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1-2 model, dimers, and clusters

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Abstract

A 1-2 model configuration is a subgraph of the hexagonal lattice, in which the edge-degree of each vertex is either 1 or 2. We prove that for any translation invariant Gibbs measure of the 1-2 model configurations on the whole-plane hexagonal lattice, almost surely there are no infinite paths. Using a measure-preserving correspondence between 1-2 model configurations on the hexagonal lattice and perfect matchings on a decorated graph, we construct an explicit translation invariant Gibbs measure for 1-2 model configurations on the bi-periodic hexagonal lattice. We prove that the behaviors of infinite clusters are different for small and large local weights, which shows the existence of a phase transition.

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Computer scientists M. Schwartz and J. Bruck ([18]) proposed the uniform 1-2 model (not-all-equal-relation), as a graphical model whose partition function (total number of configurations) can be computed by computing determinants via the holographic algorithm ([19]). A general version of the 1-2 model was explored in [15], as an application of a generalized holographic algorithm, and local statistics were computed. The idea of the holographic algorithm is to relate general vertex configurations to perfect matchings through a base change, hence it significantly enlarges the class of exactly solvable vertex models. The holographic algorithm, although very general and beautiful, turns out not to be an efficient method to solve the 1-2 model.

In this paper, we introduce a new approach to solve the 1-2 model exactly, by constructing a measure-preserving correspondence between 1-2 model configurations on the hexagonal lattice and perfect matchings on a decorated graph. With the help of such a measure-preserving correspondence, we compare the behaviors of infinite homogeneous clusters for small and large local weights by analyzing the underlying perfect matchings, and prove the existence of a phase transition (Sect. 5). It is unknown whether the phase-transition result can be proved using the more complicated holographic algorithm.

We start with the basic properties of the underlying hexagonal lattice on which the 1-2 model is defined. Let $\mathbb{H} = (V, E)$ be a hexagonal lattice. It is a bipartite graph, in the

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sense that all the vertices of $\mathbb{H}$ can be colored black and white so that black vertices are incident only to white vertices and vice versa. Let Γ be a subgroup of the automorphism group of $\mathbb{H}$, acting on $\mathbb{H}$ by translations. Note that Γ is isomorphic to $\mathbb{Z}^2$. In particular, each $\gamma \in \Gamma$ is a color-preserving automorphism of $\mathbb{H}$, i.e., it maps each white (resp. black) vertex to another white (resp. black) vertex.

A **1-2 model configuration** $\omega = (V_\omega, E_\omega)$ of $\mathbb{H}$ is a subgraph of $\mathbb{H}$, satisfying

1. $V_\omega = V$,
2. the degree (number of incident edges) for each vertex of V_ω in ω is either 1 or 2.

An example of a 1-2 model configuration on a finite graph is illustrated in Figure 1. In particular, in Figure 1, the graph is drawn in a large rhombus region. We identify the northwestern (resp. northeastern) boundary of the rhombus with the southeastern (resp. southwestern) boundary, and get a graph embedded in a torus (toroidal graph), in which each face is a hexagon. The edges of each hexagon are either blank (non-present) or black (present). The subgraph consisting of black edges in Figure 1, is a 1-2 model configuration, in the sense that the degree of each vertex on the torus is either 1 or 2. We can also consider it as a 1-2 model configuration on a large rhombus region of $\mathbb{H}$ with the periodic boundary condition (see Page 4).

A **self-avoiding path** is a sequence $v_0 e_1 v_1 \cdot \ldots \cdot v_{k-1} e_k v_k$, so that the endpoints of e_i is v_{i-1} and v_i, and $v_i \neq v_j$ for any $i \neq j$. A **loop** is a sequence $v_0 e_1 v_1 \cdot \ldots \cdot v_{k-1} e_k v_0$, so that the endpoints of e_i is v_{i-1} and v_i, and $v_i \neq v_j$ for any $i \neq j$. We can see from Figure 1 that for any 1-2 model configuration ω, the only possible connected components are either self-avoiding paths or loops. The **1-2 model** is a probability measure defined on a sample space consisting of all 1-2 model configurations. The probability measure of the 1-2 model on a finite graph, is a **Boltzmann measure**, i.e., the measure is defined in such a way that the probability of each configuration is proportional to the product of configuration weights at all vertices, which we will explain below.

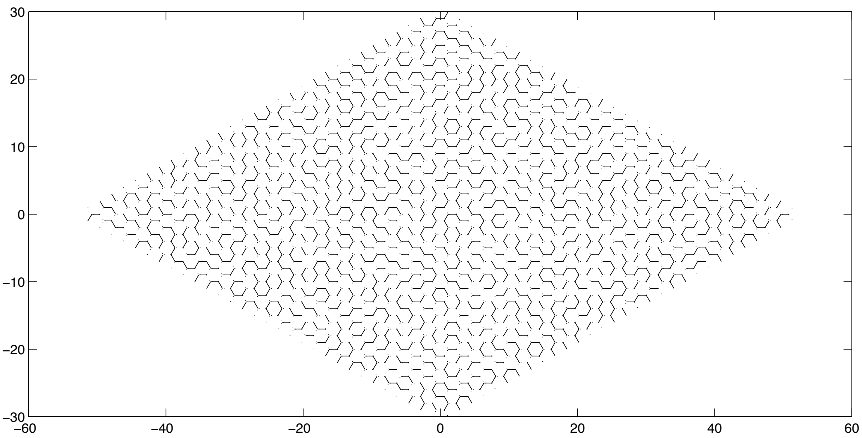

Figure 1: 1-2 model configuration

We look at a configuration ω locally at a vertex $v \in V$, and use $\omega|_v$ to denote the **local configuration** of ω at the vertex v, i.e, the intersection of E_ω with the set of incident edges of v. Each one of the 3 incident edges e_1, e_2, e_3 of v is either present in ω or not; therefore each local configuration at v can be labeled by a 3-digit binary number. Namely, each edge corresponds to a digit, a present edge corresponds to the number

"1" on the digit; while an edge corresponds to "0" if it is not present in a 1-2 model configuration. Such correspondences are illustrated in Figure 2 and Figure 3, where the thick edges are present edges, and thin edges are not present. More precisely, the correspondence between digits of a length-3 binary number and incident edges of a vertex can be described as follows: the right digit corresponds to the horizontal edge; the middle digit corresponds to the northwestern-southeastern edge; and the left digit corresponds to the northeastern-southwestern edge.

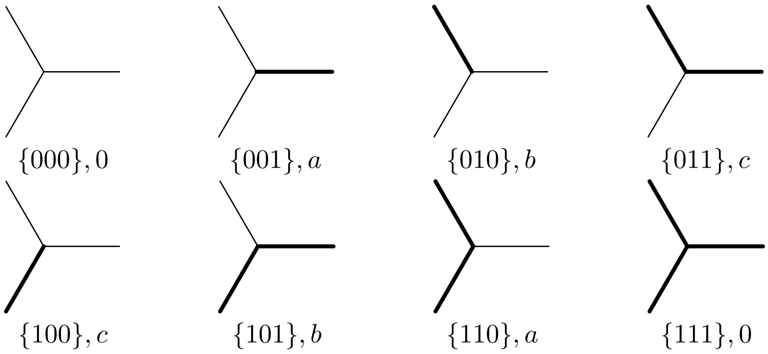

Figure 2: local configurations, binary numbers and weights at a black vertex

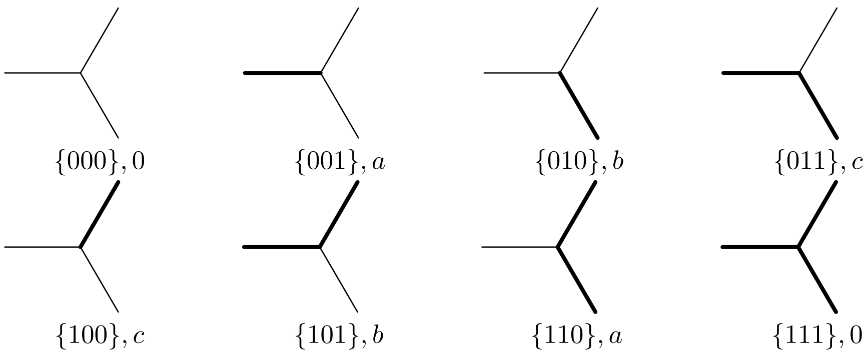

Figure 3: local configurations, binary numbers and weights at a white vertex

A non-negative number (**weight**) is associated to each local configuration at a vertex, i.e, a choice of subsets of incident edges of v, or a specific 3-digit binary number. This way we can write a dimension-8 vector indexed by the 8 different local configurations, so that the entry at a configuration, or a specific 3-digit binary number, is the weight of the configuration. For a 1-2 model, the local configuration $\{000\}$ (no incident edges are included) or $\{111\}$ (all incident edges are included) are not allowed, and we give them a weight 0. Such a vector is called the **signature** at a vertex v. The 1-2 model considered in this paper has the property that each local configuration and its complement have the same weight. Namely, the signature r_v at a vertex v has the following form

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 0 & a & b & c & c & b & a & 0 \end{array}, \tag{0.1}$$

where a, b, c are positive numbers independent of v. The 1-2 model is called **uniform** if $a = b = c$.

Let $\mathbb{H}_n = (V_n, E_n)$ be a hexagonal lattice embedded into a $n \times n$ torus (toroidal graph). The toroidal graph $\mathbb{H}_n$ is a quotient graph of $\mathbb{H}$, with a translation group $\Gamma_n \simeq \mathbb{Z}_n \times \mathbb{Z}_n$, and it is also bipartite. The total number of white vertices in $\mathbb{H}_n$ is n^2, as well as the total number of black vertices, and every vertex is a translation of any other vertex of the same color.

Note that the torus $\mathbb{T}^2$ has a nontrivial homology group $H_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$. Let γ_x, γ_y be two homology generators of $H_1(\mathbb{T}^2)$. In other words, γ_x and γ_y are two simple, essential (non-contractible), and essentially distinct (one cannot be obtained from the other by moving on the surface $\mathbb{T}^2$) cycles winding around the torus. Let E_x (resp. E_y) be all the edges of $\mathbb{H}_n$ crossed by γ_x (resp. γ_y).

Define a **partial-graph** $G_P = (V_{G_P}, E_{G_P}, HE_{G_P})$, where V_{G_P} is the set of vertices, E_{G_P} is the set of edges, so that each edge connects a pair of vertices, and HE_{G_P} is the set of half-edges, and each half-edge has exactly one vertex as its endpoint. We cut the torus along γ_x and γ_y, and obtain a planar partial-graph $\mathbb{H}'_n$. $\mathbb{H}'_n$ is almost the same as $\mathbb{H}_n$ except that for each edge in $E_x \cup E_y$ of $\mathbb{H}_n$, it corresponds to two half edges in $\mathbb{H}'_n$. We use the term partial-graph for $\mathbb{H}'_n$, because strictly speaking, $\mathbb{H}'_n$ is not a graph since each half-edge has only one endpoint. In other words $\mathbb{H}'_n = (V_n, E_n \setminus \{E_x \cup E_y\}, E_x^1 \cup E_y^1 \cup E_x^2 \cup E_y^2)$, where V_n is the vertex set, $E_n \setminus \{E_x \cup E_y\}$ is the set of edges, $E_x^1 \cup E_y^1 \cup E_x^2 \cup E_y^2$ is the set of half-edges, and half-edges in $E_x^1 \cup E_y^1$ (resp. $E_x^2 \cup E_y^2$) are in 1-to-1 correspondence with edges in $E_x \cup E_y$. More precisely, for $e = (u, v) \in E_x \cup E_y$, we have exactly two half-edges e^1 and e^2 corresponding to e, so that $e^1 \sim u,\ e^1 \in E_x^1 \cup E_y^1$, and $e^2 \sim v,\ e^2 \in E_x^2 \cup E_y^2$, where u and v are a pair of incident vertices. As $n \to \infty$, $\mathbb{H}'_n$ exhausts $\mathbb{H}$. We can also see that $\mathbb{H}'_n$ approximates $\mathbb{H}$ in the Benjamini-Schramm sense ([2]). Namely, we fix a vertex v in $\mathbb{H}$, and a vertex v'_n in $\mathbb{H}'_n$, as root vertices of $\mathbb{H}$ and $\mathbb{H}'_n$, respectively. For $r = 1, 2, ...$, let $B_{\mathbb{H}}(v, r)$ be the closed ball of radius r to v in $\mathbb{H}$, and similarly for $\mathbb{H}'_n$ and v'_n. Let k be the supreme of all r, so that there is an isomorphism from $B_{\mathbb{H}}(v, r)$ onto $B_{\mathbb{H}'_n}(v'_n, r)$ which takes v to v'_n. We set $d[(\mathbb{H}, v), (\mathbb{H}'_n, v'_n)] = 2^{-k}$, then $\lim_{n\to\infty} d(\mathbb{H}'_n, \mathbb{H}) = 0$.

