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Müntz linear transforms of Brownian motion

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Abstract

We consider a class of Volterra linear transforms of Brownian motion associated to a sequence of Müntz Gaussian spaces and determine explicitly their kernels; the kernels take a simple form when expressed in terms of Müntz-Legendre polynomials. These are new explicit examples of progressive Gaussian enlargement of a Brownian filtration. We give a necessary and sufficient condition for the existence of kernels of infinite order associated to an infinite dimensional Müntz Gaussian space; we also examine when the transformed Brownian motion remains a semimartingale in the filtration of the original process. This completes some already obtained partial answers to the aforementioned problems in the infinite dimensional case.

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1 Introduction

There has been a renewed interest in Müntz spaces which is particularly motivated by topics related to Markov inequalities and approximation theory, see for example ([5]–[7]) and the references therein. In the meanwhile, Volterra transforms with non square-integrable kernels, involving some functional spaces, provide interesting examples of noncanonical decompositions of the Brownian filtration. This motivated many studies on the topic, for instance see ([4], [10], [17], [19], [33]). Our aim in this paper is to study the class of Volterra transforms involving Gaussian spaces which are generated from sequences of Müntz polynomials. This gives new explicit examples of progressive enlargement of filtrations and interesting links with Müntz-Legendre polynomials; see ([26]–[29], [37]) for studies on this topic in more general frameworks.

To be more precise, let us fix our mathematical setting. Let $B := (B_t, t \geq 0)$ be a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\{\mathcal{F}_t^B, t \geq 0\}$ the filtration it generates. We encountered, in literature, two

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types of linear transforms of B which are of our interest in this paper. The first type consists of transforms of the form

$$T(B)_t = \int_0^t \rho(t/s)\, dB_s \tag{1.1}$$

for all $t > 0$ and some $\rho \in \mathcal{M}$, with

$$\mathcal{M} = \{\rho : [1, \infty) \to \mathbb{R} \text{ measurable function s.t. } \int_0^1 \rho^2(1/v)\, dv < \infty\}.$$

These transforms were intensively studied in [29]. In particular, we found in Proposition 15 therein a variant of Theorem 6.5 of [31] which states that $(T(B)_t, t \geq 0)$ is a semimartingale relative to the filtration of B if and only if there exists $c \in \mathbb{R}$ and $g \in \mathcal{M}$ such that

$$\rho(.) = c + \int_1^{\cdot} \frac{1}{y} g(y)\, dy.$$

The second type consists of Volterra transforms with non-square-integrable kernels which are of the form

$$T(B)_t = B_t - \int_0^t ds \int_0^s l(s, v)\, dB_v \tag{1.2}$$

for all $t > 0$, where the kernel $l : \mathbb{R}_+^2 \to \mathbb{R}$, which satisfies $l(s, v) = 0$ for $s < v$, is such that the symmetrized kernel

$$\tilde{l}(t, s) := \begin{cases} l(t, s) & \text{if} \quad s \leq t; \\ l(s, t) & \text{if} \quad s \geq t, \end{cases}$$

is continuous on $\mathbb{R}_+^2$. These transforms were studied for example in ([2], [14], [20]). Note that, in the semimartingale case with $c = 1$, the transform (1.1) becomes a Volterra transform of the form (1.2) with kernel $l(t, s) = t^{-1} g(t/s)$ for $s \leq t$ and $l(t, s) = 0$ otherwise. Conversely, all transforms of the form (1.2) which we will consider in this paper are of the form (1.1). Uniqueness when defining $T(B)$ by either (1.1) or (1.2) holds only up to a stochastic modification and we work with the continuous one.

Let $f_j(x) := x^{\lambda_j}$, $j = 1, 2, \cdots$, be a sequence of Müntz polynomials where $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ is a sequence of reals satisfying

$$\lambda_j > -1/2, \qquad j = 1, 2, \cdots . \tag{1.3}$$

These generalized polynomials are defined on $[0, \infty)$ and the value of f_j at $x = 0$ is the limit of $f_j(x)$ as $x \to 0$ from $(0, \infty)$ for $j = 1, 2, \cdots$. For each fixed $t > 0$, let us define the Müntz Gaussian spaces

$$G_t(\lambda_1, \cdots \lambda_n; B) = \text{Span}\left\{\int_0^t s^{\lambda_j}\, dB_s, j = 1, 2, \cdots n\right\} \tag{1.4}$$

and

$$G_t(\lambda_1, \lambda_2, \cdots; B) = \text{Span}\left\{\int_0^t s^{\lambda_j}\, dB_s, j = 1, 2, \cdots\right\} \tag{1.5}$$

and let $H_t(B)$ be the closed linear span of $\{B_s, s \leq t\}$. A Müntz transform of order n associated to $\lambda_1, \cdots, \lambda_n$, is a linear transform T_n of the form (1.1) such that the following two properties hold true:

(i) $(T_n(B)_t, t \geq 0)$ is a Brownian motion;

(ii) we have the orthogonal decomposition

$$H_t(B) = H_t(T_n(B)) \oplus G_t(\lambda_1, \cdots, \lambda_n; B), \qquad t > 0. \tag{1.6}$$

Following [2] and [20], if $n < \infty$ then such transforms exist. As we shall see, the transform T_n of the form (1.2) with $l(t,s) = k_n(t,s) := t^{-1}K_n(s/t)$ for $s \le t$ where $K_n(x) := \sum_{j=1}^n a_{j,n}x^{\lambda_j}$ for $0 \le x \le 1$ and $a_{1,n}, a_{2,n}, ..., a_{n,n}$ are uniquely determined by the system

$$\sum_{j=1}^{n} \frac{a_{j,n}}{\lambda_j + \lambda_k + 1} = 1, \quad k = 1, \cdots, n, \tag{1.7}$$

is a Müntz transform of order n. The latter system, when $\lambda_j = j$ for $j = 1, 2, \cdots$, was first discovered by P. Lévy, see ([32], [34]) and was further studied in [10]. Solving (1.7), we found that the sequence of Müntz polynomials $(K_n, n = 1, 2, \cdots)$ can be simply expressed in terms of Müntz-Legendre polynomials which allows to simplify the study of some of their properties. Note that T_n takes the form (1.1) with

$$\rho_n(x) = 1 - \int_1^x K_n(1/r)\frac{dr}{r}, \quad x \ge 1. \tag{1.8}$$

Thus, we are in the semimartingale case with $c = 1$ and $g(.) = -K_n(1/.)$.

