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Inequalities for permanental processes

Nathalie Eisenbaum*

Abstract

Permanental processes are a natural extension of the definition of squared Gaussian processes. Each one-dimensional marginal of a permanental process is a squared Gaussian variable, but there is not always a Gaussian structure for the entire process. The interest to better know them is highly motivated by the connection established by Eisenbaum and Kaspi, between the infinitely divisible permanental processes and the local times of Markov processes. Unfortunately the lack of Gaussian structure for general permanental processes makes their behavior hard to handle. We present here an analogue for infinitely divisible permanental vectors, of some well-known inequalities for Gaussian vectors.

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1 Introduction

A real-valued positive vector $(\psi_i, 1 \leq i \leq n)$ is a permanental vector if its Laplace transform satisfies for every $(\alpha_1, \alpha_2, ..., \alpha_n)$ in $\mathbb{R}_+^n$

$$\mathbb{E}[\exp\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_i\}] = |I + G\alpha|^{-1/\beta} \tag{1.1}$$

where I is the $n \times n$-identity matrix, α is the diagonal matrix diag$(\alpha_i)_{1\leq i\leq n}$, $G = (G(i,j))_{1\leq i,j\leq n}$ and β is a fixed positive number.
Such a vector $(\psi_i, 1 \leq i \leq n)$ is a permanental vector with kernel $(G(i,j), 1 \leq i, j \leq n)$ and index β.
Necessary and sufficient conditions for the existence of permanental vectors have been established by Vere-Jones [14].
Permanental vectors represent a natural extension of squared centered Gaussian vectors. Indeed for $\beta = 2$ and G positive definite matrix, (1.1) is the Laplace transform of a squared Gaussian vector: a vector $(\eta_1^2, \eta_2^2, ..., \eta_n^2)$ with $(\eta_1, \eta_2, ..., \eta_n)$ centered Gaussian vector with covariance G.

*CNRS, Université Pierre et Marie Curie (Paris 6), France. E-mail: nathalie.eisenbaum@upmc.fr

The recent extension of Dynkin isomorphism theorem [5] (reminded at the beginning of Section 2) to non necessarily symmetric Markov processes suggests that the path behavior of local times of Markov processes should be closely related to the path behavior of infinitely divisible permanental processes. The problem is that permanental processes are new objects of study. The original version of Dynkin isomorphism theorem connects local times of symmetric Markov processes to squared Gaussian processes. The successful uses of this identity (see [1], [12] or [3]) are mostly based on inequalities specific to Gaussian vectors such as Slepian Lemma, Sudakov inequality, or concentration inequalities. Hence the preliminary question to face, in order to exploit the extended Dynkin isomorphism theorem, seems to be the existence of analogous inequalities for permanental vectors.
Here we provide some answers to this first question. We establish in Section 2 a tool (Lemma 2.2) to stochastically compare permanental vectors with index $1/4$. The choice of the index is due to technical reasons (see Lemma 2.1), but one notes that infinitely divisible permanental processes are related to local times independently of their indexes. The obtained tool allows then to present in Section 3, inequalities analoguous to Slepian lemma for infinitely divisible permanental vectors and a weak version of Sudakov inequality in Section 4. In Section 5, some concentration inequalities are proved.

2 A tool

We will use the extension of Dynkin's isomorphism Theorem [4] to non necessarily symmetric Markov process established in [5]. Consider a transient Markov process X with state space E and Green function $g = (g(x,y),(x,y) \in E \times E)$. We have shown that there exists a permanental process $(\phi_x, x \in E)$, independent of X, with kernel g and index 2. We have proved that infinite divisibility characterizes the permanental processes admitting the Green function of a Markov process for kernel. Let a and b be elements of E. Denote by $(L^{ab}_x, x \in E)$ the process of the total accumulated local times of X conditionned to start at a and killed at its last visit to b. Then the process $(L^{aa}_x + \frac{1}{2}\phi_x, x \in E)$ has the law of the process $(\frac{1}{2}\phi_x, x \in E)$ under the probability $\frac{1}{\mathbb{E}[\phi_a]}\mathbb{E}[\phi_a, .]$.

Now let $(\psi_x, x \in E)$ denote a permanental process, independent of X, with kernel g and index β (such a process exists thanks to the infinite infinite divisibility of ϕ). Then similarly to the above relation, one shows that for every $\beta > 0$, the process $(L^{aa}_x + \frac{1}{2}\psi_x, x \in E)$ has the law of the process $(\frac{1}{2}\psi_x, x \in E)$ under the probability $\frac{1}{\mathbb{E}[\psi_a]}\mathbb{E}[\psi_a, .]$.

We start by showing the existence of a nice density with respect to the Lebesgue measure for permanental vectors with index $1/4$.

Lemma 2.1. *A permanental vector $(\psi_1, \psi_2, ..., \psi_n)$ with index $1/4$ admits a density h with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover h is C^2 with first and second derivatives converging to 0 as $|z|$ tends to 0.*

Proof: Denote by $\hat{\mu}(z)$ the Fourier transform of a permanental vector with index 2. Then one checks that : $\int_{\mathbb{R}^n} |\hat{\mu}(z)|^2 dz < \infty$. Hence $\mu * \mu * \mu * \mu$ admits a continuous density with respect to the Lebesgue measure. We note then that: $\int_{\mathbb{R}^n} |\hat{\mu}(z)|^4 |z|^2 dz < \infty$, which thanks to Proposition 28.1 in Sato's book [13] (p.190) implies that the density of μ^{*8} has a C^2 density with first and second derivatives converging to 0 as $|z|$ tends to 0. □

Let G be a $n \times n$-matrix such that there exists a permanental vector with index $1/4$ and kernel G. For any measurable function F on $\mathbb{R}^n_+$, $\mathbb{E}_G[F(\psi)]$ denotes the expectation with respect to a permanental vector with kernel G and index $1/4$. We define a functional $\mathcal{F}$

on such matrices G by setting

$$\mathcal{F}(G) = \mathbb{E}_G[F(\psi)]$$

We denote by $C_{kj}(G)$ the entry G_{kj}.
We now compute the derivatives of $\mathcal{F}$ with respect to C_{kj}. We have the following lemma.

Lemma 2.2. *Let $\psi = (\psi_{x_k})_{1\leq k\leq n}$ be a permanental vector with kernel $(G(x_k, x_j), 1 \leq k, j \leq n)$ and index $1/4$. Let F be a bounded real valued function on $\mathbb{R}^n_+$, admitting bounded second order derivatives. We have then:*

$$\frac{\partial}{\partial C_{kk}}\mathbb{E}_G[F(\frac{\psi}{2})] = 4\mathbb{E}_G[\frac{\partial F}{\partial z_k}(\frac{\psi}{2})] + \frac{1}{2}\mathbb{E}_G[\psi_{x_k}\frac{\partial^2 F}{\partial z_k^2}(\frac{\psi}{2})]. \tag{2.1}$$

Assume moreover that ψ is infinitely divisible. For $k \neq j$, we have:

$$\frac{\partial}{\partial C_{kj}}\mathbb{E}_G[F(\frac{\psi}{2})] = 4G(x_j, x_k)\mathbb{E}_G[\frac{\partial^2 F}{\partial z_k \partial z_j}(\frac{\psi}{2} + L^{x_j x_k})], \tag{2.2}$$

where $L^{x_j x_k}$ is a vector independent of ψ with the law of the total accumulated local time of an associated Markov process conditionned to start at x_j and killed at its last visit to x_k.

