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Degenerate irregular SDEs with jumps and application to integro-differential equations of Fokker-Planck type

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Abstract

We investigate stochastic differential equations with jumps and irregular coefficients, and obtain the existence and uniqueness of generalized stochastic flows. Moreover, we also prove the existence and uniqueness of L^p-solutions or measure-valued solutions for second order integro-differential equation of Fokker-Planck type.

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1 Introduction

Recently, there are increasing interests to extend the classical DiPerna-Lions theory [7] about ordinary differential equations (ODE) with Sobolev coefficients to the case of stochastic differential equations (SDE) (cf. [18, 19, 11, 32, 33, 34, 10, 22]). In [11], Figalli firstly extended the DiPerna-Lions theory to SDE in the sense of martingale solutions by using analytic tools and solving deterministic Fokker-Planck equations. In [18], Le Bris and Lions studied the almost everywhere stochastic flow of SDEs with constant diffusion coefficients, and in [19], they also gave an outline for proving the pathwise uniqueness for SDEs with irregular coefficients by studying the corresponding Fokker-Planck equations with irregular coefficients. In [32] and [34], we extended DiPerna-Lions' result to the case of SDEs by using Crippa and De Lellis' argument [6], and obtained the existence and uniqueness of generalized stochastic flows for SDEs with irregular coefficients (see also [10] for some related works). Later on, Li and Luo [22] extended Ambrosio's result [1] to the case of SDEs with BV drifts and smooth diffusion coefficients by transforming the SDE to an ODE. Moreover, a limit theorem for SDEs with discontinuous coefficients approximated by ODEs was also obtained in [26].

In this paper we are concerned with the following SDEs in $[0,1]\times\mathbb{R}^d$ with jumps:

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t + \int_{\mathbb{R}^d\setminus\{0\}} f_t(X_{t-},y)\tilde{N}(\mathrm{d}t,\mathrm{d}y), \tag{1.1}$$

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where $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ and $f : [0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions, $(W_t)_{t\in[0,1]}$ is a d-dimensional standard Brownian motion, and $N(\mathrm{d}t, \mathrm{d}y)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}^d \setminus \{0\}$ with intensity measure $\nu_t(\mathrm{d}y)\mathrm{d}t$, $\tilde{N}(\mathrm{d}t, \mathrm{d}y) := N(\mathrm{d}t, \mathrm{d}y) - \nu_t(\mathrm{d}y)\mathrm{d}t$ is the compensated Poisson random measure.

The aim of the present paper is to extend the results in [32] to the above jump SDEs with Sobolev drift b and Lipschitz σ, f. Let us now describe the motivation. Suppose that $f_t(x,y) = y$. Let $\mathscr{L}$ be the generator of SDE (1.1) (a second order integro-differential operator) given as follows: for $\varphi \in C_b^\infty(\mathbb{R}^d)$, smooth function with bounded derivatives of all orders,

$$\mathscr{L}_t\varphi(x) := \frac{1}{2}a_t^{ij}(x)\partial_i\partial_j\varphi(x) + b_t^i(x)\partial_i\varphi(x) + \int_{\mathbb{R}^d\setminus\{0\}}[\varphi(x+y) - \varphi(x) - y^i\partial_i\varphi(x)]\nu_t(\mathrm{d}y),$$

where $a_t^{ij}(x) := \sum_k \sigma_t^{ik}(x)\sigma_t^{jk}(x)$, and we have used that the repeated indices in a product are summed automatically, and this convention will be in forced throughout the present paper. Here, we assume that for any $p \geqslant 1$,

$$\int_0^1\int_{\mathbb{R}^d\setminus\{0\}} |y|^2(1+|y|^2)^p\nu_s(\mathrm{d}y)\mathrm{d}s < +\infty, \tag{1.2}$$

which is equivalent that for all $p \geqslant 1$,

$$\mathbb{E}\left|\int_0^1\int_{\mathbb{R}^d\setminus\{0\}} y\tilde{N}(\mathrm{d}t,\mathrm{d}y)\right|^p < +\infty.$$

This will be used to derive that the solution of SDE (1.1) has finite moments of *all orders*. Let now X_t be a solution of SDE (1.1). The law of X_t in $\mathbb{R}^d$ is denoted by μ_t. Then by Itô's formula (cf. [14, 2] and [17]), one sees that μ_t solves the following second order partial integro-differential equation (abbreviated as PIDE) of Fokker-Planck type in the distributional sense:

$$\partial_t\mu_t = \mathscr{L}_t^*\mu_t, \tag{1.3}$$

subject to the initial condition:

$$\lim_{t\downarrow 0}\mu_t = \text{Law of } X_0 \text{ in the sense of weak convergence}, \tag{1.4}$$

where $\mathscr{L}_t^*$ is the adjoint operator of $\mathscr{L}_t$ formally given by

$$\mathscr{L}_t^*\mu := \frac{1}{2}\partial_i\partial_j(a_t^{ij}(x)\mu) - \partial_i(b_t^i(x)\mu) + \int_{\mathbb{R}^d\setminus\{0\}}[\tau_y\mu - \mu + y^i\partial_i\mu]\nu_t(\mathrm{d}y),$$

where for a probability measure μ in $\mathbb{R}^d\setminus\{0\}$ and $y \in \mathbb{R}^d$, $\tau_y\mu := \mu(\cdot - y)$. More precisely, for any $\varphi \in C_b^\infty(\mathbb{R}^d)$,

$$\partial_t\langle\mu_t, \varphi\rangle = \langle\mu_t, \mathscr{L}_t\varphi\rangle, \tag{1.5}$$

where $\langle\mu_t, \varphi\rangle := \int_{\mathbb{R}^d}\varphi(x)\mu_t(\mathrm{d}x)$. If b and σ are not continuous, in order to make sense for (1.5), one needs to at least assume that

$$\int_0^1\int_{\mathbb{R}^d}(|b_t(x)| + |a_t(x)|)\mu_t(\mathrm{d}x)\mathrm{d}t < +\infty.$$

The following two questions are our main motivations of this paper:

(1^o) Under what less conditions on the coefficients and in what spaces or senses does the *uniqueness* for PIDE (1.3)-(1.4) hold?

(2^o) If the initial distribution μ_0 has a density with respect to the Lebesgue measure, does μ_t have a density with respect to the Lebesgue measure for any $t \in (0,1]$?

When there is no jump part and the diffusion coefficient is non-degenerate, in [3] the authors have already given rather weak conditions for the uniqueness of measure-valued solutions based upon the Dirichlet form theory. In [11], Figalli also gave some other conditions for the uniqueness of $L^1 \cap L^\infty$-solutions by proving a maximal principle. In [28], using a representation formula for the solutions of PDE (1.3) proved in [11], which is originally proved by Ambrosio [1] for continuity equation, we gave different conditions for the uniqueness of measure-valued solutions and L^p-solutions to second order degenerated Fokker-Planck equations. However, for obtaining the uniqueness for the above integro-differential equation of Fokker-Planck type (1.3), the *non-local* character of the operator $\mathscr{L}$ causes some new difficulties to analyze by the classical tools.

On the other hand, in the various *non-degenerate* cases, there have been many works devoting to the study of the absolute continuity of the law of X_t with respect to the Lebesgue measure even that the initial distribution is a dirac measure. Since we are working in a different direction (i.e., degenerate case), we do not intend to pursue this issue and only mention the recent works of [12, 16] (see also the references therein).

Now, for answering the above two questions to equation (1.3), we shall use a purely probabilistic approach. The first step is to extend the almost everywhere stochastic flow in [18, 32, 34] to SDE (1.1) so that we can solve the above question (2^o). In this extension, we need to carefully treat the jump part. Since even in the linear case, if one does not make any restriction on the jump, the law of the solution would not be absolutely continuous with respect to the Lebesgue measure (cf. [24, p.328, Example]). The next step is to prove a representation formula for the solution of (1.3) as in [11, Theorem 2.6]. This will lead to the uniqueness of PIDE (1.3) by proving the pathwise uniqueness of SDE (1.1).

