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Continuum percolation for quermass interaction model

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Abstract

Continuum percolation for Markov (or Gibbs) germ-grain models in dimension 2 is investigated. The grains are assumed circular with random radii on a compact support. The morphological interaction is the so-called Quermass interaction defined by a linear combination of the classical Minkowski functionals (area, perimeter and Euler-Poincaré characteristic). We show that percolation occurs for any coefficient of this linear combination and for a large enough activity parameter. An application to the phase transition of the multi-type Quermass interaction model is given.

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1 Introduction

The *germ-grain model* is built by unifying random convex sets– *the grains* –centred at the points– *the germs* –of a spatial point process. It is used for modelling random surfaces and interfaces, geometrical structures growing from germs, etc. For such models, *continuum percolation* refers mainly to the existence of an unbounded connected component. This phenomenon expresses some macroscopic properties of materials as permeability, conductivity, etc. Moreover, it turns out to be an efficient tool to exhibit phase transition in Statistical Mechanics [2, 4]. For these reasons, continuum percolation has been abundantly studied since the eighties and the pioneer paper of Hall [8].

When the grains are independent and identically distributed, and the germs are given by a Poisson point process (PPP), the germ-grain model is known as the *Boolean model*. In this context, continuum percolation is well-understood; see the book of Meester and Roy [13] for a very complete reference. One of the first results is the existence of a percolation threshold z^* for the intensity parameter z of the stationary PPP: provided the mean volume of the grain is finite, percolation occurs for $z > z^*$ and not for $z < z^*$.

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Because of the independence properties of the PPP, the Boolean model is sometimes caricatural for the applications in Biology or Physics. Mecke and its co-authors [11, 12] have mentioned the need of developing, via Markov or Gibbs process, an interacting germ-grain model in which the interaction would locally depend on the geometry of the set. For this purpose, let us cite the Widom-Rowlinson model [16], the area interaction process [1] and the morphological model [12]. Thus, Kendall, Van Lieshout and Baddeley suggested in [9] a generalization of the previous models, called the *Quermass Interaction Process*. In this model, the formal Hamiltonian is a linear combination of the $d+1$ fundamental Minkowski functionals in $\mathbb{R}^d$. The setting of the paper being $\mathbb{R}^2$, the formal Hamiltonian has the following expression

$$H = \theta_1 \mathcal{A} + \theta_2 \mathcal{L} + \theta_3 \chi \ , \tag{1.1}$$

where $\mathcal{A}$ is the area functional, $\mathcal{L}$ the perimeter and χ the Euler-Poincaré characteristic: the number of connected components minus the number of holes.

The existence of infinite volume Gibbs point processes in $\mathbb{R}^2$ for the Hamiltonian H has been recently proved in [3]. This paper focuses on continuum percolation for such processes.

The existence of a percolation threshold z^* for the Boolean model relies on a basic (but essential) monotonicity argument: see [13], Chapter 2.2. This argument fails in the case of Gibbs point processes with Hamiltonian H. So, no percolation threshold can be expected in our context. However, other stochastic arguments as stochastic domination or FKG lead to percolation results. In [2], Chayes et al. prove that percolation occurs for z large enough and $\theta_2 = \theta_3 = 0$. To our knowledge, the percolation phenomenon for other values of parameters $\theta_1, \theta_2, \theta_3$ has not been investigated yet.

Our main result (Theorem 3.1) states that, for any $\theta_1, \theta_2, \theta_3$ (positive or negative), percolation occurs with probability 1 for z large enough. The only assumption involves the random radii of the circular grains: they have to belong to a compact set not containing 0. The proof of this theorem is relatively easy in the case $\theta_3 = 0$. Indeed, the *local energy* $h((x,R),\omega)$– the energy variation when the grain $\bar{B}(x,R)$ is added to the configuration ω – is uniformly bounded and, by a stochastic comparison with respect to the PPP, the result follows. When $\theta_3 \neq 0$, the local energy becomes unbounded from above and below, and the previous stochastic comparison fails. So the main challenge of the present paper concerns the case $\theta_3 \neq 0$. Following Georgii and Häggström [4], our strategy is based on a classical comparison with a site percolation model. However, the complexity of the interaction H (defined in (1.1)) implies an accurate geometrical study of the produced random shapes. Indeed, an arduous control of the hole number variation, when a new grain is added, is the main technical issue. We prove essentially that this variation is moderate for a large enough set of admissible locations of grains.

Following [2, 4], we use our percolation result (Theorem 3.1) to exhibit a phase transition phenomenon for Quermass interaction model with several type of particles (Theorem 3.4).

The existence of the infinite volume Quermass-interaction process in $\mathbb{R}^d$ is not proved in general for $d > 2$. The main obstruction is that the Euler-Poincaré characteristic functional χ is not stable in this case [9]. So as soon as $\theta_3 \neq 0$, the existence is not proved. It is the main reason to restrict the paper to the case $\mathbb{R}^2$.

Our paper is organized as follows. In Section 2, the Quermass model and the main notations are introduced. The local energy $h((x,R),\omega)$ is defined in (2.3). Section 3 contains the results of the paper. Section 3.2 is devoted to the case $\theta_3 = 0$ and Section 3.3 to the phase transition result. The proof of Theorem 3.1 is developed in Section 4.

2 Quermass interaction model

2.1 Notations

We denote by $\mathcal{B}(\mathbb{R}^2)$ the set of bounded Borel sets in $\mathbb{R}^2$ with a positive Lebesgue measure. For any Λ and Δ in $\mathcal{B}(\mathbb{R}^2)$, $\Lambda \oplus \Delta$ stands for the Minkowski sum of these sets. Let $0 < R_0 \leq R_1$ be some positive reals and $\mathcal{E}$ be the product space $\mathbb{R}^2 \times [R_0, R_1]$ endowed with its natural Euclidean Borel σ-algebra $\sigma(\mathcal{E})$. For any $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, $\mathcal{E}_\Lambda$ denotes the space $\Lambda \times [R_0, R_1]$. A *configuration* ω is a subset of $\mathcal{E}$ which is locally finite with respect to its first coordinate: $\#(\omega \cap \mathcal{E}_\Lambda)$ is finite for any Λ in $\mathcal{B}(\mathbb{R}^2)$. The configuration set Ω is endowed with the σ-algebra $\mathcal{F}$ generated by the functions $\omega \mapsto \#(\omega \cap A)$ for any A in $\sigma(\mathcal{E})$.
We will merely denote by ω_Λ instead of $\omega \cap \mathcal{E}_\Lambda$ the restriction of the configuration ω (with respect to its first coordinate) to Λ. Moreover, for any (x, R) in $\mathcal{E}$, we will write $\omega \cup (x, R)$ instead of $\omega \cup \{(x, R)\}$.

A configuration $\omega \in \Omega$ can be interpreted as a marked configuration on $\mathbb{R}^2$ with marks in $[R_0, R_1]$. To each $(x, R) \in \omega$ is associated the closed ball $\bar{B}(x, R)$ (the grain) centred at x (the germ) with radius R. The germ-grain surface $\bar{\omega}$ is defined as

$$\bar{\omega} = \bigcup_{(x,R)\in\omega} \bar{B}(x, R) .$$

2.2 Quermass interaction

Let us define the Quermass interaction as in Kendall et al. [9] for the case $\mathbb{R}^2$. The energy (or Hamiltonian) of a finite configuration ω in Ω is defined by

$$H(\omega) = \theta_1 \mathcal{A}(\bar{\omega}) + \theta_2 \mathcal{L}(\bar{\omega}) + \theta_3 \chi(\bar{\omega}) , \tag{2.1}$$

where θ_1, θ_2 and θ_3 are three real numbers, and $\mathcal{A}$, $\mathcal{L}$ and χ are the three fundamental Minkowski functionals, respectively area, perimeter and Euler-Poincaré characteristic. This last one is the difference between the number of connected components and the number of holes. Recall that a hole of $\bar{\omega}$ is a bounded connected component of $\bar{\omega}^c$. Hadwiger's Theorem ensures that any functional F defined on the space of finite unions of convex compact sets, which is continuous for the Hausdorff topology, invariant under isometric transformations and additive (i.e. $F(A \cup B) = F(A) + F(B) - F(A \cap B)$) can be decomposed as in (2.1). This universal representation justifies the choice of the Quermass interaction for modelling mesoscopic random surfaces [11, 12].

The energy inside $\Lambda \in \mathcal{B}(\mathbb{R}^2)$ of any given configuration ω in Ω (finite or not) is defined by

$$H_\Lambda(\omega) = H(\omega_\Delta) - H(\omega_{\Delta\setminus\Lambda}) , \tag{2.2}$$

where Δ is any subset of $\mathbb{R}^2$ containing $\Lambda \oplus B(0, 2R_1)$. By additivity of functionals $\mathcal{A}$, $\mathcal{L}$ and χ, the difference $H_\Lambda(\omega)$ does not depend on the chosen set Δ.