The kernels described above are homogeneous of degree -1 in the sense that

$$k_n(\alpha t, \alpha s) = \alpha^{-1}k_n(t,s), \quad 0 < s \le t < \infty, \alpha > 0.$$

As a consequence, the associated Volterra transforms have a close connection to a class of stationary processes. That is, the process

$$(e^{-u/2}T_n(B)_{e^u}, u \in \mathbb{R})$$

is an Ornstein-Uhlenbeck process; the conventional value $-1/2$ of the parameter will be dropped from our notations. It has the moving average representation, m.a.r. for short,

$$S_n(W)_u := \int_{-\infty}^{u} \eta_n(u-r)\, dW_r$$

for $u \in \mathbb{R}$, where W is a Brownian motion indexed by $\mathbb{R}$ and $\eta_n \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ has the Fourier transform

$$\hat{\eta}_n(\xi) := \int e^{i\xi x}\eta_n(x)\, dx = \frac{1}{1/2 - i\xi}\prod_{j=1}^{n} \frac{\xi - ip_j}{\xi + ip_j}, \qquad \xi \in \mathbb{R}, \tag{1.9}$$

where $p_j = \lambda_j + \frac{1}{2}$ for $j = 1, 2, \cdots$. Applying then the characterization given in [30], the presence of the inner part, given by the product in (1.9), implies that the latter m.a.r. is not canonical with respect to W.

A natural question is to know whether there exists a transform T of the form (1.1) such that

$$H_t(B) = H_t(T(B)) \oplus G_t(\lambda_1, \lambda_2, \cdots; B)$$

for all $t > 0$. Partial answers are given in ([17], [21], [22]) where the authors established the existence of such transforms. In particular, for an infinite sequence Λ satisfying either $\sup \lambda_j = +\infty$ or $0 < \lambda_1 < \lambda_2 < ...$ there exists no such a transform such that $(T(B)_t, t \ge 0)$ is a semimartingale relative to the filtration of B. We see that a necessary and sufficient condition for the existence of transforms of the form (1.1) with infinite dimensional orthogonal complement is

$$\sum_{j=1}^{\infty} \frac{p_j}{p_j^2 + 1} < \infty. \tag{1.10}$$

This is the well-known Müntz-Szasz condition which is necessary and sufficient for f_1, f_2, $\cdots$, to be incomplete in $L^2[0,1]$, see for example [6]. Furthermore, $(T(B)_t, t \ge 0)$ is a semimartingale relative to the filtration of B if and only if (λ_k) is bounded and satisfies (1.10). Plainly, the latter happens if and only if $\sum_{j=1}^{\infty} p_j < \infty$.

2 Müntz Gaussian spaces and transforms

Throughout this paper, we assume that $\Lambda = \{\lambda_1, \lambda_2, \cdots\}$ is a sequence of distinct real numbers satisfying condition (1.3). Thus, the generalized Müntz polynomials $f_j(x) := x^{\lambda_j}$, for $j = 1, 2, \cdots$, lie in

$$L^2_{loc}(\mathbb{R}_+) := \{f : \mathbb{R}_+ \to \mathbb{R}; f \in L^2[0,t] \text{ for all } 0 < t < \infty\}.$$

For $t > 0$, let us introduce

$$M_{n,t} = \text{Span}\{x^{\lambda_1}, x^{\lambda_2}, \cdots, x^{\lambda_n}; x \in [0,t]\}$$

and

$$M_{\infty,t} = \text{Span}\{x^{\lambda_1}, x^{\lambda_2}, \cdots; x \in [0,t]\}$$

which are called Müntz spaces. An associated orthogonal system, known as Müntz-Legendre polynomials, is specified by $L_1(x) = x^{\lambda_1}$ and L_2, $L_3 \cdots$, described by

$$L_k(x) = \sum_{j=1}^{k} c_{j,k} x^{\lambda_j}, \quad c_{j,k} = \frac{\prod_{l=1}^{k-1}(\lambda_l + \lambda_j + 1)}{\prod_{l=1, l\neq j}^{k}(\lambda_j - \lambda_l)}, \quad k = 2, 3, \cdots, \tag{2.1}$$

see [6] and [7]; note that we use slightly different notations since we start the sequence Λ with λ_1 instead of λ_0. Recall that $L_k(1) = 1$ for $k = 1, 2, \cdots$. Next, to the linear spaces $M_{n,t}$ and $M_{\infty,t}$ we associate, respectively, the families of Müntz Gaussian spaces defined by (1.4) and (1.5). Recall that the closed linear span of $\{B_s, s \le t\}$, or the first Wiener chaos of B, is given by

$$H_t(B) = \left\{ \int_0^t f(u)\, dB_u; f \in L^2[0,t] \right\}. \tag{2.2}$$

It follows that the orthogonal complements of $G_t(\lambda_1, \cdots, \lambda_n; B)$ and $G_t(\lambda_1, \lambda_2 \cdots; B)$, in $H_t(B)$, are respectively given by

$$G_t^{\perp}(\lambda_1, \cdots, \lambda_n; B) = \left\{ \int_0^t f(u)\, dB_u; f \in L^2[0,t], \int_0^t f(s)p(s)\, ds = 0, p \in M_{n,t} \right\}$$

and

$$G_t^{\perp}(\lambda_1, \lambda_2, \cdots; B) = \left\{ \int_0^t f(u)\, dB_u; f \in L^2[0,t], \int_0^t f(s)p(s)\, ds = 0, p \in M_{\infty,t} \right\}.$$

Following [2], the transform T (resp. kernel) defined by (1.2) is called a Goursat-Volterra transform (resp. kernel) of order n if $(T(B)_t, t \ge 0)$ is a Brownian motion and there exists n linearly independent functions $g_j \in L^2_{loc}(\mathbb{R}_+)$ such that

$$H_t(B) = H_t(T(B)) \oplus \text{Span}\{\int_0^t g_j(s)\, dB_s, j = 1, 2, \cdots n\}$$

for all $t > 0$. We are now ready to determine a Goursat-Volterra transform T_n associated to the Müntz polynomials f_1, f_2, $\cdots$, f_n, in the case when n is finite.

Theorem 2.1. *Assuming that $n < \infty$ then*

$$k_n(t,s) := \begin{cases} t^{-1} K_n(s/t) & \textit{if} \quad s \le t; \\ 0 & \textit{otherwise}, \end{cases}$$

where

$$K_n(s) = \sum_{j=1}^{n} a_{j,n} s^{\lambda_j}, \quad a_{j,n} = \frac{\prod_{l=1}^{n}(\lambda_j + \lambda_l + 1)}{\prod_{l=1, l\neq j}^{n}(\lambda_j - \lambda_l)}, \quad j = 1, ..., n, \tag{2.3}$$

is a Goursat-Volterra kernel of order n. Furthermore, writing T_n for the Goursat-Volterra transform associated to k_n, the orthogonal complement of $H_t(T_n(B))$ in $H_t(B)$ is $G_t(\lambda_1, \cdots, \lambda_n; B)$ for all $t \geq 0$. Note that T_n is of the form (1.1) with ρ prescribed by (1.8).