Remark 2.2.1: Note that (2.2) completes the extended version of Dynkin isomorphism theorem presented above. Indeed this extended version involves the process L^{ab} only for $a = b$, whereas in the symmetric case, according to the isomorphism theorem for $a \neq b$, $(L^{ab} + \frac{1}{2}\eta^2)$ has the same law as $\frac{1}{2}\eta^2$ under $\frac{1}{\mathbb{E}[\eta_a\eta_b]}\mathbb{E}[\eta_a\eta_b, \,.\,]$, where η is a centered Gaussian process with covariance G.

Proof of Lemma 2.2 : Thanks to Lemma 2.1, we know that $\psi/2$ admits a nice density $h(z, G)$ with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover, we have:

$$h(z, G) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i<z,\lambda>}|I - iG\lambda|^{-4}d\lambda.$$

(2.1) We have

$$\frac{\partial}{\partial C_{kk}}|I - i\lambda G|^{-4} = -4|I - iG\lambda|^{-5}\frac{\partial}{\partial C_{kk}}|I - iG\lambda|.$$

Developing with respect to the k^{th} line and then deriving with respect to C_{kk}, gives

$$\frac{\partial}{\partial C_{kk}}|I - i\lambda G|^{-4} = 4i|I - iG\lambda|^{-5}\lambda_k|I - iG\lambda|^{kk}, \tag{2.3}$$

where for any square matrix A, we denote by $|A|^{kj}$ the determinant of the matrix obtained by deleting the k^{th} line and the j^{th} column. We remark then that

$$|I - iG\lambda|^{kk} = |I - iG\lambda| - \lambda_k\frac{\partial}{\partial\lambda_k}|I - iG\lambda|, \tag{2.4}$$

hence

$$\begin{aligned}\frac{\partial}{\partial C_{kk}}|I - iG\lambda|^{-4} &= 4i\lambda_k|I - iG\lambda|^{-4} - 4i\lambda_k^2|I - iG\lambda|^{-5}\frac{\partial}{\partial\lambda_k}|I - iG\lambda| \\ &= 4i\lambda_k|I - iG\lambda|^{-4} + i\lambda_k^2\frac{\partial}{\partial\lambda_k}|I - iG\lambda|^{-4}\end{aligned}$$

but $\frac{\partial}{\partial \lambda_k}|I-iG\lambda|^{-4} = \frac{i}{2}\mathbb{E}_G[\psi_{x_k}\, e^{\frac{i}{2}\sum_{p=1}^n \lambda_p\psi_{x_p}}]$. Consequently we obtain thanks to (2.3) and (2.4)

$$\begin{aligned}(2\pi)^n \frac{\partial h}{\partial C_{kk}}(z,G) &= \int_{\mathbb{R}^n} 4i\lambda_k e^{-i<z,\lambda>}|I-iG\lambda|^{-4}d\lambda \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^n} \lambda_k^2 e^{-i<z,\lambda>}\mathbb{E}_G[\psi_{x_k} e^{\frac{i}{2}\sum_{p=1}^n \lambda_p\psi_{x_p}}]d\lambda\end{aligned}$$

We have hence expressed $\frac{\partial h}{\partial C_{kk}}(z,G)$ in terms of the density of $(\psi_{x_1}/2,...,\psi_{x_n}/2)$ and of h_{kk} the density of $(\psi_{x_1}/2,...,\psi_{x_n}/2)$ under $\frac{1}{\mathbb{E}[\psi_{x_k}]}\mathbb{E}[\psi_{x_k},\ .\ ]$:

$$\frac{\partial h}{\partial C_{kk}}(z,G) = -4\frac{\partial h}{\partial z_k}(z,G) + 4G(x_k,x_k)\frac{\partial^2 h_{kk}}{\partial z_k^2}(z,G) \tag{2.5}$$

Performing then several integrations by parts, one finally obtains (2.1).

(2.2) For $k \neq j$, we have:

$$\frac{\partial h}{\partial C_{kj}}(z,G) = \frac{4i}{(2\pi)^n}\int_{\mathbb{R}^n} \lambda_k e^{-i<z,\lambda>}(I-iG\lambda)^{-1}_{jk}|I-iG\lambda|^{-4}d\lambda. \tag{2.6}$$

Indeed, we have

$$\frac{\partial}{\partial C_{kj}}|I-iG\lambda|^{-4} = -4|I-iG\lambda|^{-5}\frac{\partial}{\partial C_{kj}}|I-iG\lambda|$$

We develop first with respect to the j^{th} column and derive with respect to C_{kj} to obtain:

$$\begin{aligned}|I-iG\lambda| &= (-1)^{j+1}(-i\lambda_j)G_{1j}|I-iG\lambda|^{1,j} \\ &\quad +(-1)^{j+2}(-i\lambda_j)G_{2j}|I-iG\lambda|^{2,j} + ... + (-1)^{k+j}(-i\lambda_j)G_{kj}|I-iG\lambda|^{k,j} + ...\end{aligned}$$

hence

$$\frac{\partial}{\partial C_{kj}}|I-iG\lambda| = -i(-1)^{k+j}\lambda_j|I-iG\lambda|^{k,j} = -i\lambda_j(I-iG\lambda)^{-1}_{jk}|I-iG\lambda|$$

Consequently:

$$\frac{\partial h}{\partial C_{kj}}(z,G) = \frac{4i}{(2\pi)^n}\int_{\mathbb{R}^n} \lambda_k e^{-i<z,\lambda>}\ \ (I-iG\lambda)^{-1}_{jk}|I-iG\lambda|^{-4}d\lambda.$$

Since ψ is infinitely divisible, we know that there exists a diagonal matrix $D = Diag(D(i), 1 \leq i \leq n)$ with positive entries on the diagonal such that $\tilde{G} = DGD^{-1}$ is a potential matrix (see [5]). Denote by $L^{x_jx_k}$ the local time process of the Markov process X with Green function $\tilde{G}$, conditionned to start at x_j and killed at x_k. This is actually the local time process of the h-path transform of X with the function $h(x) = \tilde{G}(x,x_k)$, conditioned to start at x_j. The Green function of this last process is $(\tilde{G}(x_p,x_q)\frac{\tilde{G}(x_q,x_k)}{\tilde{G}(x_p,x_k)}, 1 \leq p,q \leq n)$. Now note that this Green function is independent of D, and is actually equal to $(G(x_p,x_q)\frac{G(x_q,x_k)}{G(x_p,x_k)}, 1 \leq p,q \leq n)$. To compute the Laplace transform of $L^{x_jx_k}$ we make use of a well-known formula (see e.g. [12] (2.173) but for the Green function $(G(x_p,x_q)\frac{G(x_q,x_k)}{G(x_p,x_k)}, 1 \leq p,q \leq n)$), which gives:

$$G(x_j,x_k)\mathbb{E}[e^{i\sum_{p=1}^n \lambda_p L_{x_p}^{x_jx_k}}] = ((I-iG\lambda)^{-1}G)_{j,k}$$

Note that : $(I-iG\lambda)^{-1} = I + (I-iG\lambda)^{-1}(iG\lambda)$. Hence for $k \neq j$:

$$(I-iG\lambda)^{-1}_{jk} = i[(I-iG\lambda)^{-1}G\lambda]_{jk} = i\lambda_k[(I-iG\lambda)^{-1}G]_{jk}.$$

We finally obtain:

$$(I - iG\lambda)^{-1}_{jk} = i\lambda_k G(x_j, x_k)\mathbb{E}[e^{i\sum_{p=1}^n \lambda_p L_{x_p}^{x_j x_k}}]. \tag{2.7}$$

