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Internal DLA in higher dimensions

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Abstract

Let $A(t)$ denote the cluster produced by internal diffusion limited aggregation (internal DLA) with t particles in dimension $d \geq 3$. We show that $A(t)$ is approximately spherical, up to an $O(\sqrt{\log t})$ error.

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In the process known as internal diffusion limited aggregation (internal DLA) one constructs for each integer time $t \geq 0$ an **occupied set** $A(t) \subset \mathbb{Z}^d$ as follows: begin with $A(0) = \emptyset$ and $A(1) = \{0\}$. Then, for each integer $t > 1$, form $A(t+1)$ by adding to $A(t)$ the first point at which a simple random walk from the origin hits $\mathbb{Z}^d \setminus A(t)$. Let $B_r \subset \mathbb{R}^d$ denote the ball of radius r centered at 0, and write $\mathbf{B}_r := B_r \cap \mathbb{Z}^d$. Let ω_d be the volume of the unit ball in $\mathbb{R}^d$. Our main result is the following.

Theorem 0.1. *Fix an integer $d \geq 3$. For each γ there exists an $a = a(\gamma, d) < \infty$ such that for all sufficiently large r,*

$$\mathbb{P}\left\{\mathbf{B}_{r-a\sqrt{\log r}} \subset A(\omega_d r^d) \subset \mathbf{B}_{r+a\sqrt{\log r}}\right\}^c \leq r^{-\gamma}.$$

We treated the case $d = 2$ in [7] (see also the overview in [6]), where we obtained a similar statement with $\log r$ in place of $\sqrt{\log r}$. Together with a Borel-Cantelli argument, these results in particular imply the following: let $D(r)$ be the Hausdorff distance between the ball B_r and the set $A(\omega_d r^d) + [-\frac{1}{2}, \frac{1}{2}]^d$ centered at points of the internal DLA cluster. Then

Corollary 0.2. *For each $d \geq 2$ there is a constant $a = a(d)$ such that*

$$\mathbb{P}\{D(r) \leq a(\log r)^\alpha \text{ eventually}\} = 1$$

where

$$\alpha = \begin{cases} 1, & d = 2 \\ \frac{1}{2}, & d \geq 3. \end{cases}$$

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These results show that internal DLA in dimensions $d \geq 3$ is extremely close to a perfect sphere: when the cluster $A(t)$ has the same size as a ball of radius r, its fluctuations around that ball are confined to the $\sqrt{\log r}$ scale (versus $\log r$ in dimension 2). A recent result of Asselah and Gaudillière [4] shows that Theorem 0.1 is sharp in the sense that $\sqrt{\log r}$ cannot be replaced by any function that is $o(\sqrt{\log r})$.

In [7] we explained that our method for $d = 2$ would also apply in dimensions $d \geq 3$ with the $\log r$ replaced by $\sqrt{\log r}$. We outlined the changes needed in higher dimensions (stating that the full proof would follow in this paper) and included a key step: Lemma A, which bounds the probability of "thin tentacles" in the internal DLA cluster in all dimensions. The purpose of this note is to carry out the adaptation of the $d = 2$ argument of [7] to higher dimensions. We remark that in [7] we used an estimate from [10] to start this iteration, while here we have modified the argument slightly so that this a priori estimate is no longer required.

One way for $A(\omega_d r^d)$ to deviate from the radius r sphere is for it to have a single "tentacle" extending beyond the sphere. The thin tentacle estimate [7, Lemma A] essentially says that in dimensions $d \geq 3$, the probability that there is a tentacle of length m and volume less than a small constant times m^d (near a given location) is at most e^{-cm^2}. By summing over all locations, one may use this to show that the length of the longest "thin tentacle" produced before time t is $O(\sqrt{\log t})$. To complete the proof of Theorem 0.1, we will have to show that other types of deviations from the radius r sphere are also unlikely.

Lemma A of [7] was also proved for $d = 2$, albeit with e^{-cm^2} replaced by $e^{-cm^2/\log m}$. However, when $d = 2$ there appear to be other more "global" fluctuations that swamp those produced by individual tentacles. (Indeed, we expect, but did not prove, that the $\log r$ fluctuation bound is tight when $d = 2$.) We bound these other fluctuations in higher dimensions via the same scheme introduced in [6, 7], which involves constructing and estimating certain martingales related to the growth of $A(t)$. It turns out the quadratic variations of these martingales are, with high probability, of order $\log t$ when $d = 2$ and of constant order when $d \geq 3$, closely paralleling what one obtains for the discrete Gaussian free field (as outlined in more detail in [7]). The connection to the Gaussian free field is made more explicit in [8].

Section 1 proves Theorem 0.1 by iteratively applying higher dimensional analogues of the two main lemmas of [7]. The lemmas themselves are proved in Section 3, which is the heart of the argument. Section 2 contains preliminary estimates about random walks that are used in Section 3.

A brief history of internal DLA fluctuation bounds

In 1986, Meakin and Deutch [13] defined a closely related process which they termed *diffusion limited annihilation*. In numerical experiments, they found that the average fluctuation (as opposed to the "worst case" fluctuation bounded by Theorem 0.1) was of order $\sqrt{\log r}$ in dimension 2 and of constant order in dimension 3. Diaconis and Fulton proposed internal DLA in its modern form in 1991 [5]. In 1992, Lawler, Bramson, and Griffeath proved that the limit shape of internal DLA from a point is the ball in all dimensions [10]. In 1995 Lawler gave a more quantitative proof, showing that the fluctuations of $A(\omega_d r^d)$ from the ball of radius r are at most of order $O(r^{1/3} \log^4 r)$ [11]. In December 2009, the present authors announced the bound $O(\log r)$ on fluctuations in dimension $d = 2$ [6] and gave an overview of the argument, making clear that the details remained to be written. In April 2010, Asselah and Gaudillière [1] gave a proof, using different methods from [6], of the bound $O(r^{1/(d+1)})$ in all dimensions, improving the Lawler bound for all $d \geq 3$. In September 2010, Asselah and Gaudillière improved this to $O((\log r)^2)$ in all dimensions $d \geq 2$ with an $O(\log r)$ bound on "inner" errors [2]. In

October 2010 the present authors proved the $O(\log r)$ bounds (announced in December 2009) for dimension $d = 2$ and outlined the proof of the $O(\sqrt{\log r})$ bound for dimensions $d \geq 3$ [7]. In November 2010, Asselah and Gaudillière gave a second proof of the $O(\sqrt{\log r})$ bound [3]. Their proof uses methods from [2] along with Lemma A of [7] to bound "outer" errors and a new large deviation bound (in some sense symmetric to Lemma A) to bound "inner" errors.

