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On a class of martingale problems on Banach spaces

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Abstract

We introduce the local martingale problem associated to semilinear stochastic evolution equations driven by a cylindrical Wiener process and establish a one-to-one correspondence between solutions of the martingale problem and (analytically) weak solutions of the stochastic equation. We also prove that the solutions of well-posed equations are strong Markov processes. We apply our results to semilinear stochastic equations with additive noise where the semilinear term is merely measurable and to stochastic reaction-diffusion equations with Hölder continuous multiplicative noise.

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1 Introduction

One of the most important tools in the study of stochastic differential equations is the theory of associated martingale problems of Stroock and Varadhan [38]. At the heart of their approach is the equivalence between solutions of stochastic differential equations (i.e. stochastic processes) and solutions of the associated martingale problem (i.e. probability measures on a function space).

This equivalence is helpful in several ways. First, it can be used to prove *existence of solutions* to stochastic differential equations by means of approximation and tightness arguments. Second, it plays an important role in proving *uniqueness of solutions* using techniques from semigroup theory or partial differential equations. Last but not least, the approach of Stroock and Varadhan yields, given existence and uniqueness of solutions, the strong Markov property of the solutions. This plays an important role in the study of further properties of the solutions, e.g. their asymptotic behavior.

In this article, we set up a theory of (local) martingale problems for stochastic evolution equations

$$dX(t) = \Big[AX(t) + F(X(t))\Big]dt + G(X(t))dW_H(t)\,, \tag{1.1}$$

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on a separable Banach space E. Here, A is the generator of a strongly continuous semigroup S on E, W_H is an H-cylindrical Wiener process where H is a separable Hilbert space and the nonlinearities $F : E \to E$ and $G : E \to \mathscr{L}(H, E)$ satisfy suitable measurability and (local) boundedness assumptions. In fact, we shall consider a slightly more general situation and allow the nonlinearities to take values in a larger Banach space $\tilde{E}$, resp. $\mathscr{L}(H, \tilde{E})$. We will make our assumptions precise in Section 3.

Martingale problems for equations of this form on 2-smoothable Banach spaces were studied by Ondreját [34]. The usual solution concept for equations of the form (1.1) is that of a mild solution which involves a stochastic convolution term. We note that to assure that this term is well-defined, one has to impose additional assumptions on the Banach space (typically geometric assumptions such as the UMD property or 2-smoothability) and/or the coefficients. This poses problems when extending the theory to general Banach spaces. Here, we overcome these problems by basing our theory on (analytically) weak solutions rather than on mild solutions.

Our approach does not only allow us to consider general Banach spaces, it also allows us to work without additional technical assumptions (such as the J-property in [34]) to ensure stochastic integrability of the occurring processes and to impose only minimal assumptions on the coefficients.

Under these minimal assumptions, we introduce the local martingale problem associated to equation (1.1) in Section 3 and establish a one-to-one correspondence between solutions of the local martingale problem and solutions of the stochastic evolution equation in Theorem 3.6. In Theorem 4.2 we prove, given existence and uniqueness of solutions, the strong Markov property for solutions of (1.1), using some abstract results about local martingale problems presented in Section 2.

Thus, Sections 2 – 4 contain the abstract theory of martingale problems on Banach spaces. In Sections 5 and 6 we discuss related results, which we believe are helpful to apply the theory.

In Section 5 we extend the Yamada-Watanabe theory [39] to the setting of Banach spaces and prove that pathwise uniqueness implies uniqueness in law (this is the uniqueness concept used in the abstract theory above) and strong existence of solutions. As in finite dimensions, pathwise uniqueness can be much easier verified than uniqueness in law in certain situations, in particular for equations with (locally) Lipschitz continuous coefficients.

In Section 6 we show that (analytically) weak and mild solutions coincide if either the coefficient G is constant, i.e. in equations with additive noise, or if the Banach space E is a UMD space. Working with mild solutions is especially helpful to prove existence of solutions, as the standard approach via approximation and tightness often uses the factorization method of [7] as a tool, which, in turn, requires a Banach space valued stochastic integral. Here, we use the Banach space valued Wiener integral, see [32], in the case of constant G and the theory of integration in UMD Banach spaces [30] in the second case. Note that this is the only section where we make use of a stochastic integral, all our abstract results do not depend on geometric assumptions on E.

Let us close this introduction by discussing applications of our theory to concrete stochastic evolution equations. Techniques inspired by martingale problems can be found frequently in the literature on infinite dimensional stochastic equations even though, more often than not, a martingale problem is not used directly. This is most apparent in the term *martingale solution* which in infinite dimensions does not refer to solutions of the martingale problem but is used synonymously for stochastically weak solutions (thus for stochastic processes). Such solutions were constructed, for example, in [6, 12, 2, 41]. Concerning uniqueness, several authors [6, 11, 40] have proved uniqueness in law for certain equations by using partial differential equations on Hilbert spaces.

Naturally, the results contained in this article can be used to prove, given well-posedness, the strong Markov property for solutions of stochastic evolution equations in arbitrary separable Banach spaces. However, the results obtained here can also be used to *establish* well-posedness of a given equation. Naturally, the proof of well-posedness of a stochastic evolution equation requires additional arguments which depend on the equation in question. Thus, the full proofs of our applications to stochastic evolution equations will be given elsewhere [20, 19]. We will, however, give a rough sketch in Section 7 and discuss how the results of this article enter the arguments.

2 Markov processes and local Martingale Problems

In this section (E, d) is a complete, separable metric space. We denote the Borel σ-algebra of E by $\mathscr{B}(E)$. The spaces of scalar-valued measurable, bounded measurable, continuous and bounded continuous functions will be denoted by $B(E)$, $B_b(E)$, $C(E)$ and $C_b(E)$ respectively. $\mathscr{P}(E)$ denotes the set of all probability measures on $(E, \mathscr{B}(E))$. For $x \in E$, the Dirac measure in x is denoted by δ_x.

By $C([0,\infty);E)$ we denote the space of all continuous, E-valued functions. The elements of $C([0,\infty);E)$ will be denoted by bold lower case letters: $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Endowed with the metric $\boldsymbol{\delta}$, defined by

$$\boldsymbol{\delta}(\mathbf{x},\mathbf{y}) := \sum_{k=1}^{\infty} 2^{-k} \sup_{t\in[0,k]} d(\mathbf{x}_t, \mathbf{y}_t) \wedge 1,$$

$C([0,\infty);E)$ is a complete, separable metric space in its own right. We denote its Borel σ-algebra by $\mathscr{B}$. It is well-known that $\mathscr{B} = \sigma(\mathbf{x}_s \,:\, s \geq 0)$, see [16, Lemma 16.1]. Here, in slight abuse of notation, we have identified $\mathbf{x}_s$ with the E-valued map $\mathbf{x} \mapsto \mathbf{x}_s$. We shall do so in what follows without further notice. The filtration generated by these 'coordinate mappings' is denoted by $\mathbb{B} := (\mathscr{B}_t)_{t\geq 0}$, i.e. $\mathscr{B}_t := \sigma(\mathbf{x}_s \,:\, s \leq t)$.

The space of probability measures on the Borel σ-algebra of $C([0,\infty);E)$ will be denoted by $\mathscr{P}(C([0,\infty);E)$. It will always be topologized by the *weak topology*, i.e. the coarsest topology for which for all bounded continuous function Φ on $C([0,\infty);E)$ the map $\mathbf{P} \mapsto \int \Phi \, d\mathbf{P}$ is continuous. It is well known that this topology is metrizable through a complete, separable metric, see [36, Section II.6], i.e. $\mathscr{P}(C([0,\infty);E))$ is a Polish space.

A probability measure $\mathbf{P}$ on $(C([0,\infty);E), \mathscr{B})$ is called a *Markov measure* if the coordinate process $(\mathbf{x}_t)_{t\geq 0}$ defined on $(C([0,\infty);E), \mathscr{B}, \mathbf{P})$ is a Markov process with respect to $\mathbb{B}$, i.e. for all $f \in B_b(E)$ and $s,t \geq 0$ we have

$$\mathbb{E}\big[f(\mathbf{x}_{t+s})\big|\mathscr{B}_t\big] = \mathbb{E}\big[f(\mathbf{x}_{t+s})\big|\mathbf{x}_t\big] \quad \mathbf{P} - a.e.,$$

where $\mathbb{E}$ denotes (conditional) expectation with respect to $\mathbf{P}$. If this equation also holds whenever t is replaced with a $\mathbb{B}$-stopping time τ which is almost surely finite, i.e. the coordinate process is a strong Markov process with respect to $\mathbb{B}$, then $\mathbf{P}$ is called a *strong Markov measure*. Here, as usual, $\mathscr{B}_\tau$ is the σ-algebra

$$\mathscr{B}_\tau := \{A \in \mathscr{B} \,:\, A \cap \{\tau \leq t\} \in \mathscr{B}_t \text{ for all}\, t \geq 0\}.$$

A *transition semigroup* is a family $\mathscr{T} := (\mathscr{T}(t))_{t\geq 0}$ of positive contractions on $B_b(E)$ such that

1. $\mathscr{T}$ is a semigroup, i.e. $\mathscr{T}(0) = I$ and $\mathscr{T}(t+s) = \mathscr{T}(t)\mathscr{T}(s)$ for all $t,s \geq 0$.

2. Every operator $\mathscr{T}(t)$ is associated with a *Markovian kernel*, i.e a map $p_t : E \times \mathscr{B}(E) \to [0,1]$ such that (i) $p_t(x,\cdot) \in \mathscr{P}(E)$ for all $x \in E$ and (ii) $p_t(\cdot, A) \in B_b(E)$ for all $A \in \mathscr{B}(E)$. That $\mathscr{T}(t)$ is associated with p_t means that $\mathscr{T}(t)f(x) = \int_E f(y)\,p_t(x,dy)$ for all $f \in B_b(E)$.

The kernels p_t themselves are referred to as *transition functions* or *transition probabilities*. The semigroup property above is equivalent with the *Chapman-Kolmogorov equations*.

A probability measure $\mathbf{P}$ on $C([0,\infty);E)$ is called *Markov measure with transition semigroup* $\mathscr{T}$ if for all $f \in B_b(E)$ and $s,t \geq 0$ we have

$$\mathbb{E}\big[f(\mathbf{x}_{t+s})\big|\mathscr{B}_t\big] = \mathbb{E}\big[f(\mathbf{x}_{t+s})\big|\mathbf{x}_t\big] = \big[\mathscr{T}(s)f\big](\mathbf{x}_t) \quad \mathbf{P} - a.e.$$

If this equation also holds whenever t is replaced with a $\mathbf{P}$-a.s. finite $\mathbb{B}$-stopping time τ, then $\mathbf{P}$ is called a *strong Markov measure with transition semigroup* $\mathscr{T}$.

The connection between martingale problems and Markovian measures is well established, see [9, Chapter 4]. However, if we want to treat stochastic evolution equations on Banach spaces, we have to consider *local* martingale problems rather than martingale problems.

Definition 2.1. *An* admissible operator *is a map $\mathscr{L}$, defined on a subset $D(\mathscr{L}) \subset C(E)$ and taking values in $B(E)$ such that for all $f \in D(\mathscr{L})$ the function $\mathscr{L}f$ is bounded on compact subsets of E.*

Given an admissible operator $\mathscr{L}$, a probability measure $\mathbf{P}$ on $C([0,\infty);E)$ is said to solve the local martingale problem for $\mathscr{L}$ *if for every $f \in D(\mathscr{L})$ the process $\mathbf{M}^f$ defined by*

$$\big[\mathbf{M}^f(\mathbf{x})\big](t) := f(\mathbf{x}_t) - f(\mathbf{x}_0) - \int_0^t \mathscr{L}f(\mathbf{x}_s)\,ds$$

is a local martingale under $\mathbf{P}$. This of course means that there exists a sequence τ_n, which may depend on f, of $\mathbb{B}$-stopping times with $\tau_n \uparrow \infty$ $\mathbf{P}$-almost surely such that the stopped processes $\mathbf{M}^f_{\tau_n}$, defined by $\mathbf{M}^f_{\tau_n}(t) := \mathbf{M}^f(t \wedge \tau_n)$, are martingales for all $n \in \mathbb{N}$.

If an initial distribution $\mu \in \mathscr{P}(E)$ is specified, we say that $\mathbf{P}$ is a solution to the local martingale problem for $(\mathscr{L},\mu)$ *to indicate that in addition to being a solution to the local martingale problem for $\mathscr{L}$, the measure $\mathbf{P}$ satisfies $\mathbf{P}(\mathbf{x}_0 \in \Gamma) = \mu(\Gamma)$ for all $\Gamma \in \mathscr{B}(E)$, i.e. under $\mathbf{P}$ the random variable $\mathbf{x}_0$ has distribution μ.*

We note that by the continuity of $t \mapsto \mathbf{x}_t$ and since $\mathscr{L}f$ is bounded on compact subsets of E, the process $\mathbf{M}^f$ is well-defined. In fact, since f is a continuous function, it follows that $\mathbf{M}^f$ is a continuous process.

The proofs of our results in Section 4 are based on the following theorem.

Theorem 2.2. *Let $\mathscr{L}$ be admissible. Suppose that for every $\mu \in \mathscr{P}(E)$ any two solutions $\mathbf{P},\mathbf{Q}$ of the local martingale problem for $(\mathscr{L},\mu)$ have the same one-dimensional distributions, i.e. for all $t \geq 0$ we have*

$$\mathbf{P}(\mathbf{x}_t \in \Gamma) = \mathbf{Q}(\mathbf{x}_t \in \Gamma) \quad \forall\,\Gamma \in \mathscr{B}(E)\,.$$

Then

1. *Every solution of the local martingale problem for $\mathscr{L}$ is a strong Markov measure.*
2. *For every $\mu \in \mathscr{P}(E)$, there is at most one solution to the local martingale problem for $(\mathscr{L},\mu)$.*

If in addition to the uniqueness assumption above for every $x \in E$ there exists a solution $\mathbf{P}_x$ to the local martingale problem for $(\mathscr{L}, \delta_x)$ and if the map $x \mapsto \mathbf{P}_x(B)$ is Borel measurable for all $B \in \mathscr{B}$, then

(3) *For every $\mu \in \mathscr{P}(E)$, there exists a solution $\mathbf{P}_\mu$ of the local martingale problem for $(\mathscr{L}, \mu)$.*

(4) *Define the operator $\mathscr{T}(t)$ by $\mathscr{T}(t)f(x) := \int f(\mathbf{x}_t)\, d\mathbf{P}_x$ for $f \in B_b(E)$. Then every solution $\mathbf{P}$ of the local martingale problem for $\mathscr{L}$ is a strong Markov measure with transition semigroup $\mathscr{T} := (\mathscr{T}(t))_{t\geq 0}$.*

Proof. This Theorem is a generalization of [9, Theorem 4.4.2] to local martingale problems. Hence, we have the added difficulty that in the definition of "solution of the local martingale problem" a sequence of stopping times appears. We only give the proof of statement (1), the other statements are derived following the proofs of the corresponding statements in [9, Theorem 4.4.2] with similar changes due to the presence of stopping times.

Let $\mathbf{P}$ be a solution of the local martingale problem for $(\mathscr{L}, \mu)$. We denote (conditional) expectation with respect to $\mathbf{P}$ by $\mathbb{E}$. Let ρ be a stopping time with $\rho < \infty$ almost surely and define the mappings Θ_ρ and $\Psi_\rho : C([0,\infty); E) \to C([0,\infty); E)$ by

$$(\Theta_\rho \mathbf{x})(t) := \mathbf{x}(t + \rho(\mathbf{x})) \quad \text{and} \quad (\Psi_\rho \mathbf{x})(t) := \mathbf{x}((t - \rho(\mathbf{x}))^+)\,.$$

Then Θ_ρ and Ψ_ρ are measurable mappings with $\Psi_\rho\Theta_\rho\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in C([0,\infty); E)$.