This paper is organized as follows: In Section 2, we collect some well known facts for later use. In Section 3, we study the smooth SDEs with jumps, and prove an a priori estimate about the Jacobi determinant of $x \mapsto X_t(x)$. In Section 4, we prove the existence and uniqueness of almost everywhere or generalized stochastic flows for SDEs with jumps and rough drifts. In Section 5, the application to second order integro-differential equations of Fokker-Planck type is presented. In this part, we only consider the constant coefficient jump.

2 Preliminaries

Throughout this paper we assume that $d \geqslant 2$. Let $\mathbb{M}_{d\times d}$ be the set of all $d \times d$-matrices. We need the following simple lemma about the differentials of determinant function.

Lemma 2.1. *Let $A = (a_{ij}), B = (b_{ij}) \in \mathbb{M}_{d\times d}$. Then the first and second order derivatives of the determinant function $\det : \mathbb{M}_{d\times d} \to \mathbb{R}$ are given by*

$$(\nabla \det)(A)(BA) := \frac{\mathrm{d}}{\mathrm{d}t} \det(A + tBA)|_{t=0} = \det(A)\mathrm{tr}(B) \tag{2.1}$$

and

$$\begin{aligned}(\nabla^2 \det)(A)(BA, BA) &:= \frac{\partial^2}{\partial t \partial s} \det(A + tBA + sBA)|_{s=t=0} \\ &= \det(A) \sum_{i,j} [b_{ii}b_{jj} - b_{ij}b_{ji}]. \end{aligned} \tag{2.2}$$

Moreover, if $|b_{ij}| \leqslant \alpha$ *for all* i, j*, then*

$$|\det(\mathbb{I}+B)-1-\mathrm{tr}(B)| \leqslant d!d^2\alpha^2(1+\alpha)^{d-2}. \tag{2.3}$$

Proof. Notice that

$$\det(A+tBA)=\det(A)\det(\mathbb{I}+tB)$$

and

$$\det(A+tBA+sBA)=\det(A)\det(\mathbb{I}+(t+s)B).$$

Formulas (2.1) and (2.2) are easily derived from the definition

$$\det(\mathbb{I}+tB):=\sum_{\sigma\in S_d}\mathsf{sgn}(\sigma)\prod_{i=1}^{d}(1_{i\sigma(i)}+tb_{i\sigma(i)}), \tag{2.4}$$

where S_d is the set of all permutations of $\{1,2,\cdots,d\}$ and $\mathrm{sgn}(\sigma)$ is the sign of σ.

As for (2.3), let $h(t):=\det(\mathbb{I}+tB)$, then $h'(0)=\mathrm{tr}(B)$ and

$$\det(\mathbb{I}+B)-1-\mathrm{tr}(B)=\int_0^1\int_0^t h''(s)\mathrm{d}s\mathrm{d}t=\int_0^1(1-s)h''(s)\mathrm{d}s.$$

Estimate (2.3) now follows from (2.4). □

The following result is due to Lepingle and Mémin [20] and taken from [25, Theorem 6].

Theorem 2.2. *(Lepingle-Mémin [20]) Let* M *be a locally square integrable martingale such that* $\Delta M > -1$ *a.s. Let* $\mathcal{E}(M)$ *be the Doléans-Dade exponential defined by*

$$\mathcal{E}(M)_t := \exp\left\{M_t - \frac{1}{2}\langle M^{\mathrm{c}}\rangle_t\right\}\times\prod_{0<s\leqslant t}(1+\Delta M_s)e^{-\Delta M_s}.$$

If for some $T>0$*,*

$$\mathbb{E}\Big[\exp\Big\{\frac{1}{2}\langle M^{\mathrm{c}}\rangle_T+\langle M^{\mathrm{d}}\rangle_T\Big\}\Big]<\infty,$$

where M^{c} *and* M^{d} *are respectively continuous and purely discontinuous martingale parts of* M*, and the angle bracket* $\langle\cdot\rangle$ *denotes the conditional quadratic variation, then* $\mathcal{E}(M)$ *is a martingale on* $[0,T]$*.*

In Sections 3 and 4, we shall deal with the general Poisson point process. Below we introduce some necessary spaces and processes. Let $(\Omega,\mathscr{F},P;(\mathscr{F}_t)_{t\geqslant 0})$ be a complete filtered probability space and $(\mathbb{U},\mathscr{U})$ a measurable space. Let $(W(t))_{t\geqslant 0}$ be a d-dimensional standard $(\mathscr{F}_t)$-adapted Brownian motion and $(p_t)_{t\geqslant 0}$ an $(\mathscr{F}_t)$-adapted Poisson point process with values in $\mathbb{U}$ and with intensity measure $\nu_t(\mathrm{d}u)\mathrm{d}t$, a σ-finite measure on $[0,1]\times\mathbb{U}$ (cf. [14]). Let $N((0,t],\mathrm{d}u)$ be the counting measure of p_t, i.e., for any $\Gamma\in\mathscr{U}$,

$$N((0,t],\Gamma):=\sum_{0<s\leqslant t}1_\Gamma(p_s).$$

The compensated Poisson random measure of N is given by

$$\tilde{N}((0,t],\mathrm{d}u):=N((0,t],\mathrm{d}u)-\int_0^t\nu_s(\mathrm{d}u)\mathrm{d}s.$$

We remark that for $\Gamma\in\mathscr{U}$ with $\int_0^t\nu_s(\Gamma)\mathrm{d}s<+\infty$, the random variable $N((0,t],\Gamma)$ obeys the Poisson distribution with parameter $\int_0^t\nu_s(\Gamma)\mathrm{d}s$.

Below, the letter C with or without subscripts will denote a positive constant whose value is not important and may change in different occasions. Moreover, all the derivatives, gradients and divergences are taken in the distributional sense.

The following lemma is a generalization of [27, Proposition 1.12, p. 476] (cf. [23, Lemma A.2]).

Lemma 2.3. *Let $L : \mathbb{U} \to \mathbb{R}$ be a measurable function satisfying that $\int_0^1\int_{\mathbb{U}} L(u)\nu_s(\mathrm{d}u)\mathrm{d}s < +\infty$ and $|L(u)| \leqslant C$. Then for any $t > 0$,*

$$\mathbb{E}\exp\left\{\sum_{0<s\leqslant t} L(p_s)\right\} = \exp\left\{\int_0^t\int_{\mathbb{U}}(e^{L(u)}-1)\nu_s(\mathrm{d}u)\mathrm{d}s\right\} < +\infty.$$

We also need the following technical lemma (cf. [34, Lemma 3.4]).

Lemma 2.4. *Let μ be a locally finite measure on $\mathbb{R}^d$ and $(X_n)_{n\in\mathbb{N}}$ be a family of random fields on $\Omega\times\mathbb{R}^d$. Suppose that X_n converges to X for $P\otimes\mu$-almost all (ω, x), and for some $p \geqslant 1$, there is a constant $K_p > 0$ such that for any nonnegative measurable function $\varphi \in L^p_\mu(\mathbb{R}^d)$,*

$$\sup_n \mathbb{E}\int_{\mathbb{R}^d}\varphi(X_n(x))\mu(\mathrm{d}x) \leqslant K_p\|\varphi\|_{L^p_\mu}. \tag{2.5}$$

Then we have:

(i). For any nonnegative measurable function $\varphi \in L^p_\mu(\mathbb{R}^d)$,

$$\mathbb{E}\int_{\mathbb{R}^d}\varphi(X(x))\mu(\mathrm{d}x) \leqslant K_p\|\varphi\|_{L^p_\mu}. \tag{2.6}$$

(ii). If φ_n converges to φ in $L^p_\mu(\mathbb{R}^d)$, then for any $N > 0$,

$$\lim_{n\to\infty}\mathbb{E}\int_{|x|\leqslant N}|\varphi_n(X_n(x)) - \varphi(X(x))|\mu(\mathrm{d}x) = 0. \tag{2.7}$$

Let φ be a locally integrable function on $\mathbb{R}^d$. For every $R > 0$, the local maximal function is defined by

$$\mathcal{M}_R\varphi(x) := \sup_{0<r<R}\frac{1}{|B_r|}\int_{B_r}\varphi(x+y)\mathrm{d}y =: \sup_{0<r<R}\frac{1}{|B_r|}\int_{B_r}\varphi(x+y)\mathrm{d}y,$$

where $B_r := \{x \in \mathbb{R}^d : |x| < r\}$ and $|B_r|$ denotes the volume of B_r. The following result can be found in [9, p.143, Theorem 3] and [6, Appendix A].