A 1-2 model configuration $\omega = (V_\omega, E_\omega, HE_\omega)$ on the partial-graph $\mathbb{H}'_n$ is a sub-partial-graph, satisfying $V_\omega = V_n$, and for each vertex v, the sum of numbers of incident edges and incident half-edges in ω is 1 or 2. We will define a mixed periodic/antiperiodic boundary condition on 1-2 model configurations of $\mathbb{H}'_n$. Namely, we consider 1-2 model configurations on $\mathbb{H}'_n$ satisfying one of the boundary conditions:

1. for any $e \in E_x \cup E_y$, the two corresponding half-edges $e^1 \in E_x^1 \cup E_y^1$, $e^2 \in E_x^2 \cup E_y^2$ of e in $\mathbb{H}'_n$ have the same configuration (either both are present or neither are present), and let $\Omega_{n,1}$ be the set of all such 1-2 model configurations;
2. for any $e_x \in E_x$, the two corresponding half-edges $e_x^1 \in E_x^1$, $e_x^2 \in E_x^2$ of e_x in $\mathbb{H}'_n$ have the opposite configuration (exactly one of them are present); for any $e_y \in E_y$, the two corresponding half-edges $e_y^1 \in E_y^1$, $e_y^2 \in E_y^2$ of e_y in $\mathbb{H}'_n$ have the same configuration, and let $\Omega_{n,2}$ be the set of all such 1-2 model configurations;
3. for any $e_x \in E_x$, the two corresponding half-edges $e_x^1 \in E_x^1$, $e_x^2 \in E_x^2$ of e_x in $\mathbb{H}'_n$ have the same configuration; for any $e_y \in E_y$, the two corresponding half-edges $e_y^1 \in E_y^1$, $e_y^2 \in E_y^2$ of e_y in $\mathbb{H}'_n$ have the opposite configuration, and let $\Omega_{n,3}$ be the set of all such 1-2 model configurations;
4. for any $e \in E_x \cup E_y$, the two corresponding half-edges $e^1 \in E_x^1 \cup E_y^1$, $e^2 \in E_x^2 \cup E_y^2$ of e in $\mathbb{H}'_n$ have the opposite configuration, and let $\Omega_{n,4}$ be the set of all such 1-2 model configurations.

Case 1 (resp. 4) is also called the periodic (resp. antiperiodic) boundary condition. Consider the state space $\Omega_n = \cup_{i=1}^4 \Omega_{n,i}$. We will construct a probability measure on the sample space Ω_n as follows. The probability of configuration $\omega \in \Omega_n$, is defined to be proportional to the product of weights of its local configurations at all the vertices, namely,

$$P_n(\omega) = \frac{\prod_{v\in V_n} w(\omega|_v)}{Z_n}, \qquad \text{if } \omega \in \Omega_n, \tag{0.2}$$

where $w(\omega|_v)$ is the weight of the local configuration $\omega|_v$, and Z_n is a normalizing constant called the **partition function** defined by

$$Z_n := \sum_{\omega\in\Omega_n} \prod_{v\in V_n} w(\omega|_v).$$

In fact, (0.2) defines a Boltzmann measure for 1-2 model configurations on the finite partial-graph $\mathbb{H}'_n$. The mixed periodic/antiperiodic boundary condition is important because it corresponds to the periodic boundary condition of perfect matchings on a torus, which we will explain in Sect. 3. In particular, for the uniform 1-2 model with $a = b = c$, the corresponding probability measure defined above is the uniform measure on the sample space Ω_n.

The definition of Gibbs measure of the 1-2 model is based upon the well-known Dobrushin-Lanford-Ruelle (DLR) definition of a Gibbs state. A Gibbs measure μ for 1-2 model configurations on the hexagonal lattice $\mathbb{H}$ is a probability measure on the sample space of all possible 1-2 model configurations (denote the sample space by Ω), so that for any finite set of vertices $\Lambda \subset V$, and μ almost every configuration s_{Λ^c} on the complement vertex set $\Lambda^c = V \setminus \Lambda$, the (conditional) measure on Λ of a configuration ω, is proportional to the product of configuration weights at all the vertices of Λ, if ω is compatible with s_{Λ^c}, i.e., the configuration ω, restricted on Λ^c (denoted by $\omega|_{\Lambda^c}$), agrees with s_{Λ^c}. If ω is not compatible with s_{Λ^c}, the (conditional) measure on Λ of ω is 0. Namely, let $\mathcal{T}_\Lambda$ be the σ-field generated by the 1-2 model configurations on Λ^c, for μ almost every s_{Λ^c}

$$\mu(\omega|_\Lambda; \mathcal{T}_\Lambda)(s_{\Lambda^c}) = \begin{cases} \frac{1}{Z_{s_{\Lambda_c}}} \prod_{v\in\Lambda} w(\omega|_v) & \text{if } \omega|_{\Lambda^c} = s_{\Lambda^c}; \\ 0 & \text{otherwise}, \end{cases}$$

where $\omega|_v$ is the configuration of ω restricted at the vertex v, i.e. $\omega|_v$ is one of the six possible 1-2 model configurations $\{001\}, \{010\}, \{011\}, \{100\}, \{101\}, \{110\}$, and $w(\cdot)$ is the weight function at a vertex. Moreover,

$$Z_{s_{\Lambda_c}} = \sum_{\{\omega:\omega|_{\Lambda^c}=s_{\Lambda^c}\}} \prod_{v\in\Lambda} w(\omega|_v).$$

Equivalently, a probability measure μ for 1-2 model configurations in Ω is a Gibbs measure if it satisfies the DLR equation: for each $A \subseteq \Omega$, $\Lambda \subseteq V$, $|\Lambda| < \infty$,

$$\mu(A) = \int_{\{s_{\Lambda_c}:\exists\xi\in A, \xi|_{\Lambda_c}=s_{\Lambda_c}\}} \mu(ds_{\Lambda_c}) \frac{1}{Z_{s_{\Lambda_c}}} \int_{\{\omega\in A:\omega|_{\Lambda_c}=s_{\Lambda_c}\}} \prod_{v\in\Lambda} w(\omega|_v).$$

Note that if the weak limit of Boltzmann measures on $\{\mathbb{H}'_n\}_{n=1}^\infty$ exists, as $n \to \infty$, the limit measure will be a Gibbs measure for 1-2 model configurations on $\mathbb{H}$.

Let μ be a probability measure on Ω, we say μ is **translation invariant**, if for any two events $E_1, E_2 \in \Omega$, so that there exists $\gamma \in \Gamma$, satisfying $E_2 = \gamma E_1$, we have $\mu(E_2) = \mu(E_1)$. We say μ is **ergodic**, if for any translation invariant event E ($E = \gamma E$, for all $\gamma \in \Gamma$), either $\mu(E) = 0$ or $\mu(E) = 1$.

A **connected set of vertices** W is a subset of V, so that for any $w_1, w_2 \in W$, there exists a path

$$u_1(= w_1), u_1, u_2, ..., u_n(= w_2),$$

satisfying $u_i \in W$, for $0 \leq i \leq n$, and u_i and u_{i-1} are incident vertices of $\mathbb{H}$, for $1 \leq i \leq n$.

Let us call a connected set of vertices, each of which has a configuration with weight a, an **a-cluster**. We will construct a measure-preserving correspondence between the 1-2 model configurations on $\mathbb{H} = (V, E)$ and the perfect matchings (dimer configurations) on a decorated graph $\mathbb{H}_\Delta = (V_\Delta, E_\Delta)$, so that for each vertex $v \in V$, the bisectors of the three angles of $\mathbb{H}$ at v are edges in E_Δ, see Sect. 3 for details. The main results of this paper can be summarized as follows:

Theorem 0.1. *Consider the 1-2 model on the hexagonal lattice $\mathbb{H}$ with signature given by (0.1). Let P_n be the Boltzmann measure on 1-2 model configurations of $\mathbb{H}'_n$ with mixed boundary conditions, defined in (0.2). The sequence of measures $\{P_n\}_{n=1}^{\infty}$ converges weakly to a translation invariant Gibbs measure P on 1-2 model configurations of $\mathbb{H}$. For the limit measure P,*

1. *Let $\mathcal{L}$ be a path of length $\ell + 1$. Assume $E_{\mathcal{L}} = \{u_1 v_1, \cdots, u_\ell v_\ell\}$ is the set of all the bisector edges of angles with two sides in $\mathcal{L}$, then*

$$P(\text{Path } \mathcal{L} \text{ appears in a } 1-2 \text{ model configuration}) = \frac{1}{2} \prod_{k=1}^{\ell} w_{u_k v_k} \left| \mathrm{Pf} K_{E_{\mathcal{L}}}^{-1} \right|$$

   where $w_{u_k v_k}$ is the weight of the edge $u_k v_k$ in $\mathbb{H}_\Delta$, and $K_{E_{\mathcal{L}}}^{-1}$ is the submatrix the inverse of the weighted adjacency matrix of $\mathbb{H}_\Delta$ with rows and columns indexed by $u_1, v_1, \cdots, u_\ell, v_\ell$. The probability that the path $\mathcal{L}$ appears in a 1-2 model configuration is the probability that $\mathcal{L}$ is contained in a path in a 1-2 model configuration; in other words, all edges in $\mathcal{L}$ are present in a 1-2 model configuration.
2. *Almost surely there are no infinite paths consisting of present edges of a 1-2 model configuration.*
3. *Fixing $b, c > 0$, when a is sufficiently small, almost surely there are no infinite a-clusters. When a is sufficiently large, almost surely there exists a unique infinite a-cluster.*

Here is the outline of the paper. In Sect. 2, we prove results on the expected number of self-avoiding paths of the 1-2 model, as well as the monotonicity of the expected number of specific local configurations with respect to local weights, following directly from the definition of the measure. In Sect. 3, we prove the almost-sure non-existence of infinite paths for any translation invariant Gibbs measure, with the help of the mass-transport principle introduced in [1]. In Sect. 4, we introduce a measure-preserving correspondence between 1-2 model configurations on the hexagonal lattice $\mathbb{H}$ and dimer configurations on a decorated lattice $\mathbb{H}_\Delta$, and prove the weak convergence of the Boltzmann measures using larger and larger tori to approximate the infinite periodic planar graph. Then we prove a closed form of the probability that a self-avoiding path appears in a 1-2 model configuration under the limit measure. In Sect. 5, we prove Part 3 of Theorem 1. The different behaviors of infinite clusters imply the existence of a phase transition.

1 Self-avoiding path and monotonicity

In this section, we prove two propositions resulting from the definition of the measure of the 1-2 model. First of all, we notice that in each 1-2 model configuration, there are two kinds of connected components: either self-avoiding paths or loops, see Figure 1. Proposition 1.1 shows how to compute the expected number of self-avoiding paths explicitly, and Proposition 1.2 shows the monotonicity of the expected number of a specific local configuration with respect to local weights.

Note that the propositions of this section are related to 1-2 model configurations on the toroidal graph $\mathbb{H}_n$, which are equivalent to 1-2 model configurations on the partial graph $\mathbb{H}'_n$ with periodic boundary conditions, i.e., configurations in $\Omega_{n,1}$ described as in Case 1.

Proposition 1.1. *Consider the 1-2 model defined on $\mathbb{H}_n$, the honeycomb lattice embedded into an $n \times n$ torus. Let σ_n denote the number of self-avoiding paths in a random 1-2 model configuration in $\mathbb{H}_n$, then*

$$\mathbb{E}\sigma_n = \frac{1}{2}n^2,$$

where the expectation is taken with respect to the Boltzmann measure of 1-2 model configurations on $\mathbb{H}_n$, or equivalently, conditional Boltzmann measure $P_n(\cdot|b_p)$ of 1-2 model configurations on $\mathbb{H}'_n$ defined in (0.2), conditional on the periodic boundary condition given by Case 1 of the mixed periodic/antiperiodic boundary condition.