Making use of (2.7), we have:

$$\frac{\partial h}{\partial C_{kj}}(z, G) = \frac{-4}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i<z,\lambda>}\ \lambda_k\lambda_j\, G(x_j, x_k)\ \mathbb{E}_G[e^{i\sum_{p=1}^n \lambda_p(\frac{\psi_{x_p}}{2} + L_{x_p}^{x_j x_k})}]d\lambda,$$

which leads to

$$\frac{\partial h}{\partial C_{kj}}(z, G) = 4G(x_j, x_k)\frac{\partial^2 h_{jk}}{\partial z_k \partial z_j}(z, G) \tag{2.8}$$

where h_{jk} is the density of the vector $(\frac{\psi(x_p)}{2} + L_{x_p}^{x_j x_k}, 1 \le p \le n)$. One finally obtains (2.2) after two integrations by parts. □

3 Slepian lemmas for permanental vectors

In view of Lemma 2.2, we see that in order to stochastically compare two permanental vectors, we better have to choose them infinitely divisible. The problem is to find a path from one vector to the other that stays in the set of infinitely divisible permanental vectors. From the definition (1.1), one remarks that for a permanental vector there is no unicity of the kernel. For an infinitely divisible permanental vector with kernel G one can always choose a nonnegative kernel. Indeed, there exists a $n \times n$-signature matrix σ such that $\sigma G \sigma$ is the inverse of a M-matrix (see [5]). We remind that a signature matrix is a diagonal matrix with its diagonal entries in $\{-1, 1\}$. A non singular matrix A is a M-matrix if its off-diagonal entries are nonpositive and the entries of A^{-1} are nonnegative. In particular all the entries of $\sigma G \sigma$ are nonnegative. We can choose $(|G(i, j)|, 1 \le i, j \le n)$ to be the kernel of ψ.
Given two inverse M-matrices, the problem becomes then to find a nice path from one to the other that stays in the set of inverse M-matrices. Unlike for positive definite matrices, linear interpolations between two inverse M-matrices are not always inverse M-matrices. This creates the limits for the use of the presented tool.
Here are some results of comparison of infinitely divisible permanental processes. The proofs are presented at the end of the section.

Lemma 3.1. *Let ψ and $\tilde{\psi}$ be two infinitely divisible permanental vectors with index $1/4$ and respective nonnegative kernels G and $\tilde{G}$ such that for every i, j*

$$G^{-1}(i, j) \ge \tilde{G}^{-1}(i, j). \tag{3.1}$$

Then for every function F on $\mathbb{R}^n_+$ such that

- $\frac{\partial^2 F}{\partial z_i \partial z_j} \ge 0$ *for every i, j such that $i \neq j$*
- $\frac{\partial F}{\partial z_i} + \frac{z_i}{4}\frac{\partial^2 F}{\partial z_i^2} \ge 0$

we have:

$$\mathbb{E}[F(\psi/2)] \le \mathbb{E}[F(\tilde{\psi}/2)].$$

The proof of Lemma 3.1 will show that (3.1) implies that for every i, j $G(i, j) \le \tilde{G}(i, j)$.

Lemma 3.2. *Let ψ and $\tilde{\psi}$ be two infinitely divisible permanental vectors with index $1/4$ and respective nonnegative kernels G and $\tilde{G}$ such that:*

$$G^{-1}(i, j) \ge \tilde{G}^{-1}(i, j) \tag{3.2}$$

for every $1 \leq i, j \leq n$*. Then for every positive* $s_1, s_2, ..., s_n$*, we have:*

$$\mathbb{P}[\cap_{i=1}^n (\psi_i > s_i)] \leq \mathbb{P}[\cap_{i=1}^n (\tilde{\psi}_i > s_i)]. \tag{3.3}$$

If moreover: $G(i,i) = \tilde{G}(i,i)$*, for every* $1 \leq i \leq n$ *then*

$$\mathbb{P}[\cap_{i=1}^n (\psi_i < s_i)] \leq \mathbb{P}[\cap_{i=1}^n (\tilde{\psi}_i < s_i)]. \tag{3.4}$$

Under the assumptions of Lemma 3.2, we obtain for example:

$$\mathbb{E}[F(\inf_{1 \leq i \leq n} \psi_i)] \leq \mathbb{E}[F(\inf_{1 \leq i \leq n} \tilde{\psi}_i)]$$

for every increasing function F on $\mathbb{R}^+$ and when moreover $G(i,i) = \tilde{G}(i,i)$ for every i, then

$$\mathbb{E}[F(\sup_{1 \leq i \leq n} \psi_i)] \geq \mathbb{E}[F(\sup_{1 \leq i \leq n} \tilde{\psi}_i)].$$

As a direct consequence of the work of Fang and Hu [8], one can stochastically compare two infinitely divisible squared Gaussian processes. Indeed let $(\eta_1, \eta_2, ..., \eta_n)$ and $(\tilde{\eta}_1, \tilde{\eta}_2, ..., \tilde{\eta}_n)$ be two centered Gaussian vectors with respective nonnegative covariance matrices G and $\tilde{G}$, such that $\eta^2 = (\eta_1^2, \eta_2^2, ..., \eta_n^2)$ and $\tilde{\eta}^2 = (\tilde{\eta}_1^2, \tilde{\eta}_2^2, ..., \tilde{\eta}_n^2)$ are infinitely divisible. We have then

$$\text{If} \quad \tilde{G}^{-1}(i,j) \geq G^{-1}(i,j) \text{ for every } i,\ j,$$

then

$$\mathbb{E}[F(\eta^2)] \geq \mathbb{E}[F(\tilde{\eta}^2)]$$

for every increasing in each variable function F on $\mathbb{R}_+^n$.
With elementary considerations, this comparison extends to permanental vectors with symmetric kernels and index $1/4$. The above lemmas can be seen as extensions of this relation to infinitely divisible permanental vectors with non symmetric kernels.

Lemma 3.3. *Let* ψ *be an infinitely divisible permanental vector with kernel* G *and index* $1/4$*. Then for every diagonal matrix* D *with nonnegative entries, there exists an infinitely divisible permanental vector* $\tilde{\psi}$ *with kernel* $(G + D)$*. Moreover for every positive* $s_1, s_2, ..., s_n$*, we have:*

$$\mathbb{P}[\cap_{i=1}^n (\psi_i < s_i)] \geq \mathbb{P}[\cap_{i=1}^n (\tilde{\psi}_i < s_i)]$$

and

$$\mathbb{P}[\cap_{i=1}^n (\psi_i > s_i)] \leq \mathbb{P}[\cap_{i=1}^n (\tilde{\psi}_i > s_i)].$$

The following lemma is an immediat consequence of the fact that infinite divisibility implies positive correlation (see [2]).