Let us end with defining the local energy $h((x, R), \omega)$ of the marked point $(x, R) \in \mathcal{E}$ (or of the associated ball $\bar{B}(x, R)$) with respect to the configuration ω:

$$h((x, R), \omega) = H_\Lambda(\omega \cup (x, R)) - H_\Lambda(\omega) , \tag{2.3}$$

for any $\Lambda \in \mathcal{B}(\mathbb{R}^2)$ containing x. Remark this definition does not depend on the choice of the set Λ. The local energy $h((x, R), \omega)$ represents the energy variation when the ball $\bar{B}(x, R)$ is added to the configuration ω.

2.3 The Gibbs property

Let Q be a reference probability measure on $[R_0, R_1]$. Without loss of generality, R_0 and R_1 can be chosen such that, for every $\varepsilon > 0$,

$$Q([R_0 + \varepsilon, R_1]) < 1 \text{ and } Q([R_0, R_1 - \varepsilon]) < 1 . \tag{2.4}$$

Let $z > 0$. Let us denote by λ the Lebesgue measure on $\mathbb{R}^2$ and by π^z the PPP on $\mathcal{E}$ with intensity measure $z\lambda \otimes Q$. Under π^z, the law of the random surface $\bar{\omega}$ is the stationary Boolean model with intensity $z > 0$ and distribution of radius Q. Finally, for any $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, let us denote by π^z_Λ the PPP on $\mathcal{E}_\Lambda$ with intensity measure $z\lambda_\Lambda \otimes Q$, where λ_Λ is the restriction of the Lebesgue measure λ to Λ.

Definition 2.1. *A probability measure P on Ω is a Quermass Process for the intensity $z > 0$ and the parameters $\theta_1, \theta_2, \theta_3$ if P is stationary (in space) and if for every Λ in $\mathcal{B}(\mathbb{R}^2)$, for every bounded positive measurable function f from Ω to $\mathbb{R}$,*

$$\int f(\omega)P(d\omega) = \int\int f(\omega'_\Lambda \cup \omega_{\Lambda^c}) \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} e^{-H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \pi^z_\Lambda(d\omega'_\Lambda) P(d\omega) , \tag{2.5}$$

where $Z_\Lambda(\omega_{\Lambda^c})$ is the partition function

$$Z_\Lambda(\omega_{\Lambda^c}) = \int\int e^{-H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} \pi^z_\Lambda(d\omega'_\Lambda) .$$

The equations (2.5)– for all $\Lambda \in \mathcal{B}(\mathbb{R}^2)$ –are called DLR for Dobrushin, Landford and Ruelle. They are equivalent to: for any $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, the law of ω_Λ under P given ω_{Λ^c} is absolutely continuous with respect to the Poisson Process π^z_Λ with the local density

$$g_\Lambda(\omega'_\Lambda | \omega_{\Lambda^c}) = \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} e^{-H_\Lambda(\omega'_\Lambda \cup \omega_{\Lambda^c})} . \tag{2.6}$$

See [15] for a general presentation of Gibbs measures and DLR equations.

The existence, the uniqueness or non-uniqueness (phase transition) of Quermass processes are difficult problems in statistical mechanics. The existence has been proved recently in [3], Theorem 2.1 for any parameters $z > 0$ and $\theta_1, \theta_2, \theta_3$ in $\mathbb{R}$. A phase transition result is proved in [2, 6, 16] for $R_0 = R_1$, $\theta_2 = \theta_3 = 0$ and for $\theta_1 = z$ large enough.

3 Results

3.1 Percolation occurs

We say that *percolation occurs* for a given configuration $\omega \in \Omega$ if the subset $\bar{\omega}$ of $\mathbb{R}^2$ contains at least one unbounded connected component. The set of configurations such that percolation occurs is a translation invariant event. Its probability, called *the percolation probability*, equals to 0 or 1 for any ergodic Quermass process. However, the Quermass processes are not necessarily ergodic (they are only stationary) and their percolation probabilities may be different from 0 and 1. Besides, in [2], Chayes et al. have built two Quermass processes, both corresponding to $\theta_2 = \theta_3 = 0$ and $\theta_1 = z$ large enough, whose percolation probabilities respectively equal to 0 and 1. Since any mixture of these two processes is still a Quermass process, the authors obtain Quermass processes whose percolation probabilities equal to any value between 0 and 1.
Our main result states that percolation occurs with probability 1 for any (ergodic or not) Quermass process whenever the intensity z is large enough.

Theorem 3.1. *Let $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$. There exists $z^* > 0$ such that for any Quermass process P associated to the parameters $\theta_1, \theta_2, \theta_3$ and $z > z^*$, percolation occurs P-almost surely.*

The proof of Theorem 3.1 is based on a discretization argument which allows to reduce the percolation problem from the (continuum) Quermass interaction model to a site percolation model on the lattice $\mathbb{Z}^2$ (up to a scale factor). This proof is rather long and technical so it is addressed in Section 4.

Let us point out here that our theorem does not claim z^* is a percolation threshold. In other words, for $z < z^*$, percolation could be lost and recovered on different successive ranges.

Another natural question involves the number of unbounded connected components. Following the classical arguments for continuum percolation, we prove that this number is almost surely equal to zero or one.

Proposition 3.2. *For any Quermass process P the number of unbounded connected component is a random variable in $\{0, 1\}$.*

Proof. It is well-known that any Gibbs measure is a mixture of extremal ergodic Gibbs measures. For each ergodic Quermass process P, the number of connected component is almost surely a constant in $\mathbb{N}\cup\{+\infty\}$. For any $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, thanks to the DLR equations (2.5), it is easy to prove that the law of ω_Λ under P is equivalent to π^z_Λ. Therefore, following the general scheme of the proof of Theorem 2.1 in [13], we show that the number of unbounded connected components is necessarily 0 or 1. □

3.2 Percolation when $\theta_3 = 0$

In the particular case $\theta_3 = 0$, Theorem 3.1 can be completed and proved in a simple way.

First, let us recall the definitions involving the stochastic domination for point processes. We follow the notations given in [5]. An event A in $\mathcal{F}$ is called increasing if for every $\omega \in A$ and any $\omega' \in \Omega$ containing ω then $\omega' \in A$ too. Let P and P' be two probability measures on Ω. We say that P is dominated by P', denoted by $P \preceq P'$, if for every increasing event $A \in \mathcal{F}$, $P(A) \leq P'(A)$. In this section, we focus our attention on the increasing event "there exists an unbounded connected component".

Let P be any Quermass process and assume $\theta_3 = 0$. Thanks to Lemma 4.12, the local energy is uniformly bounded: there exist constants C_0 and C_1 such that for any $(x, R) \in \mathcal{E}$ and $\omega \in \Omega$,

$$C_0 \leq h((x, R), \omega) \leq C_1 \ . \tag{3.1}$$

Let us mention that the basic assumption $R_0 > 0$ is crucial in the Lemma 4.12. Combining (3.1) and Theorem 1.1 in [5], we get the following stochastic dominations:

$$\pi^{ze^{-C_1}} \preceq P \preceq \pi^{ze^{-C_0}} \ .$$

Now, the (stationary) Boolean models corresponding to $\pi^{ze^{-C_1}}$ and $\pi^{ze^{-C_0}}$ admit positive and finite percolation thresholds (see [14], Chapter 3). It follows :

Proposition 3.3. *For every θ_1, θ_2 in $\mathbb{R}$, there exist constants z_0, z_1 such that for any Quermas Process P associated to parameters z, θ_1, θ_2 and $\theta_3 = 0$, percolation occurs P-almost surely if $z > z_1$ and does not occur P-almost surely if $z < z_0$.*

Proposition 3.3 improves Theorem 3.1 in the case $\theta_3 = 0$ since it ensures the existence of a subcritical regime.

It is worth pointing out here that the uniform bounds in (3.1) do not hold whenever $\theta_3 \neq 0$. Precisely, the hole number variation is not uniformly bounded from above and below.

3.3 Phase transition for multi-type Quermass Process

In this section, the multi-type Quermass interaction model is introduced and a phase transition is exhibited, i.e. the existence of several Gibbs processes for the same parameters is proved.