More references and a more general discussion of internal DLA history appear in [7].

1 Proof of Theorem 0.1

We recall the overall structure of the proof in [7]. The first step is to quantify how early or late each point joins the cluster $A(T)$. Lemma 1.1, below, then says that an early point is unlikely unless there is also a comparably late point. Lemma 1.2 says that a late point is unlikely unless there is also a *significantly* earlier point. Since $A(T)$ is a connected set of T lattice sites in $\mathbb{Z}^d$, we have $A(T) \subset \mathbf{B}_T$, which gives an upper bound on how early any point can be. Thus the region $\{m > T\}$ at the top right of Figure 1 has probability 0. The other colored rectangles in Figure 1 are unlikely by Lemmas 1.1 and 1.2, so with high probability there are no very early or late points.

To make the above outline more precise, let m and ℓ be positive real numbers. We say that $x \in \mathbb{Z}^d$ is m-early if

$$x \in A(\omega_d(|x| - m)^d),$$

where $|x| = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$, and ω_d is the volume of the unit ball in $\mathbb{R}^d$. Likewise, we say that x is ℓ-late if

$$x \notin A(\omega_d(|x| + \ell)^d).$$

Let $\mathcal{E}_m[T]$ be the event that some point of $A(T)$ is m-early. Let $\mathcal{L}_\ell[T]$ be the event that some point of $\mathbf{B}_{(T/\omega_d)^{1/d} - \ell}$ is ℓ-late. These events correspond to "outer" and "inner" deviations of $A(T)$ from circularity.

Lemma 1.1. (Early points imply late points) *Fix a dimension $d \geq 3$. For each $\gamma \geq 1$, there is a constant $C_0 = C_0(\gamma, d)$, such that for all sufficiently large T, if $m \geq C_0\sqrt{\log T}$ and $\ell \leq m/C_0$, then*

$$\mathbb{P}(\mathcal{E}_m[T] \cap \mathcal{L}_\ell[T]^c) < T^{-10\gamma}.$$

Lemma 1.2. (Late points imply early points) *Fix a dimension $d \geq 3$. For each $\gamma \geq 1$, there is a constant $C_1 = C_1(\gamma, d)$ such that for all sufficiently large T, if $m \geq \ell \geq C_1\sqrt{\log T}$ and $\ell \geq C_1((\log T)m)^{1/3}$, then*

$$\mathbb{P}(\mathcal{E}_m[T]^c \cap \mathcal{L}_\ell[T]) \leq T^{-10\gamma}.$$

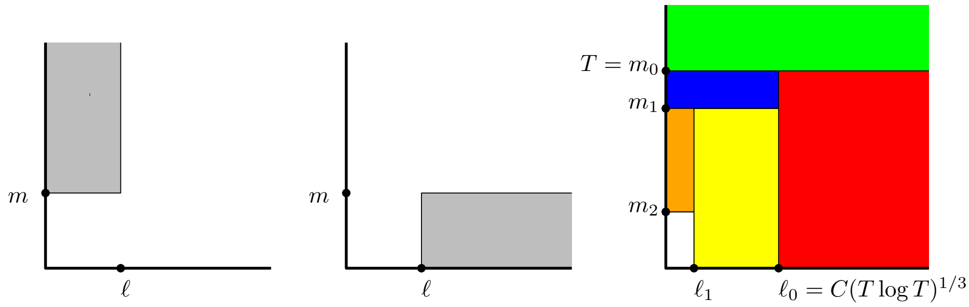

Figure 1: Let m^T be the largest m' for which $A(T)$ contains an m' early point. Let ℓ^T be the largest ℓ' for which some point of $B_{(T/\omega_d)^{1/d}-\ell'}$ is ℓ'-late. By Lemma 1.1, the pair of random variables (ℓ^T, m^T) is unlikely to belong to the semi-infinite rectangle in the left figure if $\ell \leq m/C_0$. By Lemma 1.2, (ℓ^T, m^T) is unlikely to belong to the semi-infinite rectangle in the second figure if $\ell \geq C_1((\log T)m)^{1/3}$. Theorem 0.1 will follow because the event $\{m^T > T\}$ is impossible and the other colored rectangles on the right are all (by Lemmas 1.1 and 1.2) unlikely.

We now proceed to derive Theorem 0.1 from Lemmas 1.1 and 1.2. The lemmas themselves will be proved in Section 3. Let $C = \max(C_0, C_1)$. We start with

$$m_0 = T.$$

Note that $A(T) \subset \mathbf{B}_T$, so $\mathbb{P}(\mathcal{E}_T[T]) = 0$. Next, for $j \geq 0$ we let

$$\ell_j = \max(C((\log T)m_j)^{1/3}, C\sqrt{\log T})$$

and

$$m_{j+1} = C\ell_j.$$

By induction on j, we find

$$\mathbb{P}(\mathcal{E}_{m_j}[T]) < 2jT^{-10\gamma}$$
$$\mathbb{P}(\mathcal{L}_{\ell_j}[T]) < (2j+1)T^{-10\gamma}.$$