Proof. Since each self-avoiding path has two degree-1 vertices as endpoints, and all the other vertices are of degree 2, the number of non-loop connected components is one half of the number of degree-1 vertices ($\mathcal{N}_1$). By symmetry, each configuration and its complement have the same probability, therefore

$$\mathcal{N}_1 =^{law} 2n^2 - \mathcal{N}_1.$$

Hence we have

$$\mathbb{E}\mathcal{N}_1 = n^2.$$

As a result,

$$\mathbb{E}\sigma_n = \frac{1}{2}\mathbb{E}\mathcal{N}_1 = \frac{1}{2}n^2.$$

□

Let $\mathcal{N}_a$ be the number of vertices which have a configuration of weight a (a-configuration). In other words, $\mathcal{N}_a$ is the number of vertices which have configurations 001 or 110. We call all such vertices a-vertices. Whether or not a vertex is an a-vertex depends on the specific configuration taken.

Proposition 1.2. *Fixing the size n of the torus, $\mathcal{N}_{a_2}$ stochastically dominates $\mathcal{N}_{a_1}$ with respect to the conditional measure $P_n(\cdot|b_p)$, if $a_2 > a_1$.*

Proof. It suffices to prove that for any positive integer k, $P_n(\mathcal{N}_a \geq k|b_p)$ is increasing with respect to a.

Let $1_{\mathcal{N}_a \geq k}$ be the indicator of the event that $\mathcal{N}_a \geq k$. Obviously for any $k, k' \geq 0$, $1_{\mathcal{N}_a \geq k}$ and $1_{\mathcal{N}_a \geq k'}$ have positive correlation, i.e.,

$$\mathbb{E}1_{\mathcal{N}_a \geq k} \cdot 1_{\mathcal{N}_a \geq k'} \geq \mathbb{E}1_{\mathcal{N}_a \geq k} \cdot \mathbb{E}1_{\mathcal{N}_a \geq k'},$$

where the expectation is taken with respect to the probability measure of 1-2 model configurations on the torus $\mathbb{H}_n$ given periodic boundary conditions.

We claim that $1_{\mathcal{N}_{a_1}\geq k}$ and $\left(\frac{a_2}{a_1}\right)^{\mathcal{N}_{a_1}}$ also have positive correlation. Note that

$$\begin{aligned}\left(\frac{a_2}{a_1}\right)^{\mathcal{N}_{a_1}} &= \exp\left[\mathcal{N}_{a_1}\log\left(\frac{a_2}{a_1}\right)\right]\\ &= \sum_{t=0}^{\infty}\frac{\mathcal{N}_{a_1}^t\left[\log\left(\frac{a_2}{a_1}\right)\right]^t}{t!}\\ &= \sum_{t=0}^{\infty}\frac{\left[\log\left(\frac{a_2}{a_1}\right)\right]^t}{t!}\sum_{1\leq k'\leq 2n^2}[(k')^t-(k'-1)^t]1_{\mathcal{N}_{a_1}\geq k'}.\end{aligned}$$

Therefore, since $1_{\mathcal{N}_a\geq k}$ and $1_{\mathcal{N}_a\geq k'}$ have positive correlation, $1_{\mathcal{N}_{a_1}\geq k}$ and $\left(\frac{a_2}{a_1}\right)^{\mathcal{N}_{a_1}}$ have positive correlation as well. Hence $P_n(\mathcal{N}_a\geq k|b_p)$ is increasing with respect to a.

In particular, the lemma implies that for any $\gamma>0$, $\mathbb{E}\mathcal{N}_a^\gamma$ is increasing with respect to a, since

$$\mathbb{E}\mathcal{N}_a^\gamma=\sum_{1\leq k\leq 2n^2}k^\gamma P_n(\mathcal{N}_a=k|b_p)=\sum_{1\leq k\leq 2n^2}(k^\gamma-(k-1)^\gamma)P_n(\mathcal{N}_a\geq k|b_p).$$

□

2 Nonexistence of infinite paths

In this section, we will apply the mass-transport principle ([1]) to prove the almost-sure nonexistence of infinite paths in 1-2 model configurations for any translation invariant Gibbs measure with respect to positive parameters a,b,c. Define a **semi-infinite path** to be an infinite self-avoiding path consisting of edges of $\mathbb{H}$, which has a finite endpoint and is infinite in just one direction. In other words, this infinite path has a single vertex of degree 1, while all the other (infinitely many) vertices on the path have degree 2. Next we introduce the definition of unimodularity. Before defining unimodularity, we review the definition of a quasi-transitive graph.

Let $G=(V(G),E(G))$ be a graph. Let Γ be a subgroup of the automorphism group of G. Γ is said to **act quasi-transitively on** G if there exists a finite subset $W\subseteq V(G)$ so that, for $v\in V(G)$ there exists $\alpha\in\Gamma$ satisfying $\alpha v\in W$. We call W a **fundamental domain**, and shall normally (but not invariably) take W to be minimal with this property. G is called a **quasi-transitive graph** if there exists a subgroup of the automorphism group of G acting quasi-transitively on G.

Definition 2.1. *Let Γ be a subgroup of the automorphism group of G acting quasi-transitively on G. The graph G is said to be* ***unimodular*** *with respect to Γ, if for all $u,v\in V$ in the same orbit of Γ, we have the symmetry*

$$|\mathrm{Stab}(u)v|=|\mathrm{Stab}(v)u|,$$

where $\mathrm{Stab}(u)$ is the stabilizer of the vertex u in Γ.

In our case, $\Gamma\simeq\mathbb{Z}^2$ is the subgroup of the automorphism group of $\mathbb{H}$ ($\mathbb{H}=(V,E)$) consisting of all translations of $\mathbb{H}$. Each element in Γ corresponds to a color-preserving isomorphism of $\mathbb{H}$. That is, it maps each black vertex to another black vertex, and each white vertex to another white vertex. The orbit of each black (resp. white) vertex is the set of all black (resp. white) vertices. Moreover, since the 1-2 model we considered in this paper has the same signature at all vertices, each element in Γ is also a weight-preserving transformation, i.e., it maps each local configuration at one vertex to a local configuration at another vertex with the same weight. We can see that Γ acts quasi-transitively on $\mathbb{H}$ with a fundamental domain W consisting of one black vertex and one white vertex. Obviously $\mathbb{H}$ is unimodular since $|\mathrm{Stab}(u)v| = |\mathrm{Stab}(v)u| = 1$, for all $u, v \in V$.

Let $m(u, v, \omega)$ be a non-negative function of three variables: two vertices $u, v \in V$, and the 1-2 model configuration ω. We assume $m(u, v, \omega) = 0$ unless u and v have the same color. More precisely, $m(u, v, \omega)$ is defined as follows. Each vertex u sitting in a semi-infinite path sends unit mass to the unique endpoint of this path, if the endpoint has the same color as u. Vertices not sitting in a semi-infinite path, or having a different color with the endpoint, send no mass at all. In other words, $m(u, v, \omega) = 1$ if and only if in the configuration ω there is a semi-infinite path starting at v, passing u and both u and v are black, or both u and v are white. Otherwise $m(u, v, \omega) = 0$. Obviously, $m(u, v, \omega)$ is invariant under the diagonal action of Γ, meaning that $m(u, v, \omega) = m(\gamma u, \gamma v, \gamma\omega)$, for all u, v, ω, and $\gamma \in \Gamma$. In particular, for each fixed pair of vertices (u, v), $m(u, v, \cdot)$ is a random variable.

Lemma 2.2. *Assume μ is a probability measure invariant under action of Γ. Given $m(\cdot, \cdot, \cdot)$ as above, let*

$$M(u, v) = \int_{\Omega} m(u, v, \omega) d\mu(\omega) = \mathbb{E}m(u, v, \omega),$$

for any $u, v \in V$. The expected total mass transported out of any vertex v equals the expected total mass transported into v, that is

$$\sum_{u \in V} M(v, u) = \sum_{u \in V} M(u, v) \tag{2.1}$$

Proof. This is a special case of the general mass transport principle as in Sect. 3 of [1]. Without loss of generality, assume v is black. The terms contributing to the sum are $M(u, v)$'s for which u is also black. For any $u, v \in V$, both u and v are black, there is a unique $h \in \Gamma \simeq \mathbb{Z}^2$, satisfying $u = hv$. In other words, h is a translation mapping u to v. This gives

$$\sum_{u\ black} M(v, u) = \sum_{h \in \Gamma} M(v, hv) = \sum_{h \in \Gamma} M(h^{-1}v, v) = \sum_{u\ black} M(u, v),$$

where the second equality follows from the facts that $m(\cdot, \cdot, \cdot)$ is invariant under diagonal actions of Γ, and μ is a translation invariant measure. $\square$

Lemma 2.3. *Let μ be a translation invariant measure. Then μ-a.s. there are no semi-infinite paths in 1-2 model configurations.*

Proof. Since μ is a translation invariant probability measure, Lemma 2.2 holds. The expected mass sent from a vertex is at most 1, while if semi-infinite paths exist with positive probability, then some vertex will receive infinite mass with positive probability, so that the expected mass received is infinite, contradicting (2.1). $\square$

Theorem 2.4. *For any translation invariant Gibbs measure of 1-2 model configurations with respect to positive parameters a, b, c, a.s. there are no infinite paths.*

Proof. First of all, we claim that if μ is a Gibbs measure with strictly positive parameters a, b, c, s.t. $\mu(\text{there is an infinite path}) > 0$, then $\mu(\text{there is a semi} - \text{infinite path}) > 0$.

To see that, let Ω_1 be the set of configurations with an infinite path going through the edge $[0, 1]$, so that neither 0 nor 1 is an endpoint of the infinite path. Let Ω_2 be the set of configurations obtained from Ω_1 by deleting the edge $[0,1]$. Then up to translation and rotation we may assume $\mu(\Omega_1) > 0$. From the Gibbs property we have the finite energy estimate

$$\mu(\Omega_2) \geq \left[\frac{\min(a,b,c)}{\max(a,b,c)}\right]^2 \mu(\Omega_1) > 0$$

Combining Lemma 2.3 with the claim gives: any translation invariant Gibbs measure with positive parameter a, b, c has a.s. no infinite paths. □

Lemma 2.3 and Theorem 2.4 hold for any translation invariant Gibbs measure of the 1-2 model configurations with positive parameters a, b, c. Since the measure we will construct is the weak limit of translation invariant Boltzmann measures on finite tori, by letting the tori become larger and larger to approximate the infinite periodic planar graph, (see Sect. 4), Lemma 2.3 and Theorem 2.4 apply also to the specific measure we will construct.

3 Correspondence with dimers

A **dimer configuration**, or a **perfect matching**, of a graph is a collection of edges with the property that each vertex is incident to exactly one of these edges. Each edge e is associated with a positive weight w_e. The partition function of dimer configurations on a finite graph is given by

$$Z_M := \sum_{\mathcal{D}} \prod_{e \in \mathcal{D}} w_e, \tag{3.1}$$

where the sum is over all dimer configurations of the finite graph.

The probability of a dimer configuration on a finite graph is defined to be proportional to the product of weights of included edges. Namely, let $\mathcal{D}$ be a dimer configuration, then

$$P_M(\mathcal{D}) = \frac{1}{Z_M} \prod_{e \in \mathcal{D}} w_e. \tag{3.2}$$

We will construct a measure-preserving correspondence between 1-2 model configurations on a honeycomb lattice $\mathbb{H} = (V, E)$, and the dimer configurations on a decorated graph $\mathbb{H}_\triangle = (V_\triangle, E_\triangle)$, as illustrated in Figure 8, where the left graph shows a 1-2 model configuration on the hexagonal lattice $\mathbb{H}$, and the right graph shows the corresponding dimer configuration on the decorated graph $\mathbb{H}_\triangle$. The correspondence between dimer configurations and 1-2 model configurations is partially inspired by the Fisher correspondence for the Ising model and the dimer model, see [6].