Let K be a positive integer. The K-type Quermass interaction model is defined on the space Ω_K of configurations in $\mathcal{E}_K = \mathbb{R}^2 \times [R_0, R_1] \times \{1, 2, \dots, K\}$. Each disc is now marked by a number specifying its type. We don't give the natural extension of the notations involving the sigma-field and so on.
The following Quermass energy function is defined such that all discs of a connected component have the same number. This is a non-overlapping multi-type germ-grain model. Precisely the energy of a finite configuration ω is now given by

$$H(\omega) = \theta_1 \mathcal{A}(\bar{\omega}) + \theta_2 \mathcal{L}(\bar{\omega}) + \theta_3 \chi(\bar{\omega}) + \sum_{\substack{(x,R,i),(y,R',j)\in\omega \\ i\neq j}} \phi(|x-y| - R - R') \,, \tag{3.2}$$

where ϕ is an hardcore potential equals to infinity on $]-\infty, 0]$ and zero on $]0, +\infty]$. The energy inside $\Lambda \in \mathcal{B}(\mathbb{R}^2)$ of any finite or infinite configuration ω is defined as in (2.2) with the convention $+\infty - \infty = +\infty$. The definition of the K-type Quermass process via the DLR equations follows as in Definition 2.1.

The proof of the existence of such processes is similar to the one of the existence of Quermass process. See Theorem 2.1 of [3] for more details. Here is our phase transition result:

Theorem 3.4. *For any θ_1, θ_2 and θ_3 in $\mathbb{R}$, there exists $z_0 > 0$ such that, for any $z > z_0$, there exist at least K different K-type Quermass Processes. There is a phase transition.*

We follow the scheme of the proof of Theorem 2.2 of [2] or Theorem 1.1 of [4]. It is based on a random-cluster representation (or Gray Representation) analogous to the Fortuin-Kasteleyn representation of Potts model. The existence of an unbounded connected component allows to prove the existence of a K-type Quermass process in which the density of particles of a given type, say type k, is larger than the density of the other types. It is showed by fixing the outside configuration of the finite volume Gibbs measure with the type k. In the thermodynamic limit, this type k remains dominant since the balls of the unbounded component have this type k. By symmetry of the types, we prove the existence of at least K different K-type Quermass processes.

4 Proof of Theorem 3.1

4.1 General scheme

In the following, P denotes a stationary Quermass process on Ω associated to the intensity $z > 0$ and the parameters $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$.
Let ℓ be a real number such that $\ell > 2R_1 + 2R_0$. Let us define the diamond box Δ as the interior of the convex hull of the eight points $(3\ell, 0)$, $(6\ell, 0)$, $(9\ell, 3\ell)$, $(9\ell, 6\ell)$, $(6\ell, 9\ell)$, $(3\ell, 9\ell)$, $(0, 6\ell)$ and $(0, 3\ell)$. This large octagon contains four smaller boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$ with side length ℓ; precisely $B_{\mathtt{N}} = (4\ell, 7\ell) + [0, \ell]^2$, $B_{\mathtt{S}} = (4\ell, \ell) + [0, \ell]^2$, $B_{\mathtt{E}} = (7\ell, 4\ell) + [0, \ell]^2$ and $B_{\mathtt{W}} = (\ell, 4\ell) + [0, \ell]^2$. The subscripts N, S, E and W refer to the cardinal directions. See Figure 1. Thus, let us introduce the indicator function ξ defined on Ω and equal to 1 if and only if the two following conditions are satisfied:

(C1) Each box $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$, contains at least one point of ω_Δ;

(C2) The number $N_{\mathtt{cc}}^{\Delta}(\omega)$ of connected components of $\bar{\omega}_\Delta$ having at least one ball centred in one of the boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ or $B_{\mathtt{W}}$, is equal to 1.

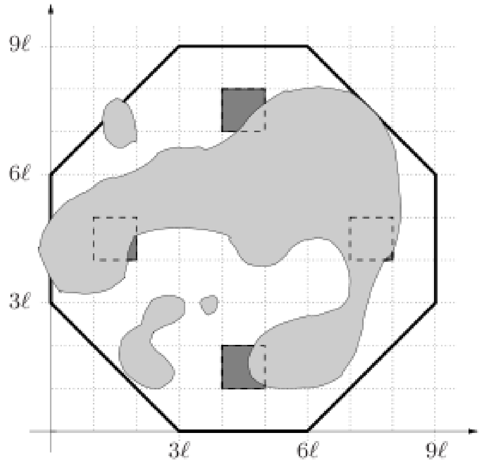

Figure 1: Here is the diamond box Δ. The light gray set represents the configuration ω restricted to Δ. The dark gray squares are the fourth cardinal boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$ with side length ℓ. On this picture, conditions (C1) and (C2) are fulfilled, i.e. $\xi(\omega) = 1$.

In other words, $\xi(\omega) = 1$ means the boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$ are connected through $\bar{\omega}_{\Delta}$.

For any $x \in (6\ell\,\mathbb{Z})^2$, let τ_x be the translation operator on the configuration set $\mathcal{E}$ defined by $(y, R) \in \tau_x\omega$ if and only if $(y + x, R) \in \omega$. Hence, we can define the translated indicator function ξ_x of ξ on the translated box $\Delta_x = x + \Delta$ by $\xi_x(\omega) = \xi(\tau_x\omega)$. Let us remark that $\xi_x(\omega)$ only depends on the restriction of the configuration ω to the box Δ_x. Moreover, thanks to the stationary character of the Quermass process P, the random variables ξ_x, $x \in (6\ell\,\mathbb{Z})^2$, are identically distributed. They are dependent too.
Let us consider $x, y \in (6\ell\,\mathbb{Z})^2$ such that $y = (6\ell, 0) + x$. The boxes Δ_x and Δ_y have in common a cardinal box, i.e. $x + B_{\mathtt{E}} = y + B_{\mathtt{W}}$. So, the condition $\xi_x(\omega) = \xi_y(\omega) = 1$ ensures that the cardinal boxes of Δ_x and Δ_y are connected together through the restriction of $\bar{\omega}$ to $\Delta_x \cup \Delta_y$. The same is true when $y = (0, 6\ell) + x$. This induces a graph structure on the vertex set $V = (6\ell\,\mathbb{Z})^2$: for any $x, y \in V$, $\{x, y\}$ belongs to the edge set E if and only if

$$y - x \in \{\pm(6\ell, 0), \pm(0, 6\ell)\} .$$

The graph (V, E) is merely the square lattice $\mathbb{Z}^2$ with the scale factor 6ℓ. The family $\{\xi_x, x \in V\}$ provides a site percolation process on the graph (V, E). It has been built so as to satisfy the following statement.

Lemma 4.1. *Let $\omega \in \Omega$ such that percolation occurs in the site percolation process $\{\xi_x, x \in V\}$. Then ω also percolates.*

Let us note that other shapes for Δ (not necessarily octagonal) are possible. The advantage of this one is that the associated graph (V, E) is merely $\mathbb{Z}^2$.

Let Π_p be the Bernoulli (with parameter p) product measure on $\{0, 1\}^V$. A stochastic domination result of Liggett et al. [10] (Theorem 1.3) allows to compare the site percolation processes induced by the family $\{\xi_x, x \in V\}$ and Π_p. Here is a version adapted to our context. Basic definitions about stochastic domination for lattice state spaces are not recalled here. They are similar to the ones presented in Section 3.2 for point processes. See also [7].

Lemma 4.2. *Let $p \in [0,1]$. Assume that, for any vertex $x \in V$,*

$$P\left(\xi_x = 1 \,|\, \xi_y : \{x,y\} \notin E\right) \geq p \ \text{ a.s.} \tag{4.1}$$

Then the distribution of the family $\{\xi_x, x \in V\}$ stochastically dominates the probability measure $\Pi_{f(p)}$, where $f : [0,1] \to [0,1]$ is a deterministic function such that $f(p)$ tends to 1 as p tends to 1.

Actually Theorem 3.1 is easily deduced from Lemmas 4.1 and 4.2. Let us first recall that in the site percolation model on the graph (V,E), there exists a threshold value $p^* < 1$ such that percolation occurs with Π_p-probability 1 whenever $p > p^*$. See the book [7], p. 25. So, let p be a real number in $[0,1]$ such that

$$f(p) > p^* \ . \tag{4.2}$$

Whenever the Quermass process P satisfies (4.1) for that p, then combining Lemmas 4.1 and 4.2 percolation occurs P-a.s. Therefore it remains to show that for any $p > 0$, hypothesis (4.1) holds for z large enough.

The next result claims that each Borel set of $\mathbb{R}^2$, sufficiently thick in some sense, contains at least one element of the configuration ω with a probability tending to 1 as the intensity z tends to infinity. It will be proved at the end of this section.