Lemma 3.4. *Let* ψ *be a* n*-dimensional infinitely divisible permanental vector with index* β *and nonnegative kernel* G*. Let* $\tilde{\psi}$ *be a* n*-dimensional permanental vector with index* β *and kernel* D *defined by*

$$D(i,j) = \begin{cases} 0 & \text{if} \quad i \neq j \\ G(i,i) & \text{if} \quad i = j \end{cases}$$

Then for every positive $s_1, s_2, ..., s_n$*, we have:*

$$\mathbb{P}[\cap_{i=1}^n (\psi_i < s_i)] \geq \mathbb{P}[\cap_{i=1}^n (\tilde{\psi}_i < s_i)]$$

and

$$\mathbb{P}[\cap_{i=1}^n (\psi_i > s_i)] \geq \mathbb{P}[\cap_{i=1}^n (\tilde{\psi}_i > s_i)].$$

Proof of Lemma 3.1: The two matrices G and $\tilde{G}$ are inverse of M-matrices: $G = c(I-P)^{-1}$ and $\tilde{G} = \tilde{c}(I-\tilde{P})^{-1}$, where c and $\tilde{c}$ are positive numbers and P and $\tilde{P}$ are convergent matrices (i.e. nonnegative matrices such that $\rho(P), \rho(\tilde{P}) < 1$). Note that c and P are not unique in the decomposition of G. One can hence choose c small enough to have : $c \leq \tilde{c}$. Consequently: $G^{-1}_{ij} \geq \tilde{G}^{-1}_{ij}$, implies that : $P_{ij} \leq \tilde{P}_{ij}$, for every i, j. For θ in $[0,1]$, define the convergent matrix $P(\theta)$ by $P(\theta)_{ij} = \theta\tilde{P}_{ij} + (1-\theta)P_{ij}$, and the constant c_θ by: $c_\theta = \theta\tilde{c} + (1-\theta)c$. Set then

$$G(\theta) = c_\theta(I - P(\theta))^{-1} \tag{3.5}$$

The matrix $G(\theta)$ is the kernel of an infinitely divisible permanental vector with index $1/4$. Set:

$$f(\theta) = \mathbb{E}_{G(\theta)}[F(\psi)] = \mathcal{F}(G(\theta))$$

We have: $f'(\theta) = \sum_{1\leq i,j\leq n} \frac{\partial \mathcal{F}}{\partial C_{ij}}(G(\theta))\frac{\partial C_{ij}(G(\theta))}{\partial \theta}$. Note that : $\frac{\partial P(\theta)}{\partial \theta}(i,j) = \tilde{P}_{ij} - P_{ij} \geq 0$. Hence for every integer k , $(P(\theta))^k(i,j)$ is an increasing function of θ. Since c_θ is also an increasing function of θ, we obtain: $\frac{\partial C_{ij}(G(\theta))}{\partial \theta} \geq 0$. Lemma 2.2 and the assumptions on F lead then to: $f'(\theta) \geq 0$. In particular: $f(0) \leq f(1)$, which means that: $\mathbb{E}_G[F(\psi)] \leq \mathbb{E}_{\tilde{G}}[F(\psi)]$. □

Proof of Lemma 3.2 (3.3) Let N be a real standard Gaussian variable and p the density with respect to the Lebesgue measure of $N1_{N<0}$. For $\varepsilon > 0$, set:

$$f_{\varepsilon,c}(x) = 1_{x>c} + 1_{x\leq c}\int_{-\infty}^{\frac{x-c}{\varepsilon}} p(y)dy. \tag{3.6}$$

As ε tends to 0, $f_{\epsilon,c}$ converges pointwise to $1_{[c,+\infty)}$. Note that on $(-\infty, c]$, $f_{\varepsilon,c}$ is C^2 with $f'_{\varepsilon,c} \geq 0$ and $f''_{\varepsilon,c} \geq 0$.
Define the function F_ε on $\mathbb{R}^n$ by

$$F_\varepsilon(z) = \Pi_{k=1}^n f_{\varepsilon,s_k}(z_k). \tag{3.7}$$

One can not directly use Lemma 2.2 for F_ε but thanks to (2.5), for any C kernel of an infinitely divisible permanental vector with index $1/4$, we have:

$$\begin{aligned}&\frac{\partial}{\partial C_{11}}\mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \\ = \quad & \int_{\mathbb{R}_+^{n-1}} \Pi_{k=2}^n f_{\varepsilon,s_k}(z_k)dz_2...dz_n \int_0^\infty f_{\varepsilon,s_1}(z_1)\{-4\frac{\partial h}{\partial z_1}(z,C) + 4C(1,1)\frac{\partial^2 h_{11}}{\partial z_1^2}(z,C)\}dz_1\end{aligned}$$

Note that we have:

$$\begin{aligned}\textstyle\int_0^\infty \quad & f_{\varepsilon,s_1}(z_1)\{-4\frac{\partial h}{\partial z_1}(z,C) + 4C(1,1)\frac{\partial^2 h_{11}}{\partial z_1^2}(z,C)\}dz_1 \\ = \quad & 4h(z,C)_{|z_1=s_1} - 4C(1,1)\frac{\partial h_{11}}{\partial z_1}(z,C)_{|z_1=s_1} \\ + \quad & \int_0^{s_1} f_{\varepsilon,s_1}(z_1)\{-4\frac{\partial h}{\partial z_1}(z,C) + 4C(1,1)\frac{\partial^2 h_{11}}{\partial z_1^2}(z,C)\}dz_1 \\ = \quad & 4f_{\varepsilon,s_1}(0)h(z,C)_{|z_1=0} + 4\int_0^{s_1} h(z,C)f'_{\varepsilon,s_1}(z_1)dz_1 - 4C_{1,1}f_{\varepsilon,s_1}(0)\frac{\partial h_{11}}{\partial z_1}(z,C)_{|z_1=0} \\ + \quad & 4C_{1,1}h_{11}(z,C)_{|z_1=0}f''_{\varepsilon,s_1}(0) + 4C_{1,1}\int_0^{s_1} h_{11}(z,C)f''_{\varepsilon,s_1}(z_1)dz_1\end{aligned}$$

by performing two integration by parts. We note then that the two densities h and h_{11} are connected as follows: $4C_{1,1}h_{11}(z,C) = z_1 h(z,C)$. One obtains in particular: $4C_{1,1}\frac{\partial h_{11}}{\partial z_1}(z,C)_{|z_1=0} = h(z,C)_{|z_1=0}$, which leads to:

$$\begin{aligned}
\tfrac{\partial}{\partial C_{11}} \quad & \mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \\
= \quad & \int_{\mathbb{R}_+^{n-1}} \Pi_{k=2}^n f_{\varepsilon,s_k}(z_k)dz_2...dz_n\{3f_{\varepsilon,s_1}(0)h(z,C)_{|z_1=0} + 4\int_0^{s_1} h(z,C)f'_{\varepsilon,s_1}(z_1)dz_1 \\
& + 4C_{1,1}\int_0^{s_1} h_{11}(z,C)f''_{\varepsilon,s_1}(z_1)dz_1\}.
\end{aligned}$$

Since $h(z,C)_{|z_1=0} = 0$, one obtains:

$$\begin{aligned}
\tfrac{\partial}{\partial C_{11}} \quad & \mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \\
= \quad & \int_{\mathbb{R}_+^{n-1}} \Pi_{k=2}^n f_{\varepsilon,s_k}(z_k)dz_2...dz_n \int_0^{s_1} h(z,C)\{4f'_{\varepsilon,s_1}(z_1) + z_1 f''_{\varepsilon,s_1}(z_1)\}dz_1
\end{aligned}$$

Consequently: $\frac{\partial}{\partial C_{11}}\mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \geq 0$. Similarly one obtains:

$$\frac{\partial}{\partial C_{kk}}\mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \geq 0, \tag{3.8}$$

for every $1 \leq k \leq n$.