We construct the graph $\mathbb{H}_\triangle$ by giving the vertex set $V_\triangle$ and edge set $E_\triangle$. First of all, $V \subset V_\triangle$. Namely, each vertex of the hexagonal lattice $\mathbb{H}$ is also a vertex of $\mathbb{H}_\triangle$. For each $v \in V$, the three incident edges of v in $\mathbb{H}$ form three angles $\alpha_{1,v}$, $\alpha_{2,v}$, $\alpha_{3,v}$. (Note that $\alpha_{i,v} = \frac{2\pi}{3}$, for $1 \leq i \leq 3$). Let $e^\triangle_{i,v}$ be the bisector of $\alpha_{i,v}$. In the graph $\mathbb{H}_\triangle$, let $e^\triangle_{i,v}$ $(1 \leq i \leq 3)$ be the three incident edges of v, i.e., $e^\triangle_{i,v} \in E_\triangle$, for $v \in V$ and $1 \leq i \leq 3$, see Figure 4 and Figure 5.

On each face of $\mathbb{H}$, we construct a gadget for $\mathbb{H}_\triangle$, which is a modified hexagon, with the topmost edge removed, and the other four vertices except the top two replaced by a triangle, see Figure 6, or see Figure 8 for a larger picture.

Note that from $\mathbb{H}$ to $\mathbb{H}_\triangle$, each face is replaced by a 14-vertex gadget. This 14-vertex gadget can be further simplified into a 12-vertex gadget. Namely, we can remove the top two vertices, and merge the two incident edges of each one of them into a single edge.

$\{011\}, c$ local dimer configuration for$\{011\}$

Figure 4: local structure of a black vertex in $\mathbb{H}$ and $\mathbb{H}_\triangle$

$\{001\}, a$ local dimer configuration for$\{001\}$

Figure 5: local structure of a white vertex in $\mathbb{H}$ and $\mathbb{H}_\triangle$

Or we may construct a 12-vertex gadget with 6 outer vertices and 6 inner vertices, so that each outer vertex is incident to a vertex of $\mathbb{H}$ and 2 inner vertices, and each inner vertex is incident to 2 outer vertices and 2 inner vertices. See Figure 7 for possible choices of 12-vertex gadgets.

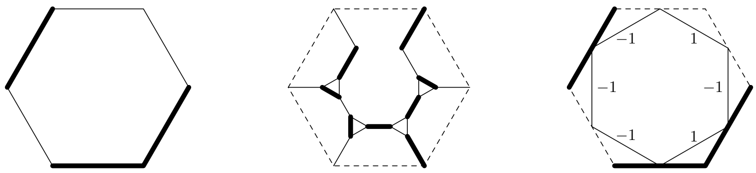

Figure 6: local structure of a face in $\mathbb{H}$, in $\mathbb{H}_\triangle$ and medial graph

We claim that there is a 2-to-1 correspondence between 1-2 model configurations on $\mathbb{H}$ and dimer configurations on $\mathbb{H}_\triangle$. Namely, at each angle of $\mathbb{H}$, if both sides of the angle have the same configuration, that is, either both of them are present, or neither of them are present, then the bisector edge is present in the dimer configuration of $\mathbb{H}_\triangle$. Otherwise the bisector is not present. Examples of this local correspondence are show in Figure 4 and Figure 5. Obviously, under the assumption that each vertex $v \in V$ has one or two incident present edges in $\mathbb{H}$, each corresponding local configuration on $\mathbb{H}_\triangle$ satisfies the condition that v has exactly one present incident edge. See Figure 4 and Figure 5.

Moreover, such a configuration on incident edges of vertices in V (bisector edges) always has a unique extension to a dimer configuration (each vertex in $V_\triangle$ has exactly one present incident edge) of $\mathbb{H}_\triangle$, because around each face of $\mathbb{H}$, there are always an even number of bisector edges of $\mathbb{H}_\triangle$ present in the configuration. See Figure 6, or see Figure 8 for a larger picture.

Note that for such a construction, two 1-2 model configurations, the union of which is the graph $\mathbb{H}$, correspond to the same dimer configuration of $\mathbb{H}_\triangle$. This is the reason why the correspondence is actually 2-to-1.

Assigning edge weights for $\mathbb{H}_\triangle$ appropriately will ensure the correspondence to be measure-preserving. For example, if ξ_v is a degree-1 local configuration at v (one incident edge of v is present) of $\mathbb{H}$, and ξ_v has weight a, we assign the same weight a to the bisector edge in $E_\triangle$ parallel to the present edge in ξ_v of E and incident to v in $\mathbb{H}_\triangle$. This way, we can assign a weight to every edge in $E_\triangle$ incident to a vertex $v \in V$. See the

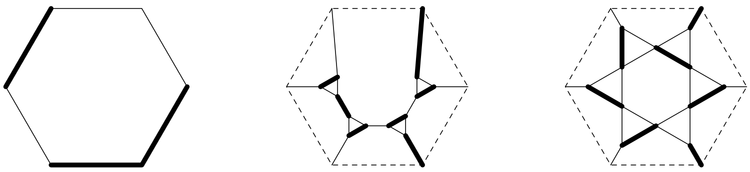

Figure 7: 12-vertex Gadgets

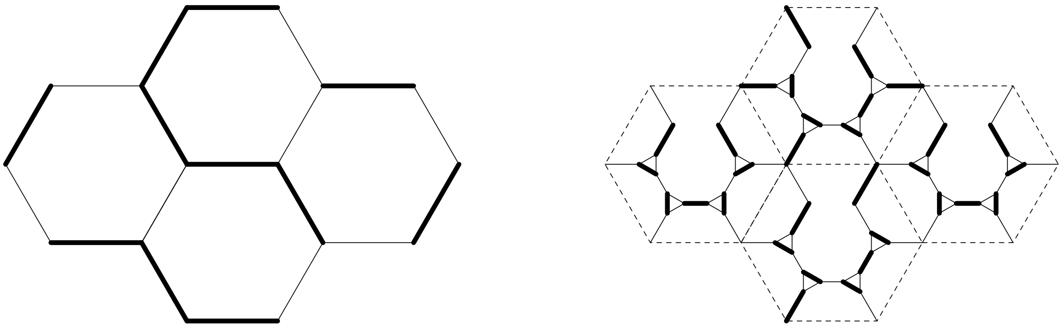

Figure 8: 1-2 model configuration and corresponding dimer configuration

right graph of Figure 10. For all the other edges in E_Δ, we assign weight 1.

We can also construct a toroidal graph $\mathbb{H}_{\Delta,n}$ for $\mathbb{H}_n$ in the same way. However, when constructing the correspondence between dimer configurations on $\mathbb{H}_{\Delta,n}$ and 1-2 model configurations on $\mathbb{H}_n$, complications might happen due to the fact that the torus has non-trivial homology group H_1. We consider an essential cycle (non-contractible) winding around the $n \times n$ torus, consisting of edges of $\mathbb{H}_n$. A necessary and sufficient condition for a dimer configuration on $\mathbb{H}_{\Delta,n}$ to correspond to a 1-2 model configuration on $\mathbb{H}_n$ can be described as follows.

Condition 3.1. *Around any essential cycle, there are an even number of bisector edges not present in the perfect matching.*

It is not hard to see that once Condition 3.1 is satisfied, before and after winding around an essential cycle, each edge has the same configuration.

We claim that checking Condition 3.1 is equivalent to checking that an even number of bisector edges are not present in a perfect matching around two homology generators. In other words, for each fixed dimer configuration on $\mathbb{H}_{\Delta,n}$, the parity of the number of non-present bisector edges around an essential cycle depends only on the homology class of the cycle. To see that, we introduce the medial graph $D\mathbb{H}_n = (V_{D,n}, E_{D,n})$ of $\mathbb{H}_n = (V_n, E_n)$. Each vertex of $D\mathbb{H}_n$ is the midpoint of an edge of $\mathbb{H}_n$, i.e., there is a 1-to-1 correspondence between $V_{D,n}$ and E_n. Two vertices in $V_{D,n}$ are connected by an edge in $D\mathbb{H}_n$ if and only if the corresponding edges in $\mathbb{H}_n$ share an edge. In other words, there is a 1-to-1 correspondence between $E_{D,n}$ and bisector edges in $E_{\Delta,n}$. The medial graph of the hexagonal lattice is the Kagome lattice, see Figure 9.

Given a dimer configuration on $\mathbb{H}_{\Delta,n}$, one associates a sign ± 1 to each edge of $D\mathbb{H}_n$ (1 if the bisector edge is present, -1 otherwise). Then the product of signs along each non-essential cycle of $D\mathbb{H}_n$ is 1. Hence the product of signs is a homology invariant, so is the number of non-present bisector edges along a cycle, see Figure 6.

However, Condition 3.1 cannot always be satisfied since it is always possible that winding around an essential cycle, an odd number of bisector edges are not present in a perfect matching. But if we consider the mixed periodic/antiperiodic boundary

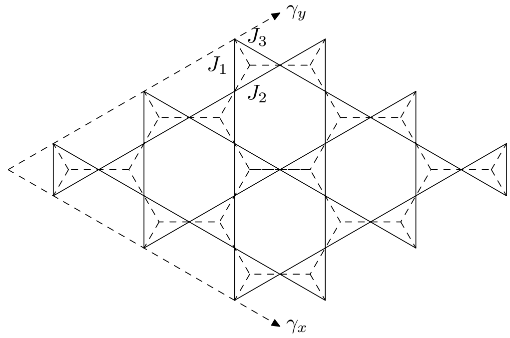

Figure 9: Kagome lattice and honeycomb lattice

condition for 1-2 model configurations on $\mathbb{H}'_n$, the constraints above can be removed. In other words, if Z_n is the partition function for 1-2 model configurations on $\mathbb{H}'_n$ with mixed periodic/antiperiodic boundary condition, and $Z_{M,n}$ is the partition function for dimer configurations on $\mathbb{H}_{\triangle,n}$, then

$$Z_{M,n} = Z_{M,n}^{00} + Z_{M,n}^{01} + Z_{M,n}^{10} + Z_{M,n}^{11},$$

where $Z_{M,n}^{00}$ (resp. $Z_{M,n}^{11}$) is the partition function of dimer configurations so that winding around two nonparallel essential cycles, both of them have an even (resp. odd) number of bisector edges that are not present; $Z_{M,n}^{01}$ is the partition function of dimer configurations so that winding around two nonparallel essential cycles, one of them has an even number of bisector edges that are not present, and the other has an odd number of bisector edges that are not present, and similarly for $Z_{M,n}^{10}$. Using the criteria described above to assign edge weights for dimers on $\mathbb{H}_{n,\triangle}$ (which inherit edge weights from $\mathbb{H}_\triangle$), we have

$$Z_n = 2Z_{M,n},$$

where the coefficient 2 comes from the fact that the correspondence from 1-2 model configurations to perfect matchings is 2-to-1. Moreover, if ξ, $\bar{\xi}$ are two 1-2 model configurations of $\mathbb{H}_n$ complement to each other, and $\mathcal{D}$ is the corresponding dimer configuration on $\mathbb{H}_{\triangle,n}$, by definition of the measures of the 1-2 model and the dimer model, we have

$$P_n(\xi) = P_n(\bar{\xi}) = \frac{1}{2}P_{M,n}(\mathcal{D})$$

where $P_{M,n}$ is the measure for perfect matchings on the toroidal graph $\mathbb{H}_{\triangle,n}$, defined as in (3.2).

Therefore, we can investigate the measure of the 1-2 model by investigating the measure of the dimer model. An important object in understanding the infinite volume limit of the periodic dimer model is the **characteristic polynomial** (see the next paragraph for a definition). To introduce the notation, first of all, we give $\mathbb{H}_\triangle$ a **clockwise-odd orientation**, so that traversing each face of $\mathbb{H}_\triangle$ clockwise gives an odd number of edges with the same orientation with the clockwise orientation. See Figure 10.