Lemma 4.3. *Let $V \in \mathcal{B}(\mathbb{R}^2)$ such that there exist $U \in \mathcal{B}(\mathbb{R}^2)$ with positive Lebesgue measure and $\varepsilon > 0$ satisfying $U \oplus \bar{B}(0, R_1 + R_0 + \varepsilon) \subset V$. Then there exists a constant $C > 0$, depending on $\lambda(U)$ and ε, such that for any configuration $\omega \in \Omega$ and for any $z > 0$,*

$$P\left(\omega_V = \emptyset \,|\, \omega_{V^c}\right) \leq C z^{-1} \ .$$

Since the Quermass process P is stationary, it is sufficient to prove (4.1) with $x = (0,0)$. So, we focus our attention on the diamond box $\Delta = \Delta_{(0,0)}$ and use Lemma 4.3 to check that condition (C1) is fulfilled in this box. Since $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$ are sufficiently thick (with side length $\ell > 2R_1 + 2R_0$), it follows

$$P\left(\omega_{B_i} = \emptyset \,|\, \omega_{\Delta^c}\right) = P\left(P\left(\omega_{B_i} = \emptyset \,|\, \omega_{B_i^c}\right) \,|\, \omega_{\Delta^c}\right) \leq C z^{-1} \ ,$$

for any $i \in \{\mathtt{N}, \mathtt{S}, \mathtt{E}, \mathtt{W}\}$. So the conditional probability that ω satisfies (C1) is larger than $1 - 4Cz^{-1}$.
The equation $N_{\mathsf{cc}}^{\Delta}(\omega) = 0$ forces the box $B_{\mathtt{N}}$ (for instance) to be empty of points of the configuration ω. Hence,

$$P\left(N_{\mathsf{cc}}^{\Delta}(\omega) = 0 \,|\, \omega_{\Delta^c}\right) \leq C z^{-1} \ .$$

Checking that condition (C2) is fulfilled in the diamond box Δ needs what we call the Connection Lemma (Lemma 4.4). This result states the conditional probability that $N_{\mathsf{cc}}^{\Delta}(\omega)$ is larger than 2 converges to 0 uniformly on the configuration outside Δ. This is the heart of the proof of Theorem 3.1. Its technical proof is given in Section 4.2.

Lemma 4.4 (The Connection Lemma)**.** *There exists a constant $C' > 0$ such that for any configuration $\omega \in \Omega$ and for any $z > 0$,*

$$P\left(N_{\mathsf{cc}}^{\Delta}(\omega) \geq 2 \,|\, \omega_{\Delta^c}\right) \leq C' z^{-1} \ . \tag{4.3}$$

The above inequalities and the Connection Lemma imply that conditions (C1) and (C2) are fulfilled in Δ with a probability tending to 1 as z tends to ∞:

$$P\left(\xi_{(0,0)}(\omega) = 1 \,|\, \omega_{\Delta^c}\right) \geq 1 - (5C + C') z^{-1} \ .$$

The hypothesis (4.1) then follows. Let y be a vertex of the graph (V, E) which is not a neighbour of $(0,0)$. By construction, the box Δ_y is included in $\Delta^c = \Delta^c_{(0,0)}$ (since Δ is an open set). This means the random variable ξ_y is measurable with respect to the σ-algebra induced by the configurations restricted to $\Delta^c_{(0,0)}$. So,

$$P\left(\xi_{(0,0)} = 1 \,|\, \xi_y : \{(0,0), y\} \notin E\right) \geq 1 - (5C + C')z^{-1} ,$$

and the hypothesis (4.1) holds with $x = (0,0)$ and any $p \in [0,1[$, provided the intensity z is large enough. This ends the proof of Theorem 3.1.

Lemma 4.3. Let $U \in \mathcal{B}(\mathbb{R}^2)$ be a bounded Borel set with positive Lebesgue measure and $V \supset U \oplus \bar{B}(0, R_1 + R_0 + \varepsilon)$. First, let us write:

$$\begin{aligned} P\left(\omega_V = \emptyset \,|\, \omega_{V^c}\right) &= \frac{1}{Z_V(\omega_{V^c})} \int_{\Omega_V} 1\!\!1_{\omega_V = \emptyset}\, e^{-H_V(\omega_V \cup \omega_{V^c})}\, \pi^z_V(d\omega_V) \\ &= \frac{e^{-z\lambda(V)}}{Z_V(\omega_{V^c})} , \end{aligned} \tag{4.4}$$

since the empty configuration has a null energy, i.e. $H_V(\omega_{V^c}) = 0$. A configuration ω whose restriction to V satisfies

$$\#\omega_{U \times [R_0, R_0 + \varepsilon]} = 1 \;\text{ and }\; \omega_{V \setminus U} = \emptyset$$

is reduced to a ball $\bar{B}(x, R)$ centred at a x in U and with a radius $R_0 < R < R_0 + \varepsilon$. Since the ball $\bar{B}(x, R)$ does not overlap $\bar{\omega}_{V^c}$, its energy $H_V((x, R) \cup \omega_{V^c})$ is easy to compute;

$$H_V((x, R) \cup \omega_{V^c}) = \theta_1 2\pi R + \theta_2 \pi R^2 + \theta_3$$

(it is not worth using inequalities of Lemma 4.12 here). So, $H_V((x, R) \cup \omega_{V^c})$ is bounded by a positive constant K only depending on parameters θ_1, θ_2, θ_3 and radius R_1. Henceforth,

$$\begin{aligned} &P\left(\#\omega_{U \times [R_0, R_0+\varepsilon]} = 1,\ \omega_{V \setminus U} = \emptyset \,|\, \omega_{V^c}\right) \\ &\qquad = \frac{1}{Z_V(\omega_{V^c})} \int_{U \times [R_0, R_0 + \varepsilon]} e^{-H_V((x,R) \cup \omega_{V^c})}\, z e^{-z\lambda(V)}\, \lambda(dx) Q(dR) \\ &\qquad \geq \frac{e^{-z\lambda(V)}}{Z_V(\omega_{V^c})} z e^{-K} \lambda(U) Q([R_0, R_0 + \varepsilon]) . \end{aligned}$$

Recall that $Q([R_0, R_0 + \varepsilon])$ is positive by (2.4). Using the identity (4.4), we finally upper-bound the conditional probability $P(\omega_V = \emptyset | \omega_{V^c})$ by

$$\left(z e^{-K} \lambda(U) Q([R_0, R_0 + \varepsilon])\right)^{-1} P\left(\#\omega_{U \times [R_0, R_0+\varepsilon]} = 1,\ \omega_{V \setminus U} = \emptyset \,|\, \omega_{V^c}\right) .$$

This proves Lemma 4.3 with $C = (e^{-K} \lambda(U) Q([R_0, R_0 + \varepsilon]))^{-1}$. □

4.2 Proof of the Connection Lemma

4.2.1 Outline

Let us recall that $N^{\Delta}_{\mathsf{cc}}(\omega)$ denotes the number of connected components of $\bar{\omega}_\Delta$ having at least one ball centred in one of the four cardinal boxes B_{N}, B_{S}, B_{E} or B_{W}. Our strategy for proving the Connection Lemma is to exhibit, for each ω such that $N^{\Delta}_{\mathsf{cc}}(\omega) \geq 2$, a deterministic set B from some family $\mathcal{B}$ of subsets of Δ such that $\omega_B = \emptyset$. Now, a uniform bound (in ω_{B^c}) for the energy $H_B((x, R) \cup \omega_{B^c})$ implies that the set B contains a point of the configuration ω with high probability when z tends to infinity.

For $x \in B$, let us denote by $\mathcal{N}_{\text{hol}}((x,R),\omega_{B^c})$ the hole number variation when the ball $\bar{B}(x,R)$ is added to the configuration ω_{B^c}. This quantity is central in our proof. Indeed, a first upper bound for the energy $H_B((x,R)\cup\omega_{B^c})$ is given by Lemma 4.12:

$$H_B((x,R)\cup\omega_{B^c}) = h((x,R),\omega_{B^c}) \le K - \theta_3\,\mathcal{N}_{\text{hol}}((x,R),\omega_{B^c}) \ , \tag{4.5}$$

where K is a positive constant only depending on parameters θ_1, θ_2, θ_3 and radii R_0, R_1. So, in order to bound from above the energy $H_B((x,R)\cup\omega_{B^c})$ it is sufficient to bound from above the number of created holes (resp. deleted holes) when θ_3 is negative (resp. positive). This is the reason why the proof of the Connection Lemma differs according to the sign of the parameter θ_3.

4.2.2 When θ_3 is negative

Let ω be a configuration and α be a positive real number. A couple $(x,R)\in\mathbb{R}^2\times[R_0,R_1]$ is said to be *good* if all the connected components of the set $\bar{\omega}_\Delta\cap\bar{B}(x,R)$ have an area larger than α. These couples are well-named because adding a ball $\bar{B}(x,R)$ to the configuration ω_Δ, with a good couple (x,R), does not create too many holes.