Thanks to (2.8), one computes:

$$\begin{aligned}
& \frac{\partial}{\partial C_{12}}\mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \\
= \quad & 4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n f_{\varepsilon,s_k}(z_k)dz_3...dz_n \int_{\mathbb{R}_+^2} f_{\varepsilon,s_1}(z_1)f_{\varepsilon,s_2}(z_2)\frac{\partial^2 h_{2,1}}{\partial z_1 \partial z_2}(z,G)dz_1dz_2 \\
= \quad & 4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n f_{\varepsilon,s_k}(z_k)dz_3...dz_n\{f_{\varepsilon,s_1}(0)f_{\varepsilon,s_2}(0)h_{2,1}(z,C)_{|z_1=z_2=0} \\
+ \quad & f_{\varepsilon,s_1}(0)\int_0^{s_2} h_{2,1}(z,C)_{|z_1=0}f'_{\varepsilon,s_2}(z_2)dz_2 + f_{\varepsilon,s_2}(0)\int_0^{s_1} h_{2,1}(z,C)_{|z_2=0}f'_{\varepsilon,s_1}(z_1)dz_1 \\
+ \quad & \int_0^{s_2}\int_0^{s_1} h_{2,1}(z,C)f'_{\varepsilon,s_1}(z_1)f'_{\varepsilon,s_2}(z_2)dz_1dz_2\}.
\end{aligned}$$

Note that: $h_{2,1}(z,C)_{|z_1=z_2=0} = h_{2,1}(z,C)_{|z_1=0} = h_{2,1}(z,C)_{|z_2=0} = 0$. Indeed, denote by L^{ab} the local time process of the Markov process associated to C conditioned to start at a and to die at its last visit to b. Then we have: $L^{ab}_a > 0$ a.s. and $L^{ab}_b > 0$ a.s. We hence obtain

$$\begin{aligned}
\tfrac{\partial}{\partial C_{12}} \quad & \mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \\
= \quad & 4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n f_{\varepsilon,s_k}(z_k)dz_3...dz_n \int_0^{s_2}\int_0^{s_1} h_{2,1}(z,C)f'_{\varepsilon,s_1}(z_1)f'_{\varepsilon,s_2}(z_2)dz_1dz_2\},
\end{aligned} \tag{3.9}$$

which leads to: $\frac{\partial}{\partial C_{12}}\mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \geq 0$.

Similarly one shows that for every $i \neq j$, $\frac{\partial}{\partial C_{ij}}\mathbb{E}_C[F_\varepsilon(\frac{\psi}{2})] \geq 0$.

One uses then the matrices $G(\theta)$ defined in (3.5) to obtain the conclusion similarly as in the proof of Lemma 3.1 by dominated convergence.

(3.4): Define the function $\tilde{F}_\epsilon$ on $\mathbb{R}_+^n$ by:

$$\tilde{F}_\varepsilon(x) = \prod_{k=1}^{n}(1 - f_{\varepsilon,s_k}(x_k)) \tag{3.10}$$

where $f_{\epsilon,c}$ is given by (3.6). Denote by $\tilde{f}_{\epsilon,c}$ the function $(1 - f_{\epsilon,c})$. As ε tends to 0, $\tilde{F}_\epsilon(x)$ converges to $\prod_{k=1}^{n} 1_{x_k<s_k}$.
For every i, we have:

$$\frac{\partial}{\partial C_{ii}}\mathbb{E}_C[\tilde{F}_\varepsilon(\frac{\psi}{2})] \leq 0. \tag{3.11}$$

Indeed thanks to (2.8), for any C kernel of an infinitely divisible permanental vector with index $1/4$, we have, making use of the computations in the proof of (3.3)

$$\begin{aligned}
& \frac{\partial}{\partial C_{11}}\mathbb{E}_C[\tilde{F}_\varepsilon(\frac{\psi}{2})] \\
= & \int_{\mathbb{R}_+^{n-1}} \Pi_{k=2}^n \tilde{f}_{\varepsilon,s_k}(z_k)dz_2...dz_n \int_0^\infty (1 - f_{\varepsilon,s_1}(z_1)\{-4\frac{\partial h}{\partial z_1}(z,C) + 4C(1,1)\frac{\partial^2 h_{11}}{\partial z_1^2}(z,C)\}dz_1 \\
= & \frac{\partial}{\partial C_{11}}\mathbb{E}_C[\prod_{k=2}^{n}\tilde{f}_{\varepsilon,s_k}(\frac{\psi_k}{2})] \\
& -\int_{\mathbb{R}_+^{n-1}} \Pi_{k=2}^n \tilde{f}_{\varepsilon,s_k}(z_k)dz_2...dz_n \int_0^{s_1} h(z,C)\{4f'_{\varepsilon,s_1}(z_1) + z_1 f''_{\varepsilon,s_1}(z_1)\}dz_1 \\
\leq & \; 0
\end{aligned}$$

since $\frac{\partial}{\partial C_{11}}\mathbb{E}_C[\prod_{k=2}^{n}\tilde{f}_{\varepsilon,s_k}(\frac{\psi_k}{2})] = 0$.

$$\begin{aligned}
& \frac{\partial}{\partial C_{12}}\mathbb{E}_C[\tilde{F}_\varepsilon(\frac{\psi}{2})] \\
= & \; 4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n \tilde{f}_{\varepsilon,s_k}(z_k)dz_3...dz_n \int_{\mathbb{R}_+^2} \tilde{f}_{\varepsilon,s_1}(z_1)\tilde{f}_{\varepsilon,s_2}(z_2)\frac{\partial^2 h_{2,1}}{\partial z_1 \partial z_2}(z,G)dz_1dz_2 \\
= & \frac{\partial}{\partial C_{12}}\mathbb{E}_C[\prod_{k=3}^{n}\tilde{f}_{\varepsilon,s_k}(\frac{\psi_k}{2})] - \frac{\partial}{\partial C_{12}}\mathbb{E}_C[\prod_{k=2}^{n}\tilde{f}_{\varepsilon,s_k}(\frac{\psi_k}{2})] - \frac{\partial}{\partial C_{12}}\mathbb{E}_C[\tilde{f}_{\varepsilon,s_1}(\frac{\psi_1}{2})\prod_{k=3}^{n}\tilde{f}_{\varepsilon,s_k}(\frac{\psi_k}{2})] \\
& +4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n \tilde{f}_{\varepsilon,s_k}(z_k)dz_3...dz_n \int_{\mathbb{R}_+^2} f_{\varepsilon,s_1}(z_1)f_{\varepsilon,s_2}(z_2)\frac{\partial^2 h_{2,1}}{\partial z_1 \partial z_2}(z,G)dz_1dz_2 \\
= & \; 4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n \tilde{f}_{\varepsilon,s_k}(z_k)dz_3...dz_n \int_{\mathbb{R}_+^2} f_{\varepsilon,s_1}(z_1)f_{\varepsilon,s_2}(z_2)\frac{\partial^2 h_{2,1}}{\partial z_1 \partial z_2}(z,G)dz_1dz_2 \\
= & \; 4C_{2,1}\int_{\mathbb{R}_+^{n-2}} \Pi_{k=3}^n \tilde{f}_{\varepsilon,s_k}(z_k)dz_3...dz_n \int_0^{s_2}\int_0^{s_1} h_{2,1}(z,C)f'_{\varepsilon,s_1}(z_1)f'_{\varepsilon,s_2}(z_2)dz_1dz_2\} \\
\geq & \; 0
\end{aligned}$$

thanks to the computations in the proof of (3.3). More generally, we obtain for every $i \neq j$

$$\frac{\partial}{\partial C_{ij}}\mathbb{E}_C[\tilde{F}_\varepsilon(\frac{\psi}{2})] \geq 0 \tag{3.12}$$

We keep definition (3.5) for $G(\theta)$. Set: $f(\theta) = \mathbb{E}_{G(\theta)}[\tilde{F}_\varepsilon(\psi)]$, and for any kernel M of a n-dimensional permanental vector with index $1/4$: $\tilde{\mathcal{F}}_\varepsilon(M) = \mathbb{E}_M[\tilde{F}_\varepsilon(\psi)]$. We have:

$$f'(\theta) = \sum_{1\leq i,j\leq n, i\neq j} \frac{\partial \tilde{\mathcal{F}}_\varepsilon}{\partial C_{ij}}(G(\theta))\frac{\partial C_{ij}(G(\theta))}{\partial \theta}$$