Now fix $A \in \mathscr{B}_\rho$ with $\mathbf{P}(A) > 0$ and define the measures $\mathbf{P}_1, \mathbf{P}_2$ on $C([0,\infty); E)$ by

$$\mathbf{P}_1(B) := \frac{\mathbb{E}\big[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{\Theta_\rho^{-1}B} | \mathscr{B}_\rho]\big]}{\mathbf{P}(A)} \quad \text{and} \quad \mathbf{P}_2(B) := \frac{\mathbb{E}\big[\mathbb{1}_A \mathbb{E}[\mathbb{1}_{\Theta_\rho^{-1}B} | \mathbf{x}(\rho)]\big]}{\mathbf{P}(A)}\,.$$

We note that under $\mathbf{P}_1$ and $\mathbf{P}_2$ the distribution of $\mathbf{x}(0)$ are identical, namely for $\Gamma \in \mathscr{B}(E)$ we have

$$\mathbf{P}_1(\mathbf{x}(0) \in \Gamma) = \mathbf{P}_2(\mathbf{x}(0) \in \Gamma) = \mathbf{P}(\mathbf{x}(\rho) \in \Gamma | A)\,.$$

Hence, if we prove that $\mathbf{P}_1$ and $\mathbf{P}_2$ solve the local martingale problem associated with $\mathscr{L}$, we can conclude from our assumption that $\mathbf{P}_1$ and $\mathbf{P}_2$ have the same one-dimensional distributions. This will then imply that for $t > 0$ and $\Gamma \in \mathscr{B}(E)$, we have

$$\begin{aligned} & \mathbf{P}_1(\mathbf{x}(t) \in \Gamma) = \mathbf{P}(A)^{-1}\mathbb{E}\big[\mathbb{1}_A \mathbb{E}[\mathbf{x}(t+\rho) \in \Gamma | \mathscr{B}_\rho]\big] \\ = \quad & \mathbf{P}_2(\mathbf{x}(t) \in \Gamma) = \mathbf{P}(A)^{-1}\mathbb{E}\big[\mathbb{1}_A \mathbb{E}[\mathbf{x}(t+\rho) \in \Gamma | \mathbf{x}(\rho)]\big]\,. \end{aligned}$$

Multiplying with $\mathbf{P}(A)$ and observing that A with $\mathbf{P}(A) > 0$ was arbitrary, it follows that $\mathbb{E}[\mathbf{x}(t+\rho) \in \Gamma | \mathscr{B}_\rho] = \mathbb{E}[\mathbf{x}(t+\rho) \in \Gamma | \mathbf{x}(\rho)]$. Since t, ρ and Γ were arbitrary, this proves that $(\mathbf{x}(t))_{t\geq 0}$ is a strong Markov process under $\mathbf{P}$.

It remains to prove that $\mathbf{P}_1$ and $\mathbf{P}_2$ solve the local martingale problem associated with $\mathscr{L}$. Fix $f \in D(\mathscr{L})$. Since $\mathbf{P}$ solves the local martingale problem, there exists a sequence τ_n of stopping times with $\tau_n \to \infty$ almost everywhere with respect to $\mathbf{P}$ such that $\mathbf{M}^f_{\tau_n}$ is a martingale under $\mathbf{P}$. We put $\sigma_n := \tau_n \circ \Psi_\rho$. Note that $\{\sigma_n \leq t\} = \Psi_\rho^{-1}\{\tau_n \leq t\} \in \mathscr{B}_t$, since τ_n is a stopping time and since $\Psi_\rho^{-1}A \in \mathscr{B}_t$ for all $A \in \mathscr{B}_t$, as is easy to see. Hence σ_n is a stopping time. Since $\Psi_\rho\Theta_\rho\mathbf{x} = \mathbf{x}$, it follows from the definition of $\mathbf{P}_1$ and $\mathbf{P}_2$ that $\sigma_n \uparrow \infty$ almost surely with respect to $\mathbf{P}_1$ and $\mathbf{P}_2$.

Now fix $t > s$ and $C \in \mathscr{B}_s$ and observe that

$$\xi(\mathbf{x}) := \Big[\big(\mathbf{M}^f_{\sigma_n}(t) - \mathbf{M}^f_{\sigma_n}(s)\big)\mathbb{1}_C\Big](\Theta_\rho\mathbf{x}) = \Big[\big(\mathbf{M}^f_{\tau_n}(t+\rho) - \mathbf{M}^f_{\tau_n}(s+\rho)\big)\mathbb{1}_{\Theta_\rho^{-1}C}\Big](\mathbf{x})$$

where $\Theta_\rho^{-1}C \in \mathscr{B}_{s+\rho}$. Since $\mathbf{M}^f_{\tau_n}$ is a continuous $\mathbf{P}$-martingale, it follows from the optional sampling theorem that $\mathbb{E}[\xi|\mathscr{B}_\rho] = 0$, and hence, since $\sigma(\mathbf{x}(\rho)) \subset \mathscr{B}_\rho$, also $\mathbb{E}[\xi|\mathbf{x}(\rho)] = 0$. Recalling the definition of $\mathbf{P}_1$ and $\mathbf{P}_2$, we see that that $\mathbf{M}^f_{\sigma_n}$ is a martingale under $\mathbf{P}_1$ and $\mathbf{P}_2$. □

Definition 2.3. *Let $\mathscr{L}$ be an admissible operator. We say that the local martingale problem for $\mathscr{L}$ is* well-posed *if for every $x \in E$, there exists a unique solution $\mathbf{P}_x$ of the local martingale problem for $(\mathscr{L}, \delta_x)$.*

We say that the martingale problem for $\mathscr{L}$ is completely well-posed*, if (i) for every $\mu \in \mathscr{P}(E)$ there exists a unique solution $\mathbf{P}_\mu$ of the local martingale problem for $(\mathscr{L}, \mu)$ and (ii) the map $x \mapsto \mathbf{P}_x(B)$ is measurable for every $B \in \mathscr{B}$.*

In the case of uniqueness, we will use the notation $\mathbf{P}_x$ resp. $\mathbf{P}_\mu$ for the solution of the local martingale problem for $(\mathscr{L}, \delta_x)$, resp. $(\mathscr{L}, \mu)$.

In Theorem 4.2, we will prove that if the martingale problem for $\mathscr{L}$ is well-posed, then it is already completely well-posed. Thus, we obtain the measurability of the map $x \mapsto \mathbf{P}_x$ and existence and uniqueness of solutions for arbitrary initial distributions μ for free.

We note that by (2) of Theorem 2.2, the uniqueness assumption in the definition of 'completely well-posed' can be weakened to uniqueness of one-dimensional marginals. Similarly, by (3) of Theorem 2.2, in the definition of 'completely well-posed' it suffices to assume existence of solutions only for degenerate initial distributions δ_x, for all $x \in E$.

By part (4) of Theorem 2.2, if the local martingale problem for $\mathscr{L}$ is completely well-posed, then there exists a transition semigroup $\mathscr{T}$ such that every solution $\mathbf{P}_\mu$ is a strong Markov measure with transition semigroup $\mathscr{T}$. This semigroup $\mathscr{T}$ is uniquely determined by $\mathscr{L}$ and will be called the *associated semigroup*.

3 Stochastic differential equations and the associated local martingale problem

We now turn our attention to the stochastic evolution equation (1.1). In order to stress the dependence on the coefficients, we will also refer to equation (1.1) as equation $[A, F, G]$. The following are our standing hypotheses on the coefficients and will be assumed in the rest of this paper.

Hypothesis 3.1. *$\tilde{E}$ is a separable Banach space and A generates a strongly continuous semigroup $S := (S(t))_{t\geq 0} = (S_t)_{t\geq 0}$ on $\tilde{E}$. H is a separable Hilbert space and W_H is an H-cylindrical Wiener process. E is a separable Banach space such that $D(A) \subset E \subset \tilde{E}$ with continuous and dense embeddings. Throughout, all Banach spaces are real. Furthermore,*

1. *$F : E \to \tilde{E}$ is strongly measurable and bounded on bounded subsets of E;*
2. *$G : E \to \mathscr{L}(H, \tilde{E})$ is H-strongly measurable, i.e. $Gh : E \to \tilde{E}$ is strongly measurable for all $h \in H$, and G is bounded on bounded subsets of E.*

Example 3.2. *Let us describe typical examples in which Hypothesis 3.1 is satisfied.*

In the easiest example, $\tilde{E} = E$ and A is the generator of a strongly continuous S on $\tilde{E}$. In applications, A is typically a differential operator and $\tilde{E}$ is an L^p-space. In that situation, it is also possible to replace E with a suitable Sobolev space or a space of continuous functions. To model equations driven by (additive or multiplicative) white noise, it is often useful to replace $\tilde{E}$ with a suitable extrapolation space, see, for example, [31].

In these situations, the semigroup S typically maps $\tilde{E}$ into E and restricts to a strongly continuous semigroup on E. Moreover, one has some control over the norms

$\|S(t)\|_{\mathscr{L}(\tilde{E},E)}$ *at* $t = 0$*. It should be noted, that we assume none of this in Hypothesis 3.1. However, later on (in Hypothesis 6.5) we will make precisely these assumptions.*

Before defining what we mean by 'a solution' of equation $[A, F, G]$, let us recall the notion of an H-cylindrical Wiener process. Let $(\Omega, \Sigma, \mathbb{F}, \mathbb{P})$ be a stochastic basis, i.e. a probability space $(\Omega, \Sigma, \mathbb{P})$ together with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$. We say that the *usual conditions* are satisfied if $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets and the filtration is right continuous.

An *H-cylindrical Wiener process (with respect to $\mathbb{F}$)* is a bounded linear operator W_H from $L^2(0,\infty;H)$ to $L^2(\Omega,\Sigma,\mathbb{P})$ with the following properties:

1. for all $f \in L^2(0,\infty;H)$ the random variable $W_H(f)$ is centered Gaussian.
2. for all $t \geq 0$ and $f \in L^2(0,\infty;H)$ with support in $[0,t]$, the random variable $W_H(f)$ is $\mathscr{F}_t$-measurable.
3. for all $t \geq 0$ and $f \in L^2(0,\infty;H)$ with support in $[t,\infty)$, the random variable $W_H(f)$ is independent of $\mathscr{F}_t$.
4. for all $f_1, f_2 \in L^2(0,\infty;H)$ we have $\mathbb{E}(W_H(f_1)W_H(f_2)) = [f_1, f_2]_{L^2(0,\infty;H)}$.

We shall write

$$W_H(t)h := W_H(\mathbb{1}_{(0,t]} \otimes h), \quad t > 0, h \in H.$$

It is easy to see that for $h \in H$ the process $W_H h := (W_H(t)h)_{t\geq 0}$ is a real-valued Brownian motion (which is standard if $\|h\|_H = 1$).

We now define the concept of a weak solution. The relation of weak solutions with other solution concepts will be discussed in Section 6.

Definition 3.3. *A tuple $\big((\Omega, \Sigma, \mathbb{F}, \mathbb{P}), W_H, \mathbf{X}\big)$, where $(\Omega, \Sigma, \mathbb{F}, \mathbb{P})$ is stochastic basis satisfying the usual conditions, W_H is an H-cylindrical Wiener process with respect to $\mathbb{F}$ and $\mathbf{X} = (X_t)_{t\geq 0}$ is a continuous, $\mathbb{F}$-progressive, E-valued process is called* weak solution *of* (1.1) *if for all $x^* \in D(A^*) \subset \tilde{E}^*$ and $t \geq 0$ we have*

$$\langle X_t, x^*\rangle = \langle X_0, x^*\rangle + \int_0^t \langle X_s, A^*x^*\rangle\, ds + \int_0^t \langle F(X_s), x^*\rangle\, ds + \int_0^t G(X_s)^*x^*\, dW_H(s)\,, \quad (3.1)$$

$\mathbb{P}$-*a.e.*

Remark 3.4. *Weak solutions are weak both in the analytic sense, i.e. we require* (3.1) *to hold only if tested against functionals $x^* \in D(A^*)$ and in the probabilistic sense, i.e. the stochastic basis and the cylindrical Wiener process are part of the solution. More appropriately, we should speak of 'analytically weak and stochastically weak solution' or 'weak martingale solution'. However, to shorten notation, we have settled on the term 'weak solution'.*

By the continuity of the paths and our assumptions in Hypothesis 3.1, the Lebesgue-integral in (3.1) is well defined. The stochastic integral in equation (3.1) is an integral of an $H \simeq H^*$-valued stochastic processes with respect to a cylindrical Wiener process. It is well known how to construct such an integral for progressive H-valued processes Φ such that $\Phi \in L^2(0,T;H)$ almost surely for all $T > 0$. Namely, if (h_k) is a (finite or countably infinite) orthonormal basis of the separable Hilbert space H and we define $\beta_k(s) := W_H(s)h_k$, then

$$\int_0^t \Phi(s)\, dW_H(s) := \sum_k \int_0^t [\Phi(s)\,,\, h_k]_H\, d\beta_k(s)\,.$$

The integral process $\mathbf{I}(t) := \int_0^t \Phi(s) dW_H(s)$ is a real-valued, continuous, local martingale with with quadratic variation $[\![\mathbf{I}]\!]_t = \int_0^t \|\Phi(s)\|_H^2 \, ds$. We also note that for an $\mathbb{F}$-stopping time τ we have almost surely $\mathbf{I}(t \wedge \tau) = \int_0^t \mathbb{1}_{[0,\tau]}(s)\Phi(s) \, dW_H(s)$ for all $t \geq 0$.

In order to shorten notation, we will say that a process $\mathbf{X}$ is a weak solution of (1.1), meaning that $\mathbf{X}$ is a continuous, progressive, E-valued process, defined on a stochastic basis $(\Omega, \Sigma, \mathbb{P}, \mathbb{F})$, satisfying the usual conditions, on which an H-cylindrical Wiener process W_H with respect to $\mathbb{F}$ is defined such that the tupel $((\Omega, \Sigma, \mathbb{F}, \mathbb{P}), W_H, \mathbf{X})$ is a weak solution of (1.1). In this case, unless stated otherwise, $\mathbb{P}$ will denote the measure on the probability space and W_H the H-cylindrical Wiener process. These remarks apply, mutatis mutandis, also for the other solution concepts that we will introduce.

Remark 3.5. *We note that the exceptional set in* (3.1) *which initially depends on x^* and t may be chosen independently of t, since the deterministic integrals as well as the stochastic integral in* (3.1) *are pathwise continuous in t.*

We now establish a one-to-one correspondence between weak solutions of equation $[A, F, G]$ and solutions of the local martingale problem for an (admissible) operator $\mathscr{L}_{[A,F,G]}$ which we call the *associated local martingale problem*.

The operator $\mathscr{L}_{[A,F,G]}$ is defined as follows.

By $\mathscr{D}$ we denote the vector space of all functions $f : E \to \mathbb{R}$ of the form

$$f(x) = \varphi(\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle)$$

where $n \in \mathbb{N}$, $\varphi \in C^2(\mathbb{R}^n)$ and $x_1^*, \ldots, x_n^* \in D(A^*)$.

For $f = \varphi(\langle \cdot, x_1^* \rangle, \ldots, \langle \cdot, x_n^* \rangle) \in \mathscr{D}$ we put

$$\begin{aligned} L_{[A,F,G]} f(x) :=& \sum_{k=1}^n \frac{\partial \varphi}{\partial u_k}(\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle) \cdot \big[\langle x, A^* x_k^* \rangle + \langle F(x), x_k^* \rangle\big] \\ &+ \frac{1}{2} \sum_{k,l=1}^n [G(x)^* x_k^* \, , \, G(x)^* x_l^*]_H \frac{\partial^2 \varphi}{\partial u_k \partial u_l}(\langle x, x_1^* \rangle, \ldots, \langle x, x_n^* \rangle) \end{aligned} \tag{3.2}$$

The operator $\mathscr{L}_{[A,F,G]}$ is defined by $D(\mathscr{L}) = \mathscr{D}$ and $\mathscr{L}_{[A,F,G]} f := L_{[A,F,G]} f$. Put $\mathscr{D}_{\min} := \big\{ \langle \cdot, x^* \rangle^j \ : \ x^* \in D(A^*), j = 1, 2 \big\}$. We will also use the operator $\mathscr{L}_{[A,F,G]}^{\min} := \mathscr{L}_{[A,F,G]}|_{\mathscr{D}_{\min}}$. We note that since F and G are bounded on bounded subsets of E, the operators $\mathscr{L}_{[A,F,G]}$ and $\mathscr{L}_{[A,F,G]}^{\min}$ are admissible. We would like to point out that the function $\mathscr{L}_{[A,F,G]} f$ can be unbounded even if φ has compact support. This is the reason for considering local martingale problems, rather than martingale problems.

