sigma \in C_b^2$ is uniformly elliptic and $b(x) = 1_{(a,b)}(x)$ where $-\infty \leq a < b \leq \infty$ then for each $t \in (0, T]$, X_t admits the density p_t which belongs to the space C^λ for any $\lambda \in (0, 1)$ and any $t \in (0, T]$.*

One may extend the above example to the multi-dimensional case in the situations where the Doss formula (see [13], Section 5.2, Proposition 2.2.1) can be applied.

6.3 An example of an exploding noise which generates more regularity

We introduce an example of a diffusion coefficient σ for which $\beta < 1$ and $t_0 = T$ in (3.2). Let $\rho \in (0, \frac{1}{2})$. Define

$$\sigma(s) = \begin{cases} \frac{1}{(T-s)^\rho} I_d; & s \in [0, T), \\ I_d; & s = T, \end{cases} \tag{6.3}$$

where I_d is an identity matrix of $\mathbf{R}^d \times \mathbf{R}^d$. Then for any $s \in (0, T]$, we have

$$\int_s^T \langle a(u)\theta, \theta \rangle du = \frac{|\theta|^2 (T-s)^{1-2\rho}}{1-2\rho}.$$

This implies that we can take $\beta = 1 - 2\rho$ at T in (3.2).

This example can be understood as a case, where there is sufficient noise near T in order to increase the regularity of the density.

Corollary 6.5. *Consider the one dimensional SDE*

$$X_t = x + \int_0^t \sigma(s) dB_s + \int_0^t b(X_s) ds, \ t \in [0, T]$$

where σ is given by (6.3) *and b satisfies* (A1) *and* (A2)*. Then assumptions* (A3) *and* (A4) *are satisfied with $\beta = 1 - 2\rho$ and therefore the density of X_T exists and it belongs to the space C^λ for any $\lambda < \frac{2\gamma}{p} + \frac{2}{\beta} - 1 - d$.*

Remark 6.6. *1. We remark that hypothesis* (A5) *is satisfied due to a classical limit argument. In fact, for any interval* $[0, t]$ *with* $t < T$ *one has weak existence and uniqueness of solutions. Then it is not difficult to see that the limit in law of* X_t *as* $t \to T$ *exists.*

2. As claimed in the introduction and in Remark 3.3.2. as ρ *gets closer to* $\frac{1}{2}$*,* β *gets closer to* 0 *and therefore the density has more derivatives.*

7 Some Conclusions and Final Comments

We have seen that if the Fourier transform of the drift term exists then its integrability (or decay at infinity) may be the determining factor in the regularity of the density. We have shown that some representative examples can be studied with our results. In particular, we believe that the irregularity of the drift coefficient plays also an important role in determining the regularity of the density.

For $d \geq 2$, we can also consider marginal densities of X_{t_0}. Our main result tells us that they are smoother than the joint density of X_{t_0}. For example, assume that (A2′) in section 6.1 holds with $\alpha > 0$. If we take $\theta = (\theta_1, 0, \ldots, 0)$ then

$$\phi(\theta_1) := \mathbf{E}\left[e^{i\theta_1 X^1_{t_0}}\right] = \mathbf{E}\left[e^{i\langle \theta, X_{t_0}\rangle}\right],$$

where $X^1_{t_0}$ is the first component of X_{t_0}. Hence $|\theta_1|^\eta |\phi(\theta_1)|$ belongs to $L^1(\mathbf{R})$ for any $\eta \in (0, d-1+\alpha+\frac{2}{\beta}-2)$. Since $d \geq 2$, $\alpha > 0$ and $\beta \leq 1$, we see that $d-1+\alpha+\frac{2}{\beta}-2 \geq 1+\alpha$. Therefore the marginal density of $X^1_{t_0}$ is differentiable even if the joint density is not. Therefore a deeper investigation on this matter is needed.

8 Appendix 1: Proofs of Auxiliary Lemmas

We recall the reader that we are assuming hypotheses (A1)-(A5) throughout this section.

8.1 Proof of Lemma 5.8

Proof. Let $n \in \mathbf{N}$, $\theta, z \in \mathbf{R}^d$ and $u \in [\tau, t_0]$, where t_0 satisfies the assumption (A4). By the definitions of I_n, (5.6), $M^{(n)}$ and (5.4), we have that

$$I_n(u, \theta, z) = \mathbb{E}\left[e^{i\langle \theta, Y_u + z\rangle} M^{(n)}_u\right] = \mathbb{E}\left[e^{i\langle \theta, Y_u + z\rangle} \int_\tau^u M^{(n-1)}_s h(s, Y_s + z) dW_s\right].$$

Since the Skorokhod integral for adapted process coincides with its Itô integral (see Proposition 1.3.11 in [19]), the duality relation (see Definition 1.3.1 in [19]) and (5.1) yield

$$\mathbb{E}\left[e^{i\langle \theta, Y_u + z\rangle} \int_\tau^u M^{(n-1)}_s h(s, Y_s + z) dW_s\right] = \mathbb{E}\left[\int_\tau^u i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_u + z\rangle} M^{(n-1)}_s ds\right].$$