Proof of Theorem 2.1. $T_n(B)$ is a Brownian motion if and only if k_n satisfies the self-reproduction property

$$k_n(t,s) = \int_0^s k_n(t,u) k_n(s,u)\, du \tag{2.4}$$

for a.e. $s \leq t$, which is found in Theorem 6.1 in [14]. This is obtained by writing $\mathbb{E}[T_n(B)_t T_n(B)_s] = s \wedge t$, differentiating and rearranging terms. But, if we set $k_n(t,s) = t^{-1}\sum_{j=1}^n a_{j,n}(s/t)^{\lambda_j}$, then (2.4) is equivalent to saying that $(a_{j,n}, j = 1, 2, \cdots, n)$ solves the linear system (1.7). To study the system, consider the n-degree polynomial

$$p_n(x) = \prod_{j=1}^n (x + \lambda_j + 1) - \sum_{k=1}^n a_{k,n} \prod_{j=1, j\neq k}^n (x + \lambda_j + 1)$$

which, of course, has at most n roots. But $p_n(x) = 0$ is equivalent to $\sum_{k=1}^n \frac{a_{k,n}}{x+\lambda_k+1} = 1$. This fact, when combined with $\lim_{x\to\infty} p_n(x)/x^n = 1$, implies that $p_n(x) = \prod_{j=1}^n (x - \lambda_j)$. Now, let us choose $m \in \{1, \cdots, n\}$ and substitute the latter product formula in the expression of $p_n(x)$. Dividing both sides by $\prod_{j\neq m}(x + \lambda_j + 1)$, rearranging terms and letting $x \to -(\lambda_m + 1)$ we obtain the expressions of $a_{1,n}$,$a_{2,n}$, $\cdots$, $a_{n,n}$. Next, k_n is a Volterra kernel because it is continuous on $\{(u,v) \in \mathbb{R}_+^2; u > v\}$ and satisfies the following integrability condition which is enough for (1.2) to be well defined. We have

$$\begin{aligned}
\int_0^t \left(\int_0^u k_n^2(u,v)\, dv \right)^{1/2} du &= \int_0^t \left(\int_0^1 k_n^2(u, ur) u\, dr \right)^{1/2} du \\
&= 2\sqrt{t} \left(\int_0^1 K_n^2(r)\, dr \right)^{1/2} \\
&= 2\sqrt{t} K_n^{1/2}(1) < +\infty,
\end{aligned}$$

where we used the homogeneity and the self-reproduction properties of k_n. Finally, we need to identify $H_t(B) \ominus H_t(T_n(B))$ for an arbitrarily fixed $t > 0$. The condition $\int_0^t f(u)\, dB_u \perp T(B)_s$ for all $s \leq t$ is equivalent to

$$\int_0^s f(r)\, dr = \int_0^s du \int_0^u k_n(u,v) f(v)\, dv. \tag{2.5}$$

If we write $k_n(u,v) = \sum_{j=1}^n \varphi_j(u) f_j(v)$ then by differentiating the latter equation we obtain the integral equation

$$\begin{aligned}
f(s) &= \int_0^s k_n(s,v) f(v)\, dv \\
&= \sum_{j=1}^n \varphi_j(s) \int_0^s f_j(v) f(v)\, dv
\end{aligned}$$

for a.e. $t > 0$, this can also be found in [28]. Clearly, if f solves it then $f(t)/\varphi_1(t)$ must be absolutely continuous with respect to the Lebesgue measure. Repeating this argument, we see that (2.5) is equivalent to an ordinary linear differential equation of degree n which should hold for a.e. $s \in [0,t]$. The functions $u \to u^{\lambda_j}$, $j = 1, \cdots, n$, being n linearly independent solutions, we conclude that $G_t(\lambda_1,\cdots,\lambda_n;B)$ is the orthogonal complement of $H_t(T_n(B))$ in $H_t(B)$ as required. Next, by using the homogeneity property of the kernel k_n and the stochastic Fubini theorem, we can write

$$\begin{aligned} T_n(B)_t &= B_t - \int_0^t \int_0^u k_n(u,v)\, dB_v\, du \\ &= \int_0^t \left(1 - \int_v^t k_n(u,v)\, du\right) dB_v \\ &= \int_0^t \left(1 - \int_1^{t/v} k_n(vr,v) v\, dr\right) dB_v \\ &= \int_0^t \rho_n(t/v)\, dB_v, \quad t \geq 0, \end{aligned}$$

where

$$\rho_n(x) = 1 - \int_1^x k_n(r,1)\, dr = 1 - \int_1^x K_n(1/r)\frac{dr}{r}.$$

□

Remark 2.2. *Since when $n < \infty$ we have $K_n \in L^2[0,1]$, by using the homogeneity property of k_n we obtain that $\int_0^1 ds \int_0^1 k_n^2(s,v)\, dv = +\infty$ i.e. $k_n \notin L^2([0,1]\times[0,1])$. The representation (1.2) is the semimartingale decomposition of $T_n(B)$ with respect to $\mathcal{F}^B$; it is noncanonical relative to the filtration of B since $\mathcal{F}_t^{T(B)} \subsetneq \mathcal{F}_t^B$ for all $t>0$. The Volterra representation of $T_n(B)$ as $T_n(B)_t = X_t - \int_0^t ds \int_0^s l(s,v)\, dX_v$, where $l : \mathbb{R}_+^2 \to \mathbb{R}$ is such that $l(s,v) = 0$ for $s < v$ and X is a Brownian motion, is not unique. Indeed, one representation is given with $X = B$ and $l = k_n$ and another one is given with $X = T_n(B)$ and $l \equiv 0$. But if we add the condition $l \in L^2([0,1]\times[0,1])$ then the representation above is unique, see [25].*

The covariance matrix

$$(m_t^n)_{lj} = \frac{t^{\lambda_l+\lambda_j+1}}{\lambda_l+\lambda_j+1}, \quad l,j = 1,2,\cdots,n,$$

of the Gaussian process $(\int_0^t f^*(s)\, dB_s)$, where $f := (f_1,\cdots,f_n)^*$ is the transpose of the row vector $(f_1,\cdots,f_n)$, has an inverse matrix which we denote by α_t^n. In fact, m_1^n is a Cauchy matrix and an explicit formula for its inverse can be found in ([18], [36]). Note also that the Goursat form of k_n given below is given in a semi-explicit form in [20]. Here we propose another method to compute the entries of α_t^n and φ.