Recall that $\mathbb{H}_{\triangle,n} = (V_{\triangle,n}, E_{\triangle,n})$ is the quotient graph of $\mathbb{H}_\triangle$ under the translation of $n\mathbb{Z} \times n\mathbb{Z}$. $\mathbb{H}_{\triangle,n}$ is a finite graph which can be embedded into an $n \times n$ torus. Obviously $\mathbb{H}_{\triangle,n}$ inherited a clockwise odd orientation from $\mathbb{H}_\triangle$. Let γ_x, γ_y be two homology generators of the torus. Multiply the weights of the edges crossed by γ_x by z (or $\frac{1}{z}$), according

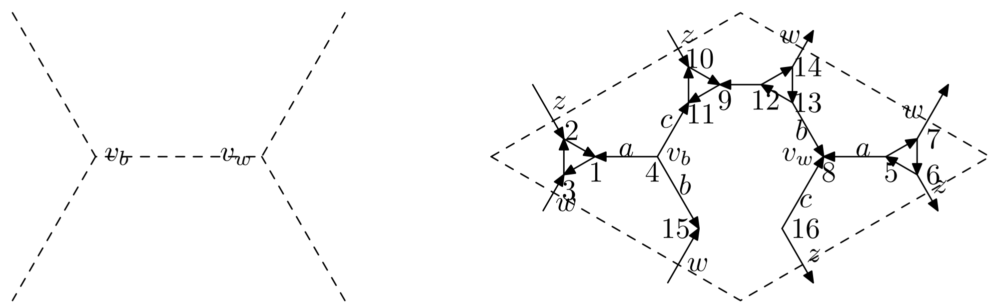

Figure 10: one fundamental domain

to its orientation, and similarly, multiply the weights of edges crossed by γ_y by w (or $\frac{1}{w}$) according to its orientation. This way we get a **modified weighted adjacency matrix** $K_n(z,w)$. More precisely, $K_n(z,w)$ is a $|V_{\Delta,n}| \times |V_{\Delta,n}|$ matrix with rows and columns indexed by all the vertices of $\mathbb{H}_{\Delta,n}$. Let $u, v \in V_{\Delta,n}$. The entry of $K_n(z,w)$ at the row corresponding to u and the column corresponding to v is 0, if u and v are not adjacent in $\mathbb{H}_{\Delta,n}$; the entry is $(-1)^{\sigma(u,v)} w_{u,v}$, if u and v are adjacent, where $w_{u,v}$ is the modified edge weight corresponding to the edge uv (namely, if the edge uv is crossed by γ_x or γ_y, the modified weight is obtained from the original weight by multiplying a factor of z, or w, or $\frac{1}{z}$, or $\frac{1}{w}$, depending on the orientation); $\sigma(u,v) = 0$, if the orientation is from u to v, and $\sigma(u,v) = 1$, if the orientation is from v to u. In particular, if $|z| = 1$ and $|w| = 1$, $K_n(z,w)$ is an anti-Hermitian matrix, in the sense that $K_n^{uv}(z,w) = -\overline{K_n^{vu}(z,w)}$, where $K_n^{uv}(z,w)$ is the entry of $K_n(z,w)$ at the row corresponding to u, and the column corresponding to v. Moreover, if $z = \pm 1$ and $w = \pm 1$, $K_n(z,w)$ is an anti-symmetric matrix, since all the entries are real. In this case, its determinant is the perfect square of its Pfaffian; hence its determinant is nonnegative. The **characteristic polynomial** $P(z,w)$ is defined to be the $\det K_1(z,w)$, and the **spectral curve** is defined to be the zero locus $P(z,w) = 0$.

The characteristic polynomial, spectral curve can be defined for a bipartite graph in a simpler way, see [13].

Lemma 3.2. *If $a, b, c > 0$, either the spectral curve of the dimer model does not intersect the unit torus $\mathbb{T}^2 := \{(z,w) \in \mathbb{C}^2 : |z| = 1, |w| = 1\}$, or the intersection is a single real point (1,1), and the intersection is of multiplicity 2. Moreover, the spectral curve intersects $\mathbb{T}^2$ at $(1,1)$ if and only if $a^2 + b^2 + c^2 = 2(ab + bc + ca)$.*

Proof. Recall that $K_1(z,w)$ is the Kasteleyn matrix for the graph embedded on the 1×1 torus, as illustrated in the right graph of Figure 10. It is a 16×16 matrix with rows and columns indexed by vertices of the graph, whose determinant can be computed explicitly. In fact, $K_1(z,w)$ has the following form:

$$K_1(z,w) = \\ \begin{pmatrix} 0 & -1 & 1 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{z} \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{w} & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{z} & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 & -\frac{1}{w} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the rows and columns of $K_1(z,w)$ are indexed by vertices from 1 to 16, as illustrated in the right graph of Figure 10. By definition, the characteristic polynomial

$$\begin{aligned} P(z,w) = \det K_1(z,w) \quad &= \quad a^4 + b^4 + c^4 + 6a^2b^2 + 6a^2c^2 + 6b^2c^2 + 4ab(c^2 - a^2 - b^2)\cos\theta & (3.3) \\ &\quad +4ac(b^2 - a^2 - c^2)\cos\phi + 4bc(a^2 - b^2 - c^2)\cos(\theta - \phi) & (3.4) \end{aligned}$$

where $z = e^{i\theta}, w = e^{i\phi}$. First of all, note that if $a^2 + b^2 > c^2$, $a^2 + c^2 > b^2$ and $b^2 + c^2 > a^2$, we have for any $(z,w) \in \mathbb{T}^2 \setminus \{(1,1)\}$,

$$P(z,w) > P(1,1) \geq 0,$$

Since $P(1,1)$ is the determinant of an anti-symmetric matrix, which is the perfect square of a real number.

If a^2, b^2, c^2 cannot be side lengths of a triangle, we consider the minimal value of the characteristic polynomial in $\mathbb{T}^2$. Without loss of generality, assume $b^2 \geq a^2 + c^2$. It suffices to consider the minimal value of $R(e^{i\theta}, e^{i\phi})$ for $(\theta, \phi) \in [0, 2\pi]^2$, where

$$R(e^{i\theta}, e^{i\phi}) = A + B\cos\theta + C\sin\theta$$

where

$$\begin{aligned} A &= 4ac(b^2 - a^2 - c^2)\cos\phi \\ B &= 4ab(c^2 - a^2 - b^2) + 4bc(a^2 - b^2 - c^2)\cos\phi \\ C &= 4bc(a^2 - b^2 - c^2)\sin\phi. \end{aligned}$$

Define

$$\begin{aligned} x &= 4ac(b^2 - a^2 - c^2); \\ y &= 4ab(c^2 - a^2 - b^2); \\ t &= 4bc(a^2 - b^2 - c^2); \end{aligned}$$

then

$$R(e^{i\theta}, e^{i\phi}) = x\cos\phi + (y + t\cos\phi)\cos\theta + t\sin\phi\sin\theta.$$

Given ϕ, the minimal value of $R(e^{i\theta}, e^{i\phi})$ with respect to $\theta \in [0, 2\pi]$ is

$$f(u) = A - \sqrt{B^2 + C^2} = xu - \sqrt{y^2 + t^2 + 2ytu};$$

where $u = \cos\phi$. Moreover,

$$\frac{\partial f(u)}{\partial u} = x - \frac{yt}{\sqrt{y^2 + t^2 + 2ytu}}.$$

The minimal value of $f(u)$ with respect to u is achieved at the point satisfying $\frac{\partial f(u)}{\partial u} = 0$, where

$$u = \frac{yt}{2x^2} - \frac{y^2 + t^2}{2yt}. \tag{3.5}$$

We claim that $(3.5) \geq 1$, when $b^2 \geq a^2 + c^2$. In fact

$$\frac{yt}{2x^2} - \frac{y^2 + t^2}{2yt} - 1 = \frac{yt}{2x^2} - \frac{(y+t)^2}{2yt}.$$

It suffices to prove that $|yt| \geq |x(y+t)|$, i.e.

$$b(a^2 + b^2 - c^2)(b^2 + c^2 - a^2) \geq (b^2 - a^2 - c^2)[a(a^2 + b^2 - c^2) + c(b^2 + c^2 - a^2)], \tag{3.6}$$

when $b^2 \geq a^2 + c^2$. Without loss of generality, assume $c \geq a$.

First of all, if $b \geq a + c$, we have

$$\begin{aligned}
&a^2 + b^2 - c^2 \geq b^2 - a^2 - c^2 \geq 0 \\
&b(b^2 + c^2 - a^2) \geq (a + c)(b^2 + c^2 - a^2) \geq a(a^2 + b^2 - c^2) + c(b^2 + c^2 - a^2) \geq 0.
\end{aligned}$$

Hence if $b \geq a + c$, (3.6) is true.

Now consider the case when $b < a + c$, but $b^2 \geq a^2 + c^2$. Let $p = \frac{a}{b}$, and $q = \frac{c}{b}$, then $p + q > 1$, and $p^2 + q^2 \leq 1$. Plugging p and q in (3.6), and expanding it if necessary, (3.6) is equivalent to

$$1 + p^5 + q^5 - (p^2 - q^2)^2 - p(1 - q^2)^2 - q(1 - p^2)^2 \geq 0. \tag{3.7}$$

Elementary algebraic transformation of left side of (3.7) gives

$$\begin{aligned}
&[(p+q)^2 - 1]^2 + (p^2 + q^2 - 1)^2 \\
&+ (1 - p^2 - q^2)(p + q - 1)[2(p+q) + 1 - (p^2 + q^2)] + 2pq(p+q)(1 - p^2 - q^2),
\end{aligned}$$

which is nonnegative when $p + q > 1$ and $p^2 + q^2 \leq 1$.

Hence when $u = \cos\phi \in [-1, 1]$, the minimal value of $A - \sqrt{B^2 + C^2}$ can only be achieved at $\cos\phi = 1$. Therefore the $P(z, w)$ can only be achieved at $z = w = 1$, but $P(1, 1) \geq 0$. It is elementary to check that if $P(1, 1) = 0$, then the zero is of multiplicity 2 by taking derivatives; and $P(1, 1) = 0$ if and only if $a^2 + b^2 + c^2 = 2(ab + bc + ca)$. □

Proposition 3.3. *Let $P_{M,n}$ denote the Boltzmann measure of the dimer configurations on $\mathbb{H}_{\Delta,n}$. As $n \to \infty$, the sequence $\{P_{M,n}\}_{n=1}^{\infty}$ converges weakly to an ergodic translation invariant Gibbs measure P_M of dimer configurations on $\mathbb{H}_\Delta$.*

Proof. The proof of convergence is exactly the same as Lemma 4.8 in [16]. The translation invariance of the measure P_M is obvious. P_M is an ergodic dimer measure because it is translation invariant and mixing. The mixing of P_M follows from the fact that the intersection of the spectral curve with $\mathbb{T}^2$ is either empty or a single real point of multiplicity 2, hence the covariance of any two event, measurable with respect to two finite set of vertices far away from each other, decays to zero either exponentially fast or polynomially fast. Moreover, P_M is a Gibbs measure since $P_{M,n}$'s are Boltzmann measures, i.e., the probability of a configuration is proportional to product of local weights. □

Let P be the measure for 1-2 model configurations on $\mathbb{H}$ satisfying

$$P(E) = P(E^*) = \frac{1}{2}P_M(E_D), \text{if } E \cap E^* = \emptyset.$$

where $E \subseteq \Omega$ is an event of 1-2 model configurations, E^* consists of all the 1-2 model configurations whose present edges are complement to the present edges of a configuration in E, and E_D is the corresponding event on dimers. Note that although P_M is an ergodic measure on dimer configurations of $\mathbb{H}_\triangle$, P might not be ergodic as a measure for 1-2 model configurations on $\mathbb{H}$. That is because the correspondence between 1-2 model configurations on $\mathbb{H}$ and dimer configurations on $\mathbb{H}_\triangle$ is 2-to-1. Although every translation invariant event of dimers on $\mathbb{H}_\triangle$ has probability 0 or 1, this may not be true for translation invariant events of the 1-2 model on $\mathbb{H}$.

The following theorem is a rephrasing of Part 1 of Theorem 0.1.

Theorem 3.4. *Using a large torus to approximate the infinite hexagonal lattice, the probability that a path $\mathfrak{L}$ occurs in 1-2 model configurations is equal to the Pfaffian of an anti-symmetric matrix, namely, $K^{-1}_{E_{\mathfrak{L}}}$, the submatrix of the infinite inverse Kasteleyn matrix with rows and columns indexed by endpoints of bisector edges in $\mathbb{H}_\triangle$ between each adjacent pair of edges along the path $\mathfrak{L}$, multiplied by one half of the product of the configuration weights.*

Proof. Consider an arbitrary self-avoiding path, consisting of edges of the hexagonal lattice $\mathbb{H}$. The event that the path occurs in a 1-2 model configuration, i.e., all the edges along the path are present in the 1-2 model configuration, corresponds to the event that all the bisector edges along the path are present in the dimer configuration of the decorated graph $\mathbb{H}_\triangle$.