Lemma 2.5. *(i) (Morrey's inequality) Let $\varphi \in L^1_{loc}(\mathbb{R}^d)$ be such that $\nabla\varphi \in L^q_{loc}(\mathbb{R}^d)$ for some $q > d$. Then there exist $C_{q,d} > 0$ and a negligible set A such that for all $x, y \in A^c$ with $|x-y| \leqslant R$,*

$$\begin{aligned}|\varphi(x)-\varphi(y)| &\leqslant C_{q,d}\cdot|x-y|\cdot\left(\frac{1}{|B_{|x-y|}|}\int_{B_{|x-y|}}|\nabla\varphi|^q(x+z)\mathrm{d}z\right)^{1/q}\\ &\leqslant C_{q,d}\cdot|x-y|\cdot(\mathcal{M}_R|\nabla\varphi|^q(x))^{1/q}.\end{aligned} \tag{2.8}$$

(ii) Let $\varphi \in L^1_{loc}(\mathbb{R}^d)$ be such that $\nabla\varphi \in L^1_{loc}(\mathbb{R}^d)$. Then there exist $C_d > 0$ and a negligible set A such that for all $x, y \in A^c$ with $|x-y| \leqslant R$,

$$|\varphi(x)-\varphi(y)| \leqslant C_d\cdot|x-y|\cdot(\mathcal{M}_R|\nabla\varphi|(x) + \mathcal{M}_R|\nabla\varphi|(y)). \tag{2.9}$$

(iii) Let $\varphi \in L^p_{loc}(\mathbb{R}^d)$ for some $p > 1$. Then for some $C_{d,p} > 0$ and any $N, R > 0$,

$$\left(\int_{B_N}(\mathcal{M}_R|\varphi|(x))^p\mathrm{d}x\right)^{1/p} \leqslant C_{d,p}\left(\int_{B_{N+R}}|\varphi(x)|^p\mathrm{d}x\right)^{1/p}. \tag{2.10}$$

3 SDEs with jumps and smooth coefficients

In this section, we consider the following SDE with jump:

$$X_t(x) = x + \int_0^t b_s(X_s(x))\mathrm{d}s + \int_0^t \sigma_s(X_s(x))\mathrm{d}W_s + \int_0^{t+}\int_{\mathbb{U}} f_s(X_{s-}(x), u)\tilde{N}(\mathrm{d}u, \mathrm{d}s), \quad (3.1)$$

where the coefficients $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d, \sigma : [0,1] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $f : [0,1] \times \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d$ are measurable functions and smooth in the spatial variable x, and satisfy that

$$\int_0^1 (|b_s(0)| + \|\nabla b_s\|_\infty)\mathrm{d}s + \int_0^1 (|\sigma_s(0)|^2 + \|\nabla\sigma_s\|_\infty^2)\mathrm{d}s < +\infty. \quad (3.2)$$

Moreover, we assume that there exist two functions $L_1, L_2 : \mathbb{U} \to \mathbb{R}_+$ with

$$0 \leqslant L_1(u) \leqslant \alpha \wedge L_2(u), \quad \int_0^1\int_{\mathbb{U}} |L_2(u)|^2(1 + L_2(u))^p \nu_s(\mathrm{d}u)\mathrm{d}s < +\infty, \quad (3.3)$$

where $\alpha \in (0,1)$ is small and $p \in (1,\infty)$ is arbitrary, and such that for all $(s,x,u) \in [0,1] \times \mathbb{R}^d \times \mathbb{U}$,

$$|\nabla_x f_s(x,u)| \leqslant L_1(u), \quad |f_s(0,u)| \leqslant L_2(u). \quad (3.4)$$

Under conditions (3.2)-(3.4) with small α (saying less than $\frac{1}{8d}$), it is well known that SDE (3.1) defines a flow of C^∞-diffeomorphisms (cf. [13, 24], [23, Theorem 1.3]).

Let

$$J_t := J_t(x) := \nabla X_t(x) \in \mathbb{M}_{d\times d}.$$

Then J_t satisfies the following SDE (cf. [13, 24]):

$$J_t = \mathbb{I} + \int_0^t \nabla b_s(X_s)J_s\mathrm{d}s + \int_0^t \nabla\sigma_s(X_s)J_s\mathrm{d}W_s + \int_0^{t+}\int_{\mathbb{U}} \nabla f_s(X_{s-}, u)J_s\tilde{N}(\mathrm{d}u, \mathrm{d}s). \quad (3.5)$$

The following lemma will be our starting point in the sequent development.

Lemma 3.1. *The Jacobi determinant* $\det(J_t)$ *has the following explicit formula:*

$$\det(J_t) = \exp A_t \cdot \exp\left\{M_t - \tfrac{1}{2}\langle M^{\mathrm{c}}\rangle_t\right\} \prod_{0<s\leqslant t}(1 + \Delta M_s)e^{-\Delta M_s} =: \exp A_t \cdot \mathcal{E}(M)_t,$$

where $A_t := A_t^{(1)} + A_t^{(2)}$ *and* $M_t := M_t^{\mathrm{c}} + M_t^{\mathrm{d}}$ *are given by (3.6), (3.7), (3.8) and (3.9) below.*

Proof. By (3.5), Itô's formula and Lemma 2.1, we have

$$\begin{aligned}\det(J_t) = 1 &+ \int_0^t \mathrm{div} b_s(X_s)\det(J_s)\mathrm{d}s + \int_0^t \mathrm{div}\sigma_s(X_s)\det(J_s)\mathrm{d}W_s \\ &+ \frac{1}{2}\sum_{i,j,k}\int_0^t [\partial_i\sigma_s^{ik}\partial_j\sigma_s^{jk} - \partial_j\sigma_s^{ik}\partial_i\sigma_s^{jk}](X_s)\det(J_s)\mathrm{d}s \\ &+ \int_0^{t+}\int_{\mathbb{U}} \Big[\det((\mathbb{I} + \nabla f_s(X_{s-}, u))J_{s-}) - \det(J_{s-})\Big]\tilde{N}(\mathrm{d}u, \mathrm{d}s) \\ &+ \int_0^t\int_{\mathbb{U}} \Big[\det((\mathbb{I} + \nabla f_s(X_{s-}, u))J_{s-}) - \det(J_{s-}) \\ &\qquad - \mathrm{div} f_s(X_{s-}, u)\det(J_{s-})\Big]\nu_s(\mathrm{d}u)\mathrm{d}s \\ =: 1 &+ \int_0^{t+} \det(J_{s-})\mathrm{d}(A_s + M_s),\end{aligned}$$

where $A_t := A^{(1)}_t + A^{(2)}_t$ is a continuous increasing process given by

$$A^{(1)}_t = \int_0^t \Big[\mathrm{div} b_s(X_s) + \tfrac{1}{2}\sum_{i,j,k}[\partial_i\sigma^{ik}_s\partial_j\sigma^{jk}_s - \partial_j\sigma^{ik}_s\partial_i\sigma^{jk}_s](X_s)\Big]\mathrm{d}s \tag{3.6}$$

and

$$A^{(2)}_t = \int_0^t\!\!\int_{\mathbb{U}} \Big[\det(\mathbb{I}+\nabla f_s(X_{s-},u)) - 1 - \mathrm{div} f_s(X_{s-},u)\Big]\nu_s(\mathrm{d}u)\mathrm{d}s; \tag{3.7}$$

and $M_t := M^{\mathrm{c}}_t + M^{\mathrm{d}}_t$ is a martingale given by

$$M^{\mathrm{c}}_t := \int_0^t \mathrm{div}\sigma_s(X_s)\mathrm{d}W_s \tag{3.8}$$

and

$$M^{\mathrm{d}}_t := \int_0^{t+}\!\!\int_{\mathbb{U}} \Big[\det(\mathbb{I}+\nabla f_s(X_{s-},u)) - 1\Big]\tilde{N}(\mathrm{d}u,\mathrm{d}s). \tag{3.9}$$

By Doléans-Dade's exponential formula (cf. [24]), we obtain the desired formula. □

Below, we shall give an estimate for the p-order moment of the Jacobi determinant. For this aim, we introduce the following function of jump size control α:

$$\beta_\alpha := (d\alpha + d!d^2\alpha^2(1+\alpha)^{d-2})^{-1}. \tag{3.10}$$

Note that

$$\lim_{\alpha\downarrow 0}\beta_\alpha = +\infty.$$

Lemma 3.2. *Let β_α be defined by (3.10), where α is from (3.3) small enough so that $\beta_\alpha > 1$. Then for any $p\in(0,\beta_\alpha)$, we have*

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d}\mathbb{E}\left(\sup_{t\in[0,1]}\det(J_t(x))^{-p}\right)\\ &\qquad\leqslant C\left(p, \int_0^1\|[\mathrm{div}b_s]^-\|_\infty\mathrm{d}s, \int_0^1\|\nabla\sigma_s\|^2_\infty\mathrm{d}s, \int_0^1\!\!\int_{\mathbb{U}} L_1(u)^2\nu_s(\mathrm{d}u)\mathrm{d}s\right),\end{aligned}$$

where for a real number a, $a^- = \min(-a,0)$, the constant C is an increasing function with respect to its arguments.