Proposition 2.3. *The kernel k_n of Theorem 2.1 satisfies*

$$k_n(t,s) = \begin{cases} \varphi^*(t)\cdot f(s) & \text{if } \; s \leq t; \\ 0 & \text{otherwise}, \end{cases}$$

where $\varphi(.) = \alpha(.)\cdot f(.)$, $\varphi_l(t) = a_{l,n} t^{-\lambda_l - 1}$, $l = 1,2,\cdots,n$ and the entries of α_t^n are given by $(\alpha_t^n)_{l,j} = a_{l,n}a_{j,n}(\lambda_l+\lambda_j+1)^{-1}\, t^{-\lambda_l-\lambda_j-1}$.

Proof of Proposition 2.3. Assume that k_n is of the given Goursat form where $\varphi_1,\varphi_2,\cdots,\varphi_n$ are unknown. By using the self-reproduction property (2.4) a little algebra gives that $\varphi(.) = \alpha(.)\cdot f(.)$. The entries of $\varphi(t)$ are identified from the expression of k_n given in Theorem 2.1. Next, from Theorem 2.2 in [2] , we know that $(\alpha_t^n, t > 0)$ is given, in terms of φ, by

$$\alpha_t^n = \int_t^\infty \varphi(u)\cdot\varphi^*(u)\, du + \alpha_\infty, \quad \varphi(t) = \alpha_t \cdot f(t), \quad t > 0.$$

But, here $\alpha_\infty^n \equiv 0$ because f_1, f_2, $\cdots$, f_n are not square-integrable over $(0,+\infty)$. Plugging in the vector φ we obtain the matrix α_t^n. □

Remark 2.4. *In terms of filtrations, for* $n<\infty$ *and* $0<T<\infty$*, we have* $\mathcal{F}_T^B=\mathcal{F}_T^{T_n(B)}\otimes\sigma\left(G_T(\lambda_1,\cdots,\lambda_n;B)\right)$*. In fact,* $\mathcal{F}_T^{T_n(B)}$ *coincides, up to null sets, with* $\sigma\{B_u^{(br)},u\le T\}$*, where* $(B_u^{(br)},u\le T)$ *is the* f*-generalized bridge over the interval* $[0,T]$*. A realization of this is given by* $B_u^{(br)}=B_u-\psi_T^*(u)\cdot\int_0^T f(s)\,dB_s$ *where* $\psi_T(u)=\alpha_T^n\cdot\int_0^u f(r)\,dr$*, for* $u<T$*. This is called a generalized bridge because* $\int_0^T f_j(s)\,dB_s^{(br)}=0$ *for* $j=1,\cdots,n$*. Note that* $T_n(B^{(br)})=T_n(B)$ *on* $(0,T)$*; we refer to [1] for more details on these processes.*

The objective of the next proposition is to show that we can express K_n in terms of Müntz-Legendre polynomials given by formula (2.1) which form an orthogonal basis of $M_{n,1}$. In the special case when $\lambda_j=j$ for all j, an integro-difference equation satisfied by ρ_n, $n=1,2,\cdots$, was discovered in [10]. The second assertion of the following result proves useful for finding the analogue of Chiu's result in the general Müntz framework.

Proposition 2.5. *Recall that the functions* L_n *and* K_n *are given by (2.1) and (2.3), respectively. The following assertions hold true.*

1) We have

$$K_n(x)=\sum_{j=1}^n(1+2\lambda_j)L_j(x),\quad x\le 1. \tag{2.6}$$

In particular, $K_n(1)=\sum_{j=1}^n(1+2\lambda_j)$*. Consequently, we have*

$$K_n(x)=x^{-\lambda_n}\frac{\partial}{\partial x}\left(x^{\lambda_n+1}L_n(x)\right)$$

and, equivalently,

$$L_n(x)=x^{-\lambda_n-1}\int_0^x s^{\lambda_n}K_n(s)\,ds.$$

Note that unlike Müntz-Legendre polynomials, the Müntz polynomial K_n *does not depend on the order of* $\lambda_1,\lambda_2,\cdots,\lambda_n$*.*

2) The sequence K_n*,* $n=1,2,\cdots$*, satisfies the integro-difference equation*

$$K_n(x)=K_{n-1}(x)+(2\lambda_n+1)x^{\lambda_n}\left(1-\int_x^1 u^{-\lambda_n-1}K_{n-1}(u)\,du\right).$$

Proof of Proposition 2.5. 1) We have $(1+2\lambda_n)c_{n,n}=a_{n,n}$ and $(1+2\lambda_n)c_{j,n}=a_{j,n}-a_{j,n-1}$ for $j=1,2,\cdots,n-1$. Thus, we can write

$$\begin{aligned}K_n(x)-K_{n-1}(x)&=a_{n,n}x^{\lambda_n}+\sum_{j=1}^{n-1}(a_{j,n}-a_{j,n-1})x^{\lambda_j}\\&=(1+2\lambda_n)c_{n,n}x^{\lambda_n}+(1+2\lambda_n)\sum_{j=1}^{n-1}c_{j,n}x^{\lambda_j}\\&=(1+2\lambda_n)L_n(x).\end{aligned}$$

Iterating, with the convention that $K_0\equiv 0$, and summing up the equations we get the first formula; $K_n(1)$ is obtained by setting $x=1$ an using $L_j(1)=1$ for $j=1,2,\cdots,n$. As a by-product formula, we note that $(\lambda_j+\lambda_n+1)c_{j,n}=a_{j,n}$ for $j\le n$. The second assertion is easily obtained by integration.

2) We quote from [7] the recurrence formula

$$x^{\lambda_n+\lambda_{n-1}+1}\left(x^{-\lambda_n}L_n(x)\right)'=\left(x^{\lambda_{n-1}+1}L_{n-1}(x)\right)'.$$

Combining this with the first assertion and simplifying yields

$$\begin{aligned}\left(x^{-\lambda_n}L_n(x)\right)' &= x^{-\lambda_n-1}K_n(x)-(2\lambda_n+1)x^{-2\lambda_n-2}\int_0^x s^{\lambda_n}K_n(s)\,ds.\\ &= x^{-\lambda_n-1}K_{n-1}(x).\end{aligned}$$

Differentiating, we find $-\lambda_n K_n(x)+xK_n'(x)=(\lambda_n+1)K_{n-1}+xK_{n-1}'$. This is nothing but a differential form of the integro-difference equation. It remains to use the first assertion on the form $K_n(x)=K_{n-1}(x)+(1+2\lambda_n)L_n(x)$ and the fact that $L_n(1)=1$ to conclude. □

Our aim now is to outline a connection between self-reproducing kernels and the classical kernel systems.