To estimate the size of ℓ_j, let $K = C^4 \log T$ and note that $\ell_j \leq \ell'_j$, where

$$\ell'_0 = (KT)^{1/3}; \quad \ell'_{j+1} = \max((K\ell'_j)^{1/3}, K^{1/2}).$$

Then

$$\ell'_j \leq \max(K^{1/3+1/9+\cdots+1/3^j} T^{1/3^j}, K^{1/2})$$

so choosing $J = \log T$ we have

$$T^{1/3^J} < 2$$

and

$$\ell_J \leq 2K^{1/2} \leq C\sqrt{\log T}.$$

Setting $T = \omega_d r^d$, $\ell = \ell_J$ and $m = m_J$, the event $\mathcal{E}_m[T] \cup \mathcal{L}_\ell[T]$ has probability at most

$$(4J+1)T^{-10\gamma} < T^{-9\gamma} < r^{-\gamma}.$$

We conclude that if a is sufficiently large, then

$$\mathbb{P}\left\{\mathbf{B}_{r-a\sqrt{\log r}} \subset A(\omega_d r^d) \subset \mathbf{B}_{r+a\sqrt{\log r}}\right\} \leq \mathbb{P}(\mathcal{E}_m[T] \cup \mathcal{L}_\ell[T]) < r^{-\gamma}$$

which completes the proof of Theorem 0.1.

2 Green function estimates on the grid

This section assembles several Green function estimates that we need to prove Lemmas 1.1 and 1.2. The reader who prefers to proceed to the heart of the argument may skip this section on a first read and refer to the lemma statements as necessary. Fix $d \geq 3$ and consider the d-dimensional grid

$$\mathcal{G} = \{(x_1, \dots, x_d) \in \mathbb{R}^d \,:\, \text{at most one } x_i \notin \mathbb{Z}\}.$$

In many of the estimates below, we will assume that a positive integer k and a $y \in \mathbb{Z}^d$ have been fixed. We write

$$\Omega = \Omega(y,k) := \mathcal{G} \cap B_{|y|+k} \backslash \{y\}.$$

For $x \in \Omega \cup \partial\Omega$, let

$$P(x) = P_{y,k}(x)$$

be the probability that a Brownian motion on the grid $\mathcal{G}$ (defined in the obvious way; see [7]) starting at x reaches y before exiting $B_{|y|+k}$. Note that P is **grid harmonic** in Ω (i.e., P is linear on each segment of $\Omega \setminus \mathbb{Z}^d$, and for each $x \in \Omega \cap \mathbb{Z}^d$, the sum of the slopes of P on the $2d$ directed edge segments starting at x is zero). Boundary conditions are given by $P(y) = 1$ and $P(x) = 0$ for $x \in (\partial\Omega) \setminus \{y\}$.

The point y plays the role that ζ played in [7], and $P_{y,k}$ plays the role of the discrete harmonic function H_ζ. One difference from [7] is that we will sometimes take $k > 1$ so that y lies inside the ball instead of on the boundary. As we explain in Section 3, this extra parameter k (in particular, the gain of a factor of k in the lower bound of Lemma 2.5(a)) is what enables the improved arithmetic in Lemma 1.2 which results in fluctuation bounds of order $\sqrt{\log T}$ instead of $\log T$ in Theorem 0.1.

To estimate P we use the discrete Green function $g(x)$, defined as the expected number of visits to x by a simple random walk started at the origin in $\mathbb{Z}^d$. The well-known asymptotic estimate for g is [14]

$$\left| g(x) - a_d |x|^{2-d} \right| \leq C|x|^{-d} \tag{2.1}$$

for dimensional constants a_d and C (i.e., constants depending only on the dimension d). We extend g to a function, also denoted g, defined on the grid $\mathcal{G}$ by making g linear on each segment between lattice points. Note that g is grid harmonic on $\mathcal{G} \setminus \{0\}$.

Throughout we use C to denote a large positive dimensional constant, and c to denote a small positive dimensional constant, whose values may change from line to line.

Lemma 2.1. *There is a dimensional constant C such that*

(a) $P(x) \leq C/(1 + |x-y|^{d-2})$.
(b) $P(x) \leq Ck(|y| + k + 1 - |x|)/|x-y|^d$, *for* $|x - y| \geq k/2$.
(c) $\max_{x \in \mathbf{B}_r} P(x) \leq Ck/(|y| - r - k)^{d-1}$ *for* $r < |y| - 2k$.

Proof. The maximum principle (for grid harmonic functions) implies $Cg(x-y) \geq P(x)$ on Ω, which gives part (a).

For part (b), let y^* be one of the lattice points nearest to $(1 + (2k + C_1)/|y|)y$. By (2.1) we can choose a dimensional constant C_1 large enough so that $g(x-y) \geq g(x-y^*)$ for all $x \in \partial B_{|y|+k}$. By the maximum principle it follows that for $x \in \Omega$ we have

$$P(x) \leq C(g(x-y) - g(x-y^*)) \tag{2.2}$$

where $C = (g(0) - g(y-y^*))^{-1}$. Indeed, both sides are grid harmonic on Ω, and the right side is nonnegative on $\partial B_{|y|+k}$.

Combining (2.1) and (2.2) yields the bound

$$P(x) \leq \frac{Ck}{|x-y|^{d-1}}, \quad \text{for } |x-y| \geq 2k.$$

Next, let $z \in \partial B_{|y|+k}$ be such that $|z-y| = 2L$, with $L \geq 2k$. The bound above implies

$$P(x) \leq \frac{Ck}{L^{d-1}}, \quad \text{for } x \in B_L(z)$$

Let z^* be one of the lattice points nearest to $(|y|+k+L+C_1)z/|z|$. Then

$$F(x) = a_d L^{2-d} - g(x-z^*)$$

is comparable to L^{2-d} on $\partial B_{2L}(z^*)$ and positive outside the ball $B_L(z^*)$ (for a large enough dimensional constant C_1 — in fact, we can also do this with $C_1 = 1$ with L large enough). It follows that

$$P(x) \leq C(k/L^{d-1})(L^{d-2})F(x)$$

on $\partial(B_{2L}(z^*) \cap \Omega)$ and hence by the maximum principle on $B_{2L}(z^*) \cap \Omega$. Moreover,

$$F(x) \leq C(|y|+k+1-|x|)/L^{d-1}$$

for x a multiple of z and $|y|+k-L \leq |x| \leq |y|+k$. Thus for these values of x,

$$P(x) \leq C(k/L)F(x) \leq Ck(|y|+k+1-|x|)/L^d$$

We have just confirmed the bound of part (b) for points x collinear with 0 and z, but z was essentially arbitrary. To cover the cases $|x-y| \leq 2k$ one has to use exterior tangent balls of radius, say $k/2$, but actually the upper bound in part (a) will suffice for us in the range $|x-y| \leq Ck$.