Proof. First of all, by (3.6), we have

$$-A^{(1)}_t \leqslant C\int_0^1(\|[\mathrm{div}b_s]^-\|_\infty + \|\nabla\sigma_s\|^2_\infty)\mathrm{d}s,$$

and by (3.7), (2.3) and (3.4),

$$-A^{(2)}_t \leqslant C\int_0^t\!\!\int_{\mathbb{U}} L_1(u)^2\nu_s(\mathrm{d}u)\mathrm{d}s.$$

Hence, for any $p\geqslant 0$, we have

$$\sup_{t\in[0,1]}\exp(-pA_t) \leqslant \exp\left(C\int_0^1(\|[\mathrm{div}b_s]^-\|_\infty + \|\nabla\sigma_s\|^2_\infty)\mathrm{d}s + C\int_0^1\!\!\int_{\mathbb{U}} L_1(u)^2\nu_s(\mathrm{d}u)\mathrm{d}s\right).$$

Thus, by Lemma 3.1, it suffices to prove that for any $p \in (0, \beta_\alpha)$,

$$\mathbb{E}\left(\sup_{t\in[0,1]} \mathcal{E}(M)_t^{-p}\right) \leqslant C\left(p, \int_0^1 \|\mathrm{div}\sigma_s\|_\infty^2 \mathrm{d}s, \int_0^1\int_{\mathbb{U}} L_1(u)^2 \nu_s(\mathrm{d}u)\mathrm{d}s\right). \tag{3.11}$$

Noting that

$$\Delta M_s := M_s - M_{s-} = \det(\mathbb{I} + \nabla f_s(X_{s-}, p_s)) - 1,$$

by (2.3) and (3.3), we have

$$\begin{aligned} |\Delta M_s| &\leqslant |\mathrm{div} f_s(X_{s-}, p_s)| + d! d^2 L_1(u)^2 (1 + L_1(u))^{d-2} \\ &\leqslant d L_1(u) + d! d^2 L_1(u)^2 (1 + L_1(u))^{d-2} \\ &\leqslant d\alpha + d! d^2 \alpha^2 (1+\alpha)^{d-2} = \beta_\alpha^{-1}. \end{aligned} \tag{3.12}$$

Fixing $q \in (p, \beta_\alpha)$, we also have

$$|\Delta(-qM)_s| = q|\Delta M_s| < 1.$$

On the other hand, we have

$$\langle M^{\mathrm{c}} \rangle_1 \leqslant \int_0^1 \|\mathrm{div}\sigma_s\|_\infty^2 \mathrm{d}s,$$

and by (3.12) and Lemma 2.3,

$$\mathbb{E}\exp(p^2 \langle M^{\mathrm{d}} \rangle_1) = \mathbb{E}\exp\left\{p^2 \sum_{0<s\leqslant 1} |\Delta M_s|^2\right\} \leqslant \mathbb{E}\exp\left\{C \sum_{0<s\leqslant 1} |L_1(p_s)|^2\right\} < +\infty.$$

Thus, by Theorem 2.2, one knows that $t \mapsto \mathcal{E}(-qM)_t$ is an exponential martingale. Observe that

$$\begin{aligned} \mathcal{E}(M)_t^{-p} &= \mathcal{E}(-qM)_t^{\frac{p}{q}} \cdot \exp\left\{\frac{(q+1)p}{2} \langle M^{\mathrm{c}} \rangle_t\right\} \cdot \prod_{0<s\leqslant t} \frac{(1+\Delta M_s)^{-p}}{(1-q\Delta M_s)^{\frac{p}{q}}} \\ &\leqslant \mathcal{E}(-qM)_t^{\frac{p}{q}} \cdot \exp\left\{C\int_0^1 \|\mathrm{div}\sigma_s\|_\infty^2 \mathrm{d}s\right\} \cdot \prod_{0<s\leqslant t} G(\Delta M_s), \end{aligned}$$

where

$$G(r) := \frac{(1+r)^{-p}}{(1-qr)^{\frac{p}{q}}}, \quad |r| \leqslant \beta_\alpha^{-1}.$$

By Hölder's inequality and Doob's inequality, we obtain that for $\gamma \in (1, \frac{q}{p})$ and $\gamma^* = \frac{\gamma}{\gamma-1}$,

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0,1]} \mathcal{E}(M)_t^{-p}\right) &\leqslant C\left(\mathbb{E}\sup_{t\in[0,1]} \mathcal{E}(-qM)_t^{\frac{\gamma p}{q}}\right)^{\frac{1}{\gamma}} \cdot \left(\mathbb{E}\prod_{0<s\leqslant 1} G(\Delta M_s)^{\gamma^*}\right)^{\frac{1}{\gamma^*}} \\ &\leqslant C\left(\mathbb{E}\mathcal{E}(-qM)_1^{\frac{\gamma p}{q}}\right)^{\frac{1}{\gamma}} \cdot \left(\mathbb{E}\prod_{0<s\leqslant 1} G(\Delta M_s)^{\gamma^*}\right)^{\frac{1}{\gamma^*}} \\ &\leqslant C\left(\mathbb{E}\prod_{0<s\leqslant 1} G(\Delta M_s)^{\gamma^*}\right)^{\frac{1}{\gamma^*}}. \end{aligned} \tag{3.13}$$

Thanks to the following limit

$$\lim_{r\downarrow 0}\frac{\log G(r)}{r^2}=\frac{p(q+1)}{2},$$

we have for some $C=C(q,p,\beta_\alpha)>0$,

$$|\log G(r)|\leqslant C|r|^2,\ \ \forall|r|\leqslant\beta_\alpha^{-1}.$$

Therefore, by Lemma 2.3,

$$\begin{aligned}\mathbb{E}\left[\prod_{0<s\leqslant 1}G(\Delta M_s)^{\gamma^*}\right]&=\mathbb{E}\exp\left\{\sum_{0<s\leqslant 1}\gamma^*\log G(\Delta M_s)\right\}\\&\leqslant\mathbb{E}\exp\left\{\sum_{0<s\leqslant 1}C|\Delta M_s|^2\right\}\\&\overset{(3.12)}{\leqslant}\mathbb{E}\exp\left\{\sum_{0<s\leqslant 1}CL_1(p_s)^2\right\}\\&\leqslant\exp\left\{C\int_0^1\!\!\int_{\mathbb{U}}L_1(u)^2\nu_s(\mathrm{d}u)\mathrm{d}s\right\}.\end{aligned}\tag{3.14}$$