Lemma 4.5. *Let $(x,R)\in\mathbb{R}^2\times[R_0,R_1]$ be a good couple. Then,*

$$\mathcal{N}_{\text{hol}}((x,R),\omega_\Delta) \le \frac{\pi R_1^2}{\alpha} \ .$$

Proof. The number of created holes when the ball $\bar{B}(x,R)$ is added to ω_Δ is smaller than the number of connected components of the set $\bar{\omega}_\Delta\cap\bar{B}(x,R)$. This can be checked by the additive property of the functional χ. Since (x,R) is good, all these connected components have an area larger than α. So, there are at most $\pi R^2/\alpha$ such connected components. □

Let us denote by $\text{Bad}(\omega_\Delta,\alpha)$ the following set:

$$\text{Bad}(\omega_\Delta,\alpha) = \{x\in\mathbb{R}^2,\ \exists R\in[R_0,R_0+\varepsilon],\ (x,R) \text{ is not good } \} \ .$$

Lemma 4.6. *The area of the set* $\text{Bad}(\omega_\Delta,\alpha)$ *tends to* 0 *as* α *and* ε *tend to* 0*, uniformly on the configuration* ω_Δ*.*

Lemma 4.6 will be proved at the end of this section. Lemmas 4.5 and 4.6 allow us to prove the Connection Lemma. First, a family of (small) non-overlapping squared boxes whose union covers the convex hull of the boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$ is needed. Precisely, for $\kappa>0$, let us consider a subset $\mathcal{B}$ of $\{v+[0,\kappa[^2, v\in\mathbb{R}^2\}$ such that for any B,B' in $\mathcal{B}$, $B\cap B'$ is empty, and

$$\text{Conv}\,(B_{\mathtt{N}},B_{\mathtt{S}},B_{\mathtt{E}},B_{\mathtt{W}}) \subset \bigcup_{B\in\mathcal{B}} B \subset \Delta \ .$$

The family $\mathcal{B}$ is made up of at most $c_\kappa = \kappa^{-2}\mathcal{A}(\Delta)$ elements.
The hypothesis $N_{\mathsf{cc}}^{\Delta}(\omega)\ge 2$ ensures the existence of two elements $(x_1,\cdot)$ and $(x_2,\cdot)$ of ω, whose centres x_1 and x_2 are in the union of the four cardinal boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$ and $B_{\mathtt{W}}$, and whose balls $\bar{B}(x_1,\cdot)$ and $\bar{B}(x_2,\cdot)$ belong to two different connected components of $\bar{\omega}$, say respectively C_1 and C_2. Let $[x_1,x_2]$ be the segment in $\mathbb{R}^2$ linking x_1 with x_2 and d be the euclidean distance on $\mathbb{R}^2$. The continuous map

$$f : x\in[x_1,x_2] \ \mapsto\ d(x,C_1) - d(x,\bar{\omega}\setminus C_1)$$

satisfies $f(x_1)<0$ and $f(x_2)>0$. So there exists a point x in $[x_1,x_2]$ such that $d(x,C_1)$ and $d(x,\bar{\omega}\setminus C_1)$ are equal (and positive). Hence, the ball $\bar{B}(x,R_0)$ does not contain any point of ω_Δ. Moreover, since x is in the convex hull of the boxes B_{N}, B_{S}, B_{E} and B_{W}, then it belongs to one box of the family $\mathcal{B}$, say B. With $\kappa < R_0/\sqrt{2}$, the box B is contained in $\bar{B}(x, R_0)$. Consequently, ω_B is empty:

$$P\left(N_{\mathrm{cc}}^{\Delta}(\omega) \geq 2 \,|\, \omega_{\Delta^c}\right) \leq \sum_{B \in \mathcal{B}} P\left(\omega_B = \emptyset \,|\, \omega_{\Delta^c}\right) \ . \tag{4.6}$$

For a given box $B \in \mathcal{B}$, let us consider the (random) set $U(\omega_{\Delta \setminus B})$ of points $x \in B$ such that, for any radius $R \in [R_0, R_0 + \varepsilon]$, the couple (x, R) is good:

$$U(\omega_{\Delta \setminus B}) = B \setminus \mathrm{Bad}(\omega_{\Delta \setminus B}, \alpha) \ .$$

Let $x \in U(\omega_{\Delta \setminus B})$ and $R \in [R_0, R_0 + \varepsilon]$. On the one hand, using (4.5), $\theta_3 \leq 0$ and Lemma 4.5, we get

$$H_B((x, R) \cup \omega_{B^c}) \leq K - \theta_3 M \ , \tag{4.7}$$

where $M = M(R_1, \alpha)$ denotes the upper bound given by Lemma 4.5. On the other hand, Lemma 4.6 implies that the area of $U(\omega_{\Delta \setminus B})$ is larger than $\kappa^2/2$ for α and ε small enough, uniformly on the configuration $\omega_{\Delta \setminus B}$. It follows:

$$\begin{aligned} P&\left(\#\omega_{B \times [R_0, R_0+\varepsilon]} = 1 \,|\, \omega_{B^c}\right) \\ &= \frac{1}{Z_B(\omega_{B^c})} \int_{B \times [R_0, R_0+\varepsilon]} e^{-H_B((x,R) \cup \omega_{B^c})} \, z e^{-z\lambda(B)} \, \lambda(dx) Q(dR) \\ &\geq \frac{z e^{-z\kappa^2}}{Z_B(\omega_{B^c})} \int_{U(\omega_{\Delta \setminus B}) \times [R_0, R_0+\varepsilon]} e^{-H_B((x,R) \cup \omega_{B^c})} \, \lambda(dx) Q(dR) \\ &\geq \frac{z e^{-z\kappa^2}}{Z_B(\omega_{B^c})} e^{-K + \theta_3 M} \, \frac{\kappa^2}{2} \, Q([R_0, R_0 + \varepsilon]) \ . \end{aligned}$$

In the previous inequality, replacing $e^{-z\kappa^2} Z_B(\omega_{B^c})^{-1}$ with the conditional probability $P(\omega_B = \emptyset | \omega_{B^c})$, we obtain

$$P\left(\omega_B = \emptyset \,|\, \omega_{B^c}\right) \leq \frac{2 \, e^{K - \theta_3 M}}{z \, \kappa^2 \, Q([R_0, R_0 + \varepsilon])} \ .$$

Finally, the Connection Lemma is deduced from the above inequality and (4.6), with the constant

$$C' = \frac{2 \, c_\kappa \, e^{K - \theta_3 M}}{\kappa^2 \, Q([R_0, R_0 + \varepsilon])} \ .$$

In order to prove Lemma 4.6, we have to locate the set $\mathrm{Bad}(\omega_\Delta, \alpha)$. Lemma 4.7 claims that the distance from any point $(x, .)$ in $\mathrm{Bad}(\omega_\Delta, \alpha)$ to $\bar{\omega}_\Delta$ is close to R_0. Let $\bar{B}(x, R)$ and $\bar{B}(y, R')$ be two balls satisfying $R \in [R_0, R_0 + \varepsilon]$, $R' \in [R_0, R_1]$ and

$$0 < \mathcal{A}(\bar{B}(x, R) \cap \bar{B}(y, R')) \leq \alpha \ .$$

Then there exists a positive function $g(\varepsilon, \alpha)$, which tends to 0 when α and ε tend to 0, such that

$$|\, d(x, \bar{B}(y, R')) - R_0 \,| \leq g(\varepsilon, \alpha) \ . \tag{4.8}$$

The function g is also allowed to depend on radii R_0 and R_1. The topological boundary $\partial\bar{\omega}_\Delta$ is composed of a finite number of arcs. Let a be one of them. This arc is a part of the boundary of a ball coming from the configuration ω_Δ, say $(y, \cdot)$. Now, we can define the circular strip $S_g(a)$ of width $2g(\varepsilon, \alpha)$ by

$$S_g(a) = \left\{ x \in \mathbb{R}^2;\ \exists y' \in a \text{ s.t. } \begin{array}{c} x = y' + \mu(y' - y) \text{ with } \mu > 0 \text{ and} \\ |\, d(x, y') - R_0 \,| \leq g(\varepsilon, \alpha) \end{array} \right\} \ .$$

Lemma 4.7. *The following inclusion holds;*

$$\mathrm{Bad}(\omega_\Delta, \alpha) \subset \bigcup_{a,\text{ arc of }\partial\bar{\omega}_\Delta} S_g(a) \ . \tag{4.9}$$

Proof. Let us consider a point x in $\mathrm{Bad}(\omega_\Delta, \alpha)$. Let $R \in [R_0, R_0 + \varepsilon]$ such that (x, R) is not good. So there exists a connected component of $\bar{\omega}_\Delta \cap \bar{B}(x, R)$ of area smaller than α. The boundary of this connected component through the open ball $B(x, R)$ is composed of a finite number of arcs, say $a_1, \dots, a_n$. Let a be one of them realizing the minima

$$d(x, a) = \min_{1 \le i \le n} d(x, a_i) \ .$$

Let $(y, \cdot)$ be the element of the configuration ω_Δ generating the arc a. Let $S(a)$ be the semi-infinite cone centred at y and with arc a (i.e. the union of semi-line $[y, y')$ for $y' \in a$). Then, x necessarily belongs to $S(a)$. Indeed, the opposite situation could lead to the existence of another arc a' satisfying $d(x, a') < d(x, a)$. To sum up, x is in the semi-infinite cone $S(a)$ and the area of $\bar{B}(x, R) \cap \bar{B}(y, \cdot)$ is positive and smaller than α. So x satisfies (4.8) and then belongs to $S_g(a)$. □