Part (c) of the lemma follows from part (b). □

The mean value property (as typically stated for continuum harmonic functions) holds only approximately for discrete harmonic functions. There are two choices for where to put the approximation: one can show that the average of a discrete harmonic function h over the discrete ball $\mathbf{B}_r$ is approximately $h(0)$, or one can find an approximation w_r to the discrete ball $\mathbf{B}_r$ such that averaging h with respect to w_r yields *exactly* $h(0)$. The divisible sandpile model of [12] accomplishes the latter. In particular, the following discrete mean value property follows from Theorem 1.3 of [12]. For the sake of completeness we include a proof in dimensions $d \geq 3$. (Although we only use the $d \geq 3$ result here, the proof below also applies in dimension 2 after replacing the Green function $g(x)$ by $-a(x)$ where a is the recurrent potential kernel for $\mathbb{Z}^2$.)

Lemma 2.2. (Exact mean value property on an approximate ball) *For each real number $r > 0$, there is a function $w = w_r : \mathbb{Z}^d \to [0,1]$ such that*

(i) $w(x) = 1$ *for all* $x \in \mathbf{B}_{r-c}$, *for a constant* c *depending only on* d.

(ii) $w(x) = 0$ *for all* $x \notin \mathbf{B}_r$.

(iii) *For any function* h *that is discrete harmonic on* $\mathbf{B}_r$,

$$\sum_{x \in \mathbb{Z}^d} w(x)(h(x) - h(0)) = 0.$$

Proof. Let $m = \omega_d (r-a)^d$, for a constant a to be chosen below. Let $\mathcal{F}$ be the set of all functions $f : \mathbb{Z}^d \to \mathbb{R}$ satisfying

$$\begin{aligned} f &\geq 0 && \text{on } \mathbb{Z}^d \\ \Delta f &\leq 1 - m\delta_0 && \text{on } \mathbb{Z}^d. \end{aligned}$$

Here $\Delta f(x) = \frac{1}{2d} \sum_{y \sim x} (f(y) - f(x))$ denotes the discrete Laplacian of f, where the sum is over the $2d$ lattice neighbors y of x; and δ_0 denotes the function that is 1 at the origin and 0 elsewhere.

Let $u : \mathbb{Z}^d \to \mathbb{R}$ be defined by

$$u(x) = \inf_{f \in \mathcal{F}} f(x)$$

and let $w = m\delta_0 + \Delta u$. (Intuitively, w is the result of starting with mass m at the origin and spreading it out by discrete balayage until every site in $\mathbb{Z}^d$ has mass at most 1.)

It is straightforward to show that $u \in \mathcal{F}$ and hence $w \leq 1$ on $\mathbb{Z}^d$. Next we show $w \geq 1_U$ where $U = \{x \in \mathbb{Z}^d | u(x) > 0\}$ is the support of u. Indeed, if for some $x \in \mathbb{Z}^d$ we have $w(x) < 1_U(x)$, then for small enough $\epsilon > 0$ we would have $u - \epsilon\delta_x \in \mathcal{F}$, contradicting the minimality of u.

To prove items (i) and (ii) we express u in terms of an obstacle problem for the discrete Laplacian. Consider the "obstacle" $\gamma : \mathbb{Z}^d \to \mathbb{R}$ given by

$$\gamma(x) = -|x|^2 - mg(x)$$

where g is the discrete Green function for $\mathbb{Z}^d$. Let Φ be the set of all functions $\phi : \mathbb{Z}^d \to \mathbb{R}$ satisfying

$$\begin{aligned} \phi &\geq \gamma && \text{on } \mathbb{Z}^d \\ \Delta \phi &\leq 0 && \text{on } \mathbb{Z}^d. \end{aligned}$$

Let

$$s(x) = \inf_{\phi \in \Phi} \phi(x).$$

Since $\Delta\gamma = -1 + m\delta_0$, a simple argument using the maximum principle shows that $u = s - \gamma$.

By the Green function estimate (2.1), we have

$$\gamma(x) = \Gamma(|x|) + O((r/|x|)^d)$$

where $\Gamma(t) := -t^2 - \frac{2}{d-2}(r-a)^d t^{2-d}$, and the constant in the error term depends only on the dimension d. In particular, there is a constant C depending only on d, such that for all $t \in [r/2, 2r]$ and all $x, y \in \partial \mathbf{B}_t$ we have $|\gamma(x) - \gamma(y)| < C$. We choose $a = 3C$.

Since $s \geq \gamma \geq \Gamma(r) - C$ on $\partial \mathbf{B}_r$ and s is superharmonic, we have $s \geq \Gamma(r) - C$ on $\mathbf{B}_r$ by the minimum principle. Since Γ is maximized at $t = r - a$, it follows that $s > \gamma$ on $\mathbf{B}_{r-a-b}$ for a constant b depending only on d. Hence for all $x \in \mathbf{B}_{r-a-b}$ we have $u(x) > 0$ and hence $w(x) = 1$, which proves (i).

To prove (ii), note that the constant function $\phi(x) \equiv \Gamma(r-a) + C$ belongs to Φ. Hence $s \leq \Gamma(r-a) + C$, which shows that $u \leq 2C$ on $\partial \mathbf{B}_{r-a}$. We will show this implies u is supported in $\mathbf{B}_{r-a+2C}$. For each $x \in U - \{0\}$ the equality $\Delta u(x) = 1$ implies that at least one neighbor y of x has $u(y) \geq u(x) + 1$; hence there is a path $x = x_0, x_1, \ldots, x_k = 0$ such that $u(x_i) > i$. If $|x| > r - a$ then this path must pass through $\partial \mathbf{B}_{r-a}$, which shows that $|x| \leq r - a + 2C$, proving (ii).