Given an arbitrary 1-2 model configuration $\mathcal{C}$ in the sample space Ω_n of $\mathbb{H}'_n$, the configuration $\mathcal{C}^*$, which occupies all the unoccupied edges and half-edges of $\mathcal{C}$ and leaves all the occupied edges and half-edges of $\mathcal{C}$ unoccupied, has exactly the same probability as the configuration $\mathcal{C}$, and corresponds to the same dimer configuration of $\mathbb{H}_{\triangle,n}$. Let $\mathfrak{L}$ be a path consisting of edges of $\mathbb{H}'_n$. We have

$$\begin{aligned} & P_n(\text{path } \mathfrak{L} \text{ appears in } 1-2 \text{ model configurations of } \mathbb{H}'_n) \\ = \; & \frac{1}{2}P_{M,n}(\text{all the bisector edges along } \mathfrak{L} \text{ are present in the dimer configurations of } \mathbb{H}_{\triangle,n}). \end{aligned}$$

Using a large torus to approximate the infinite graph, we can actually compute the probability on the right explicitly. First we consider a finite $n \times n$ torus, where n is even. Let $K_n(z,w)$ be the corresponding modified weighted adjacency matrix, then the partition function $Z_{M,n}$ is given by

$$4Z_{M,n} = |-\mathrm{Pf}K_n(1,1) + \mathrm{Pf}K_n(1,-1) + \mathrm{Pf}K_n(-1,1) + \mathrm{Pf}K_n(-1,-1)| := 4\hat{Z}_n,$$

since

$$\begin{aligned} \mathrm{Pf}K_n(1,1) &= Z^{00}_{M,n} - Z^{01}_{M,n} - Z^{10}_{M,n} - Z^{11}_{M,n} && (3.8)\\ \mathrm{Pf}K_n(1,-1) &= Z^{00}_{M,n} + Z^{01}_{M,n} - Z^{10}_{M,n} + Z^{11}_{M,n} && (3.9)\\ \mathrm{Pf}K_n(-1,1) &= Z^{00}_{M,n} - Z^{01}_{M,n} + Z^{10}_{M,n} + Z^{11}_{M,n} && (3.10)\\ \mathrm{Pf}K_n(-1,-1) &= Z^{00}_{M,n} + Z^{01}_{M,n} + Z^{10}_{M,n} - Z^{11}_{M,n}, && (3.11) \end{aligned}$$

when the vertices are ordered in K_n in an appropriate way. See [10, 11] for the proof of the formulas (3.8)-(3.11). In fact, if we change the order of vertices in K_n, the signs in (3.8)-(3.11) will change simultaneously, and the sign does not matter if we take the absolute value.

Let $\tilde{K}_n(z,w)$ be the modified weighted adjacency matrix of a graph obtained from $\mathbb{H}_{\Delta,n}$ by removing all the bisector edges along the path as well as their ending vertices. Let $E_{\mathfrak{L}}$ be the set of all bisector edges along the path. Namely,

$$E_{\mathfrak{L}} = \{u_1v_1, u_2v_2, \cdots, u_kv_k\}$$

Let $|u_i|$ be the index of the column, corresponding to the vertex u_i, in the weighted adjacency matrix. Then

$$\begin{aligned}&P_{M,n}(\text{all the edges in } \mathrm{E}_{\mathfrak{L}} \text{ are present})\\ =\quad& \frac{\prod_{1\leq i\leq k} w_{u_iv_1}\left|-\mathrm{Pf}\tilde{K}_n(1,1)+\mathrm{Pf}\tilde{K}_n(1,-1)+\mathrm{Pf}\tilde{K}_n(-1,1)+\mathrm{Pf}\tilde{K}_n(-1,-1)\right|}{4\hat{Z}_n}\end{aligned}$$

where $w_{u_iv_i}$ is the dimer edge weight for u_iv_i, assigned to be consistent with the corresponding weight of local 1-2 model configuration, see Figure 10. It is obvious that the numerator in the above equation consists of exactly those terms in $2\hat{Z}_n$ with all edge weights of $E_{\mathfrak{L}}$ appearing. Moreover,

$$\mathrm{Pf}\tilde{K}_n((-1)^\theta,(-1)^\tau) = (-1)^{\sum_i |u_i|+|v_i|}\mathrm{Pf}K^{-1}_{E_{\mathfrak{L}},n}((-1)^\theta,(-1)^\tau)\mathrm{Pf}K_n((-1)^\theta,(-1)^\tau)$$

$\theta,\tau\in\{0,1\}$, where $K^{-1}_{E_{\mathfrak{L}},n}((-1)^\theta,(-1)^\tau)$, if exists, is the submatrix of $K_n^{-1}((-1)^\theta,(-1)^\tau)$ with rows and columns indexed by vertices in $E_{\mathfrak{L}}$.

Since the planar graphs $\mathbb{H}$ and $\mathbb{H}_\Delta$ are bi-periodic, i.e., translation invariant along two nonparallel directions, we can divide them into fundamental domains, and a single fundamental domain is illustrated as in Figure 10. Since the group of translations of $\mathbb{H}$ and $\mathbb{H}_\Delta$ is isomorphic to $\mathbb{Z}^2$, we can label all the fundamental domains of by $(p,q)\in\mathbb{Z}^2$. Let (p_1,q_1,u) and (p_2,q_2,v) be two vertices, where (p_1,q_1) and (p_2,q_2) are indices of the fundamental domains, and u,v are indices of vertices in the fundamental domain $(0,0)$. In other words, if we translate the vertex (p_1,q_1,u) (resp. (p_2,q_2,v)) by $(-p_1,-q_1)$ (resp. $(-p_2,-q_2)$), we get exactly the vertex u (resp. v) in the fundamental domain $(0,0)$. Then we have

$$\begin{aligned}K^{-1}_{(p_1,q_1,u),(p_2,q_2,v)} &= \lim_{n\to\infty} K^{-1}_{n,(p_1,q_1,u),(p_2,q_2,v)}((-1)^\theta,(-1)^\tau)\\ &= \frac{1}{2\pi}\iint_{\mathbb{T}^2} z^{p_1-q_1}w^{p_2-q_2}\frac{\mathrm{Cofactor}K_1(z,w)_{u,v}}{\det K_1(z,w)}\frac{dz}{iz}\frac{dw}{iw}\end{aligned}$$

for any θ,τ in $\{0,1\}$. The convergence follows from the fact that the intersection of $\det K_1(z,w)$ and $\mathbb{T}^2$ is either empty or a single real point of multiplicity 2, and the machinery described in [3, 16]. Hence, as the size of the torus goes to infinity, we have

$$P(\text{Path } \mathfrak{L} \text{ appears in } 1-2 \text{ model configurations}) = \frac{1}{2}\prod_{1\leq i\leq k} w_{u_iv_i}\left|\mathrm{Pf}K^{-1}_{E_{\mathfrak{L}}}\right|$$

where the entries of K^{-1} are described as above. □

4 Infinite clusters

In this section, we present a proof for the existence of a phase transition by exploring the behavior of infinite clusters. At each vertex, a 1-2 model configuration has 3 possible weights, a, b or c. Let us classify all local configurations at one vertex according to their weights, into 3 categories: a-type, b-type and c-type. Note that each type of configurations actually include 2 different configurations, namely, they are complement to each other and the occupying degree of the vertex is either 1 or 2. An a-cluster (resp.

b-cluster or c-cluster) is a connected set of vertices such that each vertex in it has an a-type (resp. b-type or c-type) configuration. Note that by "connected set of vertices" here, we mean that the set of vertices are connected by edges of the hexagonal lattice $\mathbb{H}$, instead of saying that they are connected by present edges in a specific 1-2 model configuration ω. See Figure 11 for an example of an a-cluster on the toroidal graph $\mathbb{H}_n$. In Figure 11, the vertices $v_1,\ v_2,\ v_3, v_4$ form an a-cluster. Note that although an a-type configuration can be one of 2 different configurations, each a-cluster can have only one configuration for all the vertices, similarly for b-clusters and c-clusters. We fix the value of b and c, and are interested in the behavior of a-clusters as we vary the value of a. We will always use P to denote the measure obtained by torus approximation.

4.1 Exponential decay of large cluster probabilities

In this subsection, we prove the exponential decay of large cluster probabilities under the probability measure we constructed above. The idea is to prove that the probability of a single edge being present in a dimer configuration goes to zero when the corresponding parameter goes to zero, and the exponential decay of large cluster probabilities follows from the estimate of the explicit Pfaffian formula for the large cluster probability.

Lemma 4.1. *$P_{\{001\}\&\{110\}}$, the probability that an a-configuration appears at a vertex for the measure P, is continuous in a, for any $a > 0$. Moreover,*

$$\lim_{a\to\infty} P_{\{001\}\&\{110\}} = 1 \qquad \text{for any fixed } b, c > 0; \tag{4.1}$$

$$\lim_{a\to 0} P_{\{001\}\&\{110\}} = 0 \qquad \text{for any fixed } b, c > 0. \tag{4.2}$$

Proof. By construction

$$P_{\{001\}\&\{110\}} = \frac{a}{4\pi^2}\int_{|w|=1}\int_{|z|=1} \frac{Q(z,w)}{P(z,w)}\frac{dz}{iz}\frac{dw}{iw}$$

where $P(z,w)$ is the characteristic polynomial as in (3.4), and $Q(z,w)$ is the cofactor of $K(z,w)$ by removing rows and columns corresponding to endpoints of the edge e. Namely,

$$Q(z,w) = 3ab^2+3ac^2-c^3w-b^3z+a^3-\frac{a^2c}{w}-2a^2cw+b^2cw-2a^2bz+bc^2z-\frac{ab(a-cw)}{z}+\frac{abcz}{w}$$

The continuity of $P_{\{001\}\&\{110\}}$ with respect to a follows from the fact that the intersection of $P(z,w) = 0$ with $\mathbb{T}^2$ can either be empty or a single real point, and the intersection is of multiplicity 2. To see how it works, let $a_0 > 0$, and $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty} a_n = a_0$. Since we are considering the behavior of the polynomial $P(z,w)$ with a changing coefficient a, we introduce a third variable a, and write $P(z,w,a)$ by taking a into account. Similarly, we write $P_{\{001\}\&\{110\}}(a)$ for the probability that an a-configuration appears at a vertex, which changes with the coefficient a.

If $P(z,w,a_0) = 0$ does not intersect $\mathbb{T}^2$. In this case, $\frac{Q(z,w,a_n)}{P(z,w,a_n)}$ is uniformly bounded for $(z,w) \in \mathbb{T}^2$, and $n \in \mathbb{N}$. According to the dominated convergence theorem,

$$\lim_{n\to\infty} P_{\{001\}\&\{110\}}(a_n) = P_{\{001\}\&\{110\}}(a_0). \tag{4.3}$$

If $P(z,w,a_0) = 0$ has a unique zero $(1,1)$ on $\mathbb{T}^2$, and the zero is of multiplicity 2, explicit computation shows that

$$\begin{aligned} P(1,1,a) &= (a^2+b^2+c^2-2ab-2ac-2bc)^2 \\ Q(1,1,a) &= (a-b-c)(a^2+b^2+c^2-2ab-2ac-2bc) \end{aligned}$$

Therefore, if $P(z,w,a_0)$ has a zero of multiplicity 2 at $(1,1)$, then $Q(z,w,a_0)$ has a zero of multiplicity at least 1 at $(1,1)$. Hence $\frac{Q(z,w,a_0)}{P(z,w,a_0)}$ is integrable on $(z,w)\in\mathbb{T}^2$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying $\lim_{n\to\infty} a_n = a_0$, and $a_n>0$. The identity (4.3) follows from the fact that $\frac{Q(z,w,a_n)}{P(z,w,a_n)}$ converges to $\frac{Q(z,w,a_0)}{P(z,w,a_0)}$ almost everywhere on $\mathbb{T}^2$ as n goes to infinity, and that the sequence $\left\{\frac{Q(z,w,a_0)}{P(z,w,a_0)}\right\}_{n=1}^{\infty}$ is uniformly integrable.