Theorem 3.6. *Suppose that $\mathbf{X}$ is a weak solution of equation $[A, F, G]$. Then the law $\mathbf{P}$ of $\mathbf{X}$ solves the local martingale problem for $\mathscr{L}_{[A,F,G]}$.*

Conversely, if $\mathbf{P}$ solves the local martingale problem for $\mathscr{L}_{[A,F,G]}^{\min}$, then there exists a weak solution $\mathbf{X}$ of equation $[A, F, G]$ with distribution $\mathbf{P}$.

Proof. First suppose that $\mathbf{X}$ is a weak solution of equation $[A, F, G]$.

Let $f = \varphi(\langle \cdot, x_1^* \rangle, \ldots, \langle \cdot, x_n^* \rangle) \in \mathscr{D}$ and define the $\mathbb{R}^n$-valued process ξ by $\xi_k(t) = \langle X(t), x_k^* \rangle$ for all $t \geq 0$ and $k = 1, \ldots, n$. We also define $\mathbb{R}^n$-valued processes V and M by

$$V_k(t) := \int_0^t \langle X_s, A^* x_k^* \rangle + \langle F(X_s), x_k^* \rangle \, ds \quad , \quad M_k(t) := \int_0^t G(X_s)^* x_k^* \, dW_H(s),$$

for $k = 1, \ldots, n$. Note that, almost surely, V has continuous trajectories of locally bounded variation and that M is a continuous, local martingale. Since X is a weak solution, it follows that $\xi = \xi_0 + M + V$.

Itô's formula [9, Theorem 5.2.9] yields

$$\begin{aligned}
& f(X_t) - f(X_0) = \varphi(\xi_t) - \varphi(\xi_0) \\
&= \sum_{k=1}^n \int_0^t \frac{\partial \varphi}{\partial u_k}(\xi_s)\, dV_k(s) + \frac{1}{2}\sum_{k,l=1}^n \int_0^t \frac{\partial^2\varphi}{\partial u_k \partial u_l}(\xi_s)\, d[\![M_k, M_l]\!]_s \\
&\qquad + \sum_{k=1}^n \int_0^t \frac{\partial \varphi}{\partial u_k}(\xi_s)\, dM_k(s) \\
&= \int_0^t \big[L_{[A,F,G]} f\big](X_s)\, ds + \sum_{k=1}^n \int_0^t \frac{\partial \varphi}{\partial u_k}(\xi_s)\, dM_k(s)\ ,
\end{aligned}$$

for all $t \geq 0$. Here, we have used that $[\![M_k, M_l]\!]_t = \int_0^t [G(X_s)^* x_k^*, G(X_s)^* x_l^*]_H\, ds$. It thus follows that

$$f(X_t) - f(X_0) - \int_0^t [L_{[A,F,G]} f](X_s)\, ds$$

is a continuous local martingale with respect to $\mathbb{F}$. Hence, under the distribution $\mathbf{P}$ of $\mathbf{X}$, the process $\mathbf{M}^f$ is a continuous local martingale with respect to $\mathbb{B}$.

We now prove the converse. First note that if $x^* \in D(A^*)$, then for $f_1(x) = \langle x, x^*\rangle$ we have $L_{[A,F,G]} f_1(x) = \langle x, A^* x^*\rangle + \langle F(x), x^*\rangle$. Similarly, for $f_2(x) = \langle x, x^*\rangle^2$ we have $L_{[A,F,G]} f_2(x) = 2\langle x, x^*\rangle \cdot \big[\langle x, A^*x^*\rangle + \langle F(x), x^*\rangle\big] + \|G(x)^*x^*\|_H^2$. If $\mathbf{P}$ is a solution of the local martingale problem for $\mathscr{L}_{[A,F,G]}$, then under $\mathbf{P}$ the processes $\mathbf{M}^{f_1}$ and $\mathbf{M}^{f_2}$ are local martingales with respect to the canonical filtration $\mathbb{B}$. Using that the coefficients F and G are bounded on bounded subsets, an approximation argument shows that we can use $\tau_n := \inf\{t > 0 : \|\mathbf{x}(t)\| \geq n\}$ as localizing sequence for both $\mathbf{M}^{f_1}$ and $\mathbf{M}^{f_2}$. As in [17, Chapter 5, Problem 4.13] we see that the stopped processes $\mathbf{M}^{f_1}_{\tau_n}$ and $\mathbf{M}^{f_2}_{\tau_n}$ are martingales with respect to filtration $\mathbb{F} := (\mathscr{F}_t)$, where $\mathscr{F}_t$ is the augmentation of $\mathscr{B}_{t+}$ by the $\mathbf{P}$ null sets. Hence $\mathbf{M}^{f_1}$ and $\mathbf{M}^{f_2}$ are local martingales with respect to the filtration $\mathbb{F}$, which satisfies the usual conditions. It now follows from [34, Lemma 34] that under $\mathbf{P}$ the process

$$\langle \mathbf{x}_t, x^*\rangle - \langle \mathbf{x}_0, x^*\rangle - \int_0^t \langle \mathbf{x}_s, A^*x^*\rangle + \langle F(\mathbf{x}_s), x^*\rangle\, ds$$

is a continuous local martingale with quadratic variation $\int_0^t \|G(\mathbf{x}_s)^*x^*\|_H^2\, ds$. By [35, Theorem 3.1], we find an extension $(\Omega, \Sigma, \tilde{\mathbb{F}}, \mathbb{P})$ of $(C([0,\infty); E), \mathscr{B}, \mathbb{F}, \mathbf{P})$ on which a cylindrical Brownian motion W_H is defined such that for all $x^* \in D(A^*)$ we have

$$\langle \mathbf{x}_t, x^*\rangle - \langle \mathbf{x}_0, x^*\rangle - \int_0^t \langle \mathbf{x}_s, A^*x^*\rangle + \langle F(\mathbf{x}_s), x^*\rangle\, ds = \int_0^t G(\mathbf{x}_s)^* x^* dW_H(s)$$

$\mathbf{P}$-almost everywhere for all $t \geq 0$ This proves that $\mathbf{x}$, defined on this extension, is a weak solution of $[A, F, G]$. ☐

Corollary 3.7. *A measure $\mathbf{P} \in \mathscr{P}(C([0,\infty); E)$ solves the local martingale problem for $\mathscr{L}_{[A,F,G]}$ if and only if it solves the local martingale problem for $\mathscr{L}^{\min}_{[A,F,G]}$.*

Motivated by Theorem 3.6 we will say that the local martingale problem for $\mathscr{L}_{[A,F,G]}$ is the local martingale problem *associated with* equation $[A, F, G]$. We will say that equation $[A, F, G]$ is *(completely) well-posed* if the associated local martingale problem is (completely) well-posed.

4 Well-posed equations and the strong Markov property

In this section we prove that if equation $[A, F, G]$ is well-posed, then it is completely well-posed. The results of Section 2 then imply that solution of $[A, F, G]$ is a strong Markov process with transition semigroup $\mathscr{T} := (\mathscr{T}(t))_{t\geq 0}$, where $\mathscr{T}(t)f(x) = \int_E f(\mathbf{x}_t)\, d\mathbf{P}_x$.

The key step in the proof is is to show that it even suffices to consider the local martingale problem for an operator $\mathscr{L}^0_{[A,F,G]}$, defined on a countable set, cf. [9, Theorem 4.4.6].

Lemma 4.1. *There exists a countable subset $\mathscr{D}_0$ of $\mathscr{D}$ such that a measure $\mathbf{P}$ solves the local martingale problem associated for $\mathscr{L}_{[A,F,G]}$ if and only if it solves the martingale problem associated with $\mathscr{L}^0_{[A,F,G]} := \mathscr{L}_{[A,F,G]}|_{\mathscr{D}_0}$.*

Proof. Step 1: We construct the set $\mathscr{D}_0$.

First note that there exists a countable subset D of $D(A^*)$ such that for every $x^* \in D(A^*)$ there exists a sequence $(x_n^*) \subset D$ with $x_n^* \rightharpoonup^* x^*$ and $A^*x_n^* \rightharpoonup^* A^*x^*$. Here $\rightharpoonup^*$ refers to weak* convergence in $\tilde{E}^*$. To see this, first note that there is a countable set $\{z_n^* \,:\, n \in \mathbb{N}\} \subset \tilde{E}^*$ which is sequentially weak*-dense in $\tilde{E}^*$, see §21.3 (5) of [18]. Put $D := \{R(\lambda, A^*)z_n^* \,:\, n \in \mathbb{N}\}$ for some $\lambda \in \rho(A^*)$. Using that $R(\lambda, A^*)$ is $\sigma(\tilde{E}^*, \tilde{E})$-continuous as an adjoint operator, it is easy to see that D has the required properties. Replacing D with the set of all convex combinations of elements of D with rational coefficients, we may (and shall) assume that such convex combinations belong to D again.

Now choose a sequence $\varphi_n \in C^2(\mathbb{R})$ with the following properties:

1. $\varphi_n(t) = t$ for all $-n \leq t \leq n$ and $\varphi_n(t) = 0$ for $t \notin [-2n, 2n]$.
2. $\sup_n \|\varphi_n'\|_\infty, \sup_n \|\varphi_n''\|_\infty < \infty$.

We then define

$$\mathscr{D}_0 := \big\{ f = \varphi_n(\langle \cdot, x^*\rangle)^j \quad \text{for some } n \in \mathbb{N}\,,\, x^* \in D\,,\, j \in \{1,2\}\big\}.$$

Clearly, $\mathscr{D}_0$ is countable. We define $\mathscr{L}^0_{[A,F,G]} := \mathscr{L}_{[A,F,G]}|_{\mathscr{D}_0}$.

Step 2: Now let $\mathbf{P}$ be a solution of the local martingale problem for $\mathscr{L}^0_{[A,F,G]}$. We prove that $\mathbf{P}$ solves the local martingale problem for $\mathscr{L}^{\min}_{[A,F,G]}$. This finishes the proof in view of Corollary 3.7.

First note that $\mathbf{M}^f$ is a local martingale for any $f = \langle \cdot, x^*\rangle^j$, $x^* \in D\,,\, j \in \{1,2\}$. To see this, let $\sigma_n := \inf\{t > 0 \,:\, |\langle \mathbf{x}_t, x^*\rangle| \vee \|\mathbf{x}_t\| \geq n\}$ and put $f_n := \varphi_n(\langle \cdot, x^*\rangle)^j \in \mathscr{D}_0$. Clearly, $\mathbf{M}^f_{\sigma_n} = \mathbf{M}^{f_n}_{\sigma_n}$. Since $\mathbf{P}$ solves the local martingale problem for $\mathscr{L}^0_{[A,F,G]}$, the process $\mathbf{M}^{f_n}$, hence by optional sampling also $\mathbf{M}^{f_n}_{\sigma_n}$, is a local martingale under $\mathbf{P}$. Since F and G are bounded on bounded sets, $\mathbf{M}^{f_n}_{\sigma_n}$ is uniformly bounded. Thus, $\mathbf{M}^{f_n}_{\sigma_n}$ is a true martingale by dominated convergence. This proves that $\mathbf{M}^f_{\sigma_n}$ is a true martingale under $\mathbf{P}$ and hence, since $\sigma_n \uparrow \infty$ pointwise, that $\mathbf{M}^f$ is a local martingale under $\mathbf{P}$.

It remains to extend this from $x^* \in D$ to arbitrary $x^* \in D(A^*)$. To that end, fix $x^* \in D(A^*)$ and a sequence $(x_n^*) \subset D$ such that $x_n^* \rightharpoonup^* x^*$ and $A^*x_n^* \rightharpoonup^* A^*x^*$. By the uniform boundedness principle, the sequences (x_n^*) and $(A^*x_n^*)$ are bounded in $\tilde{E}^*$, say by M. For $m \in \mathbb{N}$ put $\tau_m := \inf\{t > 0 \,:\, \|\mathbf{x}(t)\| \geq m\}$.

Let us first consider $f := \langle \cdot, x^*\rangle$. Arguing as above, we see that for $f_n := \langle \cdot, x_n^*\rangle$, the stopped process $\mathbf{M}^{f_n}_{\tau_m}$ is a martingale under $\mathbf{P}$ for all $n, m \in \mathbb{N}$. Furthermore, since $L_{[A,F,G]}f_n \to L_{[A,F,G]}f$ pointwise, it follows that $\mathbf{M}^{f_n}_{\tau_m}(t) \to \mathbf{M}^f_{\tau_m}(t)$ pointwise as $n \to \infty$, for all $t \geq 0$. Since F is bounded on $\bar{B}(0, m)$, say by C_m, we find for $t > s$

$$\big|\mathbf{M}^{f_n}_{\tau_m}(\mathbf{x})(t) - \mathbf{M}^{f_n}_{\tau_m}(\mathbf{x})(s)\big| \leq (t-s)\big[m \cdot M + C_m \cdot M\big] + 2m \cdot M$$

for all $n, m \in \mathbb{N}$. Thus, applying the dominated convergence theorem to the sequence $(\mathbf{M}^{f_n}_{\tau_m}(t) - \mathbf{M}^{f_n}_{\tau_m}(s))\mathbb{1}_B$, where B is an arbitrary set in $\mathscr{B}_s$, it follows that $\int_B \mathbf{M}^{f}_{\tau_m}(t) - \mathbf{M}^{f}_{\tau_m}(s))\,d\mathbf{P} = 0$. Since $0 \le s < t$ and $B \in \mathscr{B}_s$ were arbitrary, $\mathbf{M}^{f}_{\tau_m}$ is a $\mathbb{B}$-martingale under $\mathbf{P}$. As $\tau_m \uparrow \infty$ almost surely, this proves that $\mathbf{M}^f$ is a local martingale under $\mathbf{P}$.

Next consider $f := \langle \cdot, x^* \rangle^2$. For $f_n := \langle \cdot, x_n^* \rangle^2$, the stopped process $\mathbf{M}^{f_n}_{\tau_m}$ is a martingale under $\mathbf{P}$ for all $n, m \in \mathbb{N}$. Similarly as above, one sees that for every $m \in \mathbb{N}$ the difference $|\mathbf{M}^{f_n}_{\tau_m}(t) - \mathbf{M}^{f_n}_{\tau_m}(s)|$ may be majorized by a bounded function independent of n. However, due to the term $\|G(\cdot)^* x_n^*\|_H^2$ in $L_{[A,F,G]} f_n$, the weak convergence $x_n^* \rightharpoonup^* x^*$ does not suffice to conclude that $L_{[A,F,G]} f_n \to L_{[A,F,G]} f$ pointwise. Hence we employ a different method here.