To prove (iii), let $h : \mathbb{Z}^d \to \mathbb{R}$ be discrete harmonic on $\mathbf{B}_r$ and let $H(x) = h(x) - h(0)$. Since $w = m\delta_0 + \Delta u$ is supported on $\mathbf{B}_r$, we have by summation by parts

$$\sum_{x\in\mathbb{Z}^d} w(x)H(x) = \sum_{x\in\mathbf{B}_r} \Delta u(x)H(x) = \sum_{x\in\mathbf{B}_r} u(x)\Delta H(x) = 0. \quad \square$$

The next lemma bounds sums of $P = P_{y,k}$ over discrete spherical shells and discrete balls.

Lemma 2.3. *There is a dimensional constant C such that*

(a) $\displaystyle\sum_{x\in\mathbf{B}_{r+1}\setminus\mathbf{B}_r} P(x) \le Ck$ *for all* $r \le |y| + k$.

(b) $\displaystyle\left|\sum_{x\in\mathbf{B}_r} (P(x) - P(0))\right| \le Ck$ *for all* $r \le |y|$.

(c) $\displaystyle\left|\sum_{x\in\mathbf{B}_{|y|+k}} (P(x) - P(0))\right| \le Ck^2$.

Proof. Part (a) follows from Lemma 2.1: Take the worst shell, when $r = |y|$. Then the sum over x satisfying $|x - y| \le k$ and $|y| \le |x| \le |y| + 1$ is bounded by Lemma 2.1(a)

$$\int_0^k s^{2-d}s^{d-2}ds = k$$

(volume element on disk with thickness 1 and radius k in $\mathbb{Z}^{d-1}$ is $s^{d-2}ds$.) For the remaining portion of the shell, Lemma 2.1(b) has numerator $k(|y| + k - |y|) = k^2$, so that

$$\int_k^\infty k^2 s^{-d}s^{d-2}ds = k.$$

Next, for part (b), let w_r be as in Lemma 2.2. Since P is discrete harmonic in $\mathbf{B}_{|y|}$, we have for $r \le |y|$

$$\sum_{x\in\mathbb{Z}^d} w_r(x)(P(x) - P(0)) = 0.$$

Since w_r equals the indicator $\mathbf{1}_{\mathbf{B}_r}$ except on the annulus $\mathbf{B}_r \setminus \mathbf{B}_{r-c}$, and $|w_r| \le 1$, we obtain

$$\begin{aligned}\left|\sum_{x\in\mathbf{B}_r} (P(x) - P(0))\right| &\le \sum_{x\in\mathbf{B}_r\setminus\mathbf{B}_{r-c}} |w_r(x)|\,|P(x) - P(0)| \\ &\le \sum_{x\in\mathbf{B}_r\setminus\mathbf{B}_{r-c}} (P(x) + P(0)) \\ &\le Ck.\end{aligned}$$

In the last step we have used part (a) to bound the first term; the second term is bounded by Lemma 2.1(b), which says that $P(0) \le Ck/|y|^{d-1}$.

Part (c) follows by splitting the sum over $\mathbf{B}_{|y|+k}$ into k sums over spherical shells $\mathbf{B}_{|y|+j} \setminus \mathbf{B}_{|y|+j-1}$ for $j = 1, \dots, k$, each bounded by part (a), plus a sum over the ball $\mathbf{B}_{|y|}$, bounded by part (b). $\square$

Fix $\alpha > 0$, and consider the level set

$$U = \{x \in \mathcal{G} \mid g(x) > \alpha\}.$$

For $x \in \partial U$, let $p(x)$ be the probability that a Brownian motion started at the origin in $\mathcal{G}$ first exits U at x.

Lemma 2.4. *Choose α so that ∂U does not intersect $\mathbb{Z}^d$. For each $x \in \partial U$, the quantity $p(x)$ equals the directional derivative of $g/2d$ along the directed edge in U starting at x.*

Proof. We use a discrete form of the divergence theorem

$$\int_U \operatorname{div} V = \sum_{\partial U} \nu_U \cdot V. \tag{2.3}$$

where V is a vector-valued function on the grid, and the integral on the left is a one-dimensional integral over the grid. The dot product $\nu_U \cdot V$ is defined as $e_j \cdot V(x - 0e_j)$, where e_j is the unit vector pointing toward x along the unique incident edge in U. To define the divergence, for $z = x + te_j$, where $0 \le t < 1$ and $x \in \mathbb{Z}^d$, let

$$\operatorname{div} V(z) := \frac{\partial}{\partial x_j} e_j \cdot V(z) + \delta_x(z) \sum_{j=1}^{d} (e_j \cdot V(x + 0e_j) - e_j \cdot V(x - 0e_j)).$$

If f is a continuous function on U that is C^1 on each connected component of $U - \mathbb{Z}^d$, then the gradient of f is the vector-valued function

$$V = \nabla f = (\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_d)$$

with the convention that the entry $\partial f/\partial x_j$ is 0 if the segment is not pointing in the direction x_j. Note that ∇f may be discontinuous at points of $\mathbb{Z}^d$.

Let $G = -g/2d$, so that $\operatorname{div} \nabla G = \delta_0$. If u is grid harmonic on U, then $\operatorname{div} \nabla u = 0$ and

$$\operatorname{div} (u \nabla G - G \nabla u) = u(0)\delta_0.$$

Indeed, on each segment this is the same as $(uG' - u'G)' = u'G' - u'G' + uG'' - u''G = 0$ because u and G are linear on segments. At lattice points u and G are continuous, so the divergence operation commutes with the factors u and G and gives exactly one nonzero delta term, the one indicated.