For $i \neq j$, we have: $\frac{\partial C_{ij}(G(\theta))}{\partial \theta} \geq 0$. Besides thanks to (3.12): $\frac{\partial \tilde{\mathcal{F}}_\varepsilon}{\partial C_{ij}}(G(\theta)) \geq 0$. We obtain: $f' \geq 0$ on $[0,1]$. Hence: $f(0) \leq f(1)$, which means for every $\varepsilon > 0$

$$\mathbb{E}\tilde{F}_\varepsilon(\psi)] \leq \mathbb{E}[\tilde{F}_\varepsilon(\tilde{\psi})]$$

By letting ε tend to 0, we finally obtain:

$$\mathbb{E}[\cap_{i=1}^n(\psi_i < s_i)] \leq \mathbb{E}[\cap_{i=1}^n(\tilde{\psi}_i < s_i)]. \square$$

Proof of Lemma 3.3: First we use the fact that G is an inverse M-matrix hence for every diagonal matrix D, $(G+D)$ is still an inverse M-matrix (see e.g. [10]). Then for θ in $[0,1]$, define the M-matrix $G(\theta)$ by: $G(\theta) = \theta G + (1-\theta)(G+D)$, and the associated function f on $[0,1]$:

$$f(\theta) = \mathbb{E}_{G(\theta)}[\tilde{F}_\varepsilon(\psi)],$$

where $\tilde{F}_\varepsilon$ is defined by (3.10). Thanks to (3.11), one obtains the first inequality by letting ε tend to 0. The second one is obtained similarly with $\tilde{F}_\varepsilon$ replaced by F_ε (defined by (3.7)). One concludes thanks to (3.8). $\square$

4 A weak Sudakov inequality

Let $(\eta_x)_{x\in E}$ be a centered Gaussian process with covariance function G. Define d on $E \times E$ by

$$d_\eta(x,y) = (G(x,x) + G(y,y) - 2G(x,y))^{1/2} = \mathbb{E}[(\eta_x - \eta_y)^2]^{1/2},$$

then d_η is a pseudo-distance on E.
Suppose that there exists a finite subset S of E such that for every distinct x and y elements of S, $d_\eta(x,y) > u$, then according to Sudakov inequality

$$\mathbb{E}[\sup_{x\in S} \eta_x] \geq \frac{1}{17} u \sqrt{\log |S|}. \tag{4.1}$$

We consider now a kernel $G = (G(x,y), (x,y) \in E \times E)$, such that G is a *bipotential*. This means that both G and G^t are Green functions of transient Markov processes. This is equivalent (see [6]) to the assumption that for any finite subset S of E, both $G_{|_{S\times S}}$ and $G^t_{|_{S\times S}}$ are inverse of diagonally dominant M-matrices (a matrix $(A_{ij})_{1\leq i,j\leq n}$ is diagonally dominant if for every i, $\sum_{j=1}^n A_{ij} \geq 0$).
For $(\psi_x, x \in E)$ permanental process with index $1/4$ admitting G for kernel, define the function d_G on $E \times E$ by

$$d_G(x,y) = (G(x,x) + G(y,y) - G(x,y) - G(y,x))^{1/2}. \tag{4.2}$$

As a consequence of [6], we know that d_G is a pseudo-distance on E. When there is no ambiguity, d_G will be denoted by d.
Following [11], we define $\mathbb{E}[\sup_{x\in E} \psi_x]$ as being $\sup\{\mathbb{E}[\sup_{x\in F} \psi_x], F$ finite subset of $E\}$.

Lemma 4.1. *Let $(\psi_x, x \in E)$ be a permanental process with a kernel G and index $1/4$. Assume that:*

(1) G is a bipotential and that for every x in S, $G(x,x) = 1$.

(2) S is a finite subset of E such that for every distinct x and y elements of S: $G(x,y) \leq a$.

Set: $u = (2-2a)^{1/2}$. Then for every x,y in S: $d(x,y) \geq u$ and

$$\mathbb{E}[\sup_{x\in S} \sqrt{\psi_x}] \;\geq\; \frac{1}{17\sqrt{2}}\; u\; \sqrt{\log |S|}.$$

Up to a multiplicative constant, permanental processes associated to Lévy processes satisfy (1).

The proof of Lemma 4.1 is based on the following lemma.

Lemma 4.2. *Let G be the inverse of a diagonally dominant M-matrix. Then for every diagonal matrix D with nonnegative entries, $G+D$ is still the inverse of a diagonally dominant M-matrix.*

Indeed, one already knows that $G+D$ is an inverse M-matrix. This is Theorem 1.6 in [10]. Making use of its proof, one easily shows that the M-matrix $(G+D)^{-1}$ is diagonally dominant.

Proof of Lemma 4.1: We simply write: $S=\{1,2,...,n\}$. Set: $a=\sup_{i\neq j}G(i,j)$. We have: $a\leq 1$.
Define the kernel $\tilde{G}$ on $S\times S$ as follows: $\tilde{G}(i,i)=1$ and for $i\neq j$, $\tilde{G}(i,j)=a$.
Set: $G(\theta)=\theta G+(1-\theta)\tilde{G}$. For every θ in $[0,1]$, $G(\theta)$ is a potential. Indeed, $(\theta G+(1-\theta)Diag(b-a))$ is a Green function (thanks to Lemma 4.2). Since this is also true for its transpose, it remains a potential if we add the nonnegative constant $(1-\theta)a$ to each entry (see e.g. [6]).
We use now the functions $\tilde{F}_\varepsilon$ defined by (3.10) to define $\tilde{H}_\varepsilon((y_i)_{1\leq i\leq n})=\tilde{F}_\varepsilon((\sqrt{y_i})_{1\leq i\leq n}))$, and set: $f(\theta)=\mathbb{E}_{G(\theta)}[\tilde{H}_\varepsilon(\frac{\psi}{2})]$. We compute $f'(\theta)$.

$$f'(\theta)=\sum_{1\leq k,j\leq n}\frac{\partial}{\partial C_{kj}}\mathbb{E}_{G(\theta)}[\tilde{H}_\varepsilon(\frac{\psi}{2})]\ \frac{\partial G(\theta)(k,j)}{\partial\theta}$$

Thanks to Lemma 2.2, we have: $\frac{\partial}{\partial C_{kj}}\mathbb{E}_{G(\theta)}[\tilde{H}_\varepsilon(\frac{\psi}{2})]\geq 0$. Besides note that that: $\frac{\partial G(\theta)(k,j)}{\partial\theta}=G(k,j)-\tilde{G}(k,j)$. We obtain:

$$f'(\theta)=\sum_{1\leq k,j\leq n}\frac{\partial}{\partial C_{kj}}\mathbb{E}_{G(\theta)}[\tilde{H}_\varepsilon(\frac{\psi}{2})](G(k,j)-\tilde{G}(k,j))\leq 0.$$

Consequently we have for every $\varepsilon>0$:

$$\mathbb{E}_G[\tilde{H}_\varepsilon(\frac{\psi}{2})]\leq\mathbb{E}_{\tilde{G}}[\tilde{H}_\varepsilon(\frac{\psi}{2})]$$

and in particular as ε tends to 0, one obtains:

$$\mathbb{E}_G[\sup_{x\in S}\sqrt{\psi_x}]\geq\mathbb{E}_{\tilde{G}}[\sup_{x\in S}\sqrt{\psi_x}].\tag{4.3}$$