Estimate (3.11) now follows by combining (3.13) and (3.14). □

Let $X_t^{-1}(\omega,x)$ be the inverse of the mapping $x\mapsto X_t(\omega,x)$. In order to give an estimate for $\det(\nabla X_t^{-1}(x))$ in terms of $\det(J_t(x))=\det(\nabla X_t(x))$, we shall use a trick due to Cruzeiro [5] (see also [4, 34, 10]). Below, let

$$\mu(\mathrm{d}x):=\frac{\mathrm{d}x}{(1+|x|^2)^d}.\tag{3.15}$$

We write

$$\mathcal{J}_t(\omega,x):=\frac{(X_t(\omega,\cdot))_\sharp\mu(\mathrm{d}x)}{\mu(\mathrm{d}x)},\ \ \mathcal{J}_t^-(\omega,x):=\frac{(X_t^{-1}(\omega,\cdot))_\sharp\mu(\mathrm{d}x)}{\mu(\mathrm{d}x)},$$

which means that for any nonnegative measurable function φ on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d}\varphi(X_t(\omega,x))\mu(\mathrm{d}x)=\int_{\mathbb{R}^d}\varphi(x)\mathcal{J}_t(\omega,x)\mu(\mathrm{d}x),\tag{3.16}$$

$$\int_{\mathbb{R}^d}\varphi(X_t^{-1}(\omega,x))\mu(\mathrm{d}x)=\int_{\mathbb{R}^d}\varphi(x)\mathcal{J}_t^-(\omega,x)\mu(\mathrm{d}x).\tag{3.17}$$

It is easy to see that for almost all ω and all $(t,x)\in[0,1]\times\mathbb{R}^d$,

$$\mathcal{J}_t(\omega,x)=1/\mathcal{J}_t^-(\omega,X_t^{-1}(\omega,x))\tag{3.18}$$

and

$$\mathcal{J}_t^-(x)=\frac{(1+|x|^2)^d}{(1+|X_t(x)|^2)^d}\det(J_t(x)).\tag{3.19}$$

Remark 3.3. *The measure μ in (3.15) is chosen so that $\mu(\mathbb{R}^d)<+\infty$ and it is convenient to use Itô's formula as shown below, which also plays a crucial role in the proof of Theorem 3.5. Of course, one can use other weighted functions, but other choices would lead to complicated conditions on b,σ and f (see [34]).*

We need the following estimate:

Lemma 3.4. *For any $p \geqslant 1$, we have*

$$\sup_{x\in\mathbb{R}^d}\mathbb{E}\left(\sup_{t\in[0,1]}\frac{(1+|X_t(x)|^2)^p}{(1+|x|^2)^p}\right)$$
$$\leqslant C\left(p,\int_0^1\left\|\frac{|b_s(x)|}{1+|x|}\right\|_\infty \mathrm{d}s,\int_0^1\left\|\frac{|\sigma_s(x)|}{1+|x|}\right\|_\infty^2 \mathrm{d}s,\int_0^1\int_{\mathbb{U}} L_2(u)^2(1+L_2(u))^{4p-2}\nu_s(\mathrm{d}u)\mathrm{d}s\right),$$

where the constant C is an increasing function with respect to its arguments.

Proof. Letting $h(x) := (1+|x|^2)^p$, by Itô's formula, we have

$$\begin{aligned} h(X_t)-h(x) &= \int_0^t (b_s^i\partial_i h)(X_s)\mathrm{d}s + \int_0^t (\sigma_s^{ik}\partial_i h)(X_s)\mathrm{d}W_s^k + \frac{1}{2}\int_0^t (\partial_i\partial_j h\cdot\sigma_s^{ik}\sigma_s^{jk})(X_s)\mathrm{d}s \\ &\quad + \int_0^t\int_{\mathbb{U}}(h(X_{s-}+f_s(X_{s-},u))-h(X_{s-})-f_s^i(X_{s-},u)\partial_i h(X_{s-}))\nu_s(\mathrm{d}u)\mathrm{d}s \\ &\quad + \int_0^{t+}\int_{\mathbb{U}}(h(X_{s-}+f_s(X_{s-},u))-h(X_{s-}))\tilde{N}(\mathrm{d}u,\mathrm{d}s). \end{aligned}$$

By elementary calculations, one has that for all $x\in\mathbb{R}^d$,

$$C_1(1+|x|)^{2p}\leqslant h(x)\leqslant C_2(1+|x|)^{2p} \tag{3.20}$$

and

$$|\partial_i h(x)|\leqslant\frac{Ch(x)}{1+|x|}\leqslant C(1+|x|)^{2p-1},\quad |\partial_i\partial_j h(x)|\leqslant\frac{Ch(x)}{(1+|x|)^2}\leqslant C(1+|x|)^{2p-2}.$$

On the other hand, by Taylor's formula, we have

$$|h(x+y)-h(x)|\leqslant|y^i\partial_i h(x+\theta_1 y)|$$

and

$$|h(x+y)-h(x)-y^i\partial_i h(x)|\leqslant|y^iy^j\partial_i\partial_j h(x+\theta_2 y)|/2,$$

where $\theta_1,\theta_2\in(0,1)$. Thus, for $p\geqslant 1$, we have

$$\begin{aligned} |h(x+f_s(x,u))-h(x)| &\leqslant |f_s(x,u)|\cdot(1+|x+\theta_1 f_s(x,u)|)^{2p-1} \\ &\overset{(3.4)}{\leqslant} (L_2(u)+L_1(u)|x|)(1+L_2(u)+(1+L_1(u))|x|)^{2p-1} \\ &\overset{(3.3)}{\leqslant} L_2(u)(1+L_2(u))^{2p-1}(1+|x|)^{2p} \\ &\overset{(3.20)}{\leqslant} L_2(u)(1+L_2(u))^{2p-1}h(x) \end{aligned}$$

and

$$|h(x+f_s(x,u))-h(x)-f_s^i(x,u)\partial_i h(x)|\leqslant L_2(u)^2(1+L_2(u))^{2p-2}h(x).$$

Using the above estimates, if we let

$$\ell_1(s):=\left\|\frac{|b_s(x)|}{1+|x|}\right\|_\infty,\quad \ell_2(s):=\left\|\frac{|\sigma_s(x)|}{1+|x|}\right\|_\infty,$$

and

$$\ell_3(s):=\int_{\mathbb{U}} L_2(u)^2(1+L_2(u))^{4p-2}\nu_s(\mathrm{d}u),$$

then, by Burkholder's inequality and Young's inequality, we have

$$\mathbb{E}\left(\sup_{s\in[0,t]} h(X_s)\right) \leqslant h(x) + C\int_0^t (\ell_1(s)+\ell_2^2(s))\mathbb{E}h(X_s)\mathrm{d}s + \mathbb{E}\left(\int_0^t \ell_2^2(s)h(X_s)^2\mathrm{d}s\right)^{1/2}$$
$$+ C\int_0^t \ell_3(s)\mathbb{E}h(X_s)\mathrm{d}s + C\mathbb{E}\left(\int_0^t \ell_3(s)h(X_s)^2\mathrm{d}s\right)^{1/2}$$
$$\leqslant h(x) + C\int_0^t (\ell_1(s)+\ell_2^2(s)+\ell_3(s))\mathbb{E}h(X_s)\mathrm{d}s + \frac{1}{2}\mathbb{E}\left(\sup_{s\in[0,t]} h(X_s)\right),$$

which leads to

$$\mathbb{E}\left(\sup_{s\in[0,t]} h(X_s)\right) \leqslant h(x) + C\int_0^t (\ell_1(s)+\ell_2^2(s)+\ell_3(s))\mathbb{E}h(X_s)\mathrm{d}s.$$

Hence, by Gronwall's inequality, we obtain

$$\mathbb{E}\left(\sup_{s\in[0,1]} h(X_s)\right) \leqslant Ch(x).$$

The proof is complete. □

Combining Lemmas 3.2 and 3.4, we obtain that

Theorem 3.5. *Let β_α be defined by (3.10), where α is from (3.3) small enough so that $\beta_\alpha > 1$. Then for any $p \in (0, \beta_\alpha)$,*

$$\mathbb{E}\left(\sup_{t\in[0,1]} \int_{\mathbb{R}^d} |\mathcal{J}_t(x)|^{p+1}\mu(\mathrm{d}x)\right) \leqslant C,$$

where the constant C is inherited from Lemmas 3.2 and 3.4.