Let $u(y)$ be the probability that Brownian motion on U started at y first exits U at x. Then $p(x) = u(0)$. Since u is grid-harmonic on U, we have $\operatorname{div} \nabla u = 0$ on U, hence by the divergence theorem

$$u(0) = \int_U \operatorname{div} (u \nabla G - G \nabla u) = \sum_{\partial U} u \, \nu_U \cdot \nabla G.$$

Since u vanishes on $\partial U \setminus \{x\}$, the only nonzero term in the sum on the right side is $\nu_U \cdot \nabla G(x)$. Since ∂U does not intersect $\mathbb{Z}^d$, this term equals the directional derivative of $g/2d$ along the directed edge in U starting at x. ☐

Next we establish some lower bounds for P.

Lemma 2.5. *There is a dimensional constant $c > 0$ such that*

(a) $P(0) \ge ck/|y|^{d-1}$.

(b) Let $k = 1$, and $z = (1 - \frac{2m}{|y|})y$ for $0 < m < |y|/2$. Then

$$\min_{x \in \mathbf{B}(z,m)} P(x) \geq c/m^{d-1}.$$

Proof. By the maximum principle, there is a dimensional constant $c > 0$ such that

$$P(x) \geq c(g(x-y) - a_d(k/2)^{2-d})$$

for $x \in B_{k/2}(y)$. In particular,

$$P(x) \geq ck^{2-d} \quad \text{for all } \; |x-y| \leq k/4$$

Now consider the region

$$U = \{x \in \mathcal{G} : g(x) > a_d s^{2-d}\}$$

where s is chosen so that $|s - (|y| - k/8)| < 1/2$ and all of the boundary points of U are non-lattice points. (A generic value of s in the given range will suffice.)

By (2.1), this set is within unit distance of the ball of radius $|y| - k/8$. Let $p(z)$ represent the probability that a Brownian motion on the grid starting from the origin first exits U at $z \in \partial U$. Thus

$$u(0) = \sum_{z \in \partial U} u(z)p(z) \tag{2.4}$$

for all grid harmonic functions u in U.

Take any boundary point of $z \in \partial U$. Take the nearest lattice point z^*. Let z_j be a coordinate of z largest in absolute value. Then $|z_j| \geq |z|/d$. The rate of change of $|x|^{2-d}$ in the jth direction near z has size $\geq 1/d|z|^{d-1}$, which is much larger than the error term $C|z|^{-d}$ in (2.1). It follows that on the segment in that direction, where the function $g(x) - a_d(|y| - k/8)^{2-d}$ changes sign, its derivative is bounded below by $1/2d|z|^{d-1}$. In other words, by Lemma 2.4, within distance 2 of every boundary point of $z \in \partial U$ there is a point $z' \in \partial U$ for which $p(z') \geq c/|y|^{d-1}$. There are at least ck^{d-1} such points in the ball $\mathbf{B}_{k/4}(y)$ where the lower bound for P was ck^{2-d}, so

$$P(0) \geq ck^{2-d}k^{d-1}/|y|^{d-1} = ck/|y|^{d-1}.$$

Next, the argument for Lemma 2.5(b) is nearly the same. We are only interested in $k = 1$. It is obvious that for points x within constant distance of y (and unit distance from the boundary at radius $|y| + 1$) the values of $P(x)$ are bounded below by a positive constant. We then bound $P((1 - 2m/|y|)y)$ from below using the same argument as above, but with Green's function for a ball of radius comparable to m. Finally, Harnack's inequality says that the values of $P(x)$ for x in the whole ball of size m around this point $(1 - 2m/|y|)y$ are comparable. □

3 Proofs of main lemmas

The proofs in this section make use of the martingale

$$M(t) = M_{y,k}(t) := \sum_{x \in A_{y,k}(t)} (P_{y,k}(x) - P_{y,k}(0))$$

where $P_{y,k}$ is the grid harmonic function defined in Section 2, and $A_{y,k}(t)$ is the modified internal DLA cluster in which particles are stopped if they exit Ω.

As in [7], we take the time parameter t to be real-valued: starting at each integer time n, a particle is released from the origin and performs Brownian motion on the grid $\mathcal{G}$ until reaching a point in $(\mathcal{G} \setminus \Omega) \cap (\mathbb{Z}^d \setminus A(n))$. By applying a deterministic time change to the Brownian motion we can ensure that this happens before time $n+1$, so only one particle is active at any given time. The choice of continuous time is convenient for applying the martingale representation theorem, but it is not essential for the argument: One can embed the discrete time martingale $M(n)$ into a Brownian motion using Skorohod's theorem, and estimate the elapsed time $(M(n+1)-M(n))^2$.

We view $A_{y,k}(t)$ as a multiset: points on the boundary of Ω where many stopped particles accumulate are counted with multiplicity in the sum defining M. In addition to these stopped particles, the set $A_{y,k}(t)$ contains one more point, the location of the currently active particle performing Brownian motion on $\mathcal{G}$.

Recall that $P = P_{y,k}$ and $M = M_{y,k}$ depend on k, which is the distance from y to the boundary of Ω. We will choose $k=1$ for the proof of Lemma 1.1, and $k = a\ell$ for a small constant a in the proof of Lemma 1.2. Taking $k>1$ is one of the main differences from the argument in [7]. The factor of k in the lower bound of Lemma 2.5(a) results in the bound $M(T_1) \le -k^2$ on the event that y is ℓ-late, and consequently the weaker hypothesis $\ell \ge C_1((\log T)m)^{1/3}$ suffices at the end of the proof of Lemma 1.2 (compare to [7, Lemma 13] where the power was $1/2$ instead of $1/3$).

Proof of Lemma 1.1. The proof follows the same method as [7, Lemma 12]. We highlight here the changes needed in dimensions $d \ge 3$. We use the discrete harmonic function $P(x)$ with $k=1$. Fix $z \in \mathbb{Z}^d$, let $r = |z|$ and $y = (r+2m)z/r$. Let

$$T_1 = \lceil \omega_d (r-m)^d \rceil$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$. If z is m-early, then $z \in A(T_1)$; in particuar, this means that $r \ge m$, so that $r+m$, $r+2m$ are all comparable to r. Since $k=1$, we have by Lemmas 2.1(c) and 2.5(a)

$$P(0) \approx 1/r^{d-1},$$

where $\approx$ denotes equivalence up to a constant factor depending only on d.