Now, $\tilde{G}$ is a covariance matrix, the corresponding vector ψ is the half sum of eight iid squared centered Gaussian vectors with covariance $\tilde{G}$. Denote by $\tilde{\eta}$ a centered Gaussian vector with covariance $\tilde{G}$. We have:

$$\mathbb{E}_G[\sup_{x\in S}\sqrt{\psi_x}]\quad\geq\quad\frac{1}{\sqrt{2}}\mathbb{E}[\sup_{x\in S}|\tilde{\eta}_x|]\tag{4.4}$$

Note that for every distinct i and j in S:

$$d_{\tilde{G}}(i,j)=\mathbb{E}[(\tilde{\eta}_i-\tilde{\eta}_j)^2]^{1/2}=(2-2a)^{1/2}=u.$$

Sudakov inequality (4.1) gives:

$$\mathbb{E}[\sup_{i\in S}\tilde{\eta}_i]\geq\frac{1}{17}u\sqrt{\log|S|}.$$

Consequently, we have obtained thanks to (4.4)

$$\mathbb{E}_G[\sup_{x\in S}\sqrt{\psi_x}]\geq\frac{1}{17\sqrt{2}}u\sqrt{\log|S|}.\square$$

5 Concentration inequalities for permanental processes

Here is a well-known concentration inequality for Gaussian vectors. There exists a universal constant K such that for every centered Gaussian vector $(\eta_i)_{1\leq i\leq n}$

$$\mathbb{P}[|\sup_{1\leq i\leq n}\eta_i-\mathbb{E}[\sup_{1\leq i\leq n}\eta_i]|\geq Ky\sigma]\leq 2e^{-y^2} \tag{5.1}$$

where $\sigma=\sup_{1\leq i\leq n}\mathbb{E}[\eta_i^2]^{1/2}$.
The following two subsections present partial extensions of (5.1) to infinitely disivible permanental vectors.

5.1 Sub-gaussiannity

According to [6] given a bipotential $(G(x,y),(x,y)\in E\times E)$, $\frac{G+{}^tG}{2}$ is positive definite. Set $G_1=(\frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}},(x,y)\in E\times E)$, then $\frac{G_1+G_1^t}{2}$ is also positive definite. Let $\eta=(\eta_x)_{x\in E}$ be a centered Gaussian process with covariance $\frac{G_1+G_1^t}{2}$. Define d_1 on $E\times E$ by

$$d_1(x,y)=(2-G_1(x,y)-G_1(y,x))^{1/2}=\mathbb{E}[(\eta_x-\eta_y)^2]^{1/2}.$$

Then d_1 is a pseudo-distance on E.
Note that G_1 is the kernel of an infinitely divisible permanental process.

Proposition 5.1. *Let $(\psi_t)_{t\in E}$ be a permanental process with kernel G_1 and index $2/d$ with d integer and $1\leq d\leq 8$. We have then*

$$\mathbb{E}_{G_1}[\sup_{s,t\in E}|\sqrt{\psi_t}-\sqrt{\psi_s}|]\leq K\mathbb{E}[\sup_{t\in E}\eta_t] \tag{5.2}$$

where K is an universal constant.
Moreover for every finite subset T of E, and every $u>0$, we have:

$$\mathbb{P}_{G_1}[\sup_{s,t\in T}|\sqrt{\psi_t}-\sqrt{\psi_s}|>K(\mathbb{E}[\sup_{t\in T}\eta_t]+u)]\leq(\exp\{\frac{u^2}{50\rho^2}\}-1)^{-1} \tag{5.3}$$

where $\rho=\sup_{t,s\in T}d_1(s,t)$.

Proof: Denote by $||.||$, the euclidian norm in $\mathbb{R}^d$. Note that for every s,t in T: $(\sqrt{\frac{\psi_t}{G(t,t)}},\sqrt{\frac{\psi_s}{G(s,s)}})$ has the same law as $\frac{1}{\sqrt{2}}(||\tilde{\eta}_t||,||\tilde{\eta}_s||)$ with $(\tilde{\eta}_t,\tilde{\eta}_s)=((\eta_t^{(k)})_{1\leq k\leq d},(\eta_s^{(k)})_{1\leq k\leq d})$, where the couples $(\eta_t^{(k)},\eta_s^{(k)})$, $1\leq k\leq d$, are i.i.d. with a centered gaussian law such that $\mathbb{E}[\eta_t^{(k)}]=1$, $\mathbb{E}[\eta_s^{(k)}]=1$ and $\mathbb{E}[\eta_t^{(k)}\eta_s^{(k)}]=\sqrt{G_1(t,s)G_1(s,t)}$. Hence

$$|\sqrt{\frac{\psi_t}{G(t,t)}}-\sqrt{\frac{\psi_s}{G(s,s)}}|\leq\frac{1}{\sqrt{2}}||\tilde{\eta}_t-\tilde{\eta}_s||=\frac{1}{\sqrt{2}}(\sum_{k=1}^d|\eta_t^{(k)}-\eta_s^{(k)}|^2)^{1/2}.$$

One obtains for every $\lambda>0$

$$\mathbb{E}_{G_1}[\exp(\lambda^2|\sqrt{\psi_t}-\sqrt{\psi_s}|^2])]\leq\mathbb{E}[\exp(\frac{\lambda^2}{2}\sum_{k=1}^d|\eta_t^{(k)}-\eta_s^{(k)}|^2)]\leq\mathbb{E}[\exp(\frac{\lambda^2}{2}|\eta_t^{(1)}-\eta_s^{(1)}|^2)]^d$$

and consequently :

$$\mathbb{E}[\exp(\lambda^2|\sqrt{\psi_t}-\sqrt{\psi_s}|^2])]\leq(1-\lambda^2\delta^2(s,t))^{-d/2}$$

where $\delta(s,t) = (2 - 2\sqrt{G_1(s,t)G_1(t,s)})^{1/2}$. Choosing $\lambda = \frac{1}{5\delta(s,t)}$, one obtains

$$\mathbb{E}_{G_1}[\exp(\frac{\sqrt{\psi_t} - \sqrt{\psi_s}}{5\delta(s,t)})^2] \leq 2,$$

which implies that $(\sqrt{\psi_t} - \mathbb{E}(\sqrt{\psi_t}), t \in T)$ and $(\sqrt{\psi_t} - \sqrt{\psi_s}, t \in T)$ are subgaussian relative to the scale 5δ.

One obviously has: $d_1(s,t) \leq \delta(s,t)$. But note also that: $\delta(s,t) \leq \sqrt{2} d_1(s,t)$. Indeed for every a, b in $[0,1]$

$$a + b - 2\sqrt{ab} \leq \sqrt{a} + \sqrt{b} - 2\sqrt{ab} = \sqrt{a}(1 - \sqrt{b}) + \sqrt{b}(1 - \sqrt{a}) \leq 2 - \sqrt{a} - \sqrt{b}$$

and hence

$$a + b - 2\sqrt{ab} \leq 2 - (a + b).$$

Add 2 to each member of the previous inequality and obtain

$$2 - 2\sqrt{ab} \leq 2(2 - a - b).$$

Consequently $(\sqrt{\psi_t} - \mathbb{E}(\sqrt{\psi_t}), t \in T)$ and $(\sqrt{\psi_t} - \sqrt{\psi_s}, t \in T)$ are also subgaussian relative to the scale $5\sqrt{2} d_1$. Proposition 5.1 is then a direct consequence of [11] Chapter 11, p.316, Theorem 11.18 and Theorem 12.8. □

5.2 Lévy measure of infinitely divisible permanental vectors

The following concentration inequalities for infinitely divisible permanental vectors are a consequence of a remarkable property of their Lévy measure.