Proof. The estimate follows from

$$\int_{\mathbb{R}^d} |\mathcal{J}_t(x)|^{p+1}\mu(\mathrm{d}x) \overset{(3.18)(3.17)}{=} \int_{\mathbb{R}^d} |\mathcal{J}_t^-(x)|^{-p}\mu(\mathrm{d}x)$$
$$\overset{(3.19)}{=} \int_{\mathbb{R}^d} \frac{(1+|X_t(x)|^2)^{dp}}{(1+|x|^2)^{dp}} \det(J_t(x))^{-p}\mu(\mathrm{d}x),$$

Hölder's inequality and Lemmas 3.2 and 3.4. □

4 SDEs with jumps and rough drifts

We first introduce the following notion of generalized stochastic flows (cf. [19, 32, 34]).

Definition 4.1. *Let $X_t(\omega, x)$ be a $\mathbb{R}^d$-valued measurable stochastic field on $[0,1]\times\Omega\times\mathbb{R}^d$. For a locally finite measure μ on $\mathbb{R}^d$, we say X a μ-**almost everywhere stochastic flow** or **generalized stochastic flow** of SDE (3.1) if*

(A) *there exist a $p \geqslant 1$ and a constant $K_p > 0$ such that for any nonnegative measurable function $\varphi \in L^p_\mu(\mathbb{R}^d)$,*

$$\sup_{t\in[0,1]} \mathbb{E}\int_{\mathbb{R}^d} \varphi(X_t(x))\mu(\mathrm{d}x) \leqslant K_p\|\varphi\|_{L^p_\mu}; \tag{4.1}$$

(B) *for μ-almost all $x \in \mathbb{R}^d$, $t \mapsto X_t(x)$ is a càdlàg and $(\mathscr{F}_t)$-adapted process and solves equation (3.1).*

The main result of this section is:

Theorem 4.2. *Assume that for some $q > 1$,*

$$|\nabla b| \in L^1([0,1]; L^q_{loc}(\mathbb{R}^d)), \quad [\mathrm{div} b]^-, |\nabla\sigma|^2, \frac{|b|}{1+|x|}, \frac{|\sigma|^2}{1+|x|^2} \in L^1([0,1]; L^\infty(\mathbb{R}^d)),$$

and for some functions $L_i : \mathbb{U} \to [0, +\infty), i = 1, 2$ satisfying (3.3), and all $(s, u) \in [0,1] \times \mathbb{U}$ and $x, y \in \mathbb{R}^d$,

$$|f_s(x,u) - f_s(y,u)| \leqslant L_1(u)|x-y|, \quad |f_s(0,u)| \leqslant L_2(u). \tag{4.2}$$

Let $\mu(\mathrm{d}x) = (1+|x|^2)^{-d}\mathrm{d}x$ and let β_α be defined by (3.10), where α is from (3.3) small enough so that $\beta_\alpha > \frac{1}{q-1}$. Then there exists a unique μ-almost everywhere stochastic flow to SDE (3.1) with any $p \geqslant q$ in (4.1).

Remark 4.3. *Let $b(x) = \frac{x}{|x|}1_{x\neq 0}$. It is easy to check that $\mathrm{div} b(x) = \frac{d-1}{|x|}$ and $|\nabla b| \in L^p_{loc}(\mathbb{R}^d)$ provided that $p \in [1, d)$.*

Let $\chi \in C^\infty(\mathbb{R}^d)$ be a nonnegative cutoff function with

$$\|\chi\|_\infty \leqslant 1, \quad \chi(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases} \tag{4.3}$$

Let $\rho \in C^\infty(\mathbb{R}^d)$ be a nonnegative mollifier with support in $B_1 := \{|x| \leqslant 1\}$ and $\int_{\mathbb{R}^d} \rho(x)\mathrm{d}x = 1$. Set

$$\chi_n(x) := \chi(x/n), \quad \rho_n(x) := n^d \rho(nx)$$

and define

$$b^n_s := b_s * \rho_n \cdot \chi_n, \quad \sigma^n_s := \sigma_s * \rho_n, \quad f^n_s(\cdot, u) = f_s(\cdot, u) * \rho_n. \tag{4.4}$$

Remark 4.4. *Since σ and f are Lipschitz continuous, it does not need to cut off σ^n and f^n by multiplying χ_n as seen in the following lemma.*

The following lemma directly follows from the definitions and the property of convolutions.

Lemma 4.5. *For some $C > 0$ independent of n, we have*

$$\int_0^1 \|[\mathrm{div} b^n_s]^-\|_\infty \mathrm{d}s \leqslant \int_0^1 \|[\mathrm{div} b_s]^-\|_\infty \mathrm{d}s + C \int_0^1 \left\| \frac{|b_s(x)|}{1+|x|} \right\|_\infty \mathrm{d}s$$

$$\int_0^1 \left\| \frac{|b^n_s(x)|}{1+|x|} \right\|_\infty \mathrm{d}s \leqslant C \int_0^1 \left\| \frac{|b_s(x)|}{1+|x|} \right\|_\infty \mathrm{d}s$$

$$\int_0^1 \|\nabla\sigma^n_s\|^2_\infty \mathrm{d}s \leqslant \int_0^1 \|\nabla\sigma_s\|^2_\infty \mathrm{d}s,$$

$$\int_0^1 \left\| \frac{|\sigma^n_s(x)|}{1+|x|} \right\|^2_\infty \mathrm{d}s \leqslant C \int_0^1 \left\| \frac{|\sigma_s(x)|}{1+|x|} \right\|^2_\infty \mathrm{d}s$$

and

$$|\nabla_x f^n_s(x,u)| \leqslant L_1(u), \quad |f^n_s(0,u)| \leqslant 2L_2(u).$$

Proof. Noticing that

$$\mathrm{div} b^n_s(x) = (\mathrm{div} b_s) * \rho_n(x) \cdot \chi_n(x) + b^i_s * \rho_n(x) \cdot \partial_i \chi_n(x),$$

by the elementary inequality $(a+b)^- \leqslant a^- + |b|$, we have

$$[\mathrm{div} b^n_s(x)]^- \leqslant [\mathrm{div} b_s]^- * \rho_n(x) + |b^i_s| * \rho_n(x) \cdot |\partial_i \chi_n(x)|. \tag{4.5}$$

Moreover, observing that

$$|\partial_i \chi_n(x)| = \frac{|(\partial_i \chi)(x/n)|}{n} \leqslant \frac{C 1_{n \leqslant |x| \leqslant n+1}}{1+|x|} \leqslant \frac{C}{1+|x|},$$

we have

$$|b^i_s| * \rho_n(x) \cdot |\partial_i \chi_n(x)| \leqslant C \int_{\mathbb{R}^d} \frac{|b_s(x-y)|}{1+|x|} \rho_n(y) \mathrm{d}y \leqslant 2C \int_{\mathbb{R}^d} \frac{|b_s(x-y)|}{1+|x-y|} \rho_n(y) \mathrm{d}y. \tag{4.6}$$

Combining (4.5) and (4.6), we obtain the first estimate. The others are similar. □

Let $X^n_t(x)$ be the stochastic flow of C^∞-diffeomorphisms to SDE (3.1) associated with coefficients (b^n, σ^n, f^n).

Lemma 4.6. *let β_α be defined by (3.10), where α is from (3.3) small enough so that $\beta_\alpha > 1$. Then for any $p > 1 + \frac{1}{\beta_\alpha}$, there exists a constant $C_p > 0$ such that for all nonnegative function $\varphi \in L^p_\mu(\mathbb{R}^d)$,*

$$\sup_n \mathbb{E} \left(\sup_{t \in [0,1]} \int_{\mathbb{R}^d} \varphi(X^n_t(x)) \mu(\mathrm{d}x) \right) \leqslant C_p \|\varphi\|_{L^p_\mu}. \tag{4.7}$$

Proof. The estimate follows from

$$\int_{\mathbb{R}^d} \varphi(X^n_t(x)) \mu(\mathrm{d}x) = \int_{\mathbb{R}^d} \varphi(x) \mathcal{J}^n_t(x) \mu(\mathrm{d}x) \leqslant \|\varphi\|_{L^p_\mu} \left(\int_{\mathbb{R}^d} |\mathcal{J}^n_t(x)|^{\frac{p}{p-1}} \mu(\mathrm{d}x) \right)^{1-\frac{1}{p}},$$

and Theorem 3.5 and Lemma 4.5. □

Lemma 4.7. *For any $n, m > 4/\delta > 0$, we have*

$$\frac{|z + f^n_t(x+z,u) - f^m_t(x,u)|^2 - |z|^2}{|z|^2 + \delta^2} \leqslant 4(L_1(u) + L_1(u)^2).$$