Proposition 2.6. *For each fixed $t>0$, the kernel system associated to $M_{n,t}$ is given by $g_{n,t}(u,v)=\frac{1}{t}\sum_{l=1}^{n}(1+2\lambda_l)L_l(\frac{u}{t})L_l(\frac{v}{t})$ for $0<u,v\leq t$. Letting $u\to t$ we get that $k_n(t,s)=g_{n,t}(t,s)=\frac{1}{t}\sum_{l=1}^{n}(1+2\lambda_l)L_l(\frac{s}{t})$ for $0<s\leq t<\infty$.*

Proof of Proposition 2.6. The kernel system is given by $g_{n,t}(u,v)=\sum_{k=1}^{n}q_{k,t}^n(u)q_{k,t}^n(v)$ where $(q_{k,t}^n, n=1,\cdots,n)$ is an orthonormal sequence that generates $M_{n,t}$. This is a reproducing kernel in the sense that, for any $Q_t\in M_{n,t}$, we have

$$Q_t(u)=\int_0^t g_{n,t}(u,v)Q_t(v)\,dv.$$

Exploiting homogeneity, we easily check that the sequence $(q_{j,t}^n(x), x\in[0,t]; j=1,2,\cdots,n)$ defined by $q_{m,t}^n(u):=\sum_{k=1}^{m}c_{k,m}(t)u^{\lambda_k}=\sqrt{(1+2\lambda_m)/t}L_m(u/t)$ satisfies the requirements. We conclude using continuity and the fact that $L_n(1)=1$. □

3 Connection to stationary Ornstein-Uhlenbeck processes

We discuss here a question tackled in [22]; this consists of determining a necessary and sufficient condition for the existence of transforms of the form (1.1) or (1.2) with an infinite dimensional orthogonal complement associated to Λ. Let us recall some excerpts from [29] and [31] on linear transforms of Brownian motions and stationarity. If two semimartingales W and B are related by

$$W_u=\begin{cases} B_1+\int_1^{e^u}\frac{dB_s}{\sqrt{s}} & \text{if} \quad u\geq 0;\\ B_1-\int_{e^u}^{1}\frac{dB_s}{\sqrt{s}} & \text{if} \quad u\leq 0,\end{cases} \tag{3.1}$$

and, equivalently, by

$$B_t=\int_{-\infty}^{\log t}e^{r/2}\,dW_r, \quad t>0, \tag{3.2}$$

then it is easily checked that B is standard Brownian motion if and only if W is a Brownian motion indexed by $\mathbb{R}$ i.e. W is a centered continuous Gaussian process with independent increments such that $\mathbb{E}\left[(W_u-W_v)^2\right]=|u-v|$ for all u and $v\in\mathbb{R}$. Furthermore, we have $\int_0^\infty\varphi(s)\,dB_s=\int_{\mathbb{R}}V\varphi(r)\,dW_r$ for $\varphi\in L^2(\mathbb{R}_+)$ where the isometry $V:L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$ is defined by $V\varphi(u)=e^{u/2}\varphi(e^u)$. We need to introduce the mapping $U:C(\mathbb{R}_+,\mathbb{R})\to C(\mathbb{R},\mathbb{R})$ which is specified by $U\varphi(u)=e^{-u/2}\varphi(e^u)$, for $u\in\mathbb{R}$, and denote by U^{-1} its inverse operator. Keeping in mind that T is defined by (1.1) and setting $\Theta=U\circ T$, we clearly have that $\Theta(B)=S(W)$ where the transform S is defined by

$$S(W)_u=\int_{-\infty}^{u}\eta(u-v)\,dW_v, \quad u\in\mathbb{R}, \tag{3.3}$$

with $\eta(u) = \mathbf{1}_{u>0} U\rho(u)$. Plainly, $T(B)$ is a Brownian motion if and only if $\Theta(B)$ is a stationary Ornstein-Uhlenbeck process. Moreover, for some $f \in L^2_{loc}(\mathbb{R}_+)$ we have

$$H_t(T(B)) \perp \int_0^t f(s)\,dB_s, \quad t > 0,$$

if and only if

$$\mathcal{H}(S(W))_u \perp \int_{-\infty}^u V(f)(r)\,dW_r, \quad u \in \mathbb{R},$$

where $\mathcal{H}(S(W))_u$ stand for the closed linear span of $\{S(W)_r, r \le u\}$. The focus in the next result is on the m.a.r. of $\Theta_n(B) := (U \circ T_n(B)_u, u \in \mathbb{R})$ in case when $n < \infty$.

Proposition 3.1. *Assume that $n < \infty$ and T_n is the Goursat-Volterra transform of Theorem 2.1. The process $\Theta_n(B)$ has the m.a.r. (3.3) where W is given by (3.1) and $\eta := \eta_n$ has the Fourier transform given by*

$$\hat{\eta}_n(\xi) = (1/2 - i\xi)^{-1}\Pi_n(\xi), \quad \xi \in \mathbb{R}, \tag{3.4}$$

where

$$\Pi_n(\xi) := \prod_{j=1}^n \frac{\xi - ip_j}{\xi + ip_j}$$

and $p_j = \frac{1}{2} + \lambda_j$ for $j = 1, \cdots, n$.

Proof of Proposition 3.1. Using Theorem 2.1 and the recalls above, we see that formula (3.3) holds with $\eta_n(t) = \mathbf{1}_{\{t>0\}} U \circ \rho_n(t)$ and $\eta = \eta_n$. Note that $\eta_n \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Now, for $\xi \in \mathbb{R}$, we have

$$\begin{aligned}
\hat{\eta}_n(\xi) &= \int_0^\infty e^{i\xi t} e^{-t/2} \rho_n(e^t)\,dt \\
&= \int_0^\infty e^{-(1/2 - i\xi)t} \left(1 - \int_1^{e^t} K_n(1/r)(1/r)\,dr\right) dt \\
&= (1/2 - i\xi)^{-1} - \int_1^\infty \left(\int_{\ln r}^\infty e^{-(1/2-i\xi)t} dt\right) K_n(1/r)(1/r)\,dr \\
&= \frac{1}{1/2 - i\xi}\left(1 - \sum_{j=1}^n \frac{a_{j,n}}{p_j - i\xi}\right)
\end{aligned}$$

where we used Fubini theorem for the third equality and condition (1.3) to justify the last equality. The last term is now evaluated by using the obvious decomposition

$$\prod_{j=1}^n \frac{x - \lambda_j}{x + \lambda_j + 1} = 1 - \sum_{j=1}^n \frac{a_{j,n}}{x + \lambda_j + 1}, \quad x \neq -\lambda_j - 1, \quad j = 1, 2, \cdots, n.$$

Note that the latter decomposition allows as well to resolve the system (1.7). $\square$

Our aim now is to look for the analogue of Proposition 3.1 when $n = +\infty$. Observe that for the transform (3.3) to be well defined we merely need $\eta \in L^2(\mathbb{R}_+)$ and we can even take $\eta \in L^2_{\mathbb{C}}(\mathbb{R}_+)$. Of course, we need then to work with the Fourier-Plancherel transform instead of the Fourier transform. We recall that this is connected to the Hardy class H^2_+ of holomorphic functions H in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ such that $\sup_{b>0} \int_{\mathbb{R}} |H(a + ib)|\,da < \infty$, see [12]. We gather in the following result some well known results which are mostly taken from [21] and [29]; for completeness a full proof will be given.