Theorem 5.2. *Let $\psi = (\psi_i)_{1 \leq i \leq n}$ be an infinitely divisible vector with kernel $(G(i,j), 1 \leq i,j \leq n)$ and index 2. Then for any Lipschitz function f with constant α with respect to the norm $||x|| = \sum_{i=1}^n |x_i|$, every $y \geq 0$, we have*

$$\mathbb{P}[|f(\psi) - \mathbb{E}[f(\psi)]| > y] \leq 2 \exp\{-\frac{1}{8} \min(\frac{4y^2}{\alpha M^2 n^2}, \frac{y}{\alpha M n})$$

where $M = \sup_{1 \leq i \leq n} G(i,i)$.

One obtains for example with the function $f(u) = \sum_{i=1} |u_i|$, the following inequalities

$$\mathbb{P}[|\frac{\sum_{i=1}^n (\psi_i - \mathbb{E}[\psi_i])}{n}]| \geq M\ y] \leq 2 \exp\{-\frac{1}{8} \min(y^2, \frac{y}{2})\} \tag{5.4}$$

and

$$\mathbb{P}[|\sup_{1 \leq i \leq n} |\frac{\psi_i}{G(i,i)} - \frac{\psi_1}{G(1,1)}| - \mathbb{E}[\sup_{1 \leq i \leq n} |\frac{\psi_i}{G(i,i)} - \frac{\psi_1}{G(1,1)}|]| \geq 2x] \leq 2 \exp\{-\frac{1}{8} \min(\frac{2x^2}{n^2}, \frac{x}{n})\}. \tag{5.5}$$

Proof: We use a result of Houdré (Corollary 2 in [9]) for infinitely divisible vectors X without Gaussian component such that $\mathbb{E}[||X||^2] < \infty$, and a Lévy measure ν on $\mathbb{R}^n$ such that for every $k \geq 3$

$$\int_{\mathbb{R}^n} ||u||^k \nu(du) \leq \frac{C^{k-2} k!}{2} \int_{\mathbb{R}^n} ||u||^2 \nu(du), \tag{5.6}$$

for some $C > 0$, where $||.||$ is the euclidian norm. For such vectors and any Lipschitz function f with constant α, we have then thanks to Corollary 2 in [9], for every $x \geq 0$

$$\mathbb{P}[|f(X) - \mathbb{E}[f(X)]| \geq x] \leq 2 \exp\{-\frac{1}{8} \min(\frac{2x^2}{\alpha \nu(2)}, \frac{x}{\alpha C})$$

with $\nu(2) = \int_{\mathbb{R}^n} ||u||^2 \nu(du)$.
One can actually choose another norm than the euclidian norm and keep the same result. We choose to take the norm : $||u|| = \sum_{i=1}^n |u_i|$.
We check now that (5.6) is satisfied. The expression of the Lévy measure ν of $(\frac{\psi_i}{2})_{1\leq i\leq n}$ has been established in [5]. Indeed, this permanental vector is infinitely divisible hence there exists a transient Markov process $(Y_t, t \geq 0)$ with state space $\{1, 2, ..., n\}$ and finite Green function equal to $(D(i)G(i,j)D(j)^{-1}, 1 \leq i,j \leq n)$ (D is a positive function), such that for every i

$$u_i \nu(du) = \frac{G(i,i)}{2} \mathbb{E}^{ii}[L^j \in du_j, 1 \leq j \leq n]$$

where $(L^j, 1 \leq j \leq n)$ is the total accumulated local time process of Y and for $1 \leq i,j \leq n$, $\mathbb{E}^{ij}$ is the expectation under the condition that Y starts at i and is killed at its last visit to j. In particular: $\mathbb{E}^{ij}[L^k] = G(i,k)\frac{G(k,j)}{G(i,j)}$.
Denote by $\nu(k)$ the quantity $\int_{\mathbb{R}^n} ||u||^k \nu(du)$. We have:

$$\nu(2) = \nu((\sum_{i=1}^n u_i)^2) = \nu(\sum_{j=1}^n u_j(\sum_i u_i)) = \frac{1}{2}\sum_{j=1}^n G(j,j)\mathbb{E}^{jj}[\sum_{i=1}^n L^i]$$

thanks to the definition of ν. We hence obtain

$$\nu(2) = \frac{1}{2}\sum_{1\leq i,j\leq n} G(i,j)G(j,i) \tag{5.7}$$

Similarly we have:

$$\nu(k+1) = \frac{1}{2}\sum_{i=1}^n G(i,i)\mathbb{E}^{ii}[(\sum_{j=1}^n L^j)^k]. \tag{5.8}$$

For every i and for $k \geq 2$, we have thanks to Kac's moment formula

$$\begin{aligned} G(i,i)\mathbb{E}^{ii}[(\sum_{j=1}^n L^j)^k] &= G(i,i) \sum_{(p_1,p_2,\dots,p_k)\in\{1,\dots,n\}^k} \mathbb{E}^{ii}[\prod_{m=1}^k L^{p_m}] \\ &= \sum_{(p_1,p_2,\dots,p_k)\in\{1,\dots,n\}^k} \sum_{\sigma\in\mathcal{S}_k} G(i,p_{\sigma(1)})G(p_{\sigma(1)},p_{\sigma(2)})....G(p_{\sigma(k-1)},p_{\sigma(k)})G(p_{\sigma(k)},i) \end{aligned}$$

where $\mathcal{S}_k$ is the set of permutations of $(1, 2, ..., k)$.
Note that this computations are independent of D.
We use now the following necessary property for kernels of infinitely divisible permanental vectors. For every i, j, k in $\{1, ..., n\}$, we have

$$G(i,j)G(j,k) \leq G(i,k)G(j,j). \tag{5.9}$$

Indeed, denote by T_k the first hitting time of k by Y, then:
$\mathbb{P}^{ij}[T_k < \infty] = \frac{\mathbb{E}^{ij}[L^k]}{\mathbb{E}^{kj}[L^k]} \leq 1$, which leads to (5.9).
Hence for $k \geq 2$

$$\begin{aligned} G(i,i)\mathbb{E}^{ii}[(\sum_{j=1}^n L^j)^k] &\leq \sum_{(p_1,p_2,\dots,p_k)\in\{1,\dots,n\}^k} \sum_{\sigma\in\mathcal{S}_k} G(i,p_{\sigma(k)})G(p_{\sigma(k)},i)\prod_{m=1}^{k-1} G(p_{\sigma(m)},p_{\sigma(m)}) \\ &\leq M^{k-1}\sum_{j=1}^n G(i,j)G(j,i)|A(k,j)|, \end{aligned} \tag{5.10}$$

where $A(k,j) = \{(p_1,p_2,...,p_k)\times\sigma \in \{1,...,n\}^k\times\mathcal{S}_k \ : p_{\sigma(k)} = j\}$ and $M = \sup_{1\leq i\leq n} G(i,i)$.

Note that for a fixed k, the sets $A(k,j), 1 \leq j \leq n$ form a partition of $\{(p_1, p_2, ..., p_k) \times \sigma \in \{1, ..., n\}^k \times \mathcal{S}_k\}$. Hence: $\sum_{j=1}^n |A(k,j)| = n^k\, k!$, which leads to: $|A(k,j)| = n^{k-1}\, k!$, since $|A(k,j)|$ is independent of j. This leads with (5.8) and (5.10) to

$$\nu(k+1) \leq M^{k-1} n^{k-1} k!\, \nu(2)$$

which gives

$$\nu(k) \leq (Mn)^{k-2}(k-1)!\, \nu(2).$$

Choosing $C = Mn$, we see that condition (5.6) is satisfied. We then remark that $\nu(2) \leq \frac{1}{2}(Mn)^2$. □

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