Proof of Lemma 4.6. Let a be an arc of the boundary $\partial\bar{\omega}_\Delta$. Some geometrical considerations allow to bound the area of the circular strip $S_g(a)$:

$$\mathcal{A}(S_g(a)) \le 4g(\varepsilon, \alpha)\mathrm{length}(a) \ ,$$

where $\mathrm{length}(a)$ denotes the length of the arc a. We deduce from this bound and Lemmas 4.7 and 4.11:

$$\begin{aligned}
\mathcal{A}(\mathrm{Bad}(\omega_\Delta, \alpha)) &\le \sum_{a \text{ arc of } \partial\bar{\omega}_\Delta} \mathcal{A}(S_g(a)) \\
&\le 4g(\varepsilon, \alpha) \sum_{a \text{ arc of } \partial\bar{\omega}_\Delta} \mathrm{length}(a) \\
&\le 4g(\varepsilon, \alpha)\mathcal{L}_{\Delta'}(\bar{\omega}_\Delta) \quad \text{with } \Delta' = \Delta \oplus B(0, R_1) \\
&\le 4g(\varepsilon, \alpha) \frac{\mathcal{A}(\Delta' \oplus B(0, R_0))}{R_0} \ .
\end{aligned}$$

This latter upper bound does not depend on the configuration ω_Δ. So, this ends the proof of Lemma 4.6. □

4.2.3 When θ_3 is positive

In this section, it is still assumed that $N_{cc}^{\Delta}(\omega)$ is larger than 2. But this time, our aim consists in bounding from above the number of deleted holes when the ball $\bar{B}(x, R)$, $x \in B$, is added to the configuration ω_{B^c}. The existence of a suitable set B comes from Lemma 4.8.

Lemma 4.8. *Assume $N_{cc}^{\Delta}(\omega) \ge 2$. There exist $\rho > 0$ (which does not depend on ω) and $O = O(\omega) \in \Delta$ such that:*

(i) *O is in $\mathrm{Conv}\,(B_N, B_S, B_E, B_W) \oplus B(0, \frac{3}{2}R_0)$;*

(ii) *$B(O, \rho R_0) \cap \omega$ is empty;*

(iii) *$B(O, (1 + \rho)R_0)$ does not (totally) contain any hole of $\bar{\omega}$.*

Let us first explain how to prove the Connection Lemma from Lemma 4.8. As in Section 4.2.2, we need the family $\mathcal{B}$ of non-overlapping squared boxes of length side κ. But here, $\mathcal{B}$ is required to cover a little bit more, i.e.

$$\text{Conv}\,(B_{\mathtt{N}}, B_{\mathtt{S}}, B_{\mathtt{E}}, B_{\mathtt{W}}) \oplus B(0, \frac{3}{2}R_0) \subset \bigcup_{B\in\mathcal{B}} B \;, \tag{4.10}$$

and parameters κ and ε are chosen small enough so that

$$\sqrt{2}\kappa + \varepsilon < \rho R_0 \tag{4.11}$$

(where ρ is given by Lemma 4.8). Thanks to statement (i) and (4.10), the point O belongs to a box $B \in \mathcal{B}$. Thanks to (ii), (iii) and (4.11), ω_B is empty and $\bar{\omega}_{B^c}$ has no hole in $\mathbf{B} := B \oplus B(0, R_0 + \varepsilon)$. Hence,

$$P\left(N^{\Delta}_{\text{cc}}(\omega) \geq 2 \,|\, \omega_{\Delta^c}\right) \leq \sum_{B\in\mathcal{B}} P\left(P\left(\omega_B = \emptyset \,|\, \omega_{B^c}\right) 1\!\!1_{\bar{\omega}_{B^c} \text{ has no hole in } \mathbf{B}} \,|\, \omega_{\Delta^c}\right) \;. \tag{4.12}$$

Let us pick a box $B \in \mathcal{B}$, a couple $(x, R) \in B \times [R_0, R_0 + \varepsilon]$ and assume that $\bar{\omega}_{B^c}$ has no hole in $\mathbf{B}$. Then, no hole is deleted when $\bar{B}(x, R)$ is added to ω_{B^c}. So, the hole number variation $\mathcal{N}_{\text{hol}}((x, R), \omega_{B^c})$ is non negative. Combining with $\theta_3 \geq 0$ and (4.5), the energy $H_B((x, R) \cup \omega_{B^c})$ is smaller than K and we finish the proof of the Connection Lemma as in Section 4.2.2. First,

$$P\left(\#\omega_{B\times[R_0,R_0+\varepsilon]} = 1 \,|\, \omega_{B^c}\right) \geq \frac{ze^{-z\kappa^2}}{Z_B(\omega_{B^c})} e^{-K}\, \kappa^2\, Q([R_0, R_0 + \varepsilon]) \;.$$

Thus, replacing $e^{-z\kappa^2} Z_B(\omega_{B^c})^{-1}$ by the conditional probability $P(\omega_B = \emptyset|\omega_{B^c})$, we get

$$P\left(\omega_B = \emptyset \,|\, \omega_{B^c}\right) \leq \frac{e^K}{z\, \kappa^2\, Q([R_0, R_0 + \varepsilon])} \;.$$

Finally, the Connection Lemma is deduced from the above inequality and (4.12), with

$$C' = \frac{c_\kappa\, e^K}{\kappa^2\, Q([R_0, R_0 + \varepsilon])} \;,$$

where c_κ still denotes the number of boxes contained in the family $\mathcal{B}$.

Now, let us find a point O and a radius $\rho > 0$ satisfying the three properties of Lemma 4.8. The same method as in Section 4.2.2, based on the hypothesis $N^{\Delta}_{\text{cc}}(\omega) \geq 2$, ensures the existence of a point O' in the convex hull of the $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$, $B_{\mathtt{W}}$'s, such that

$$\mathbf{d} := d(O', C_1) = d(O', \bar{\omega}_\Delta \setminus C_1) > 0 \tag{4.13}$$

where C_1 denotes a connected component of $\bar{\omega}_\Delta$ counting by $N^{\Delta}_{\text{cc}}(\omega)$. Two cases will be considered in the following. In the first one– $\mathbf{d} \geq \frac{1}{2}R_0$ –the connected components of $\bar{\omega}_\Delta$ are far away from O'. So are their holes. Then, the choice $O = O'$ is appropriate. In the second case– $\mathbf{d} \leq \frac{1}{2}R_0$ –we exhibit a region close to O' without hole and choose a suitable point O inside. About the radius ρ, it will be proved in the sequel that any positive real number such that

$$(1+\rho)^2 \,<\, 1 + \frac{1}{4} \;, \tag{4.14}$$

$$\sqrt{7}(1+\rho) - \frac{7}{4} \,<\, 1 \tag{4.15}$$

and

$$\left(1-\frac{\sqrt{7}}{4}+\rho\right)^2+\left(\frac{3}{2}-\sqrt{(1-\rho)^2-\left(1-\frac{\sqrt{7}}{4}+\rho\right)^2}\right)^2 < \left(\sqrt{3}-1-\rho\right)^2 \ , \qquad (4.16)$$

is suitable. For instance, $\rho = 0.01$ satisfies these three conditions.

Case 1: $\mathbf{d} \geq \frac{1}{2}R_0$.
By construction, O' is in the convex hull of the boxes $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$, $B_{\mathtt{W}}$ and its distance to any point x in ω_Δ is larger than $R_0 + \mathbf{d}$ from . So, it satisfies properties (i) and (ii) of Lemma 4.8. Now, let us consider a hole T of $\bar{\omega}_\Delta$. Assume in a first time that O' does not belong to T. By (4.14) and Lemma 4.13,

$$d(O',T)^2 \geq \left(1+\frac{1}{4}\right)R_0^2 \geq (1+\rho)^2R_0^2 \ .$$

This means that the hole T is outside the ball $B(O',(1+\rho)R_0)$. Now, assume that O' is in T. Since O' is equidistant from two connected components of $\bar{\omega}_\Delta$ then one of them is inside the hole T. Hence, T is too large to be totally covered by the ball $B(O',(1+\rho)R_0)$. Consequently, O' also satisfies (iii).