For any $(z,w)\in\mathbb{T}^2$, and fixed b,c,

$$\lim_{a\to\infty}\frac{aQ(z,w,a)}{P(z,w,a)}=1, \tag{4.4}$$

and the convergence of the left side of (4.4) is uniform for $(z,w)\in\mathbb{T}^2$, which gives (4.1).

When $b\neq c$,

$$\lim_{a\to 0}\frac{Q(z,w,a)}{P(z,w,a)}=\frac{-zb^3+wb^2c+zbc^2-wc^3}{((wb^2-2zbc+wc^2)(zb^2-2wbc+zc^2))/(wz)} \tag{4.5}$$

If $b\neq c$, the denominator of (4.5) has no zeros on $\mathbb{T}^2$. Again the order of the integral and the limit can be changed according to the dominated convergence theorem. As a result, when $b\neq c$, $\lim_{a\to 0} P_{\{001\}\&\{110\}}(a)=0$.

When $b=c$, note that the $P(z,w,0)$ has a zero of multiplicity 2 at $(1,1)$, and $Q(z,w,0)$ has a zero of multiplicity at least 1 at $(1,1)$. Similar arguments show the convergence to 0 of the probability $P_{\{001\}\&\{110\}}(a)$ as a tends to 0.

□

Lemma 4.2. *Let $\mathcal{S}$ be a set of N vertices of $\mathbb{H}$, where N is a large integer. Then*

1. *fixing b,c, when a is sufficiently large, there exists $\eta>0$, so that*

$$P(\text{no vertices in } \mathcal{S} \text{ have } a\text{−configurations})\leq e^{-\eta N},$$

moreover, $\eta\to\infty$, as $b,c\to 0$;

2. *fixing b,c, when a is sufficiently small, there exists $\beta>0$, so that*

$$P(\text{all vertices in } \mathcal{S} \text{ have } a\text{−configurations})\leq e^{-\beta N},$$

moreover, $\beta\to\infty$, as $a\to 0$.

Proof. We prove Part 1 here; Part 2 is very similar. Recall that P depends on $a:b:c$. If we fix b,c, and let a be sufficiently large, this is the same as when we fix a and $\frac{b}{c}$, and let b,c be sufficiently small. We will prove Part 1 of the lemma under the latter equivalent assumption.

If no vertices in $\mathcal{S}$ have a-configurations, each vertex in $\mathcal{S}$ can have either a b-configuration or a c-configuration. Let Δ_N be the set of all such configurations on $\mathcal{S}$, then $|\Delta_N|\leq 4^N$, moreover,

$$P(\text{no vertices in } \mathcal{S} \text{ have } a-\text{configurations})=\sum_{\mathcal{C}\in\Delta_N}P(\mathcal{C}).$$

Let $\mathcal{C}$ be an arbitrary fixed configuration in Δ_N. Without loss of generality, assume $\mathcal{S}$ contains at least $\frac{N}{2}$ black vertices. We can label all the black vertices in $\mathcal{S}$ by v_k, where k is an integer satisfying $1\leq k\leq\left[\frac{N}{2}\right]$. Let $u_k^{(1)}$, $u_k^{(2)}$ be the endpoints of the corresponding edge of $\mathbb{H}_\triangle$ at v_k. Namely, if v_k has a b-configuration (resp. c-configuration) in $\mathcal{C}$, then $u_k^{(1)}u_k^{(2)}$ is a b-edge (resp. c-edge), i.e., a bisector edge in $\mathbb{H}_\triangle$ whose presence in dimer configurations corresponds to a b-configuration (resp. c-configuration) in $\mathbb{H}_\triangle$. Then the configuration $\mathcal{C}$ occurs in 1-2 model configurations of $\mathbb{H}$ only if all the edges $e_k = u_k^{(1)} u_k^{(2)}$ are present in the dimer configuration of $\mathbb{H}$. Define D to be a square matrix with rows and columns labeled by all the $u_k^{(1)}, u_k^{(2)}$'s as follows

$$D(u_k^{(p)}, u_l^{(q)}) = \begin{cases} bK^{-1}(u_k^{(p)}, u_l^{(q)}) & \text{If both } e_k \text{ and } e_l \text{ are } b\text{−edges} \\ \sqrt{bc}K^{-1}(u_k^{(p)}, u_l^{(q)}) & \text{If exactly one of } e_k \text{ and } e_l \text{ is a } b\text{−edge} \\ cK^{-1}(u_k^{(p)}, u_l^{(q)}) & \text{If both } e_k \text{ and } e_l \text{ are } c\text{−edges} \end{cases}$$

Then according to the results in Sect. 3,

$$P(\mathcal{C}) \leq |\mathrm{Pf} D|$$

Note that $|D(u_k^{(1)}, u_k^{(2)})| = P_b$, or P_c, depending on whether the edge e_k is a b-edge or a c-edge. P_b (P_c) is the probability that a b-configuration (c-configuration) appears at a vertex. Since $P_a + P_b + P_c = 1$, by Lemma 4.1, we have

$$\lim_{b,c \to 0} P_b + P_c = 0.$$

Moreover, since when we fix γ, a, $\gamma \neq 1$ and b, c are sufficiently small, the spectral curve has no zeros on $\mathbb{T}^2$, the entries $D(u, v)$ decay exponentially to 0 when $|u - v| \to \infty$. Namely,

$$|D(u,v)| \leq e^{-\beta|u-v|}, \text{where } \beta > 0, \text{and } |u-v| \text{ is large.} \tag{4.6}$$

Moreover, the exponential decay rate β can be chosen as a uniform constant as the parameters a, b, c are away from the spectral curve. Now let us consider

$$\mathrm{Pf} D = \sum_{\substack{\sigma \in S_N \\ \sigma(1) < \sigma(3) < \cdots < \sigma(N-1) \\ \sigma(2i-1) < \sigma(2i)}} \mathrm{sgn}(\sigma) \prod_{i=1}^{\frac{N}{2}} D(w_{\sigma(2i-1)}, w_{\sigma(2i)}),$$

where S_N is the symmetric group, and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ, and $w_{2i-1} = u_i^{(1)}$, $w_{2i} = u_i^{(2)}$. Let V_0 denote the set of vertices in $\{u_k^{(1)}, u_k^{(2)}\}_k$, we have

$$\begin{aligned} &\sum_{\substack{\sigma \in S_N \\ \sigma(1) < \sigma(3) < \cdots < \sigma(N-1) \\ \sigma(2i-1) < \sigma(2i)}} \left| \prod_{i=1}^{\frac{N}{2}} D(w_{\sigma(2i-1)}, w_{\sigma(2i)}) \right| \\ \leq &\left(\sum_{w \in V_0, w \neq w_1} |D(w_1, w)| \right)^{\frac{N}{2}} \end{aligned}$$

Note that if w is a neighboring vertex of w_1, then $|D(w, w_1)|$ is the probability that the edge ww_1 appears in the dimer configuration of $\mathbb{H}_\triangle$; when b and c are sufficiently small, this probability is small, and approaches 0 as $b, c \to 0$. When we consider the event that all the vertices in $\mathcal{S}$ has b-configuration or c-configuration, the probability of the event is bounded above by the probability of the event that all the vertices in $\mathcal{S}' \subseteq \mathcal{S}$ have b-configuration or c-configuration.

For any $C_1 > 0$, there is $c_0 > 0$, so that for any $\mathcal{S}$, there is $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \geq c_0|\mathcal{S}|$, and the points in $\mathcal{S}$ are at pairwise distance at least C_1. In $\mathcal{S}'$, there are $O(k^2)$ points within distance kC_1 of any given point, it follows that

$$\sum_{w \in V_0(\mathcal{S}'), w \neq w_1} |D(w_1, w)| \leq \max\{P_b, P_c\} + O\left(\sum_{k \geq 1} k e^{-\beta k C_1}\right), \tag{4.7}$$

and the righthand side of (4.7) can be made smaller than 1 by taking b, c small enough and C_1 large enough. Then the lemma follows. □

4.2 Percolation

In this subsection, we explore the consequences of the exponential decay of large cluster probabilities. One consequence is the existence of a phase transition: i.e., fixing b, c, when a is sufficiently small, almost surely there are no infinite a-clusters, and when a is large, almost surely there is a unique infinite a-cluster. Another consequence is the continuity of the mean cluster size with respect to a when a is small.

Let t_n denote the number of animals (connected sets of vertices) including the origin with cardinality (total number of vertices) n. The following lemma says that t_n grows at most exponentially in n. The proof follows from a standard argument of Kesten (Lemma 5.1 of [7]).

Lemma 4.3. *$t_n \leq 10^n$, for all n.*

Proof. We consider independent site percolation on the hexagonal lattice $\mathbb{H}$ and let μ be the corresponding product measure. We have

$$P(\text{the open cluster at the origin has size } n) = \sum_b t_{n,b} p^n (1-p)^b$$

where b is the number of boundary vertices, and $t_{n,b}$ is the number of animals including the origin with n vertices and b boundary vertices. Here by boundary vertices of a vertex set we mean vertices not in the set themselves but incident to vertices in the set. Note that $t_{n,b} = 0$ unless $1 \leq b \leq 3n$. We have

$$1 \geq \sum_b t_{n,b} p^n (1-p)^b \geq \sum_b t_{n,b} p^n (1-p)^{3n}$$

Hence

$$\sum_b t_{n,b} \leq [p(1-p)^3]^{-n}$$

for all p. We choose p to maximize $p(1-p)^3$, and obtain

$$t_n = \sum_b t_{n,b} \leq \left(\frac{256}{27}\right)^n$$

and the lemma follows. □

Let $\mathcal{C}_{a,v_0}$ denote the a-cluster passing a fixed vertex v_0, and $|\mathcal{C}_{a,v_0}|$ denote the cardinality of $\mathcal{C}_{a,v_0}$. Then we have the following theorem:

Theorem 4.4. *Fixing b, c, when a is sufficiently small, almost surely there are no infinite a-clusters.*

Proof. First we claim that

$$P(|\mathcal{C}_{a,v_0}| \geq n) \leq e^{-n\rho},$$

when a is sufficiently small and $b \neq c$. Here ρ is a positive constant independent of n. In fact,

$$P(|\mathcal{C}_{a,v_0}| \geq n) = \sum_{k=n}^{\infty} P(|\mathcal{C}_{a,v_0}| = k) = \sum_{k=n}^{\infty} \sum_{A_k} P(\mathcal{C}_{a,v_0} = A_k),$$

where A_k is an animal including v_0, with cardinality k, and the inner sum is over all the animals of cardinality k including v_0. By Lemma 4.2, we have

$$P(\mathcal{C}_{a,v_0} = A_k) \leq e^{-\beta k},$$

when a is sufficiently small and $b \neq c$. Moreover, $\beta \to \infty$, as $a \to 0$. Hence we can choose a to be sufficiently small so that $e^{-\beta} < \frac{1}{20}$. By Lemma 4.3, we have

$$P(|\mathcal{C}_{a,v_0}| \geq n) \leq \sum_{k=n}^{\infty} \left(\frac{1}{2}\right)^k \to 0, \qquad \text{as } n \to \infty,$$

and the theorem follows. $\square$

Define the mean size of the cluster at v_0 as follows

$$\chi = \sum_{n=1}^{\infty} nP(|\mathcal{C}_{a,v_0}| = n).$$

We have the following proposition

Proposition 4.5. *Fixing b, c, the mean size χ of the a-cluster at the origin is a continuous function of a, when a is sufficiently small.*

Proof.