First we control the quadratic variation

$$S(t) = \lim_{\substack{0=t_0 \le \ldots \le t_N = t \\ \max(t_i - t_{i-1}) \to 0}} \sum_{i=1}^{N} (M(t_i) - M(t_{i-1}))^2$$

on the event $\mathcal{E}_{m+1}[T]^c$ that there are no $(m+1)$-early points by time T. As in [7, Lemma 9], there are independent standard Brownian motions $\widetilde{B}^0, \widetilde{B}^1, \ldots$ such that each increment $(S(n+1)-S(n))\mathbf{1}_{\mathcal{E}_{m+1}[T]^c}$ is bounded above by the first exit time of $\widetilde{B}^n$ from the interval $[-a_n, b_n]$, where

$$\begin{aligned} a_n &= P(0) \approx \frac{1}{r^{d-1}} \\ b_n &= \max_{|x| \le (n/\omega_d)^{1/d} + m + 1} P(x) \le \frac{1}{[r+2m-((n/\omega_d)^{1/d}+m+1)]^{d-1}}. \end{aligned}$$

Here we have used Lemma 2.1(b) in the bound on b_n.

Unlike in dimension 2, we will use the large deviation bound for Brownian exit times [7, Lemma 5] with $\lambda = cm^2$ instead of $\lambda = 1$. Here c is a constant depending only on d. Note that $b_n \le 1/m^{d-1}$, for all $n \le T_1$, so this is a valid choice of λ in all dimensions

$d \geq 3$ (that is, the hypothesis $\sqrt{\lambda}(a_n + b_n) \leq 3$ of [7, Lemma 5] holds). We obtain

$$\begin{aligned}\log \mathbb{E}\left[e^{\lambda S(T_1)} 1_{\mathcal{E}_{m+1}[T]^c}\right] &\leq \sum_{n=1}^{T_1} 10\lambda a_n b_n \\ &\leq \int_1^{T_1} \lambda \frac{C}{r^{d-1}} \frac{1}{(r+m-(n/\omega_d)^{1/d}-1)^{d-1}} dn \\ &\leq \int_1^{r} \lambda \frac{C}{r^{d-1}} \frac{1}{(r+m-j-1)^{d-1}} j^{d-1} dj \\ &\leq \int_1^{r} \frac{C\lambda \, dj}{(r+m-j-1)^{d-1}} \leq C\lambda/m^{d-2}.\end{aligned}$$

Note that the last step uses $d \geq 3$. Taking $\lambda = cm^2$ for small enough c we obtain

$$\mathbb{E}\left[e^{cm^2 S(T_1)} 1_{\mathcal{E}_{m+1}[T]^c}\right] \leq e^{m^2/m^{d-2}} \leq e^m.$$

Therefore, by Markov's inequality,

$$\mathbb{P}(\{S(T_1) > 1/c\} \cap \mathcal{E}_{m+1}[T]^c) \leq e^{m-m^2} < T^{-20\gamma}. \tag{3.1}$$

Fix $z \in \mathbf{B}_T$ and $t \in \{1, \ldots, T\}$, and let $Q_{z,t}$ be the event that $z \in A(t) \setminus A(t-1)$ and z is m-early and no point of $A(t-1)$ is m-early. This event is empty unless $(t/\omega_d)^{1/d} + m \leq |z| \leq (t/\omega_d)^{1/d} + m + 1$; in particular, the first inequality implies $t \leq T_1$. We will bound from below the martingale $M(t)$ on the event $Q_{z,t} \cap \mathcal{L}_\ell[T]^c$. With no ℓ-late point, the ball $\mathbf{B}_{r-m-\ell-1}$ is entirely filled by time t. Lemma 2.3(b) shows that the sites in this ball contribute at most a constant to $M(t)$ (recall that $k = 1$). The thin tentacle estimate [7, Lemma A] says that except for an event of probability e^{-cm^2}, there are order m^d sites in $A(t)$ within the ball $\mathbf{B}(z, m)$. By Lemma 2.5(b), P is bounded below by c/m^{d-1} on this ball, so these sites taken together contribute order m to $M(t)$. Each of the remaining terms in the sum defining $M(t)$ is bounded below by $-P(0)$, and there are at most ℓr^{d-1} sites in $A(t) \setminus \mathbf{B}_{r-m-\ell-1}$. So these terms contribute at least

$$-\ell r^{d-1}(1/r^{d-1}) = -\ell \geq -m/C$$

which cannot overcome the order m term. Thus

$$\mathbb{P}(Q_{z,t} \cap \{M_\zeta(t) < m/C\} \cap \mathcal{L}_\ell[t]^c) < e^{-cm^2}. \tag{3.2}$$

We conclude that

$$\begin{aligned}\mathbb{P}(Q_{z,t} \cap \mathcal{L}_\ell[T]^c) \leq \mathbb{P}(Q_{z,t} \cap \{S(t) > 1/c\}) \\ + \mathbb{P}(Q_{z,t} \cap \{M(t) < m/C\} \cap \mathcal{L}_\ell[t]^c) \\ + \mathbb{P}(\{S(t) \leq 1/c\} \cap \{M(t) \geq m/C\}).\end{aligned}$$

The first two terms are bounded by (3.1) and (3.2). Since $M(t) = B(S(t))$ for a standard Brownian motion B, the final term is bounded by

$$\mathbb{P}\left\{\sup_{0 \leq s \leq 1/c} B(s) \geq m/C\right\} < e^{-c(m/C)^2/2} < T^{-20\gamma}. \quad \square$$

Proof of Lemma 1.2. Fix $y \in \mathbb{Z}^d$, and let $L[y]$ be the event that y is ℓ-late. Set $k = a\ell$ in the definition of $P = P_{y,k}$. Here $a > 0$ is a small dimensional constant chosen below. Note that the hypotheses on m and ℓ imply that ℓ is at least of order $\sqrt{\log T}$; after choosing a, we take the constant C_1 appearing in the statement of the lemma large enough so that $k^2 > 1000\gamma \log T$.