Now Fubini's theorem and the independent increment property of Y (see Lemma 5.2) yield

$$\begin{aligned}\mathbb{E}\left[\int_\tau^u i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_u + z\rangle} M^{(n-1)}_s ds\right] &= \int_\tau^u \mathbb{E}\left[i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_u + z\rangle} M^{(n-1)}_s\right] ds \\ &= \int_\tau^u \mathbb{E}\left[e^{i\langle \theta, Y_u - Y_s\rangle}\right] \mathbb{E}\left[i\langle \theta, b(Y_s + z)\rangle e^{i\langle \theta, Y_s + z\rangle} M^{(n-1)}_s\right] ds.\end{aligned}$$

Finally, calculating $\mathbb{E}[e^{i\langle \theta, Y_u - Y_s\rangle}]$ via the transition density $p_{s,u}$ and noting that $p_{s,u}$ is symmetric (see Lemma 5.2), we see that

$$\mathbb{E}\left[e^{i\langle \theta, Y_u - Y_s\rangle}\right] = (2\pi)^d \mathscr{F} p_{s,u}(\theta).$$

□

8.2 Proof of Lemma 5.9

Proof. Let $d \in \mathbf{N}$, $z, \theta \in \mathbf{R}^d$ and $n \in \mathbf{Z}_+$. Assume that ν and μ be positive constants which satisfy that $\nu + \mu > \frac{d}{2}$ and fix through the proof any $\varepsilon \in (0, \min\{\nu + \mu - \frac{d}{2}, \mu\})$.

By the definition of g (see (A2)) we have

$$g\left(|z| + \sqrt{C_0 \log(1+|\eta|^2)}\right)^n \leq 2^{(m-1)n} C^n \left\{1 + |z|^m + \left(C_0 \log(1+|\eta|^2)\right)^{\frac{m}{2}}\right\}^n.$$

From this estimate, we see that there exists some positive constant $\tilde{C}_n$ which depends only on ε, m, d, n and C_0 such that

$$\sup_{\eta \in \mathbf{R}^d} \frac{g\left(|z| + \sqrt{C_0 \log(1+|\eta|^2)}\right)^n}{(1+|\eta|^2)^{\varepsilon}} \leq \tilde{C}_n g(|z|)^n.$$

Hence we have

$$\int_{\mathbf{R}^d} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{g\left(|z| + \sqrt{C_0 \log(1+|\eta|^2)}\right)^n}{(1+|\eta|^2)^{\nu+\mu}} d\eta \leq \tilde{C}_n g(|z|)^n \int_{\mathbf{R}^d} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta.$$

Therefore, to prove Lemma 5.9, it is enough to show that there exists a positive constant C such that

$$\int_{\mathbf{R}^d} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta \leq \frac{C}{(1+|\theta|^2)^{\nu}}. \tag{8.1}$$

To begin with, we divide the integral in two integrals on the sets $\Gamma := \{\eta \in \mathbf{R}^d ; |\eta - \theta| \geq \frac{|\theta|}{2}\}$ and Γ^c. After that we estimate each integral. Now we estimate the integral on Γ. By the definition of Γ, we see that

$$\int_{\Gamma} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta \leq \frac{1}{(1+\frac{|\theta|^2}{4})^{\nu}} \int_{\mathbf{R}^d} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta.$$

Since $\nu + \mu - \varepsilon > \frac{d}{2}$, the above integral is finite and hence there exists some positive constant C which depends only on ν, μ, ε and d such that

$$\int_{\Gamma} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta \leq \frac{C}{(1+|\theta|^2)^{\nu}}. \tag{8.2}$$

Now we estimate the integral on Γ^c. Since $\mu - \varepsilon > 0$ and $|\eta| > \frac{|\theta|}{2}$ holds for any $\eta \in \Gamma^c$, we have

$$\begin{aligned} &\int_{\Gamma^c} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta \\ &\leq \frac{1}{(1+\frac{|\theta|^2}{4})^{\nu}} \int_{\Gamma^c} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\mu-\varepsilon}} d\eta. \end{aligned}$$

Define the set $B := \{\eta \in \mathbf{R}^d ; |\eta - \theta| > |\eta|\}$ and then we obtain that

$$\begin{aligned} &\int_{\Gamma^c} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\mu-\varepsilon}} d\eta \\ &= \int_{\Gamma^c \cap B} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\mu-\varepsilon}} d\eta + \int_{\Gamma^c \cap B^c} \frac{1}{(1+|\eta-\theta|^2)^{\nu}} \frac{1}{(1+|\eta|^2)^{\mu-\varepsilon}} d\eta \\ &\leq \int_{\Gamma^c \cap B} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta + \int_{\Gamma^c \cap B^c} \frac{1}{(1+|\eta-\theta|^2)^{\nu+\mu-\varepsilon}} d\eta \\ &\leq 2 \int_{\mathbf{R}^d} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta. \end{aligned}$$

Therefore we have

$$\int_{\Gamma^c} \frac{1}{(1+|\eta-\theta|^2)^\nu} \frac{1}{(1+|\eta|^2)^{\nu+\mu-\varepsilon}} d\eta \leq \frac{2C}{(1+|\theta|^2)^\nu}, \tag{8.3}$$

where C is the same constant which appeared in (8.2).