Theorem 3.2. *Assuming that W is a Brownian motion indexed by $\mathbb{R}$ and B is a standard Brownian motion satisfying (3.1) and (3.2) then the following assertions are equivalent.*

(1) Λ *satisfies (1.10).*

(2) There exists a transform S of the form (3.3) associated to Λ such that $(S(W)_u, u \in \mathbb{R})$ is an Ornstein-Uhlenbeck process and

$$\mathcal{H}_u(W) = \mathcal{H}_u(S(W)) \oplus \textit{Span}\{\int_{-\infty}^{u} e^{p_j u}\, dW_u, j = 1, 2, \cdots\} \tag{3.5}$$

for all $u \in \mathbb{R}$ where $\mathcal{H}(S(W))_u$ stand for the closed linear span of $\{S(W)_r, r \leq u\}$.

(3) There exists a transform T of the form (1.1) such that $(T(B)_t, t \geq 0)$ is a standard Brownian motion and

$$H_t(B) = H_t(T(B)) \oplus \textit{Span}\{\int_{0}^{t} s^{\lambda_j}\, dB_s, j = 1, 2, \cdots\} \tag{3.6}$$

for all $t > 0$.

Proof of Theorem 3.2. (1) $\Leftrightarrow$ (2) Assuming that equation (1.10) is not satisfied then by Müntz-Szasz theorem, see e.g. [6], the sequence (f_k) is complete in $L^2[0,t]$ for all $t > 0$. It follows that the sequence $(e^{p_j x}, j = 1, 2, \cdots)$ is total in $L^2(-\infty, a]$ for all a real. Hence $H_u(W) = \text{Span}\{\int_{-\infty}^{u} e^{p_j s}\, dW_s, j = 1, 2, \cdots\}$ which shows that it is not possible to construct a transform S satisfying (3.5) such that $S(W)$ is an Ornstein-Uhlenbeck process. Conversely, condition (1.10) ensures the convergence of the infinite product (3.7). It is seen in Theorem 2 of [21], see also ([29], p. 60), that under the condition (1.10) the function $H : \mathbb{R} \to \mathbb{C}$ defined by

$$H(\xi) = \frac{1}{1/2 - i\xi} \prod_{j=1}^{\infty} \frac{\xi - ip_j}{\xi + ip_j} \frac{|1 - p_j|}{1 - p_j} \tag{3.7}$$

is the Fourier-Plancherel transform of a function $\eta_\infty \in L^2_{\mathbb{C}}(\mathbb{R}_+, dx)$, where dx is the Lebesgue measure; this follows from the fact that $H \in H^2_+$. Note that η_∞ is real-valued since $\overline{H(\xi)} = H(-\bar{\xi})$ for $\xi \in \mathbb{R}$. Let S_∞ be defined by (3.3) with $\eta = \eta_\infty$. The process $(S_\infty(W)_u, u \in \mathbb{R})$ is a continuous stationary Gaussian process with spectral measure $(2\pi)^{-1}|H(\xi)|^2 = (2\pi)^{-1}(\xi^2 + 1/4)^{-1}$ and covariance function

$$\begin{aligned}
\mathbb{E}\left[S_\infty(W)_u S_\infty(W)_v\right] &= \int_{-\infty}^{u\wedge v} \eta_\infty(u - r)\eta_\infty(v - r)\, dr \\
&= \int_{0}^{\infty} \eta_\infty(r)\eta_\infty(u \vee v - u \wedge v - r)\, dr \\
&= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{\eta}_\infty(\xi)|^2 e^{i(u\vee v - u\wedge v)\xi}\, d\xi \\
&= \frac{2}{\pi} \int_{-\infty}^{+\infty} e^{i(u\vee v - u\wedge v)\xi} \frac{d\xi}{1 + 4\xi^2} \\
&= e^{-\frac{1}{2}|u-v|}
\end{aligned}$$

for u and $v \in \mathbb{R}$. Thus, $S_\infty(W)$ is a stationary Ornstein-Uhlenbeck process. Now, for all $u < v$ and $j = 1, 2, \cdots$, $S_\infty(W)_u$ is independent of $\int_{-\infty}^{v} e^{p_j r}\, dW_r$ since

$$\begin{aligned}
\mathbb{E}\left[S_\infty(W)_u \int_{-\infty}^{v} e^{p_j r}\, dW_r\right] &= \int_{-\infty}^{u} e^{(\lambda_j + 1/2)r} \eta_\infty(u - r)\, dr \\
&= e^{p_j u} \int_{0}^{\infty} e^{-p_j s} \eta_\infty(s)\, ds = 0
\end{aligned}$$

where the last equality is obtained from the fact ip_j is a zero point of H.
(2) $\Leftrightarrow$ (3) Assuming (2), we can set $\rho_\infty = U^{-1}\eta_\infty$, where η_∞ is as above, and define B by (3.2). Since $\eta_\infty \in L^2(\mathbb{R}_+)$, we clearly have that $\rho_\infty \in \mathcal{M}$. Let us now define T_∞ by (1.1) where ρ_∞ and B are as prescribed above. Clearly, $T_\infty(B)$ is a standard Brownian motion. Furthermore, we have

$$H_t(T_\infty(B)) \perp \int_0^t u^q \, dB_u$$

for all $t > 0$, with $q > -1/2$, if and only if

$$\mathcal{H}(S_\infty(W))_u \perp \int_{-\infty}^u e^{pr} \, dW_r$$

for all $u \in \mathbb{R}$, with $p = q + 1/2$. Conversely, by reversing the steps we see that (3) implies (2). $\square$