$$P(|\mathcal{C}_{a,v_0}| = n) = \sum_{A_n} P(\mathcal{C}_{a,v_0} = A_n),$$

where the sum is over all the choices of size-n animals including v_0, $P(\mathcal{C}_{a,v_0} = A_n)$ is the probability of the event that all vertices in A_n have a-configurations, while all the vertices on the boundary of A_n do not have a-configurations. Hence

$$P(|\mathcal{C}_{a,v_0}| = n) = \sum_{A_n} \sum_{b} P(\mathcal{C}_{a,v_0} = A_n, \mathcal{C}_{\partial A_n} = b) \tag{4.8}$$

where b is any possible configuration (either a b-configuration or a c-configuration at each vertex) on ∂A_n, the boundary vertices of A_n. By Lemma 4.3, for each fixed n, (4.8) is a finite sum. According to the dimer representation of the 1-2 model, $P(\mathcal{C}_{a,v_0} = A_n, \mathcal{C}_{\partial A_n} = b)$ is the same as the probability that an a-bisector edge is present in the perfect matching at all the vertices of A_n, while a b-bisector edge, or a c-bisector edge is present in the perfect matching at each vertex of ∂A_n. This is the product of edge weights, multiplied by the Pfaffian of a submatrix $K^{-1}_{A_n \cup \partial A_n}$ of the inverse Kasteleyn matrix, and $K^{-1}_{A_n \cup \partial A_n}$ is indexed by the endpoints of the specified a, b, c edges in the dimer graph. When the spectral curve does not intersect the unit torus $\mathbb{T}^2$, each entry of the inverse matrix is a continuous function in a. For fixed n, $K^{-1}_{A_n \cup \partial A_n}$ is a matrix of finite order, hence $\mathrm{Pf} K^{-1}_{A_n \cup \partial A_n}$ is continuous in a, so is $P(|C_{a,v_0}| = n)$, since there are only finitely many configurations for each fixed n, according to Lemma 4.3. By definition,

$$\chi = \sum_{n=1}^{\infty} nP(|\mathcal{C}_{a,v_0}| = n),$$

which is the limit of a sequence of continuous functions. When a is sufficiently small, there exists a positive number α, such that

$$nP(|\mathcal{C}_{a,v_0}| = n) \leq nP(|\mathcal{C}_{a,v_0}| \geq n) \leq e^{-\alpha n},$$

by Theorem 12. Hence the sequence of continuous functions converges uniformly in any closed interval, as a result, χ is continuous in a if a is sufficiently small so that the spectral curve does not intersect $\mathbb{T}^2$ and that the probability of a size-n a-cluster decays exponentially at the origin, as $n \to \infty$. □

Define an $\overline{a}$**-cluster** to be a connected set of vertices, none of which have a-configurations. We have the following proposition regarding the behavior of the $\overline{a}$-cluster at the origin.

Proposition 4.6. *Fixing b, c, when a is small, the probability that an infinite $\bar{a}$-cluster appears at the origin is strictly positive.*

Proof. Let $\mathcal{C}_{\overline{a}}(\omega)$ be the largest $\overline{a}$-cluster including the origin in the configuration ω. Let Λ_k be the $k \times k$ box of $\mathbb{H}$ centered at the edge connecting $(0,0)$ and $(0,1)$. Let U_k be the event that no vertices in Λ_k have a-configurations. Namely,

$$U_k = \{\omega \in \Omega : \Lambda_k \subseteq \mathcal{C}_{\overline{a}}(\omega)\}.$$

For any fixed integer k, $P(U_k) > 0$.

For any simply connected vertex set $D \subseteq V$, let $\partial D \subseteq E$, consisting of all the edges connecting a vertex in D and a vertex outside D. Let $\hat{D} \subseteq V \setminus D$, consisting of all the vertices in $V \setminus D$ incident to an edge in ∂D. If D is a finite set, we have

$$|\hat{D}| \leq |\partial D| \leq 3|\hat{D}|,$$

here $|\cdot|$ is the cardinality of a set.

Let $\mathcal{T}$ be the dual triangular lattice of $\mathbb{H}$. Let $(\partial D)^*$ be the set of edges of $\mathcal{T}$ consisting of all the dual edges of edges in ∂D. Obviously $|\partial D| = |(\partial D)^*|$.

We consider the conditional probability $P(|\mathcal{C}_{\overline{a}}(\omega)| < \infty | U_k)$. It is not hard to see that conditional on U_k, $\mathcal{C}_{\overline{a}}(\omega)$ is finite if the following event F_S occurs: there exists a simply connected set S satisfying

1. $\Lambda_k \subseteq S$, $|S| < \infty$;
2. All the vertices in $\hat{S}$ have a-configurations.

Note that $(\partial S)^*$ is a connected set of edges of $\mathcal{T}$, surrounding the origin. Using a similar argument as in Lemma 4.3, it is easy to see that the total number of connected edge sets of $\mathcal{T}$ surrounding the origin with cardinality h, is bounded above by θ^h, when $h \geq H_0$. Here H_0 and θ are positive constants.

Therefore, we have

$$\begin{aligned} P(|\mathcal{C}_{\overline{a}}(\omega)| = \infty) &\geq P(|\mathcal{C}_{\overline{a}}(\omega)| = \infty|\mathcal{U}_k)P(\mathcal{U}_k) \\ &= [1 - P(|\mathcal{C}_{\overline{a}}(\omega)| < \infty|\mathcal{U}_k)]P(\mathcal{U}_k) \\ &\geq [1 - \sum_{h\geq 2k} \sum_{\{\hat{S}:|\partial S|=h\}} P(\text{all vertices in } \hat{\mathrm{S}} \text{ have } a-\text{config.}|\mathcal{U}_k)]P(\mathcal{U}_k) \\ &\geq P(\mathcal{U}_k) - \sum_{h\geq 2k} \sum_{\{\hat{S}:|\partial S|=h\}} P(\text{all vertices in } \hat{\mathrm{S}} \text{ have } a-\text{config.}) \\ &\geq P(\mathcal{U}_k) - \sum_{\{h:h\geq 2k\}} \theta^h e^{-\beta h} \end{aligned}$$

where the last inequality follows from Lemma 4.2. For fixed k, when a is small, $P(\mathcal{U}_k)$ is continuous in a, and is 1 if $a = 0$. Hence we can choose a sufficiently small such that, $P(\mathcal{U}_k) > \frac{3}{4}$, and $e^{-\beta}\theta < \frac{1}{2}$, then

$$P(|\mathcal{C}_{\overline{a}}(\omega)| = \infty) \geq \frac{3}{4} - \left(\frac{1}{2}\right)^{2k-1} > 0,$$

and the proposition follows. □

Proposition 4.7. *When a is sufficiently large, the probability that an infinite a-cluster appears at the origin is strictly positive.*

Proof. First of all, let Λ be a fixed finite set of vertices, and consider the probability that no vertices in Λ have a-configurations. Since each vertex of Λ may take either a b-configuration or a c-configuration, there are $2^{|\Lambda|}$ possible (dimer) configurations at most. For each configuration ξ_Λ, according to Lemma 4.2, we have

$$P(\xi_\Lambda) \leq e^{-\eta|\Lambda|},$$

when a is sufficiently large. Then using similar techniques as in Proposition 4.6, Proposition 4.7 can be proved. □

Recall that $\mathcal{T}$ is the dual triangular lattice of the hexagonal lattice $\mathbb{H}$. An **a-interface** is a connected set of edges of $\mathcal{T}$, in which every edge separates a pair of vertices in $\mathbb{H}$; one has an a-configuration and the other does not have an a-configuration. The union of all a-interfaces on the plane (or on the torus) for each random 1-2 model configuration forms a closed polygon configuration for the triangular lattice $\mathcal{T}$, i.e., at each vertex there are an even number of incident present edges, see Figure 11 - an illustration of a-interfaces for a 1-2 model configuration on a 3×3 torus.

Lemma 4.8. *When a is sufficiently large, almost surely there are no infinite a-interfaces.*

Proof. By definition, associated to a large a-interface is a large set of vertices, none of which have a-configurations. When a is sufficiently large, for any fixed large set, the probability that no vertices in the set have a-configurations decays exponentially with respect the the size of the set (Lemma 4.2). Then using similar argument as in Theorem 4.4 gives the result. □

Recall that an a-cluster is a connected set of vertices in which every vertex has an a-configuration. However, since there are two different a-configurations, either $\{100\}$ or $\{011\}$, obviously there are two types of a-clusters: either a $\{100\}$-cluster, a connected set of vertices where all the vertices have $\{100\}$ configurations or $\{011\}$-cluster, a connected set of vertices where all the vertices have $\{011\}$ configurations. Note that in one

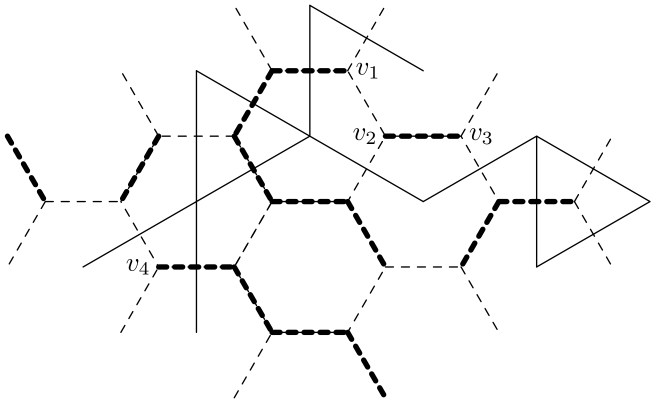

Figure 11: Interface

connected set of vertices where all the vertices have a-configurations, it is not possible to have both $\{100\}$ configurations and $\{011\}$ configurations, since this way there will be a vertex with three incident edges present in the configuration, which is a contradiction to the law that each vertex can only have one or two incident edges. Therefore, every a-cluster is either a $\{100\}$-cluster or a $\{011\}$-cluster.

Theorem 4.9. *When a is sufficiently large, almost surely there exists exactly one infinite a-cluster; in particular, the two types of infinite a-clusters, $\{100\}$ and $\{011\}$ cannot coexist.*

Proof. First of all, we prove that conditional on the existence of infinite a-clusters, it is almost surely the case that the number of infinite a-clusters is exactly 1. Let $\mathcal{S}$ be an maximal infinite a-cluster, i.e., none of the vertices in $V \setminus \mathcal{S}$ adjacent to a vertex in $\mathcal{S}$ have a-configurations. Recall that V is the vertex set of the whole plane hexagonal lattice $\mathbb{H}$. Define $T_{\mathcal{S}}$ be a set of edges of $\mathcal{T}$, each of which is the dual edge of an edge of $\mathbb{H}$ connecting a vertex in $\mathcal{S}$ and a vertex not in $\mathcal{S}$. Then the connected components of $T_{\mathcal{S}}$ are a-interfaces by definition. By Lemma 15, $T_{\mathcal{S}}$ has no infinite connected components, hence $V \setminus \mathcal{S}$ is a union of disconnected finite sets. Hence if infinite a-clusters exist, there is exactly one infinite a-cluster almost surely.

Secondly, since the measure P_M is ergodic, and the event that there exists an infinite a-cluster is translation-invariant, either almost surely there exists an infinite a-cluster or almost surely there are no infinite a-clusters. By Proposition 4.7 and the discussion above, we conclude that almost surely there exists a unique infinite a-cluster, when a is sufficiently large.

□

Remark. Corollary 4.4 and Theorem 4.9 imply that the system undergoes a phase transition. Figure 12 is a picture for 1-2 model configurations with large a. It is obtained using the Markov chain Monte Carlo simulation with parameter $a = 5, b = c = 1$.

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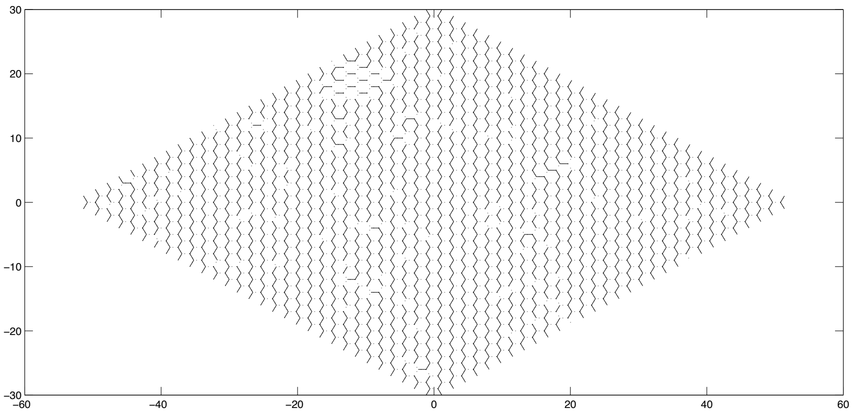

Figure 12: Large a configuration

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