We fix $0 \le s < t$ and $m \in \mathbb{N}$. The dominated convergence theorem yields weak convergence

$$\int_s^t \mathbb{1}_{[0,\tau_m]}(r) G(\mathbf{x}_r)^* x_n^*\,dr \rightharpoonup \int_s^t \mathbb{1}_{[0,\tau_m]}(r) G(\mathbf{x}_r)^* x^*\,dr \quad \text{in } L^2(C([0,\infty); E), \mathbf{P}; H)\,.$$

Hence $\int_s^t \mathbb{1}_{[0,\tau_m]}(r) G(\mathbf{x}_r)^* x^*\,dr$ belongs to the weak closure of the tail sequence

$$\Big(\int_s^t \mathbb{1}_{[0,\tau_m]}(r) G(\mathbf{x}_r)^* x_n^*\,dr \Big)_{n \ge N},$$

for any $N \in \mathbb{N}$. By the Hahn-Banach theorem, it belongs to the strong closure of that tail, whence we find vectors y_N^*, belonging to the convex hull the sequence $(x_n^*)_{n \ge N}$, such that we have *strong* convergence

$$\int_s^t \mathbb{1}_{[0,\tau_m]}(r) G(\mathbf{x}_r)^* y_N^*\,dr \to \int_s^t \mathbb{1}_{[0,\tau_m]}(r) G(\mathbf{x}_r)^* x^*\,dr \quad \text{in } L^2(C([0,\infty); E), \mathbf{P}; H)\,.$$

After passing to a subsequence, we may assume that this convergence holds pointwise $\mathbf{P}$-a.e. Note that $y_N^* \rightharpoonup^* x^*$, as y_N^* belongs to the tail $(x_n^*)_{n \ge N}$. Hence it follows that

$$\mathbf{M}^{g_N}_{\tau_m}(t) - \mathbf{M}^{g_N}_{\tau_m}(s) \to \mathbf{M}^{f}_{\tau_m}(t) - \mathbf{M}^{f}_{\tau_m}(s)$$

pointwise $\mathbf{P}$-almost everywhere. Here, $g_N := \langle \cdot, y_N^* \rangle^2$.

Note that we may assume without loss of generality that y_N^* is a convex combination of the $(x_n^*)_{n \ge N}$ with rational coefficients. Hence, $y_N \in D$ and thus $g_N \in \mathscr{D}_0$, implying that $\mathbf{M}^{g_N}_{\tau_m}$ is a martingale under $\mathbf{P}$ for all $N \in \mathbb{N}$. Now, similarly as above, the dominated convergence theorem shows that $\mathbf{M}^{f}_{\tau_m}$ is a martingale under $\mathbf{P}$ for all $m \in \mathbb{N}$. This finishes the proof. $\square$

Now the announced result about the equivalence of well-posedness and complete well-posedness follows similar to the finite-dimensional case, cf. [16, Theorem 21.10].

Theorem 4.2. *Suppose that the local martingale problem for $\mathscr{L}_{[A,F,G]}$ is well-posed. Then it is completely well-posed. Consequently, all weak solutions of equation $[A, F, G]$ are strong Markov processes with a common transition semigroup $\mathscr{T}$.*

Proof. We first prove the measurability of the map $x \mapsto \mathbf{P}_x$. Consider the set $V := \{\mathbf{P}_x : x \in E\}$. We claim that V is a Borel subset of $\mathscr{P}(C([0,\infty); E))$. Indeed, by well-posedness, $V = V_1 \cap V_2$, where V_1 is the set of all probability measures with degenerate initial distributions and V_2 is the set of all solutions to the martingale problem.

Since the map $\mathbf{P} \mapsto \mathbf{P} \circ \mathbf{x}(0)^{-1}$ is measurable from $\mathscr{P}(C([0,\infty); E))$ to $\mathscr{P}(E)$, the measurability of V_1 follows from [16, Lemma 1.39].

By Lemma 4.1, $\mathbf{P} \in V_2$ if and only if $\mathbf{M}^f$ is a local martingale under $\mathbf{P}$ for all $f \in \mathscr{D}_0$. With $\tau_n := \inf\{t > 0 : \|\mathbf{x}(t)\| \geq n\}$, this is equivalent with

$$\int_B \mathbf{M}^f(t \wedge \tau_n)\, d\mathbf{P} = \int_B \mathbf{M}^f(s \wedge \tau_n)\, d\mathbf{P} \quad \forall\, s < t,\, B \in \mathscr{B}_s\,,\, n \in \mathbb{N}.$$

However, using continuity of $t \mapsto \mathbf{x}(t)$ and the fact that the σ-algebra $\mathscr{B}_s$ is countably generated for all $s > 0$, we see that $\mathbf{M}^f$ is a local martingale under $\mathbf{P}$ whenever the above equality holds for $n \in \mathbb{N}, s, t \in \mathbb{Q}$ with $s < t$ and B in a countable subset of $\mathscr{B}_s$. Hence the set V_2 is determined by countably many 'measurable relations' and hence measurable. It follows that V is measurable as claimed.

Now define the map $\Phi : V \to E$ by defining $\Phi(\mathbf{P})$ as the unique x such that $\mathbf{P} \circ \mathbf{x}_0^{-1} = \delta_x$. Clearly, Φ is injective. Furthermore, Φ is measurable as the composition of the measurable map $\mathbf{P} \circ \mathbf{x}_0^{-1}$ and the inverse of the map $x \mapsto \delta_x$, which establishes a homeomorphism between E and the range of that map. By the Kuratowski Theorem, see [36, Section 1.3], the inverse Φ^{-1} is measurable, i.e. $x \mapsto \mathbf{P}_x$ is a measurable map from E to $\mathscr{P}(C([0,\infty); E))$

It remains to prove the uniqueness of solutions with arbitrary initial distributions μ for the martingale problem for $\mathscr{L}_{[A,F,G]}$. The existence of solutions with general initial distributions will then follow from Theorem 2.2.

To that end, assume that $\mathbf{P}$ solves the local martingale problem for $\mathscr{L}_{[A,F,G]}$ and that $\mathbf{x}(0)$ has distribution $\mu \in \mathscr{P}(E)$. Let $\mathbf{Q} : E \times \mathscr{B} \to [0,1]$ be a regular conditional probability (under $\mathbf{P}$) for $\mathscr{B}$ given $\mathbf{x}_0$. Then

$$\mathbf{P}(A) = \int_E \mathbf{Q}(x, A)\, d\mu(x) \quad \forall\, A \in \mathscr{B}\,.$$

Now let $t > s \geq 0$ and $B \in \mathscr{B}_s$ be given. Then, for $f \in \mathscr{D}$, we have

$$\int_B \mathbf{M}^f(t \wedge \tau_n) - \mathbf{M}^f(s \wedge \tau_n)\, d\mathbf{Q}(x, \cdot) = \int_{B \cap \{\mathbf{x}(0) = x\}} \mathbf{M}^f(t \wedge \tau_n) - \mathbf{M}^f(s \wedge \tau_n)\, d\mathbf{P} = 0$$

for μ-almost every x. We note that the null-set outside of which this equation holds depends on t, s, n, B and the function f. However, arguing as above, we see that for fixed f, there exists a null-set $N(f)$, such that the above equation holds outside $N(f)$ for *all* $t > s, n \in \mathbb{N}$ and $B \in \mathscr{B}_s$. Putting $N := \bigcup_{f \in \mathscr{D}_0} N(f)$, it follows that outside of N, the above holds for all $t > s, n \in \mathbb{N}, B \in \mathscr{B}_s$ and $f \in \mathscr{D}_0$. This implies that for μ-a.e. x the measure $\mathbf{Q}(x, \cdot)$ solves the local martingale problem for $\mathscr{L}^0_{[A,F,G]}$ and hence, by Lemma 4.1, the local martingale problem for $\mathscr{L}_{[A,F,G]}$. By well-posedness, $\mathbf{Q}(x, \cdot) = \mathbf{P}_x(\cdot)$ for μ-a.e. x. Hence we have

$$\mathbf{P}(A) = \int_E \mathbf{P}_x(A)\, d\mu(x) \quad \forall\, A \in \mathscr{B}\,, \tag{4.1}$$

This shows that uniqueness of solutions of the local martingale problem for $(\mathscr{L}, \delta_x)$ for all $x \in E$ implies uniqueness of the solution of the local martingale problem for $(\mathscr{L}, \mu)$ for arbitrary initial distribution μ. □

We end this section by establishing a result which allows us to construct solutions to equation $[A, F, G]$ from solutions of approximate equations $[A, F_n, G_n]$.

Lemma 4.3. *Suppose we are given sequences $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ which satisfy the assumptions of Hypothesis 3.1, are continuous and are uniformly bounded on bounded sets. Furthermore, assume that $F_n(x)$ converges to $F(x)$ in $\tilde{E}$ and $G_n(x)$ converges to $G(x)$ in $\mathscr{L}(H, \tilde{E})$, both convergences being uniform on the compact subsets of E.*

If $\mathbf{P}_n$ solves the martingale problem associated with equation $[A, F_n, G_n]$ and if the sequence $(\mathbf{P}_n)_{n\in\mathbb{N}}$ is tight, then any accumulation point of the sequence solves the martingale problem associated with $[A, F, G]$.

Proof. For a number $M \in \mathbb{R}$ we put $\tau_M := \inf\{t > 0 : \|\mathbf{x}_t\| \geq M\}$. Now fix $0 \leq s_1 < \cdots < s_N \leq s < t$, $N \in \mathbb{N}$, and for $j = 1, \ldots, N$ functions $h_j \in C_b(E)$ and $f = \varphi(\langle\cdot, x_1^*\rangle, \ldots, \langle\cdot, x_m^*\rangle) \in \mathscr{D}$.

We define $\Phi_n : C([0,\infty); E) \to \mathbb{R}$ by

$$\Phi_n(\mathbf{x}) := \Big[f(\mathbf{x}_{t\wedge\tau_M}) - f(\mathbf{x}_{s\wedge\tau_M}) - \int_s^t \mathbb{1}_{[0,\tau_M]}(r)\big(L_n f\big)(\mathbf{x}_r)\,dr\Big] \cdot \prod_{j=1}^N h_j(\mathbf{x}_{s_j}),$$

where $L_n := L_{[A,F_n,G_n]}$. Similarly, we define the function Φ, replacing L_n with $L := L_{[A,F,G]}$.

Using the assumption that F_n and G_n are uniformly bounded on bounded subsets, it is easy to see that the sequence Φ_n is uniformly bounded.

The assumptions on the convergence of F_n and G_n imply that $L_n f$ converges to Lf, uniformly on the compact subsets of E. Now let a compact subset $\mathscr{C}$ of $C([0,\infty); E)$ be given. By the Arzelà-Ascoli theorem, there exists a compact subset K of E such that $\mathbf{x}_r \in K$ for all $0 \leq r \leq t$, whenever $\mathbf{x} \in \mathscr{C}$. Let $C := \prod_{j=1}^n \|h_k\|_\infty$. Given $\varepsilon > 0$, pick n_0 such that $|L_n f(x) - Lf(x)| \leq \varepsilon$ for all $x \in K$, whenever $n \geq n_0$. Then, for $\mathbf{x} \in \mathscr{C}$ and $n \geq n_0$ we have

$$|\Phi_n(\mathbf{x}) - \Phi(\mathbf{x})| \leq \int_s^t \mathbb{1}_{[0,\tau_M]}(r)|L_n f(\mathbf{x}_r) - Lf(\mathbf{x}_r)|\,dr \cdot C \leq |t - s|\varepsilon C,$$

proving that Φ_n converges to Φ uniformly on compact subsets of $C([0,\infty); E)$.

Now let $\mathbf{P}$ be an accumulation point of the sequence $(\mathbf{P}_n)$. Passing to a subsequence, we may assume that $\mathbf{P}_n$ converges weakly to $\mathbf{P}$. In particular, the sequence $(\mathbf{P}_n)$ is tight. Thus, given $\varepsilon > 0$, we find a compact set $\mathscr{C}$ of $C([0,\infty); E)$ such that $2c\mathbf{P}_n(\mathscr{C}^c) \leq \varepsilon$, where c is such that $\|\Phi_n\|_\infty \leq c$. It follows that

$$\Big|\int \Phi\,d\mathbf{P} - \int \Phi_n\,d\mathbf{P}_n\Big| \leq \Big|\int \Phi\,d\mathbf{P} - \int \Phi\,d\mathbf{P}_n\Big| + \varepsilon + \sup_{\mathbf{x}\in\mathscr{C}} |\Phi(\mathbf{x}) - \Phi_n(\mathbf{x})|.$$

To conclude that $\int \Phi\,d\mathbf{P} = \lim_{n\to\infty} \int \Phi_n\,d\mathbf{P}_n = 0$, it remains to prove that $\int \Phi\,d\mathbf{P}_n$ converges to $\int \Phi\,d\mathbf{P}$. We know that $\mathbf{P}_n$ converges weakly to $\mathbf{P}$. Unfortunately, the function Φ is not continuous. However, it is continuous at all points $\mathbf{y}$ at which the map $\mathbf{x} \mapsto \tau_M(\mathbf{x})$ is continuous. Moreover, it can be proved that the set of all M such that $\mathbf{P}(\{\mathbf{y} : \tau_M \text{ is discontinous at } \mathbf{y}\}) > 0$ is countable, see [13, Lemma 3.5 and 3.6] (see also Sections VI.2 and VI.3 of [15]). We can thus find a number M such that Φ is continuous except for a $\mathbf{P}$-null set. As is well known, see [1, Cor. 8.4.2], this together with the weak convergence of the $\mathbf{P}_n$ suffices to conclude that $\int \Phi d\mathbf{P}_n \to \int \Phi\,d\mathbf{P}$, as desired and it follows that $\int \Phi\,d\mathbf{P} = 0$.

Since the sampling points (s_j) and s, t as well as the functions h_j were arbitrary, it follows from a monotone class argument that

$$f(\mathbf{x}_{t\wedge\tau_M}) - f(\mathbf{x}_{0\wedge\tau_M}) - \int_0^t \mathbb{1}_{[0,\tau_M]}(r)Lf(\mathbf{x}_r)\,dr$$

is a martingale under $\mathbf{P}$. Since f was arbitrary, and we can pick a sequence $M_k \uparrow \infty$ such that the above is true, we have proved that $\mathbf{P}$ solves the local martingale problem associated with equation $[A, F, G]$. ☐

As a corollary, we obtain a sufficient condition for the Feller property of the associated transition semigroup.

Corollary 4.4. *Assume that equation* $[A, F, G]$ *is well-posed and that* F *and* G *are continuous. We denote by* $\mathscr{T}$ *the transition semigroup for the associated martingale problem for* $\mathscr{L}_{[A,F,G]}$ *and by* $\mathbf{P}_\mu$ *the unique solution of the local martingale problem for* $(\mathscr{L}_{[A,F,G]}, \mu)$*. The following are equivalent*

1. *The map* $\mu \mapsto \mathbf{P}_\mu$ *is continuous from* $\mathscr{P}(E)$ *to* $\mathscr{P}(C([0,\infty); E))$ *where both are endowed with their respective weak topology.*
2. *If* $x_n \to x$ *in* E*, then the set* $\{\mathbf{P}_{x_n} : n \in \mathbb{N}\}$ *is tight.*

In this case, the semigroup $\mathscr{T}$ *has the Feller property, i.e.* $\mathscr{T}(t)f \in C_b(E)$ *for all* $f \in C_b(E)$.

Proof. (1) $\Rightarrow$ (2): If $x_n \to x$ then $\delta_{x_n} \to \delta_x$ weakly. In particular, $\{\delta_{x_n} : n \in \mathbb{N}\}$ is relatively weakly compact. By (1) the set $\{\mathbf{P}_{x_n} : n \in \mathbb{N}\}$ is relatively weakly compact hence tight.

(2) $\Rightarrow$ (1): Let $x_n \to x$. By (2), $\{\mathbf{P}_{x_n} : n \in \mathbb{N}\}$ is tight. By Lemma 4.3 any accumulation point of the $\mathbf{P}_{x_n}$ must solve the local martingale problem for $\mathscr{L}_{A,F,G}$. Since every accumulation point also must have initial distribution δ_x, well-posedness implies that the only accumulation point is $\mathbf{P}_x$. Now a subsequence-subsequence argument yields that $\mathbf{P}_{x_n}$ converges weakly to $\mathbf{P}_x$. This proves that the map $x \mapsto \mathbf{P}_x$ is continuous from E to $\mathscr{P}(C([0,\infty); E))$.

It follows from the proof of uniqueness in Theorem 2.2, namely from equation (4.1), that

$$\int \Phi \, d\mathbf{P}_\mu = \int_E \int \Phi \, d\mathbf{P}_x \, d\mu(x),$$

for all bounded, continuous functions Φ on $C([0,\infty); E)$. With this representation the continuity of $\mu \mapsto \mathbf{P}_\mu$ follows.