The estimates (8.2) and (8.3) imply (8.1) and this completes the proof. □

8.3 Proof of Lemma 5.10

We turn to the proof of Lemma 5.10.

Proof. Let $n \in \mathbf{N}$, $\theta, z \in \mathbf{R}^d$ and $s \in [\tau, t_0]$ be as in the statement of the Lemma 5.10. Define

$$\mathbb{E}[M_s^{(n-1)}](y) := \mathbb{E}[M_s^{(n-1)}|Y_s = y].$$

Then we have

$$\mathbb{E}\left[i\langle\theta, \vartheta(Y_s+z)\rangle e^{i\langle\theta, Y_s+z\rangle} M_s^{(n-1)}\right] = \int_{\mathbf{R}^d} i\langle\theta, \vartheta(y+z)\rangle e^{i\langle\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy.$$

Apply the Fourier inversion theorem for $\vartheta(y+z)$, in order to obtain

$$\begin{aligned}
&\int_{\mathbf{R}^d} i\langle\theta, \vartheta(y+z)\rangle e^{i\langle\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy \\
&= \sum_{j=1}^d i\theta_j \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} e^{i\langle y+z, \xi\rangle} \mathcal{F}\vartheta_j(\xi) d\xi\right) e^{i\langle\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy.
\end{aligned}$$

Now, Fubini's theorem yields

$$\begin{aligned}
&\sum_{j=1}^d i\theta_j \int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} e^{i\langle y+z, \xi\rangle} \mathscr{F}\vartheta_j(\xi) d\xi\right) e^{i\langle\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy \\
&= \sum_{j=1}^d i\theta_j \int_{\mathbf{R}^d} \mathscr{F}\vartheta_j(\xi) \left(\int_{\mathbf{R}^d} e^{i\langle\xi+\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy\right) d\xi.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_{\mathbf{R}^d} e^{i\langle\xi+\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy &= \mathbb{E}\left[e^{i\langle\xi+\theta, Y_s+z\rangle} M_s^{(n-1)}\right] \\
&= I_{n-1}(s, \xi+\theta, z),
\end{aligned}$$

we see that

$$\begin{aligned}
&\sum_{j=1}^d i\theta_j \int_{\mathbf{R}^d} \mathscr{F}\vartheta_j(\xi) \left(\int_{\mathbf{R}^d} e^{i\langle\xi+\theta, y+z\rangle} \mathbb{E}[M_s^{(n-1)}](y) p_{\tau,s}(y) dy\right) d\xi \\
&= \int_{\mathbf{R}^d} i\langle\theta, \mathscr{F}\vartheta(\xi)\rangle I_{n-1}(s, \xi+\theta, z) d\xi.
\end{aligned}$$

Now the change of variables with $\eta = \theta + \xi$ yields our desired equality. □

8.4 Proof of a tail estimate

Lemma 8.1. *There exists a positive constant C which depends only on $\beta \in (0,1]$ and T such that*

$$|\theta| \int_{\frac{t_0+\tau}{2}}^{t_0} e^{-|\theta|^2 (t_0-s)^\beta} ds \leq C(1+|\theta|^2)^{\frac{1}{2}-\frac{1}{\beta}}. \tag{8.4}$$

Proof. The change of variables with $u = |\theta|^{\frac{2}{\beta}}(t_0 - s)$ and $t_0 \leq T$ implies that

$$|\theta| \int_{\frac{t_0+\tau}{2}}^{t_0} e^{-|\theta|^2 (t_0-s)^\beta} ds = |\theta|^{1-\frac{2}{\beta}} \int_0^{\frac{(t_0-\tau)|\theta|^{\frac{2}{\beta}}}{2}} e^{-u^\beta} du \leq |\theta|^{1-\frac{2}{\beta}} \int_0^{T|\theta|^{\frac{2}{\beta}}} e^{-u^\beta} du.$$

We now obtain the rate at which the last term converges to zero as $|\theta|$ tends to infinity because $\int_0^{T|\theta|^{\frac{2}{\beta}}} e^{-s^\beta} ds$ is uniformly bounded as a function of $|\theta|$. When $|\theta| \leq 1$, the bound in (8.4) can be obtained as the integral converges to zero at the rate $|\theta|^{\frac{1}{\beta}}$. □

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