Case 2: $\mathbf{d} \leq \frac{1}{2}R_0$.
Let $\bar{B}(x_1,R_{x_1})$ be a ball of the connected component C_1 on which the distance $d(O',C_1)$ is reached. Let us consider the point y_1 on the segment $[O',x_1]$ satisfying $\bar{B}(y_1,R_0)$ is included in $\bar{B}(x_1,R_{x_1})$ and

$$d(O',\bar{B}(y_1,R_0)) = d(O',\bar{B}(x_1,R_{x_1})) = d(O',C_1) = \mathbf{d} \ .$$

In the same way, let us consider a point y_2 such that $\bar{B}(y_2,R_0)$ is included in $\bar{\omega}_\Delta \setminus C_1$ and

$$d(O',\bar{B}(y_2,R_0)) = d(O',\bar{\omega}_\Delta \setminus C_1) = \mathbf{d} \ .$$

The region without hole, mentioned at the beginning of the current section and which we need, is built from points y_1 and y_2. See Figure 2. Let $\mathcal{D}$ be the infinite line passing by y_1 and y_2. Thus, let us consider two infinite lines $\mathcal{D}'$ and $\mathcal{D}''$ parallel to $\mathcal{D}$ and such that

$$d(\mathcal{D}',\mathcal{D}) = d(\mathcal{D}'',\mathcal{D}) = \frac{\sqrt{7}}{4}R_0$$

(say O' and $\mathcal{D}'$ are on the same side of the line $\mathcal{D}$). We denote by $\mathcal{H}$ the intersection of the convex hull of balls $\bar{B}(y_1,R_0)$ and $\bar{B}(y_2,R_0)$ with the strip delimited by $\mathcal{D}'$ and $\mathcal{D}''$. On Figure 2, the border of $\mathcal{H}$ is drawn in bold.

Lemma 4.9. *With the previous notations and hypotheses, there is no hole in $\mathcal{H}$.*

Proof of Lemma 4.9. The closest hole T to the segment $[y_1,y_2]$ is obtained by pressing a ball with radius R_0 against $\bar{B}(y_1,R_0)$ and $\bar{B}(y_2,R_0)$. If l denotes the distance between T and $[y_1,y_2]$ then $2l$ is the distance between the center of this pressing ball and $[y_1,y_2]$. Pythagoras Theorem gives $(2l)^2+(R_0+h)^2 = (2R_0)^2$ in which h denotes

$$h := \frac{1}{2}d(y_1,y_2) - R_0 \ \leq \ \mathbf{d} \ .$$

In the worst case, $h = \frac{1}{2}R_0$. Hence, l is always larger than $\frac{\sqrt{7}}{4}R_0$, which is the distance between $\mathcal{D}$ and $\mathcal{D}'$. To complete the proof, let us add there is no hole in the balls $\bar{B}(y_1,R_0)$ and $\bar{B}(y_2,R_0)$ since they are totally covered by $\bar{\omega}_\Delta$. □

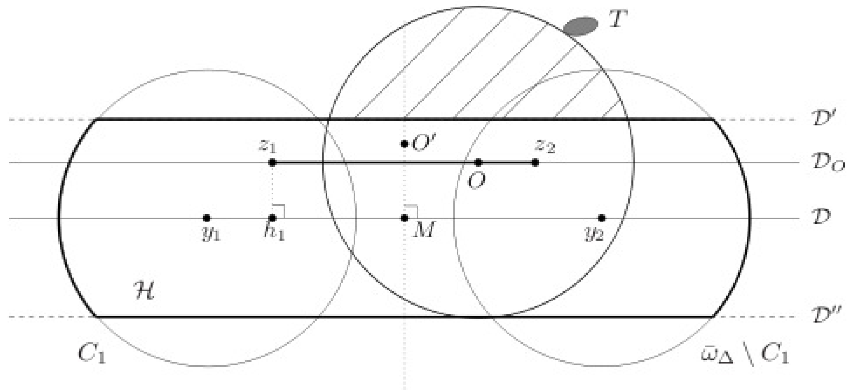

Figure 2: The balls $\bar{B}(y_1, R_0)$ and $\bar{B}(y_2, R_0)$ are respectively contained in the connected component C_1 and in $\bar{\omega}_\Delta \setminus C_1$. From these balls a point O is built and a real number $\rho > 0$ is exhibited, satisfying together the three properties of Lemma 4.8.

The idea to conclude the proof can be summed up as follows. The region $\mathcal{H}$ is sufficiently thick to contain strictly more than half of a ball with radius $(1+\rho)R_0$. Hence, the part of this ball outside $\mathcal{H}$ (this is the hatched region on Figure 2) has a diameter smaller than $2R_0$. Thanks to Lemma 4.15, it is possible to choose the center O of this ball so that $\bar{B}(O, (1+\rho)R_0) \cap \mathcal{H}^c$ does not contain any hole.
Let $\mathcal{D}_O$ be the infinite line parallel to $\mathcal{D}''$, at distance $(1+\rho)R_0$ from $\mathcal{D}''$ and on the same side as $\mathcal{D}$ of the line $\mathcal{D}''$. It follows from (4.15) that the line $\mathcal{D}_O$ is trapped between $\mathcal{D}$ and $\mathcal{D}'$. Let M be the center of the segment $[y_1, y_2]$. Let us denote by $[z_1, z_2]$ the following segment:

$$[z_1, z_2] := \bar{B}(M, (1-\rho)R_0) \cap \mathcal{D}_O \ .$$

See Figure 2. We are going to choose the point O on the segment $[z_1, z_2]$. To do it, some geometrical results about the previous construction are needed. They will be proved at the end of the section:

Lemma 4.10. *With the previous notations and hypotheses, the following statements hold:*

- (a) *for* $i = 1, 2$, $d(O', z_i) \leq \frac{3}{2}R_0$;
- (b) $[z_1, z_2] \oplus B(0, \rho R_0) \subset B(O', R_0 + \mathbf{d})$;
- (c) *for* $i = 1, 2$, $d(y_i, z_i) \leq (\sqrt{3} - 1 - \rho)R_0$.

By convexity and statement (a), any point of the segment $[z_1, z_2]$ is at distance from O' larger than $\frac{3}{2}R_0$. Moreover, O' is in the convex hull of the $B_{\mathtt{N}}$, $B_{\mathtt{S}}$, $B_{\mathtt{E}}$, $B_{\mathtt{W}}$'s. Then, any point of $[z_1, z_2]$ satisfies the property (i) of Lemma 4.8.
By construction of the point O', the ball $B(O', R_0 + \mathbf{d})$ does not contain any point of ω. So does the set $[z_1, z_2] \oplus B(0, \rho R_0)$ thanks to statement (b). This means that any point of the segment $[z_1, z_2]$ satisfies the property (ii) of Lemma 4.8.
Combining statement (c) with $i = 1$ and Lemma 4.14, we check there is no hole of $\bar{\omega}_\Delta \setminus C_1$ in the ball $B(z_1, (1+\rho)R_0)$. Let us run the center of a ball with radius $(1+\rho)R_0$ along the segment $[z_1, z_2]$ from z_1 to z_2 while this ball does not meet any hole of $\bar{\omega}_\Delta \setminus C_1$. Two cases can be distinguished.

- This running ball does not meet any hole and its centre runs to z_2. Then, the ball $B(z_2,(1+\rho)R_0)$ does not contain any hole of $\bar{\omega}_\Delta \setminus C_1$. It does not contain any hole of C_1 either thanks to statement (c) with $i=2$ and Lemma 4.14. In this case, $O=z_2$ satisfies the property (iii) of Lemma 4.8.
- This running ball meets a hole (see Figure 2): let O be the corresponding center (at the meeting) and T be the corresponding hole of $\bar{\omega}_\Delta \setminus C_1$. Let us remark that, as previously, the ball $B(O,(1+\rho)R_0)$ does not still contain any hole of $\bar{\omega}_\Delta \setminus C_1$. To prove it, denote by $\mathcal{C}$ the part of this ball outside $\mathcal{H}$:

  $$\mathcal{C} := B(O,(1+\rho)R_0) \cap \mathcal{H}^{\mathrm{c}} \ .$$

  On the one hand, thanks to (4.15) the diameter of $\mathcal{C}$ is smaller than $2R_0$. So $\mathcal{C}$ is incuded in $T \oplus B(O,2R_0)$. On the other hand, by Lemma 4.15, the set $T \oplus B(O,2R_0)$ does not contain any other hole of C_1. So does for $\mathcal{C}$. Since there is no hole in $\mathcal{H}$, this point O satisfies the property (iii) of Lemma 4.8.