Remark 3.3. *When we outlined the connection with stationary processes, we could have considered $W^{(\alpha)}$ satisfying $\int_0^\infty \varphi(s)\, dB_s = \int_{\mathbb{R}} V^{(\alpha)}\varphi(r)\, dW_r^{(\alpha)}$ for $\varphi \in L^2(\mathbb{R}_+)$, for some $\alpha > 0$, where the isometry $V^{(\alpha)} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is defined by $V\varphi(u) = \sqrt{\alpha} e^{\alpha u/2}\varphi(e^{\alpha u})$, i.e. $dW^{(\alpha)} = \alpha^{-1/2} e^{-\alpha u/2} dB(e^{\alpha u})$ with $W_0^{(\alpha)} = B_1$. But, we need to use $U^{(\alpha)}(\phi)(u) = \alpha^{-1/2} e^{\alpha u/2}\phi(e^{\alpha u})$ instead of U. The authors used this transformation with $\alpha = 2$ in [21] and [22]. Of course, the conclusions are the same up to working with $p_j = 2\lambda_j + 1$ instead of $p_j = \lambda_j + 1/2$.*

Theorem 3.4. *There exists a transform of the form (1.1) such that $(T_\infty(B)_t, t \geq 0)$ is a Brownian motion satisfying (3.6) and is a semimartingale in $\mathcal{F}^B$ if and only if (λ_k) is bounded and satisfies the Müntz-Szasz condition (1.10) i.e. $\sum_{j=1}^\infty p_j < \infty$.*

Proof of Theorem 3.4. By Theorem 3.2 there exists a transform of the form (1.1) such that (3.6) holds and $(T(B), t \geq 0)$ is a Brownian motion if and only if condition (1.10) is satisfied. Assuming that (1.10) is satisfied, let us check that the semimartingale property cannot hold if there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that $\lambda_{n_k} \to \infty$. For this, let us quote an argument, from [17] and [22], to show that we necessarily have $\int_1^\infty \rho'(u)^2\, du = +\infty$ in this case. Using the fact that $T(B)_1$ is independent of $\int_0^1 u^{\lambda_{n_k}}\, dB_u$ and a change of variables, we obtain $\int_0^1 u^{\lambda_{n_k}} \rho(1/u)\, du = \int_1^\infty u^{-(\lambda_{n_k}+2)}\rho(u)\, du = 0$. This, when combined with integration by parts, yields

$$\begin{aligned}
\int_1^\infty u^{-(\lambda_{n_k}+1)}\rho'(u)\, du &= \int_1^\infty u^{-(\lambda_{n_k}+1)}\rho'(u)\, du - (1+\lambda_{n_k})\int_1^\infty u^{-(\lambda_{n_k}+2)}\rho(u)\, du \\
&= \left[\rho(u) u^{-(\lambda_{n_k}+1)}\right]_1^\infty = -\rho(1).
\end{aligned}$$

By using the Cauchy-Schwartz inequality, we obtain $(1 + 2\lambda_{n_k})|\rho(1)|^2 \leq \int_1^\infty \rho'(u)^2\, du$ which implies that $\int_1^\infty \rho'(u)^2 du = \int_0^1 \rho'(1/v)^2 v^{-2}\, dv = +\infty$. If $T(B)$ were an $\mathcal{F}_t^B$ semimartingale then, by applying Theorem 6.5 of [31], we would have the existence of g such that $\rho(t) = c + \int_1^{\cdot} y^{-1} g(y)\, dy$ with $g \in \mathcal{M}$ and $c \neq 0$. But then $\rho'(y) = g(y)/y$, $y > 1$, and we should have $\int_0^1 g^2(1/v)\, dv = \int_0^1 \rho'(1/v)^2 v^{-2}\, dv < \infty$. This contradicts the fact that $\int_1^\infty \rho'(v)^2\, dv = +\infty$.

Let us now examine the case where (λ_k) satisfies (1.10) and is bounded. i.e. $\sum_1^\infty (1 + 2\lambda_j) < \infty$. We shall first show that K_n converges as $n \to \infty$ in $L^2[0,1]$. Using Proposition

2.5, for any positive integers $n > m$, we can write

$$\begin{aligned}\int_0^1 \left(K_n(u) - K_m(u)\right)^2 du &= \int_0^1 \left(\sum_{j=m+1}^{n} (1+2\lambda_j) L_j(u)\right)^2 du \\ &= \sum_{j=m}^{n} (1+2\lambda_j) \sum_{k=m+1}^{n} (1+2\lambda_k) \int_0^1 L_j(u) L_k(u)\, du \\ &= \sum_{j=m+1}^{n} (1+2\lambda_j) \to 0, \quad \text{as } m,n\to\infty,\end{aligned}$$

where we have used the fact that L_1, L_2, $\cdots$, are orthogonal and $\int_0^1 L_j^2(r)\, dr = 1/(1+2\lambda_j)$ for $j = 1, \cdots, n$. This shows that (K_n) is a Cauchy sequence in $L^2[0,1]$. Hence, it must converge to a limit which we denote by K. With $\rho_n(.) = 1 - \int_1^{\cdot} K_n(1/r) r^{-1}\, dr$, $n = 1, 2\cdots$, let us show that $\rho_n(1/.) \to \rho(1/.)$ in $L^2[0,1]$ where

$$\rho = 1 - \int_1^{\infty} K(1/r) r^{-1}\, dr. \tag{3.8}$$

To this end, we quote from [11] the following variant of Hardy inequality. For any $g \in L^2[0,1]$, we have

$$\int_0^1 \left(\int_u^1 g(r) r^{-1} dr\right)^2 du \le 4 \int_0^1 g^2(u)\, du.$$

Now, we can write

$$\begin{aligned}\int_0^1 \left(\rho_n(1/v) - \rho(1/v)\right)^2 dv &= \int_0^1 \left(\int_1^{1/v} (K_n(1/z) - K(1/z)) z^{-1}\, dz\right)^2 dv \\ &= \int_0^1 \left(\int_v^1 (K_n(r) - K(r)) r^{-1}\, dr\right)^2 dv \\ &\le 4\int_0^1 (K_n(u) - K(u))^2\, du \to 0 \quad \text{a } n\to\infty,\end{aligned}$$

which is our claim. It follows that

$$T_n(B)_t = \int_0^t \rho_n(t/s)\, dB_s \to \int_0^t \rho(t/s)\, dB_s =: T(B)_t \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}).$$