Proof. Noticing that by the property of convolutions and (4.2),

$$\begin{aligned} |f^n_t(x+z,u) - f^m_t(x,u)| &\leqslant |f^n_t(x,u) - f_t(x,u)| + |f^m_t(x,u) - f_t(x,u)| \\ &\quad + |f^n_t(x+z,u) - f^n_t(x,u)| \\ &\leqslant L_1(u)|z| + L_1(u)(n^{-1} + m^{-1}), \end{aligned}$$

we have

$$\begin{aligned} &\frac{|z + f^n_t(x+z,u) - f^m_t(x,u)|^2 - |z|^2}{|z|^2 + \delta^2} \\ &\leqslant \frac{2|z|(|z| + (n^{-1} + m^{-1}))L_1(u) + (|z| + (n^{-1} + m^{-1}))^2 L_1(u)^2}{|z|^2 + \delta^2} \\ &\leqslant 2[1 + \delta^{-1}(n^{-1} + m^{-1})](L_1(u) + L_1(u)^2), \end{aligned}$$

which yields the desired estimate. □

We now prove the following key estimate.

Lemma 4.8. *For any* $R > 1$*, there exist constants* $C_1, C_2 > 0$ *such that for all* $\delta \in (0,1)$ *and* $n, m > 4/\delta$*,*

$$\mathbb{E}\int_{G_R^{n,m}} \sup_{t\in[0,1]} \log\left(\frac{|X_t^n(x) - X_t^m(x)|^2}{\delta^2} + 1\right)\mu(\mathrm{d}x)$$
$$\leqslant C_1 + \frac{C_2}{\delta}\int_0^1 \left(\|b_s^n - b_s^m\|_{L^q(B_R)} + \|\sigma_s^n - \sigma_s^m\|^2_{L^{2q}(B_R)}\right)\mathrm{d}s, \quad (4.8)$$

where $\mu(\mathrm{d}x) = (1+|x|^2)^{-d}\mathrm{d}x$ *and*

$$G_R^{n,m}(\omega) := \left\{x \in \mathbb{R}^d : \sup_{t\in[0,1)} |X_t^n(\omega,x)| \vee |X_t^m(\omega,x)| \leqslant R\right\}.$$

Proof. Set

$$Z_t^{n,m}(\omega,x) := X_t^n(\omega,x) - X_t^m(\omega,x)$$

and

$$F_t^{n,m}(\omega,x,u) := f_t^n(X_{t-}^n(\omega,x),u) - f_t^m(X_{t-}^m(\omega,x),u).$$

If there are no confusions, we shall drop the variable "x" below. Note that

$$Z_t^{n,m} = \int_0^t (b_s^n(X_s^n) - b_s^m(X_s^m))\mathrm{d}s + \int_0^t (\sigma_s^n(X_s^n) - \sigma_s^m(X_s^m))\mathrm{d}W_s$$
$$+ \int_0^{t+}\int_{\mathbb{U}} F_s^{n,m}(u)\tilde{N}(\mathrm{d}u,\mathrm{d}s).$$

By Itô's formula, we have

$$\log\left(\frac{|Z_t^{n,m}|^2}{\delta^2} + 1\right) =: I_1^{n,m}(t) + I_2^{n,m}(t) + I_3^{n,m}(t) + I_4^{n,m}(t) + I_5^{n,m}(t) + I_6^{n,m}(t),$$

where

$$I_1^{n,m}(t) := 2\int_0^t \frac{\langle Z_s^{n,m}, b_s^n(X_s^n) - b_s^m(X_s^m)\rangle_{\mathbb{R}^d}}{|Z_s^{n,m}|^2 + \delta^2}\mathrm{d}s,$$
$$I_2^{n,m}(t) := 2\int_0^t \frac{\langle Z_s^{n,m}, (\sigma_s^n(X_s^n) - \sigma_s^m(X_s^m))\mathrm{d}W_s\rangle_{\mathbb{R}^d}}{|Z_s^{n,m}|^2 + \delta^2},$$
$$I_3^{n,m}(t) := \int_0^t \frac{\|\sigma_s^n(X_s^n) - \sigma_s^m(X_s^m)\|^2}{|Z_s^{n,m}|^2 + \delta^2}\mathrm{d}s,$$
$$I_4^{n,m}(t) := -2\int_0^t \frac{|(\sigma_s^n(X_s^n) - \sigma_s^m(X_s^m))^{\mathrm{t}} \cdot Z_s^{n,m}|^2}{(|Z_s^{n,m}|^2 + \delta^2)^2}\mathrm{d}s,$$
$$I_5^{n,m}(t) := \int_0^{t+}\int_{\mathbb{U}} \left(\log\frac{|Z_{s-}^{n,m} + F_s^{n,m}(u)|^2 + \delta^2}{|Z_{s-}^{n,m}|^2 + \delta^2} - \frac{|Z_{s-}^{n,m} + F_s^{n,m}(u)|^2 - |Z_{s-}^{n,m}|^2}{|Z_{s-}^{n,m}|^2 + \delta^2}\right)\nu_s(\mathrm{d}u)\mathrm{d}s,$$
$$I_6^{n,m}(t) := \int_0^{t+}\int_{\mathbb{U}} \log\frac{|Z_{s-}^{n,m} + F_s^{n,m}(u)|^2 + \delta^2}{|Z_{s-}^{n,m}|^2 + \delta^2}\tilde{N}(\mathrm{d}u,\mathrm{d}s).$$

For $I_1^{n,m}(t)$, we have

$$\sup_{t\in[0,1]}|I_1^{n,m}(t)| \leqslant 2\int_0^1 \frac{|b_s^n(X_s^n) - b_s^n(X_s^m)|}{\sqrt{|Z_s^{n,m}|^2 + \delta^2}}\mathrm{d}s + \frac{2}{\delta}\int_0^1 |b_s^n(X_s^m) - b_s^m(X_s^m)|\mathrm{d}s$$
$$=: I_{11}^{n,m} + I_{12}^{n,m}.$$

Noting that

$$G_R^{n,m}(\omega) \subset \{x : |X_t^n(\omega,x)| \leqslant R\} \cap \{x : |X_t^m(\omega,x)| \leqslant R\}, \quad \forall t \in [0,1),$$

we have

$$\begin{aligned}
\mathbb{E}\int_{G_R^{n,m}} |I_{12}^{n,m}(x)|\mu(\mathrm{d}x) &\leqslant \frac{2}{\delta}\mathbb{E}\int_0^1\!\!\int_{\mathbb{R}^d} |1_{B_R}(b_s^n - b_s^m)|(X_s^m(x))\mu(\mathrm{d}x)\mathrm{d}s \\
&\overset{(4.7)}{\leqslant} \frac{C}{\delta}\int_0^1 \|1_{B_R}(b_s^n - b_s^m)\|_{L^q_\mu}\mathrm{d}s \\
&\leqslant \frac{C}{\delta}\int_0^1 \|b_s^n - b_s^m\|_{L^q(B_R)}\mathrm{d}s. \qquad (4.9)
\end{aligned}$$

For $I_{11}^{n,m}$, in view of $\mu(\mathrm{d}x) \leqslant \mathrm{d}x$, we have

$$\begin{aligned}
&\mathbb{E}\int_{G_R^{n,m}} |I_{11}^{n,m}(x)|\mu(\mathrm{d}x) \\
&\overset{(2.9)}{\leqslant} C\mathbb{E}\int_0^1\!\!\int_{G_R^{n,m}} (\mathcal{M}_{2R}|\nabla b_s^n|(X_s^n(x)) + \mathcal{M}_{2R}|\nabla b_s^n|(X_s^m(x)))\mu(\mathrm{d}x)\mathrm{d}s \\
&\overset{(4.7)}{\leqslant} C\int_0^1 \left(\int_{B_R} (\mathcal{M}_{2R}|\nabla b_s^n|(x))^q\mu(\mathrm{d}x)\right)^{1/q}\mathrm{d}s \\
&\overset{(2.10)}{\leqslant} C\int_0^1 \|\nabla b_s^n\|_{L^q(B_{3R})}\mathrm{d}s \leqslant C\int_0^1 \|\nabla b_s\|_{L^q(B_{3R})}\mathrm{d}s. \qquad (4.10)
\end{aligned}$$