If (1) or, equivalently, (2) is satisfied, then the Feller property of $\mathscr{T}$ follows from the identity $\mathscr{T}(t)f(x) = \int f \circ \pi_t \, d\mathbf{P}_x$ and the fact that $f \circ \pi_t$ is a bounded, continuous function on $C([0,\infty); E)$. □

5 Yamada-Watanabe theory

In view of Theorem 3.6, the uniqueness requirement for the local martingale problem associated with (1.1) is equivalent with the requirement that whenever $\mathbf{X}_1$ and $\mathbf{X}_2$ are weak solutions of (1.1), possibly defined on different probability spaces, such that $X_1(0)$ and $X_2(0)$ have the same distribution μ, then $\mathbf{X}_1$ and $\mathbf{X}_2$ have the same distribution as $C([0,\infty); E)$-valued random variables. In this situation, one says that *uniqueness in law* or *uniqueness in distribution* holds.

In some cases, in particular in the case of Lipschitz continuous coefficients, it is easier to verify a different notion of uniqueness.

Definition 5.1. *We say that* pathwise uniqueness *holds for solutions of equation* (1.1) *if whenever* $((\Omega, \Sigma, \mathbb{F}, \mathbb{P}), W_H, \mathbf{X}_j)$ *are weak solution of* (1.1) *for* $j = 1, 2$ *with* $X_1(0) = X_2(0)$ *almost surely, then* $\mathbb{P}(X_1(t) = X_2(t) \, \forall \, t \geq 0) = 1$.

A classical result of Yamada and Watanabe [39] asserts that in the case where $E = \mathbb{R}^d$ and W_H is a finite dimensional Brownian motion, i.e. H is finite-dimensional, pathwise uniqueness implies uniqueness in law. Pathwise uniqueness also has other far-reaching consequences, most notably, it implies the *strong* existence of solutions.

Definition 5.2. *We say that a weak solution* $((\Omega,\Sigma,\mathbb{F},\mathbb{P}),W_H,\mathbf{X})$ exists strongly *if* $\mathbf{X}$ *is adapted to the filtration* $\mathbb{G} := (\mathscr{G}_t)_{t\geq 0}$*, where* $\mathscr{G}_t$ *denotes the augmentation of* $\sigma(X(0), W_H h_k(s) \,:\, s \leq t, k \in I)$*. Here,* $(h_k)_{k\in I}$ *is a finite or countably infinite orthonormal basis of* H*.*

A priori, strong existence of solutions is a mere measurability requirement. This requirement captures the idea that the information needed to construct a solution to a stochastic differential equation is already contained in the initial datum and the Wiener process. Of particular importance in applications is the fact that given pathwise uniqueness solutions can be constructed on a *given* stochastic basis and with respect to a *given* H-cylindrical Wiener process, see Corollary 5.4.

Ondreját [33] has generalized the Yamada-Watanabe results to the situation where E is a 2-smoothable Banach space. One of the main difficulties he had to overcome was to prove that distributional copies of solutions are again solutions. As he was working with the concept of mild solutions, this required a detailed study of the distributions of Banach space valued stochastic integrals. In our situation, with the concept of weak solutions, the proof is easier and can in fact be reduced to the finite dimensional situation.

Theorem 5.3. *Pathwise uniqueness for* (1.1) *implies uniqueness in law. Moreover, every solution of* (1.1) *exists strongly.*

For the convenience of the reader, we include a full proof which follows closely the proof in the finite dimensional situation. It is also possible to show that our situation fits into the abstract framework considered in [22] and to obtain Theorem 5.3 from the results proved there.

Proof. Let two weak solutions $((\Omega_j,\Sigma_j,\mathbb{F}_j,\mathbb{P}_j),W_H^j,\mathbf{X}_j)$ of equation (1.1) be given such that $X_1(0)$ and $X_2(0)$ have the same distribution μ. We first define distributional copies of these two solutions on a common stochastic basis.

To that end, we fix an orthonormal Basis $(h_n)_{n\in\mathbb{N}}$ (the case where H is finite dimensional is similar) of H and define the measure $\mathbf{P}_j$ on the Borel σ-algebra of

$$\tilde{\Omega} := C([0,\infty);E)\times E\times C([0,\infty);\mathbb{R}^\infty),$$

viewed as the countable product of Polish spaces, as the image of $\mathbb{P}_j$ under the map

$$\omega_j \mapsto \big(X_j(\cdot,\omega_j)-X_j(0,\omega_j), X_j(0,\omega_j), (H_H^j(\cdot,\omega_j)h_n)_{n\in\mathbb{N}}\big)$$

A typical element of $\tilde{\Omega}$ will be denoted by $(\mathbf{y},x_0,\mathbf{w})$. Note that the projection of $\mathbf{P}_j$ to $C([0,\infty);\mathbb{R}^\infty)$ is the countable product of Wiener measure; we denote this measure by $\mathbb{W}$. Thus, under $\mathbf{P}_j$, the random element $(x_0,\mathbf{w})$ has distribution $\mu\otimes\mathbb{W}$.

We let $\mathbf{Q}_j$ be a regular conditional distribution of $\mathbf{y}$ given $(x_0,\mathbf{w})$ under $\mathbf{P}_j$, i.e. $\mathbf{Q}_j(x_0,\mathbf{w},\cdot)$ is a probability measure on $\mathscr{B}(C([0,\infty);E))$ for all $x_0\in E$,$\mathbf{w}\in C([0,\infty);\mathbb{R}^\infty)$ and given sets $A\in\mathscr{B}(C([0,\infty);E))$, $B\in\mathscr{B}(E)$ and $C\in\mathscr{B}(C([0,\infty);\mathbb{R}^\infty))$, we have

$$\mathbf{P}_j(A\times B\times C) = \int_{B\times C}\mathbf{Q}_j(x_0,\mathbf{w},A)\,d(\mu\otimes\mathbb{W})(x_0,\mathbf{w}).$$

We now define distributional copies of the solutions on a common probability space. We put

$$\Omega := C([0,\infty);E)\times C([0,\infty);E)\times E\times C([0,\infty);\mathbb{R}^\infty),$$

and denote a canonical element of Ω by $(\mathbf{y}_1,\mathbf{y}_2,x_0,\mathbf{w})$. We define the measure $\mathbf{P}$ on the Borel σ-algebra Σ of Ω by

$$\mathbf{P}(A\times B\times C\times D) := \int_{C\times D}\mathbf{Q}_1(x_0,\mathbf{w},A)\mathbf{Q}_2(x_0,\mathbf{w},B)\,d(\mu\otimes\mathbb{W})(x_0,\mathbf{w}).$$

Finally, we define $\mathscr{G}_t := \sigma(x_0, \mathbf{y}_1(s), \mathbf{y}_2(s), \mathbf{w}(s) : s \leq t)$, $\mathscr{F}_t$ as the augmentation of $\mathscr{G}_{t+}$ by the $\mathbf{P}$-null sets and set $\mathbb{F} := (\mathscr{F}_t)_{t\geq 0}$. As in the finite dimensional case, see [14, Lemma IV.1.2], we see that for every $k \in \mathbb{N}$ the k-th component $\mathbf{w}_k$ of $\mathbf{w}$ is a Brownian motion with respect to $\mathbb{F}$.

As $\mathbf{w}_k$ and $\mathbf{w}_l$ are independent for $k \neq l$, we can define an H-cylindrical Wiener process with respect to $\mathbb{F}$ by setting, for $f \in L^2(0, \infty; H)$

$$W_H(f) := \sum_{k=1}^{\infty} \int_0^{\infty} [f(t), h_k]_H \, d\mathbf{w}_k(t).$$

We claim that $((\Omega, \Sigma, \mathbb{F}, \mathbf{P}), W_H, x_0 + \mathbf{y}_j)$ is a weak solution of equation (1.1) for $j = 1, 2$. We will write $\mathbf{x}_j := x_0 + \mathbf{y}_j$ for $j = 1, 2$. To prove the claim, let $x^* \in D(A^*)$ be fixed. Using the measurability of F and G, as well as the continuity of the functionals x^* resp. A^*x^*, it follows from the definitions above that the joint distribution of

$$\big(\langle X_j(0), x^*\rangle, \langle F(X_j(\cdot)), x^*\rangle, \langle X_j(\cdot), A^*x^*\rangle, ([G(X_j(\cdot))^*x^*, h_k])_{k\in\mathbb{N}}, (W_H^j(\cdot)h_k)_{k\in\mathbb{N}}\big)$$

under $\mathbb{P}_j$ is the same as that of

$$\big(\langle x_0, x^*\rangle, \langle F(\mathbf{x}_j(\cdot)), x^*\rangle, \langle \mathbf{x}_j(\cdot), A^*x^*\rangle, ([G(\mathbf{x}_j(\cdot))^*x^*, h_k])_{k\in\mathbb{N}}, (W_H(\cdot)h_k)_{k\in\mathbb{N}}\big)$$

under $\mathbf{P}$. Thus, for fixed $t \geq 0$, we infer as in the finite dimensional situation that for $j = 1, 2$ and every $n \in \mathbb{N}$ the distribution of

$$\begin{aligned} Z_{j,n}(t) := X_j(t) - \langle X_j(0), x^*\rangle - \int_0^t \langle X_j(s), A^*x^*\rangle \, ds - \int_0^t \langle F(X_j(s)), x^*\rangle \, ds \\ - \sum_{k=1}^{n} \int_0^t [G(X_j(s))^*x^*, h_k] \, dW_H^j(s)h_k \end{aligned}$$

under $\mathbb{P}_j$ is the same as that of

$$\begin{aligned} \mathbf{z}_{j,n}(t); = \mathbf{x}_j(t) - \langle \mathbf{x}_j(0), x^*\rangle - \int_0^t \langle \mathbf{x}_j(s), A^*x^*\rangle \, ds - \int_0^t \langle F(\mathbf{x}_j(s)), x^*\rangle \, ds \\ - \sum_{k=1}^{n} \int_0^t [G(\mathbf{x}_j(s))^*x^*, h_k] \, dW_H^j(s)h_k \end{aligned}$$

under $\mathbf{P}$. Since $\mathbf{X}_j$ is a solution of equation (1.1), $Z_{j,n}(t) \to 0$ $\mathbb{P}_j$-almost surely as $n \to \infty$, hence $\mathbf{z}_{j,n}$ converges to 0 in distribution and thus $\mathbf{P}$-almost surely. Since $t \geq 0$ and $x^* \in D(A^*)$ were arbitrary, this proves that $\mathbf{x}_j$ is indeed a weak solution.

As $x_0 + \mathbf{y}_1$ and $x_0 + \mathbf{y}_2$ are weak solutions defined on the same stochastic basis and with respect to the same H-cylindrical Wiener process, pathwise uniqueness implies that $x_0 + \mathbf{y}_1 = x_0 + \mathbf{y}_2$ $\mathbf{P}$-almost surely. This, in turn, implies that the random elements $\mathbf{X}_j$ have the same distribution.

As for the strong existence of solutions, define for $x_0 \in E$ and $\mathbf{w} \in C([0, \infty); \mathbb{R}^\infty)$ the measure $\mathbf{R}(x_0, \mathbf{w}, \cdot)$ on the Borel σ-algebra $\mathscr{S}$ of $C([0, \infty); E) \times C([0, \infty); E)$ as the product of $\mathbf{Q}_1(x_0, \mathbf{w}, \cdot)$ and $\mathbf{Q}_2(x_0, \mathbf{w}, \cdot)$. Then, for sets $G \in \mathscr{S}$, $C \in \mathscr{B}(E)$ and $D \in \mathscr{B}(C([0, , \infty); \mathbb{R}^\infty)$ we have

$$\mathbf{P}(G \times C \times D) = \int_{C\times D} \mathbf{R}(x_0, \mathbf{w}, G) \, d(\mu \otimes \mathbb{W})(x_0, \mathbf{w}).$$

Now consider $\Lambda := \{(\mathbf{y}_1, \mathbf{y}_2) : \mathbf{y}_1 = \mathbf{y}_2\}$. It follows from the first part of the proof that $R(x_0, \mathbf{w}, \Lambda) = 1$ for $(\mu \otimes \mathbb{W})$-almost every $(x_0, \mathbf{w})$, say outside the set $N \in \mathscr{B}(E) \otimes$

$\mathscr{B}(C([0,\infty);E))$ with $(\mu\otimes\mathbb{W})(N)=0$. Using Fubini's theorem, we find for $(x_0,\mathbf{w})\in N^c$

$$1=\mathbf{R}(x_0,\mathbf{w},\Lambda)=\int_{C([0,\infty);E)}\mathbf{Q}_1(x_0,\mathbf{w},\{\mathbf{y}\})\,\mathbf{Q}_2(x_0,\mathbf{w},d\mathbf{y}).$$

As all measures involved in this equation are probability measures, this can only happen if $\mathbf{Q}_1(x_0,\mathbf{w},\{\mathbf{y}_0\})=\mathbf{Q}_2(x_0,\mathbf{w},\{\mathbf{y}_0\})=1$ for a certain $\mathbf{y}_0=\Phi(x_0,\mathbf{w})\in C([0,\infty);E)$.

A straightforward generalization of the proof in the finite-dimensional case, see [17, Section 5.3.D], shows that the map $\Phi:E\times C([0,\infty);\mathbb{R}^\infty)\to C([0,\infty);E)$ is $\mathscr{B}(E)\otimes\mathscr{B}(C([0,\infty);\mathbb{R}^\infty))/\mathscr{B}([0,\infty);E)$-measurable. Moreover, if we define $\mathscr{H}_t$ as the augmentation of $\mathscr{B}(E)\otimes\sigma(\mathbf{w}(s):s\le t)$ by the $\mu\otimes\mathbb{W}$-null sets and $\mathscr{I}_t:=\sigma(\mathbf{y}(s):s\le t)$, then Φ is $\mathscr{H}_t/\mathscr{I}_t$-measurable for every $t>0$.

By what was done so far, $x_0+\mathbf{y}_j=x_0+\Phi(x_0,\mathbf{w})$ $\mathbf{P}$-almost surly. Thus, for $j=1,2$, we have $\mathbf{X}_j=X_j(0)+\Phi(X_j(0),(W_H^j(\cdot)h_n)_{n\in\mathbb{N}})$ $\mathbb{P}_j$-almost surely. The measurability properties of Φ now imply that the solution $((\Omega_j,\Sigma_j,\mathbb{F},\mathbf{P}_j),W_H^j,\mathbf{X}_j)$ exists strongly for $j=1,2$. □

As a consequence of pathwise uniqueness, we find solutions of equation (1.1) on a *given* probability space and with respect to a *given* H-cylindrical Wiener process.

Corollary 5.4. *Assume that pathwise uniqueness holds for equation $[A,F,G]$ and that for some $\mu\in\mathscr{P}(E)$, there exists a weak solution of $[A,F,G]$ with initial distribution μ. Then, given any stochastic basis $(\Omega,\Sigma,\mathbb{F},\mathbf{P})$ on which an H-cylindrical Wiener process W_H with respect to $\mathbb{F}$ is defined and on which an $\mathscr{F}_0$-measurable random variable ξ with distribution μ is defined, there exists a process $\mathbf{X}$ such that $((\Omega,\Sigma,\mathbf{P}),\mathbb{F},W_H,\mathbf{X})$ is a weak solution of equation $[A,F,G]$ with $X(0)=\xi$.*

Proof. Let $((\Omega',\Sigma',\mathbf{P}'),\mathbb{F}',W_H',\mathbf{X}')$ be a weak solution of $[A,F,G]$ with $X'(0)\sim\mu$. The proof of Theorem 5.3 yields that $X'=X(0)+\Phi(X'(0),(W_H'(\cdot)h_n)_{n\in\mathbb{N}})$. We put $X:=\xi+\Phi(\xi,(W_H(\cdot)h_n)_{n\in\mathbb{N}})$.