Similar arguments show that $\eta_n \to \eta$ in $L^2(\mathbb{R}_+)$, with $\eta(t) = \mathbf{1}_{t>0} U \circ \rho(t)$, and $S_n(W)_u \to \int_{-\infty}^u \eta(u-v)\, dW_v =: S(W)_u$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. We need to show that $\eta = \eta_\infty$ where η_∞ is the Fourier inverse of the function H defined by (3.7). Applying Plancherel Theorem, with $\hat{\eta}_n$ prescribed by (3.4), we see that $\hat{\eta}_n \to \hat{\eta}$ in $L^2_{\mathbb{C}}(\mathbb{R})$. But by Theorem 13.12 in [35], we know that

$$\Pi_n(\xi) \to \Pi_\infty(\xi) := \prod_{j=1}^{\infty} \frac{\xi - ip_j}{\xi + ip_j} \quad \text{as} \quad n\to\infty,$$

uniformly on compact subsets of $\mathbb{R}\setminus\{0\}$. It follows that $\hat{\eta}_n \to H$, as $n\to\infty$, uniformly on compact subsets of $\mathbb{R}\setminus\{0\}$. We conclude that necessarily $\hat{\eta} = H$ a.e. which implies that $\eta = \eta_\infty$ a.e.. It follows from the proof of Theorem 3.2 that $S_\infty(W)$ is an Ornstein-Uhlenbeck process which implies that $T_\infty(B)$ is a standard Brownian motion. Due to the fact that ρ is given by (3.8), Proposition 15 on p. 69 of [29] shows that $T(B)$ is a semimartingale with respect to $\mathcal{F}^B$. □

Definition 3.5. *A Müntz transform is a transform T of the form (1.1) such that $(T(B)_t, t \geq 0)$ is a standard Brownian motion for any Brownian motion $(B_t, t \geq 0)$ and the orthogonal decomposition $H_t(B) = H_t(T(B)) \oplus G_t(\lambda_1, \cdots, \lambda_n; B)$, for some sequence of reals $-1/2 < \lambda_1, \cdots, \lambda_n$ and $n \in \{1, 2, \cdots\}$, holds for all $t > 0$. We call n the order of the transform. The corresponding kernel $\rho(./.)$ (or k_n in the semimartingale case) is called a Müntz kernel of order n.*

Remark 3.6. *As a by-product of the discussion for the order to be infinite we mention the following result. Let φ be a $C^\infty([0,1])$ function satisfying $|\varphi^{(m)}| \leq M$ for all m, where M is some positive constant. Then φ is a solution to the integral equation $\varphi(u) = \int_0^1 \varphi(uv)\varphi(v)\,dv$, defined on $[0,1]$, if and only if $\varphi(.) = k_n(1,.) = K_n(.)$ where $\lambda_j = j$ for $j \geq 0$ and n is some finite positive integer.*

Remark 3.7. *For $n \in \{1, 2, \cdots\}$, k_n and T_n as above, introduce the notations $T_n^{(0)} = Id$, $T_n^{(1)} = T_n$ and $T_n^{(m)} = T_n^{(m-1)} \circ T_n$, for $m \geq 2$, where $\circ$ stands for the composition rule for the iterated transforms. We clearly have for m positive integer*

$$\cdots \mathcal{F}_t^{T_n^{(m+1)}(B)} \subsetneq \mathcal{F}_t^{T_n^{(m)}(B)} \subsetneq \cdots \mathcal{F}_t^{T_n}(B) \subsetneq \mathcal{F}_t(B).$$

Furthermore, since we are in the homogeneous case, we can show that the decomposition

$$\mathcal{F}_t^B = \bigotimes_{k=1}^{\infty} \sigma\left(\int_0^t u^{\lambda_j}\, dT_n^{(k)}(B)_u, 1 \leq j \leq n\right)$$

holds true. Here, by $\mathcal{F} \otimes \mathcal{G}$, for two σ-algebras $\mathcal{F}$ and $\mathcal{G}$, we mean $\mathcal{F} \vee \mathcal{G}$ with independence between $\mathcal{F}$ and $\mathcal{G}$. It follows that Müntz transforms are strongly mixing and ergodic. We also refer to [29] for a proof of this, in a more general framework, which uses the connection to stationarity.

Example 3.8. *For $r > 0$, let us set $\lambda_j = (j^{-r} - 1)/2$ and so $p_j = j^{-r}/2$, $j = 1, 2, \cdots$. For n positive integer, we obtain*

$$a_{k,n} = \frac{2}{k^r} \prod_{j=1, j\neq k}^{n} \frac{j^r + k^r}{j^r - k^r}, \quad k = 1, 2, \cdots n.$$

The hyperharmonic series $\sum_1^\infty (1 + 2\lambda_j) = \sum_1^\infty j^{-r}$ converges if and only if $r > 1$ which, by Theorem 3.4, is the necessary and sufficient condition for the existence of an associated Müntz transform of infinite order. If $r > 1$ then

$$H(\xi) = \frac{1}{1/2 - i\xi} \prod_{k=1}^{\infty} \frac{k^r - i/(2\xi)}{k^r + i/(2\xi)}, \quad \xi \in \mathbb{R}.$$

If furthermore r is a an integer then

$$H(\xi) = -(1/2 - i\xi)^{-1} \prod_{j=1}^{2r} \Gamma(-(i/(2\xi))^{1/r} \omega_{2r}^j)^{(-1)^{j+1}}$$

where we have used the relationship

$$\prod_{j\geq 1} \frac{j^r - z^r}{j^r + z^r} = -\prod_{j=1}^{2r} \Gamma(-z\omega_{2r}^j)^{(-1)^{j+1}},$$

with $\omega_{2r} = \exp(\pi i/r)$, which is valid for $z \notin \{0, 1, 2, \cdots\}$ and is found in ([5], pp. 6-7). Since the residue of $\Gamma(z)$ at $z = -k$ is $(-1)^k/k!$, we have

$$(k^r - z^r)\Gamma(-z) \to \frac{(-1)^k}{k!} r k^{r-1} \quad \text{as} \quad z \to k.$$

It follows that

$$\prod_{j=1, j\neq k} \frac{j^r - k^r}{j^r + k^r} = (-1)^{k+1} \frac{2k(k!)}{r} \prod_{j=1}^{2r-1} \Gamma(-k\omega_{2r}^j)^{(-1)^{j+1}}$$

which leads to

$$\begin{aligned} a_{k,n} &\to \frac{2}{k^r} \left\{ (-1)^{k+1} \frac{2k(k!)}{r} \prod_{j=1}^{2r-1} \Gamma(-k\omega_{2r}^j)^{(-1)^{j+1}} \right\}^{-1} \\ &= (-1)^{k+1} \frac{r}{k^{r+1} k!} \prod_{j=1}^{2r-1} \Gamma(-k\omega_{2r}^j)^{(-1)^j} \end{aligned}$$

as $n \to \infty$.

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