Proof of Lemma 4.10. The infinite line $\mathcal{D}$ splits $\bar{B}(M,R_0)$ into two half-balls; let $\mathcal{V}$ be the one containing the segment $[z_1,z_2]$. Since

$$d(O',y_1) = d(O',y_2) = R_0 + \mathbf{d} \ ,$$

the half-ball $\mathcal{V}$ is included in the ball with center O' and radius $R_0+\mathbf{d}$. There are two consequences from this inclusion. First, the points z_1 and z_2 which are in $\mathcal{V}$, are also in the ball $\bar{B}(O',R_0+\mathbf{d})$. This implies, for $i=1,2$

$$d(O',z_i) \leq \frac{3}{2}R_0 \ ,$$

i.e. statement (a). Second, the balls $\bar{B}(z_i,\rho R_0)$ which are included in $\mathcal{V}$, are also included in $\bar{B}(O',R_0+\mathbf{d})$. So is the set $[z_1,z_2] \oplus B(0,\rho R_0)$ by convexity. Statement (b) is proved. It remains to prove statement (c). Let us introduce the orthogonal projection h_1 of z_1 to the infinite line $\mathcal{D}$ (see Figure 2). Using $d(M,z_1)=(1-\rho)R_0$, $d(h_1,z_1)=(1+\rho-\frac{\sqrt{7}}{4})R_0$ and $\mathbf{d} \leq \frac{1}{2}R_0$, we get

$$d(y_1,z_1) \leq \sqrt{\left(1-\frac{\sqrt{7}}{4}+\rho\right)^2 + \left(\frac{3}{2}-\sqrt{(1-\rho)^2-\left(1-\frac{\sqrt{7}}{4}+\rho\right)^2}\right)^2} \ R_0 \ .$$

Thanks to (4.16), statement (c) follows. □

4.3 Proofs of geometrical lemmas

Lemma 4.11. *Let Δ be a bounded closed convex set. For any configuration ω, let us denote by $\mathcal{L}_\Delta(\bar{\omega})$ the perimeter of $\bar{\omega}$ viewed through Δ:*

$$\mathcal{L}_\Delta(\bar{\omega}) = \mathcal{L}(\bar{\omega} \cap \Delta) - \mathrm{length}(\partial\Delta \cap \bar{\omega}),$$

where $\mathrm{length}(\partial\Delta \cap \bar{\omega})$ *denotes the length of the boundary of Δ which is inside the set $\bar{\omega}$. Then,*

$$\mathcal{L}_\Delta(\bar{\omega}) \leq \frac{\mathcal{A}(\Delta \oplus B(0,R_0))}{R_0} \ .$$

Proof. The boundary of $\bar{\omega}$ viewed through Δ corresponds to a finite union of arcs, say $(a_i)_{1\leq i\leq n}$. For each arc a_i, coming from the ball $B(x_i,R_i)$, we consider the circular strip $S(a_i)$ of width R_0 defined by

$$S(a_i) = \left\{ x \in \mathbb{R}^2;\ \exists x' \in a_i \text{ s.t. } \begin{array}{c} x = x' + \mu(x_i - x') \text{ with } \mu > 0 \\ \text{and } d(x,x') < R_0 \end{array} \right\} \ .$$

Let us notice that the sets $(S(a_i))_{1\leq i\leq n}$ are disjoint. Indeed, let suppose that there exists $x \in S(a_i) \cap S(a_j)$ for some $i \neq j$. Without restriction, we can assume that the distance between x and a_i is smaller than or equal to the distance between x and a_j. Let y be the point on a_i such that this distance is equal to $|y - x|$. Then, y has to be strictly included in the ball $B(x_j, R_j)$ which contradicts the fact that y is on the boundary of $\bar{\omega}$.
This allows to compare the sum of the areas of $(S(a_i))_{1\leq i\leq n}$ with $\mathcal{A}(\bar{\omega}_{\Delta\oplus B(0,R_0)})$:

$$\begin{aligned}
\mathcal{L}_\Delta(\bar{\omega}) = \sum_{i=1}^{n} \text{length}(a_i) &\leq \frac{1}{R_0}\sum_{i=1}^{n} \mathcal{S}(a_i) \\
&\leq \frac{1}{R_0}\mathcal{A}(\bar{\omega}_{\Delta\oplus B(0,R_0)}) \\
&\leq \frac{\mathcal{A}(\Delta \oplus B(0,R_0))}{R_0} .
\end{aligned}$$

□

Lemma 4.12. *Let Δ be a bounded subset of $\mathbb{R}^2$, ω be a configuration on Δ and (x, R) be an element of $\Delta \times [R_0, R_1]$. Let us denote by $\mathcal{A}((x, R), \omega)$ the area variation when the ball $\bar{B}(x, R)$ is added to the configuration $\bar{\omega}$:*

$$\mathcal{A}((x, R), \omega) = \mathcal{A}((x, R) \cup \omega) - \mathcal{A}(\omega) .$$

In the same way, we consider the perimeter variation $\mathcal{L}((x, R), \omega)$ and the connected component number variation $\mathcal{N}_{\text{cc}}((x, R), \omega)$. The following inequalities hold.

$$0 \leq \mathcal{A}((x, R), \omega) \leq \pi R_1^2 . \tag{4.17}$$

$$-\frac{\pi(R_1 + R_0)^2}{R_0} \leq \mathcal{L}((x, R), \omega) \leq 2\pi R_1 . \tag{4.18}$$

$$-\frac{(R_1 + 2R_0)^2}{R_0^2} \leq \mathcal{N}_{\text{cc}}((x, R), \omega) \leq 1 . \tag{4.19}$$

Proof. Inequalities (4.17), upper bounds of (4.18) and (4.19) are obvious. The border length of $\bar{\omega}$ which is lost when the ball $\bar{B}(x, R)$ is adding can be interpreted as the perimeter of $\bar{\omega}$ viewed through $\bar{B}(x, R)$, i.e. as $\mathcal{L}_{\bar{B}(x,R)}(\bar{\omega})$. Thanks to Lemma 4.11, it is smaller than

$$\frac{\mathcal{A}(\bar{B}(x, R) \oplus B(0, R_0))}{R_0} \leq \frac{\pi(R_1 + R_0)^2}{R_0} .$$

This gives the lower bound of (4.18). It remains to prove the lower bound for $\mathcal{N}_{\text{cc}}((x, R), \omega)$. The number of deleted connected components when $\bar{B}(x, R)$ is adding to $\bar{\omega}$, is smaller than the number of non-overlapping balls with radius R_0 that we can put inside the ball $\bar{B}(x, R + 2R_0)$. By an area argument, this number is smaller than

$$\frac{\pi(R_1 + 2R_0)^2}{\pi R_0^2} .$$

□

Lemma 4.13. *Let $\mathcal{C}$ be a connected component of $\bar{\omega}_\Delta$ and T be a hole of $\mathcal{C}$. Any point $x \in \mathbb{R}^2$ such that $x \notin \mathcal{C}$ and $x \notin T$ satisfies*

$$d(x, T)^2 \geq d(x, \mathcal{C})^2 + 2d(x, \mathcal{C})R_0 .$$

Proof. Let us consider a connected component $\mathcal{C}$, a hole T and a point x satisfying the assumptions of the lemma. Let y be a point of the closure of T such that $d(x,T) = |x-y|$. Necessarily, y is on the boundary of two balls $B(z,R)$ and $B(z',R')$ of $\mathcal{C}$. Since x belongs neither to $\mathcal{C}$ nor to T, at least one of $z-y$ or $z'-y$ has a nonnegative scalar product with $x-y$. Say $z-y$. Given $|x-z|$ and $|y-z|$, the distance $|x-y|$ is minimal when the vectors $z-y$ and $x-y$ are orthogonal. Hence, using $|x-z| \geq d(x,\mathcal{C}) + R_0$ and $|y-z| \geq R_0$, it follows from Pythagoras Theorem that

$$d(x,T)^2 \geq (d(x,\mathcal{C}) + R_0)^2 - R_0^2 \ ,$$

which concludes the proof. □

The following result is a consequence of Lemma 4.13.

Lemma 4.14. *Let $\mathcal{C}$, $\mathcal{C}'$ be two connected components of $\bar{\omega}_\Delta$. Let $\bar{B}(x,R)$ be a ball of $\mathcal{C}$ and T' be a hole of $\mathcal{C}'$ which does not contain $\bar{B}(x,R)$. Then,*

$$d(x,T') \geq \sqrt{3}R_0 \ .$$

Lemma 4.15. *Let T and T' be two holes respectively of two connected components $\mathcal{C}$ and $\mathcal{C}'$ of $\bar{\omega}_\Delta$. If $T \not\subset T'$ and $T' \not\subset T$ then*

$$d(T,T') \geq 2R_0 \ .$$

Proof. Let T and T' be two holes satisfying the assumption of the lemma. We denote by x and y two points belonging respectively to the closure of T and T' such that $d(T,T') = |x-y|$. The point x (respectively y) belongs to the boundary of two balls $B(z,R)$ and $B(z',R')$ of $\mathcal{C}$ (respectively $B(w,r)$ and $B(w',r')$ of $\mathcal{C}'$). An analysis, as in the proof of Lemma 4.13, shows that the distance $|x-y|$ is minimal in the situation where $R = R' = r = r' = R_0$ and $\{z,z',w,w'\}$ is a parallelogram with length side $2R_0$. Then the points x and y are at the middle of two opposite sides and the result follows. □

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