Case 1. $1 \le |y| \le 2k$. Then $P(0) \approx 1/|y|^{d-2}$. Let

$$T_1 = \lfloor \omega_d (|y| + \ell)^d \rfloor$$

With $a_n = P(0)$ and $b_n = 1$, we have $S(n+1) - S(n) \le \tau_n$, where τ_n is the first exit time of the Brownian motion $\widetilde{B}^n$ from the interval $[-a_n, b_n]$. (Note that because we take $b_n = 1$, the indicator $\mathbf{1}_{\mathcal{E}_{m+1}[T]^c}$ is not needed here as it was in the proof of Lemma 1.1.) We obtain

$$\log \mathbb{E} e^{S(T_1)} \le \sum_{t=1}^{T_1} \log \mathbb{E} e^{\tau_n} \le T_1 P(0).$$

Let $Q = T_1 P(0)$. By Markov's inequality, $\mathbb{P}(S(T_1) > 2Q) \le e^{-Q}$.

On the event $L[y]$, the site y is still not occupied at time T_1. Accordingly, the largest $M(T_1)$ can be is if $A_{y,k}(T_1)$ fills the whole ball $\mathbf{B}_{|y|+k}$ (except for y), and then the rest of the particles will have to collect on the boundary where P is zero. The contribution from $\mathbf{B}_{|y|+k}$ is at most Ck^2 by Lemma 2.3(c). The number of particles stopped on the boundary is at least

$$T_1 - 2\omega_d (|y| + k)^d \ge \frac{T_1}{2}.$$

Therefore, on the event $L[y]$ we have

$$M(T_1) \le Ck^2 - \frac{T_1}{2} P(0). \tag{3.3}$$

Note that $Q := T_1 P(0) \approx (|y| + \ell)^d / |y|^{d-2} \ge \ell^d / (k/2)^{d-2}$, so by taking $a = k/\ell$ sufficiently small, we can ensure that the right side of (3.3) is at most $-Q/4$. Also, $Q \ge \ell^2 \ge 1000\gamma \log T$. Since $M(t) = B(S(t))$ for a standard Brownian motion B, we conclude that

$$\begin{aligned} \mathbb{P}(L[y]) &\le \mathbb{P}(S(T_1) > 2Q) + \mathbb{P}\left\{ \inf_{0 \le s \le 2Q} B(s) \le -Q/4 \right\} \\ &\le e^{-Q} + e^{-(Q/4)^2/4Q} \\ &< T^{-20\gamma}. \end{aligned}$$

Case 2. $|y| \ge 2k$. Then by Lemma 2.1(c) with $r = 1$, and Lemma 2.5(a), we have $P(0) \approx k/s^{d-1}$. First take

$$T_0 = \lfloor \omega_d (|y| + k - 3m)^d \rfloor$$

(or $T_0 = 0$ if $|y| + k - 3m \le 0$). As in the previous lemma (but taking $\lambda = 1$ instead of $\lambda = cm^2$) we have

$$\log \mathbb{E} \left[e^{S(T_0)} 1_{\mathcal{E}_m[T]^c} \right] \le C \frac{k}{|y|^{d-1}} \int_0^{T_0} \frac{dn}{\left(|y| + k - (n/\omega_d)^{1/d} \right)^{d-1}} \le Ck/m^{d-2}.$$

Since $d \ge 3$ and $m \ge k/a$, the right side is $\le C$. By Markov's inequality,

$$\mathbb{P}(\{S(T_0) > C + k^2\} \cap \mathcal{E}_m[T]^c) < e^{-k^2} < T^{-20\gamma}.$$

Now since

$$(T_1 - T_0) P(0) \approx m|y|^{d-1} (k/|y|^{d-1}) = km$$

we have

$$\log \mathbb{E} e^{S(T_1) - S(T_0)} \le Ckm.$$

Thus (since $km \geq k^2$)

$$\mathbb{P}(\{S(T_1) > 2Ckm\} \cap \mathcal{E}_m[T]^c) < 2T^{-20\gamma}. \tag{3.4}$$

As in case 1, the martingale $M(T_1)$ is largest if the ball $\mathbf{B}_{|y|+k}$ is completely filled, and in that case the total contribution of sites in this ball is at most Ck^2. On the event $L[y]$, the number of particles stopped on the boundary of Ω at time T_1 is at least

$$T_1 - \#\mathbf{B}_{|y|+k} \geq \omega_d((|y|+\ell)^d - (|y|+k+C)^d) \approx \ell |y|^{d-1}.$$

Each such particle contributes $-P(0) \approx -k/|y|^{d-1}$ to $M(T_1)$, for a total contribution of order $-k\ell = -k^2/a$. Taking a sufficiently small we obtain $M(T_1) \leq Ck^2 - k^2/a \leq -k^2$. We conclude that

$$\mathbb{P}(L[y] \cap \mathcal{E}_m[T]^c) \leq \mathbb{P}(\{S(T_1) > 2Ckm\} \cap \mathcal{E}_m[T]^c) + \\ + \mathbb{P}(\{S(T_1) \leq 2Ckm\} \cap \{M(T_1) \leq -k^2\}).$$

The first term is bounded above by (3.4), and the second term is bounded above by

$$\mathbb{P}\left\{ \inf_{s \leq 2Ckm} B(s) \leq -k^2 \right\} \leq e^{-k^4/4Ckm} < T^{-20\gamma}.$$

Hence $\mathbb{P}(L[y] \cap \mathcal{E}_m[T]^c) < 3T^{-20\gamma}$. Since $\mathcal{L}_\ell[T]$ is the union of the events $L[y]$ for $y \in \mathcal{B} := \mathbf{B}_{(T/\omega_d)^{1/d}-\ell}$, summing over $y \in \mathcal{B}$ completes the proof. □

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