Then the distribution of $(X'(0),\mathbf{X}',(W_H'(\cdot)h_n)_{n\in\mathbb{N}})$ under $\mathbf{P}'$ is the same as the distribution of $(\xi,\mathbf{X},(W_H(\cdot)h_n)_{n\in\mathbb{N}})$ under $\mathbf{P}$. Arguing as in the first part of the proof of Theorem 5.3, it follows that $((\Omega,\Sigma,\mathbf{P}),\mathbb{F},W_H,\mathbf{X})$ is a weak solution of equation $[A,F,G]$ with $X(0)=\xi$. □

6 Stochastic integration and mild solutions

We now address the question whether weak solutions of (1.1) are also mild solutions, i.e. for all $t\ge 0$ the $\mathscr{L}(H,E)$-valued process $s\mapsto S_{t-s}G(X_s)$ is stochastically integrable (in a sense to be made precise below) and we have, almost surely,

$$X_t=X_0+\int_0^t S_{t-s}F(X_s)\,ds+\int_0^t S_{t-s}G(X_s)\,dW_H(s)\,. \tag{6.1}$$

Having mild solutions, rather than weak solutions, has many advantages. In particular, one can make use of the *factorization method* [7]. The factorization method is useful to prove continuity of the paths of solutions which we have assumed throughout and also to establish the tightness assumption in Lemma 4.3, thus enabling us to construct solutions to stochastic differential equations.

In this section, we prove the equivalence of weak and mild solutions under additional assumptions on either equation $[A,F,G]$ or the state space E. As an intermediate step, we first consider *weakly mild solutions* in which we only require (6.1) to hold when tested against functionals $x^*\in\tilde{E}^*$.

6.1 Weakly mild solutions

Definition 6.1. *A tuple $\big((\Omega,\Sigma,\mathbb{F},\mathbb{P}),W_H,\mathbf{X}\big)$, where $(\Omega,\Sigma,\mathbb{F},\mathbb{P})$ is stochastic basis satisfying the usual conditions, W_H is an H-cylindrical Wiener process with respect to $\mathbb{F}$ and $\mathbf{X}$ is a continuous, $\mathbb{F}$-progressive, E-valued process is called a* weakly mild solution *of* (1.1) *if for all $x^*\in\tilde{E}^*$ and $t\geq 0$ we have*

$$\langle X_t,x^*\rangle = \langle S_tX_0,x^*\rangle + \int_0^t \langle S_{t-s}F(X_s),x^*\rangle\,ds + \int_0^t G(X_s)^*S_{t-s}^*x^*dW_H(s). \tag{6.2}$$

$\mathbb{P}$-a.e.

Remark 6.2. *By our assumptions on the coefficients A,F and G, the Lebesgue-integral and the stochastic integral in* (6.2) *are well-defined for all $t\geq 0$ and $x^*\in E^*$.*

Indeed, the map $(s,\omega)\mapsto F(X(s,\omega))$ is measurable as a composition of two measurable maps. Hence, it is the limit of a sequence of simple functions f_n almost everywhere with respect to $ds\otimes\mathbb{P}$. Thus

$$\langle S(t-\cdot)F(X),x^*\rangle = \lim\langle f_n,S(t-\cdot)^*x^*\rangle \quad ds\otimes\mathbb{P}-a.e.$$

We have $\langle f_n,S(t-\cdot)^*x^*\rangle = \sum_{j=1}^{N_n}\mathbb{1}_{A_{jn}}\langle x_{jn},S(t-\cdot)^*x^*\rangle$ for certain measurable sets A_{jn} and vectors $x_{jn}\in\tilde{E}$ and this is measurable since $s\mapsto\langle x,S(t-s)^*x^*\rangle$ is continuous for all $x\in\tilde{E}$ and $x^*\in\tilde{E}^*$. Hence $\langle S(t-\cdot)F(X),x^*\rangle$ is the limit of measurable functions $ds\otimes\mathbb{P}$ almost everywhere and thus measurable. In view of the continuity of the paths of $\mathbf{X}$, the boundedness of F on bounded sets and the boundedness of S on finite time intervals, it follows that for almost all ω the function $s\mapsto\langle S(t-s)F(X(s,\omega)),x^*\rangle$ is bounded, hence integrable.

The stochastic integral can be dealt with similarly, using the series expansion

$$G(X(s,\omega))^*S(t-s)^*x^* = \sum_k \langle G(X(s,\omega))h_k,S(t-s)^*x^*\rangle_H h_k$$

where (h_k) is a finite or countably infinite orthonormal basis of H.

We now prove that the notions 'weak solution' and 'weakly mild solution' are equivalent. Under additional assumptions which ensure that the stochastic convolution is well-defined, variations of this result (for mild solutions) have been proved in various settings, see [8, Theorem 5.4], [32, Theorem 7.1] or [37, Proposition 3.3]. Assuming that G is constant or that E is a UMD Banach space, in the following subsection we prove that weakly mild solutions are mild solutions. In particular, it *follows* that the stochastic convolution is well-defined.

We note that the adjoint semigroup S^* may not be strongly continuous, which causes technical difficulties. To overcome these, we will use results about the $\odot$*-dual semigroup* $S^\odot$. We recall some basic definitions and properties and refer the reader to [27] for more information.

Define $\tilde{E}^\odot := \overline{D(A^*)}$. Then $\tilde{E}^\odot$ is a closed, weak*-dense subspace of $\tilde{E}^*$ which is invariant under the adjoint semigroup. The restriction of the adjoint semigroup to $\tilde{E}^\odot$, denoted by $S^\odot$, is strongly continuous. In fact, one can prove that $\tilde{E}^\odot=\{x^*\in\tilde{E}^* : t\mapsto S(t)^*x^*$ is strongly continuous$\}$. We denote by $A^\odot$ the generator of $S^\odot$. Note that $A^\odot$ is exactly the part of A^* in $\tilde{E}^\odot$.

Proposition 6.3. *The weak and the weakly mild solutions of* (1.1) *coincide.*

Proof. First assume that $\mathbf{X}$ is a weak solution. For $n\in\mathbb{N}$, define

$$\tau_n := \inf\{t>0 : \|X(t)\|\geq n\}.$$

Since $\mathbf{X}$ is a weak solution, we have for $x^* \in D(A^*)$ and $t \geq 0$

$$\begin{aligned}\langle X_{t\wedge\tau_n}, x^*\rangle = \langle X_{0\wedge\tau_n}, x^*\rangle + \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\langle X_s, A^*x^*\rangle\, ds \\ + \int_0^t \mathbb{1}_{[0,\tau_n]}(s)\langle F(X_s), x^*\rangle\, ds + \int_0^t \mathbb{1}_{[0,\tau_n]}(s)G(X_s)^*x^*\, dW_H(s)\end{aligned}$$

almost surely. In view of Remark 3.5, we may (and shall) assume that the exceptional set does not depend on t. Below, we will suppress the statement '$\mathbb{P}$-almost surely'.

Fix $t > 0$ and let $f \in C^1([0,t])$ and $x^* \in D(A^*)$. Putting $\varphi := f \otimes x^*$, Itô's formula yields

$$\begin{aligned}\langle X_{t\wedge\tau_n}, \varphi(t)\rangle = \langle X_{0\wedge\tau_n}, \varphi(0)\rangle + \int_0^t \langle X_{s\wedge\tau_n}, \varphi'(s)\rangle\, ds + \int_0^{t\wedge\tau_n} \langle X_s, A^*\varphi(s)\rangle\, ds \\ + \int_0^{t\wedge\tau_n} \langle F(X_s), \varphi(s)\rangle\, ds + \int_0^t \mathbb{1}_{[0,\tau_n]}(s)G(X_s)^*\varphi(s)\, dW_H(s)\,.\end{aligned} \tag{6.3}$$

By linearity, the above equation holds for $\varphi = \sum_{k=1}^N f_k \otimes x_k^*$ where $f_k \in C^1([0,t])$ and $x_k^* \in D(A^*)$. Since $D(A^\odot)$ is a Banach space with respect to the graph norm, so is $C^1([0,t];D(A^\odot))$. Functions of the form $\varphi := \sum_{k=1}^n f_k \otimes x_k^*$ with $f_k \in C^1([0,t])$ and $x_k^* \in D(A^\odot)$ for $1 \leq k \leq n$ are dense in $C^1([0,t];D(A^\odot))$ and hence an approximation argument shows that (6.3) holds for all $\varphi \in C^1([0,t];D(A^\odot))$.

Now let $x^* \in D((A^\odot)^2)$ and $\varphi(s) = S_{t-s}^*x^*$. Then $\varphi \in C^1([0,t];D(A^\odot))$ with $\varphi'(s) = -S_{t-s}^*A^*x^*$. We note that $\int_0^t \langle X_{s\wedge\tau_n}, \varphi'(s)\rangle\, ds = \int_0^{t\wedge\tau_n} \langle X_s, \varphi'(s)\rangle\, ds + \int_{t\wedge\tau_n}^t \langle X_{\tau_n}, \varphi'(s)\rangle\, ds$, where the last term is zero if $\tau_n \geq t$. Thus equation (6.3) yields for this φ

$$\begin{aligned}\langle X_{t\wedge\tau_n}, x^*\rangle = \langle S_tX_{0\wedge\tau_n}, x^*\rangle + \int_0^{t\wedge\tau_n} \langle S_{t-s}F(X_s), x^*\rangle\, ds \\ - \int_{t\wedge\tau_n}^t \langle S_{t-s}X_{\tau_n}, A^*x^*\rangle\, ds + \int_0^t \mathbb{1}_{[0,\tau_n]}G(X_s)^*S_{t-s}^*x^*\, dW_H(s)\,.\end{aligned} \tag{6.4}$$

We next want to extend (6.4) to arbitrary $x^* \in \tilde{E}^*$. The term $\int_{t\wedge\tau_n}^t \langle S_{t-s}X_{\tau_n}, A^*x^*\rangle$ is obviously not well-defined for arbitrary $x^* \in \tilde{E}^*$. However, using the well-known fact that for $0 \leq a < b$ and $x \in \tilde{E}$ the integral $\int_a^b S(s)x\, ds$ belongs to the domain of the generator A and $A\int_a^b S(s)x\, ds = S(b)x - S(a)x$, it follows that

$$\int_{t\wedge\tau_n}^t \langle S_{t-s}X_{\tau_n}, A^*x^*\rangle\, ds = \langle S_{t-t\wedge\tau_n}X_{\tau_n} - X_{\tau_n}, x^*\rangle.$$

Since $D((A^\odot)^2)$ is sequentially weak*-dense in $\tilde{E}^*$, given $z^* \in \tilde{E}^*$, we find a sequence $x_k^* \in D((A^\odot)^2)$ such that $x_k^* \rightharpoonup^* z^*$. Arguing similar as in the proof of Lemma 4.1, we find a sequence y_m^* in the convex hull of the (x_k^*) such that $y_m^* \rightharpoonup^* z^*$ and

$$\mathbb{1}_{[0,\tau_n]}(\cdot)G(X(\cdot))^*S(t-\cdot)^*y_m^* \to \mathbb{1}_{[0,\tau_n]}(\cdot)G(X(\cdot))^*S(t-\cdot)^*z^*$$

in $L^2(\Omega)$. Thus, since $\mathbb{E}\big|\int_0^t \Phi(s)\, dW_H(s)\big|^2 = \|\Phi\|^2_{L^2(\Omega;L^2([0,t];H))}$ we see that

$$\int_0^t \mathbb{1}_{[0,\tau_n]}G(X(s))^*S(t-s)^*y_m^*\, dW_H(s) \to \int_0^t \mathbb{1}_{[0,\tau_n]}G(X(s))^*S(t-s)^*z^*\, dW_H(s)$$

in $L^2(\Omega;L^2(0,t;H))$. Passing to a subsequence, we may assume that we have convergence almost everywhere. Moreover, since (6.4) also holds for $x^* = y_m^*$, for all $m \in \mathbb{N}$, noting that

$$\mathbb{1}_{[0,\tau_n]}(s)\big|\langle S(t-s)F(X(s)), y_m^*\rangle\big| \leq \mathbb{1}_{[0,\tau_n]}(s)Me^{\omega(t-s)}B_n \cdot \sup_{m\in\mathbb{N}} \|y_m^*\|,$$

where M and ω are such that $\|S(t)\| \leq Me^{\omega t}$ for $t \geq 0$ and $B_n := \sup\{\|F(x)\| : \|x\| \leq n\}$, is follows from dominated convergence that $\int_0^{t\wedge\tau_n} \langle S_{t-s}F(X_s), y_m^*\rangle \, ds$ converges to $\int_0^{t\wedge\tau_n} \langle S_{t-s}F(X_s), z^*\rangle \, ds$ almost surely. It altogether we see that

$$\begin{aligned} \langle X_{t\wedge\tau_n}, z^*\rangle &= \langle S_t X_{0\wedge\tau_n}, z^*\rangle + \int_0^{t\wedge\tau_n} \langle S_{t-s}F(X_s), z^*\rangle \, ds \\ &+ \langle X_{\tau_n} - S_{t-t\wedge\tau_n}X_{\tau_n}, z^*\rangle + \int_0^t \mathbb{1}_{[0,\tau_n]} G(X_s)^* S_{t-s}^* z^* \, dW_H(s) \, . \end{aligned} \tag{6.5}$$

Upon letting $n \to \infty$, (6.2) is proved for arbitrary $x^* = z^*$.

We now prove the converse and assume that $\mathbf{X}$ is a weakly mild solution of (3.1). Fix $x^* \in D(A^*)$ and $t > 0$. Then for $0 < s < t$ we have

$$\begin{aligned} \langle X_s, A^*x^*\rangle = \langle S_s X_0, A^*x^*\rangle &+ \int_0^s \langle S_{s-r}F(X_r), A^*x^*\rangle \, dr \\ &+ \int_0^s G(X_r)^* S_{s-r}^* A^*x^* \, dW_H(r) \end{aligned} \tag{6.6}$$

almost surely. We note that the exceptional set may depend s. However, all terms in this equation are jointly measurable in s and ω. Hence, the left-hand side and the right-hand side of (6.6) are equal as elements of $L^0((0,t); L^0(\Omega))$. By the canonical isomorphism $L^0((0,t); L^0(\Omega)) \simeq L^0(\Omega; L^0(0,t))$, there exists a set $N \subset \Omega$ with $\mathbb{P}(N) = 0$ such that outside N equation (6.6) holds as an equation in $L^0(0,t)$, i.e. for almost every $s \in (0,t)$, where the exceptional set may depend on ω. Next note that by the continuity of the paths, the local boundedness of S and the boundedness of F on bounded sets, the first three terms are, as functions of s, $\mathbb{P}$-almost surely bounded on $(0,t)$ and hence belong to $L^1(0,t)$. Possibly enlarging N, we may assume that outside N equation (6.6) holds as an equation in $L^1(0,t)$. Integrating from 0 to t, we find that, $\mathbb{P}$-almost surely, we have

$$\begin{aligned} \int_0^t \langle X_s, A^*x^*\rangle \, ds = \int_0^t \langle S_s X_0, A^*x^*\rangle \, ds &+ \int_0^t \int_0^s \langle S_{s-r}F(X_r), A^*x^*\rangle \, dr \, ds \\ &+ \int_0^t \int_0^s G(X_r)^* x^* S_{s-r}^* A^*x^* \, dW_H(r) \, ds \, . \end{aligned} \tag{6.7}$$

Recall that for $x^* \in D(A^*)$ we have $\int_0^t S(s)^*A^*x^* \, ds = S(t)^*x^* - x^*$ for all $t \geq 0$. Here, the integral has to be understood as weak*-integral. Using this, we obtain, pathwise,

$$\int_0^t \langle S_s X_0, A^*x^*\rangle \, ds = \Big\langle X_0, \int_0^t S_s^* A^*x^* \, ds \Big\rangle = \langle X_0, S_t^*x^* - x^*\rangle = \langle S_t X_0 - X_0, x^*\rangle.$$

Using Fubini's theorem, we have

$$\begin{aligned} \int_0^t \int_0^s \langle S_{s-r}F(X_r), A^*x^*\rangle \, dr \, ds &= \int_0^t \Big\langle F(X_r), \int_r^t S_{s-r}^* A^*x^* \Big\rangle \, ds \, dr \\ &= \int_0^t \langle S_{t-r}F(X_r), x^*\rangle \, dr - \int_0^t \langle F(X_r), x^*\rangle \, dr \end{aligned}$$

pathwise. Using the stochastic Fubini theorem [29, Theorem 3.5], it follows that

$$\begin{aligned} \int_0^t \int_0^s G(X_r)^* S_{s-r}^* A^*x^* \, dW_H(r) \, ds &= \int_0^t \int_r^t G(X_r)^* S_{s-r}^* A^*x^* \, ds \, dW_H(r) \\ &= \int_0^t G(X_r)^* S_{t-r}^* x^* \, dW_H(r) - \int_0^t G(X_r)^* x^* \, dW_H(r) \end{aligned}$$

$\mathbb{P}$-almost surely.

Plugging these three identities into (6.7) and using that $\mathbf{X}$ is a mild solution, (3.1) follows. □

Since all terms appearing in (3.1) are almost surely continuous, there is no problem in writing an equation for the stopped process $\langle X_{t\wedge\tau}, x^*\rangle$ and we did this in the proof of Proposition 6.3. On the other hand, for weakly mild solutions, the integrand in the stochastic integral changes with t, causing problems to obtain an equation for the stopped process. In [3, Appendix], this problem was solved under the assumption that the stochastic convolution is almost surely continuous. In the proof of Proposition 6.3, we have shown that for a weak solution, (6.5) holds for all $x^* \in \tilde{E}^*$. Given a stopping time τ, we can repeat the arguments with τ_n replaced with $\tau_n \wedge \tau$ to obtain

Corollary 6.4. *If $\mathbf{X}$ is a weak (equivalently, weakly mild) solution of (1.1) and τ is a stopping time, then for all $t \geq 0$ and $x^* \in \tilde{E}^*$ we have*

$$\begin{aligned}\langle X_{t\wedge\tau}, x^*\rangle = \langle S_t X_{0\wedge\tau}, x^*\rangle + \int_0^{t\wedge\tau} \langle S_{t-s}F(X_s), x^*\rangle\, ds \\ + \langle X_\tau - S_{t-t\wedge\tau}X_\tau, x^*\rangle \mathbb{1}_{\{\tau<\infty\}} + \int_0^t \mathbb{1}_{[0,\tau]}(s) G(X_s)^* S_{t-s}^* x^*\, dW_H(s)\,.\end{aligned} \tag{6.8}$$

almost surely.

The question arises whether (6.2) can be extended to hold for all $x^* \in E^*$. This is indeed the case under the following additional assumption.

Hypothesis 6.5. *Assume Hypothesis 3.1, that $S(t) \subset \mathscr{L}(\tilde{E}, E)$ for all $t > 0$ and that for $x \in \tilde{E}$ the E-valued map $t \mapsto S(t)x$ is continuous on $(0,\infty)$. Furthermore, assume that for all $t > 0$ the function $(0,t) \ni s \mapsto \|S(s)\|_{\mathscr{L}(\tilde{E},E)}$ is square integrable.*

Assuming Hypothesis 6.5, a slight variation of the arguments in Remark 6.2 shows that in this case the integrals in (6.2) are well-defined for $x^* \in E^*$.

Corollary 6.6. *Assume that Hypothesis 6.5 holds. If $\mathbf{X}$ is a weak (equivalently, weakly mild) solution of (1.1), then (6.2) and (6.8) hold for all $x^* \in E^*$.*

Proof. Define

$$V := \{x^* \in E^* \,:\, \text{(6.2) holds a.e.}\,\}\,.$$

By Proposition 6.3, $\tilde{E}^* \subset V$ and hence V is weak*-dense in E^*. The claim is proved once we show that V is weak*-closed in E^*. By the Krein-Smulyan theorem (see, e.g., §21.10 (6) of [18]), V is weak*-closed in E^* if and only if $B_V := \{x^* \in V \,:\, \|x^*\|_{E^*} \leq 1\}$ is weak*-closed in E^*. However, since the weak*-topology is metrizable on bounded sets, it suffices to prove that B_V is sequentially weak*-closed.

Using Hypothesis 6.5, this can be proved similarly as when extending equation (6.4) from $x^* \in D((A^\odot)^2)$ to arbitrary $x^* \in \tilde{E}^*$ in the proof of Proposition 6.3. The proof for (6.8) is similar. □

6.2 Mild solutions

We begin by recalling some facts about stochastic integration of operator-valued processes. For time being, B denotes a general separable Banach space and H a separable Hilbert space. We also fix a stochastic basis $(\Omega, \Sigma, \mathbb{F}, \mathbb{P})$ satisfying the usual condition on which an H-cylindrical Wiener process with respect to $\mathbb{F}$ is defined.

An *elementary process* is a process $\Phi : [0,T]\times\Omega\to\mathscr{L}(H,B)$ of the form

$$\Phi(t,\omega)=\sum_{n=1}^{N}\sum_{m=1}^{M}\mathbb{1}_{(t_{n-1},t_n]\times A_{mn}}(t,\omega)\sum_{k=1}^{K}h_k\otimes x_{kmn}\,,$$

where $0\leq t_0<\cdots<t_N\leq T$, $A_{1n},\cdots,A_{Mn}\in\mathscr{F}_{t_{n-1}}$ are disjoint for all n and the vectors $h_1,\cdots,h_K$ are orthonormal in H. If Φ does not depend on ω we also say that Φ is an *elementary function*. For an elementary process, the *stochastic integral* $\int_0^T\Phi(t)\,dW_H(t)$ is defined by

$$\int_0^T\Phi(t)\,dW_H(t):=\sum_{n=1}^{N}\sum_{m=1}^{M}\mathbb{1}_{A_{mn}}\sum_{k=1}^{K}\big[W_H(t_n)h_k-W_H(t_{n-1})h_k\big]x_{kmn}$$

Now let $\Phi:[0,T]\times\Omega\to\mathscr{L}(H,B)$ be an H-strongly measurable and adapted process which belongs to $L^2(0,T;H)$ scalarly, i.e. $\Phi^*x^*\in L^0(\Omega;L^2(0,T;H))$ for all $x^*\in B^*$. Then Φ is called *stochastically integrable* (on $(0,T)$) if there exists a sequence Φ_n of elementary processes and an $C([0,T];E)$-valued random variable η such that

1. $\langle\Phi_nh,x^*\rangle\to\langle\Phi h,x^*\rangle$ in $L^0(\Omega;L^2(0,T))$ for all $h\in H$ and $x^*\in B^*$ and
2. We have
$$\eta(\cdot)=\lim_{n\to\infty}\int_0^{\cdot}\Phi_n(t)\,dW_H(t)\quad\text{in }L^0(\Omega;C([0,T];B))\,.$$

In this case, η is called the stochastic integral of Φ and we write $\int_0^t\Phi(t)\,dW_H(t):=\eta(t)$. In the case where Φ does not depend on ω, we also require that the approximating sequence Φ_n does not depend on ω.

Having defined stochastic integrability, we can now define what we mean by a *mild solution*.

Definition 6.7. *A tuple* $((\Omega,\Sigma,\mathbb{F},\mathbb{P}),W_H,\mathbf{X})$ *where* $(\Omega,\Sigma,\mathbb{F},\mathbb{P})$ *is stochastic basis satisfying the usual conditions,* W_H *is an* H*-cylindrical Wiener process with respect to* $\mathbb{F}$ *and* $\mathbf{X}$ *is a continuous,* $\mathbb{F}$*-progressive,* E*-valued process is called a* mild solution *of* (1.1) *if for all* $t\geq0$ *the function* $s\mapsto S(t-s)G(X(s))$ *is stochastically integrable and* (6.1) *holds almost surely.*

It is clear from the definition of stochastic integrability, that every mild solution of equation $[A,F,G]$ is also a weakly mild solution of $[A,F,G]$ and thus, by Proposition 6.3, also a weak solution of $[A,F,G]$. Moreover, if $\mathbf{X}$ is a mild solution, then (6.2) even holds for all $x^*\in E^*$ (rather than for $x^*\in\tilde{E}^*$) and the exceptional set outside of which (6.2) holds can be chosen independently of x^*. We also note that if $\mathbf{X}$ is a weak (hence a weakly mild) solution and it is known a priori that $s\mapsto S(t-s)G(X(s))$ is stochastically integrable, then $\mathbf{X}$ is a mild solution.

The obvious question is whether for a weak solution $\mathbf{X}$ the process $s\mapsto S_{t-s}G(X_s)$ is automatically stochastically integrable. As we shall see, this is indeed the case in two important cases. The proof relies on a characterization of stochastic integrability of a process Φ. Let us first discuss the case of $\mathscr{L}(H,B)$-valued *functions*, which was considered in [32]. It was proved there that a function $\Phi:[0,T]\to\mathscr{L}(H,B)$ is stochastically integrable if and only if there exists an B-valued random variable ξ such that

$$\langle\xi,x^*\rangle=\int_0^T\Phi(s)^*x^*\,dW_H(s).\tag{6.9}$$

This, in turn, is equivalent with Φ representing a γ-Radonifying operator. We write $\gamma(L^2(0,T;H),B)$ for the space of γ-Radonifying operators from $L^2(0,T;H)$ to B. For the definition of γ-Radonifying operators and more information, we refer to the survey article [28]. That Φ *represents* an operator $R \in \gamma(L^2(0,T;H),B)$ means that for all $x^* \in B^*$ the function $t \mapsto \Phi^*(t)x^*$ belongs to $L^2(0,T;H)$ and we have

$$\langle Rf, x^* \rangle = \int_0^T [f(t), \Phi^*(t)x^*]_H \, dt \quad \forall\, f \in L^2(0,t;H)\,,\; x^* \in B^*. \tag{6.10}$$

Note that if Φ is H-strongly measurable, then the operator R is uniquely determined by Φ.

Using the results of [32], we obtain for (1.1) with additive noise:

Proposition 6.8. *Assume Hypotheses 3.1 and 6.5 and that $G \in \mathscr{L}(H, \tilde{E})$ is constant. Then the weak, the weakly mild and the mild solutions of* (1.1) *coincide. Furthermore, if there exist solutions, the function $s \mapsto S_{t-s}G$ represents an element of $\gamma(L^2(0,t;H),E)$ for all $t > 0$.*

Proof. Let $\mathbf{X}$ be a weak (equivalently, a weakly mild) solution of (1.1). If no such solution exists, there is nothing to prove since every mild solution is also a weakly mild solution.

Arguing as Remark 6.2, using that as a consequence of Hypothesis 6.5 the map $s \mapsto \langle x, S_{t-s}^* x^* \rangle$ is continuous even for $x^* \in E^*$ and $x \in \tilde{E}$, we see that $(s,\omega) \mapsto \langle S(t-s)F(X(s,\omega)), x^* \rangle$ is measurable for all $x^* \in E^*$. By Hypothesis 6.5, $\|S_s\|_{\mathscr{L}(\tilde{E},E)}$ is majorized on $(0,t)$ by a square integrable function, say g. Hence, by the boundedness of F on bounded sets we have

$$\|S_{t-s}F(X(s,\omega))\| \le g(t-s) \sup_{r \in (0,t)} \|F(X(r,\omega)\| \in L^1(0,t).$$

This implies that $\int_0^t S_{t-s}F(X_s)\, ds$ can be defined pathwise as an E-valued Bochner integral. Furthermore, this integral is a weakly measurable function of ω. Since E is separable, $\int_0^t S_{t-s}F(X_s)\, ds$ is a strongly measurable function of ω by the Pettis measurability theorem. Consequently, $\xi := X_t - S_t X_0 - \int_0^t S_{t-s}F(X_s)\, ds$ is an E-valued random variable. Since $\mathbf{X}$ is a weakly mild solution, (6.9) holds for $T := t, \Phi : s \mapsto S_{t-s}G$ and all $x^* \in E^*$ by Corollary 6.6. The claim follows from the results of [32]. $\square$

Let us now return to our discussion of stochastic integrability in a general separable Banach space B. In order to have a powerful integration theory for $\mathscr{L}(H,B)$-valued *processes*, we need an additional geometric assumption on B. Of particular importance are the so-called *UMD Banach spaces*. For the definition of UMD spaces and more information, we refer to the survey article [4]. We here confine ourselves to note that every Hilbert space is a UMD space as are the reflexive L^p and Sobolev spaces.

The importance of the UMD property for stochastic integration is that it allows for so-called decoupling, see [10, 23]. Roughly speaking, this enables us to replace the cylindrical Wiener process W_H by an independent copy $\tilde{W}_H$ and thus use the results of [32] pathwise. This program was carried out in [30] and yields a similar characterization of stochastic integrability as in [32] in the case of processes which belong scalarly to $L^p(\Omega; L^2(0,T;H))$. We recall that $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E))$ is said to *belong to* $L^p(\Omega; L^2(0,T;H))$ *scalarly*, if for every $x^* \in E^*$ the function $t \mapsto \Phi^*(t,\omega)x^*$ belongs to $L^2(0,T;H)$ for almost every ω and the map $\omega \mapsto \Phi^*(\cdot,\omega)x^*$ belongs to $L^p(\Omega; L^2(0,T;H))$.

It is proved in [30] that an H-strongly measurable and adapted process $\Phi : [0,T] \times \Omega \to \mathscr{L}(H,E)$ which belongs to $L^p(\Omega; L^2(0,T;H))$ scalarly is stochastically integrable if and only if there is a random variable $\xi \in L^p(\Omega; E)$ such that (6.9) holds for all $x^* \in E^*$. This in turn is the case if and only if Φ represents a random variable $R \in L^p(\Omega; \gamma(L^2(0,T;H),E))$. Here 'represents' means that (6.10) holds for almost every ω.

A characterization of stochastic integrability for processes Φ which belong scalarly to $L^0(\Omega; L^2(0,T;H))$ is also contained in [30], however, in this characterization one needs information about the whole integral process $\int_0^\cdot \Phi(s)dW_H(s)$; when dealing with weakly mild solutions, such information is not available, whence this characterization cannot be used for our purposes. Therefore, in the proposition below, we use a stopping time argument to reduce to the $L^p(\Omega)$-case.

Proposition 6.9. *Assume Hypotheses 3.1 and 6.5 and that E is a UMD Banach space. Then the weak, the weakly mild and the mild solutions of* (1.1) *coincide. Furthermore, if $\mathbf{X}$ is a weak solution, then for all $t \geq 0$ the function $s \mapsto S_{t-s}G(X_s)$ represents an element of the space $L^0(\Omega, \gamma(L^2(0,t;H), E))$.*

Proof. Let $\mathbf{X}$ be a weak (equivalently, a weakly mild) solution of (1.1). If no weak solution exists, there is nothing to prove.

For $n \in \mathbb{N}$ and define $\tau_n := \inf\{s > 0 : \|X_s\| \geq n\}$. Fix $t > 0$. Arguing similar as in the proof of Proposition 6.8, we see that

$$\xi_n := X_{t\wedge\tau_n} - (X_{\tau_n} - S_{t-t\wedge\tau_n}X_{\tau_n})\mathbb{1}_{\{\tau_n<\infty\}} - S_tX_{0\wedge\tau_n} - \int_0^t \mathbb{1}_{[0,\tau_n]}S_{t-s}F(X_s)\,ds$$

is a well-defined, bounded, E-valued random variable. It follows from Corollary 6.6, that for $x^* \in E^*$,

$$\langle \xi_n, x^*\rangle = \int_0^t \mathbb{1}_{[0,\tau_n]}G(X_s)^*S_{t-s}^*x^*\,dW_H(s)\,.$$

almost surely. Since $\mathbf{X}$ has continuous paths and G is bounded on bounded subsets, $\Phi_n : s \mapsto \mathbb{1}_{[0,\tau_n]}S_{t-s}G(X_s)$ belongs to $L^\infty(\Omega; L^2(0,t;H))$ scalarly. Hence, by [30, Theorem 5.9], Φ_n is stochastically integrable and

$$\begin{aligned} X_{t\wedge\tau_n} = \; & S_tX_{0\wedge\tau_n} + X_{\tau_n} - S_{t-t\wedge\tau_n}X_{\tau_n} \\ & + \int_0^{t\wedge\tau_n} S_{t-s}F(X_s)\,ds + \int_0^t \mathbb{1}_{[0,\tau_n]}S_{t-s}G(X_s)\,dW_H(s)\,. \end{aligned} \tag{6.11}$$

Furthermore, Φ_n represents an element of $L^p(\Omega; \gamma(L^2(0,t;H),E))$ for all $p \geq 1$. Now let N be a set with $\mathbb{P}(N) = 0$ such that for $\omega \notin N$ the map $s \mapsto \Phi_n(s,\omega)$ represents an element $R_n(\omega)$ of $\gamma(L^2(0,t;H),E)$. Such a set exists by [30, Lemma 2.7].

Note that by the continuity of the paths, $\Phi_n(s,\omega) = \Phi(s,\omega) := S_{t-s}G(X(s,\omega))$ for all $s \in (0,t)$ and $n \geq n_0 = n_0(\omega)$. It follows that$\Phi(s,\omega)$ represents an element $R(\omega)$ of $\gamma(L^2(0,t;H),E)$ for all $\omega \notin N$. Since $R_n(\omega) \to R(\omega)$ for all $\omega \notin N$, R is a strongly measurable $\gamma(L^2(0,t;H),E)$-valued random variable. Furthermore, R is represented by Φ. By [30, Theorem 5.9], Φ is stochastically integrable and [30, Theorem 5.5] shows that

$$\int_0^t \Phi_n(s)\,dW_H(s) \to \int_0^t \Phi(s)\,dW_H(s) \quad \text{in } L^0(\Omega;E)\,.$$

On the other hand,

$$\xi_n \to X(t) - S(t)X(0) - \int_0^t S(t-s)F(X(s))\,ds$$

pointwise a.e. and hence in $L^0(\Omega;E)$. Thus, letting $n \to \infty$ in (6.11) finishes the proof. □

7 Applications

We end this article by discussing some examples of stochastic partial differential equations where the results of this article can be applied.

7.1 Equations with measurable semilinear term and additive noise

In [20], we are concerned with the following equation

$$dX(t) = \big[AX(t) + F(X(t)\big] + GdW_H(t) \tag{7.1}$$

where $E, \tilde{E}, H$ and A are as in Hypothesis 3.1, the semilinear term $F : E \to E$ is bounded and measurable, W_H is an H-cylindrical Wiener process and $G \in \mathscr{L}(H, \tilde{E})$. In the case where $F \equiv 0$, this is an Ornstein-Uhlenbeck equation, which is well understood. If the Ornstein-Uhlenbeck equation associated with (7.1), i.e. equation $[A, 0, G]$ is well-posed, the associated transition semigroup $\mathscr{T}_{\mathsf{ou}}$ is known explicitly. Namely,

$$\mathscr{T}_{\mathsf{ou}}(t)f(x) = \int_E f(S(t)x + y)\, d\mathscr{N}_{Q_t}(y)$$

where $\mathscr{N}_Q$ denotes the centered Gaussian measure with covariance operator Q and $Q_t : E^* \to E$ is given as

$$Q_t x^* := \int_0^t S(s)GG^*S(s)^*x^*\, ds.$$

By H_{Q_t}, we denote the reproducing kernel Hilbert space associated with Q_t. In [20], the following theorem is proved.

Theorem 7.1. *Let $E, \tilde{E}, H$ and A as in Hypothesis 3.1, $G \in \mathscr{L}(H, \tilde{E})$ and assume that also Hypothesis 6.5 is satisfied. Moreover, assume that the Ornstein-Uhlenbeck equation $[A, 0, G]$ is well-posed and that $S(t)E \subset H_{Q_t}$ for all $t > 0$ with*

$$\int_0^T \|S(t)\|_{\mathscr{L}(E, H_{Q_t})}\, dt < \infty \tag{7.2}$$

for all $T > 0$. Then for every bounded, measurable $F : E \to E$ equation (7.1) *is well-posed. The solutions are strong Markov processes with a strong Feller transition semigroup.*

This extends earlier results from [6, 11, 12] where the corresponding equation was studied for bounded and continuous (resp. bounded and weakly continuous) F under similar assumptions in the case where $E = \tilde{E}$ is a Hilbert space. The assertion that (7.1) is well-posed even for bounded *measurable* F appears to be new even in the case of Hilbert spaces since existence of solutions cannot be inferred from the Girsanov theorem, as G is, in general, not invertible.

The assumption that (7.2) holds implies that the transition semigroup $\mathscr{T}_{\mathsf{ou}}$ is strongly Feller and is satisfied in many important examples, for example for the one-dimensional stochastic heat equation driven by space-time white noise, i.e. A is the L^p-realization of the Dirichlet Laplacian on the interval $(0, 1)$ and for $p \leq 2$ we set the operator G is the injection from $L^2(0, 1)$ to $L^p(0, 1)$. In the case $p > 2$ we set $\tilde{E} = L^2(0, 1)$ and G the identity. It is also possible to consider the stochastic heat equation on $C_0(0, 1)$. More examples, which include equations in higher space dimension, more general differential operators and different noise terms are discussed in [20].

The proof of Theorem 7.1 is based on Theorem 3.6, and we prove existence and uniqueness of solutions of the associated local martingale problem. The actual proof of existence and uniqueness is then given using semigroup theory. In view of Theorem 4.2, the strong Markov property for solutions follows automatically once we have established well-posedness of $[A, F, G]$.

The first step to prove uniqueness for solutions of (7.1) is to prove a Miyadera-Voigt type perturbation result for strongly Feller semigroups. For the generator $\mathscr{A}_{\mathsf{ou}}$

of the Ornstein-Uhlenbeck semigroup $\mathscr{T}_{\mathsf{ou}}$, this result can be used to show that $\mathscr{A}_{\mathsf{pert}}$, defined by $\mathscr{A}_{\mathsf{pert}}u(x) := \mathscr{A}_{\mathsf{ou}}u(x) + \langle F(x), \nabla u(x)\rangle$, generates a strongly Feller semigroup $\mathscr{T}_{\mathsf{pert}}$. A detailed analysis of the operator $\mathscr{A}_{\mathsf{pert}}$ shows that a probability measure $\mathbf{P}$ on $C([0,\infty);E)$ solves the local martingale problem associated with equation $[A,F,G]$ if and only if it solves the true martingale problem (in the sense of [9]) for the operator $\mathscr{A}_{\mathsf{pert}}$. Thus a well-known result [9, Theorem 4.4.1] yields that the one-dimensional distributions of a solution $\mathbf{P}$ of the martingale problem for $\mathscr{A}_{\mathsf{pert}}$ are determined by the distribution of $\mathbf{x}(0)$ under $\mathbf{P}$ and the semigroup $\mathscr{T}_{\mathsf{pert}}$. By Theorem 2.2, this implies uniqueness in law for the solutions of equation (7.1). Moreover, if solutions exist, then the associated transition semigroup is $\mathscr{T}_{\mathsf{pert}}$, which is strongly Feller.

It thus remains to prove existence of solutions. If F is additionally Lipschitz continuous, then solutions can be constructed using Banach's fixed point theorem in a standard way. Thus, for bounded, Lipschitz continuous F, equation (7.1) is well-posed. To extend the existence result to general bounded, measurable F, a refinement of Lemma 4.3 is used. Indeed, making use of the strong Feller property, it can be proved that if F_n is a sequence of bounded measurable functions such that equation $[A,F_n,G]$ is well-posed for every n and the sequence F_n is uniformly bounded and converges pointwise to the bounded function F, then also equation $[A,F,G]$ is well-posed. The tightness of the solutions to the local martingale problem for $[A,F_n,G]$ can be proved using that these measures are distributions of mild solutions of the equation. Using the approximation result, well-posedness of (7.1) can be extended from bounded, Lipschitz continuous F to bounded, measurable F via a monotone class argument.

7.2 Stochastic reaction-diffusion systems with Hölder continuous multiplicative noise

Reaction-diffusion systems and stochastic perturbations of them play an important role in applications in chemistry, biology and physics [25]. In an abstract form, a stochastic reaction-diffusion system takes the form (1.1), where the state space E is a Banach space of $\mathbb{R}^r$-valued functions, defined on a domain $\mathscr{O} \subset \mathbb{R}^d$. Typically, the reaction term F is a vector of composition operators with polynomial entries.

Such systems with locally Lipschitz continuous multiplicative noise where studied in [5]. In the case where the noise term G is merely Hölder continuous, only partial results are available and, to the best of our knowledge, only for $r = 1$, i.e. a single reaction-diffusion equation rather than a system. In [2], existence of solutions for such an equation was proved under an additional boundedness assumption on G. However, a uniqueness result is missing, except for the case of locally Lipschitz continuous G.

In [19], we prove pathwise uniqueness and strong existence of solutions for a class of stochastic reaction-diffusion equations with Hölder continuous multiplicative noise. Let us here present an example which fits into the framework of [19] and explain how results of this article are used in the proof of existence and uniqueness.

Let $\mathscr{O} \subset \mathbb{R}^d$ be an open domain with Lipschitz boundary. Moreover, we let $a_1 = (a_{ij}^{(1)}), a_2 = (a_{ij}^{(2)}) \in L^\infty(\mathscr{O};\mathbb{R}^{d\times d})$ be symmetric and uniformly elliptic, i.e. there exists $\eta > 0$ such that for all $\xi \in \mathbb{R}^d$ and almost all $x \in \mathscr{O}$ we have

$$\sum_{i,j=1}^d a_{ij}^{(l)}(x)\xi_i\xi_j \geq \eta|\xi|^2$$

for $l = 1,2$. Let R_1, R_2 be Hilbert-Schmidt operators on $L^2(\mathscr{O})$ such that R_j is diagonalized by an orthonormal basis $(e_n^{(j)})_{n\in\mathbb{N}}$ of $L^2(\mathscr{O})$ which consists of functions in $C(\overline{\mathscr{O}})$ and satisfies $\sum_{n=1}^\infty \|R_j e_n^{(j)}\|_\infty^2 < \infty$ for $j = 1,2$. Finally, we let $g_1, g_2 : \mathbb{R} \to \mathbb{R}$ be of linear growth and locally $\frac{1}{2}$-Hölder continuous. We consider the following stochastic reaction-diffusion system

$$\left\{\begin{array}{lcl} du_1(t) & = & \big[\mathrm{div}(a_1\nabla u_1(t)) + u_1(t) - u_1(t)^3 + u_2(t)\big]dt + g_1(u_1(t))R_1 dW_1(t) \\ du_2(t) & = & \big[\mathrm{div}(a_2\nabla u_2(t)) + u_1(t) - u_2(t)\big]dt + g_2(u_2(t))R_2 dW_2(t) \end{array}\right. \tag{7.3}$$

complemented with conormal boundary conditions.

To reformulate the above system in our abstract framework, we set $\tilde{E} = E = C(\overline{\mathscr{O}}) \times C(\overline{\mathscr{O}})$ and $A = \mathrm{diag}(A_1, A_2)$, where A_j is the $C(\overline{\mathscr{O}})$-realization of the differential operator $\div(a_j\nabla\cdot)$ under conormal boundary conditions. We set $H = L^2(\mathscr{O}) \times L^2(\mathscr{O})$. By the assumption on R_j, for $h \in L^2(\mathscr{O})$ we find that $R_j h \in C(\overline{\mathscr{O}})$. We may thus define $G : E \to \mathscr{L}(H, E)$ by

$$[G(u,v)h](x) := (g_1(u(x))R_1 h_1(x), g_2(u(x))R_2 h_2(x))$$

for $h_1, h_2 \in L^2(\mathscr{O})$ and $x \in \overline{\mathscr{O}}$. The reaction term F is given by $[F(u,v)](x) := (u(x) - u(x)^3 + v(x), u(x) - v(x))$. This reaction Term is of Fitzhugh-Nagumo type and equations with this reaction term are generic excitable systems [25].

In [19] we prove

Theorem 7.2. *Under the assumptions above, equation* (7.3) *is well-posed on the state space $E = C(\overline{\mathscr{O}}) \times C(\overline{\mathscr{O}})$. The solutions exist strongly, they are pathwise unique and strong Markov processes.*

The proof of Theorem 7.2 is in spirit rather different from the proof of well-posedness of (7.1), insofar as we work directly with solutions of the equation, rather than with solutions of the associated local martingale problem. In the proof, we use the equivalence of weak and mild solutions. Indeed, in the proof of pathwise uniqueness, we use weak solutions, whereas in the proof of existence of solutions, we use mild solutions. We also employ the Yamada-Watanabe theory from Section 5.

The proof of pathwise uniqueness is an adaption of the proof of [39, Theorem 1]. The main difficulty in extending the proof from the finite-dimensional setting to an infinite dimensional setting is to handle the differential operators involved in (7.3). In [19], we use the concept of a weak solution and test solutions against functionals $x^* = (\lambda R(\lambda, A_1)^*\delta_x, 0)$, resp. $x^* = (0, \lambda R(\lambda, A_2)^*\delta_x)$, where A_j are the realizations of of the differential operator $\div(a_j\nabla\cdot)$ on $C(\overline{\mathscr{O}})$. This approach should be compared with [26], where pathwise uniqueness was proved for stochastic heat equations on $\mathscr{O} = \mathbb{R}^d$, namely

$$du(t) = \Delta u(t) + \sigma(u(t))dW(t),$$

where Δ is the Laplacian on $\mathbb{R}^d$, W is a colored noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is γ-Hölder continuous, where the allowed value of γ depends on the noise W. To prove pathwise uniqueness in [26], the authors convolute solutions of the stochastic heat equation with a mollifier φ_n. In their variational framework, this yields the term $u * \Delta\varphi_n$ in the equation for the resulting process. It is then used that, as a consequence of its translation invariance, the Laplacian commutes with convolutions, i.e. we have $u * (\Delta\varphi_n) = \Delta(u * \varphi_n)$. This is no longer true for differential operators with nonconstant coefficients as in (7.3).

Let us also note that a recent result [24] for the stochastic heat equation that in the case of $d = 1$ shows that we cannot hope for pathwise uniqueness in the case of space-time white noise.

Note that by Theorem 5.3, pathwise uniqueness implies uniqueness in law, hence the strong Markov property of solutions follows from Theorem 2.2 once we have established existence of solutions. To that end, we approximate the function f in the reaction term and the functions g_1, g_2 with bounded functions by cutting off the functions. Existence of solutions for the approximate problems with bounded coefficients and deterministic initial values follows from the results of [2]. We could then use Lemma 4.3 to infer existence of solutions for the limit problem (7.3). However, in [19] we choose a different approach and use that, as a consequence of pathwise uniqueness and Corollary 5.4, the approximate solutions can be realized on a common stochastic basis and with respect to a common H-cylindrical Wiener process. This allows us to adopt the strategy from [5, 21] to prove existence of solutions. Indeed, as the approximate solutions exist on a common stochastic basis and are pathwise unique, they can be 'glued together' to a 'maximal solution' of equation (7.3). To prove existence of solutions in the sense used here, we have to prove that the 'maximal solution' exists globally. By the results of [21], to that end, we have to prove uniform boundedness of the approximate solutions in $L^p(\Omega; C([0,T]; E))$ for a suitable $p > 1$, all $T > 0$ and p-integrable initial data. As the approximate solutions are also *mild* solutions, the uniform boundedness can be proved using estimates for deterministic and stochastic convolutions, see [31].

We note that, in comparison with [2], in Theorem 7.2 we do not need that the term G is bounded. Moreover, with the above arguments, we initially prove existence of solutions only for initial data with a certain integrability, thus in particular for deterministic initial data. However, by Theorem 2.2, we automatically obtain existence of solutions for all initial distributions.

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