For $I_2^{n,m}(t)$, set

$$\tau_R^{n,m}(\omega,x) := \inf\Big\{t \in [0,1] : |X_t^n(\omega,x)| \vee |X_t^m(\omega,x)| > R\Big\},$$

then it is easy to see that

$$G_R^{n,m}(\omega) = \{x \in \mathbb{R}^d : \tau_R^{n,m}(\omega,x) = 1\}.$$

By Burkholder's inequality and Fubini's theorem, we have

$$\begin{aligned}
&\mathbb{E}\int_{G_R^{n,m}} \sup_{t\in[0,1]} |I_2^{n,m}(t,x)|\mu(\mathrm{d}x) \\
&\leqslant \int_{\mathbb{R}^d} \mathbb{E}\left(\sup_{t\in[0,\tau_R^{n,m}(x)]}\left|\int_0^t \frac{\langle Z_s^{n,m}(x),(\sigma_s^n(X_s^n(x)) - \sigma_s^m(X_s^m(x)))\mathrm{d}W_s\rangle_{\mathbb{R}^d}}{|Z_s^{n,m}(x)|^2+\delta^2}\right|\right)\mu(\mathrm{d}x) \\
&\leqslant C\int_{\mathbb{R}^d} \mathbb{E}\left[\int_0^{\tau_R^{n,m}(x)} \frac{|Z_s^{n,m}(x)|^2|\sigma_s^n(X_s^n(x)) - \sigma_s^m(X_s^m(x))|^2}{(|Z_s^{n,m}(x)|^2+\delta^2)^2}\mathrm{d}s\right]^{\frac{1}{2}}\mu(\mathrm{d}x) \\
&\leqslant C\mu(\mathbb{R}^d)^{\frac{1}{2}}\left[\mathbb{E}\int_0^1\!\!\int_{|X_s^n(x)|\vee|X_s^m(x)|\leqslant R} \frac{|\sigma_s^n(X_s^n(x)) - \sigma_s^m(X_s^m(x))|^2}{|Z_s^{n,m}(x)|^2+\delta^2}\mu(\mathrm{d}x)\mathrm{d}s\right]^{\frac{1}{2}}.
\end{aligned}$$

As the treatment of $I_1^{n,m}(t)$, by Lemma 4.6, we can prove that

$$\begin{aligned}
&\mathbb{E}\int_{G_R^{n,m}} \sup_{t\in[0,1]} |I_2^{n,m}(t,x)|\mu(\mathrm{d}x) \\
&\qquad\leqslant \left(C\int_0^1 \|\nabla\sigma_s\|^2_{L^{2q}(B_{R+1})}\mathrm{d}s + \frac{C}{\delta}\int_0^1 \|\sigma_s^n - \sigma_s^m\|^2_{L^{2q}(B_R)}\mathrm{d}s\right)^{\frac{1}{2}}, \qquad (4.11)
\end{aligned}$$

and similarly,

$$\mathbb{E}\int_{G_R^{n,m}}\sup_{t\in[0,1]}|I_3^{n,m}(t,x)|\mu(\mathrm{d}x)$$
$$\leqslant C\int_0^1\|\nabla\sigma_s\|^2_{L^{2q}(B_{R+1})}\mathrm{d}s+\frac{C}{\delta}\int_0^1\|\sigma^n_s-\sigma^m_s\|^2_{L^{2q}(B_R)}\mathrm{d}s. \tag{4.12}$$

Since $I_4^{n,m}(t)$ is negative, we can drop it. For $I_5^{n,m}(t)$, by Lemma 4.7 and the elementary inequality

$$|\log(1+r)-r|\leqslant C|r|^2,\quad r\geqslant -\tfrac{1}{2},$$

we have

$$\mathbb{E}\int_{G_R^{n,m}}\sup_{t\in[0,1]}|I_5^{n,m}(t,x)|\mu(\mathrm{d}x)\leqslant C\int_0^1\!\!\int_{\mathbb{U}}L_1(u)^2\nu_s(\mathrm{d}u)\mathrm{d}s. \tag{4.13}$$

For $I_6^{n,m}(t)$, as in the treatment of $I_2^{n,m}(t)$ and $I_5^{n,m}(t)$, we also have

$$\mathbb{E}\int_{G_R^{n,m}}\sup_{t\in[0,1]}|I_6^{n,m}(t,x)|\mu(\mathrm{d}x)\leqslant C\left(\int_0^1\!\!\int_{\mathbb{U}}L_1(u)^2\nu_s(\mathrm{d}u)\mathrm{d}s\right)^{\frac{1}{2}}. \tag{4.14}$$

Combining (4.9)-(4.14), we obtain (4.8). □

We are now in a position to give

Proof of Theorem 4.2. Set

$$\Phi^{n,m}(x):=\sup_{t\in[0,1]}|X^n_t(x)-X^m_t(x)|$$

and

$$\Psi^{n,m}_\delta(x):=\log\left(\frac{\Phi^{n,m}(x)^2}{\delta^2}+1\right).$$

We have

$$\mathbb{E}\int_{\mathbb{R}^d}\Phi^{n,m}(x)\mu(\mathrm{d}x)=\mathbb{E}\int_{(G_R^{n,m})^c}\Phi^{n,m}(x)\mu(\mathrm{d}x)+\mathbb{E}\int_{G_R^{n,m}}\Phi^{n,m}(x)\mu(\mathrm{d}x),$$

where $G_R^{n,m}$ is defined as in Lemma 4.8. By Lemmas 3.4 and 4.5, the first term is less than

$$\frac{1}{\sqrt{R}}\mathbb{E}\int_{\mathbb{R}^d}\left(\sup_{t\in[0,1]}|X^n_t(x)|^{\frac{3}{2}}+\sup_{t\in[0,1]}|X^m_t(x)|^{\frac{3}{2}}\right)\mu(\mathrm{d}x)\leqslant\frac{C\int_{\mathbb{R}^d}(1+|x|^2)^{\frac{3}{4}-d}\mathrm{d}x}{\sqrt{R}}\leqslant\frac{C}{\sqrt{R}},$$

where C is independent of n,m and R, and $d\geqslant 2$.

For the second term, we make the following decomposition:

$$\mathbb{E}\int_{G_R^{n,m}}\Phi^{n,m}(x)\mu(\mathrm{d}x)=\mathbb{E}\int_{G_R^{n,m}\cap\{\Psi_\delta^{n,m}\geqslant\eta\}}\Phi^{n,m}(x)\mu(\mathrm{d}x)$$
$$+\mathbb{E}\int_{G_R^{n,m}\cap\{\Psi_\delta^{n,m}<\eta\}}\Phi^{n,m}(x)\mu(\mathrm{d}x)=:I_1^{n,m}+I_2^{n,m}.$$

For $I_1^{n,m}$, by Hölder's inequality, Lemma 3.4 and (4.8), we have

$$I_1^{n,m}\leqslant\frac{C_R}{\sqrt{\eta}}\left(\mathbb{E}\int_{G_R^{n,m}}\Psi_\delta^{n,m}(x)\mu(\mathrm{d}x)\right)^{\frac{1}{2}}$$
$$\leqslant\frac{C_R}{\sqrt{\eta}}+\frac{C_R}{\sqrt{\delta\eta}}\left(\int_0^1\!\Big(\|b^n_s-b^m_s\|_{L^q(B_R)}+\|\sigma^n_s-\sigma^m_s\|^2_{L^{2q}(B_R)}\Big)\,\mathrm{d}s\right)^{\frac{1}{2}}.$$

For $I_2^{n,m}$, noticing that if $\Psi_\delta^{n,m}(x) \leqslant \eta$, then $\Phi^{n,m}(x) \leqslant \delta\sqrt{e^\eta - 1}$, we have

$$I_2^{n,m} \leqslant C\delta\sqrt{e^\eta - 1}.$$

Combining the above calculations, we obtain that

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^d} \Phi^{n,m}(x)\mu(\mathrm{d}x) &\leqslant \frac{C}{\sqrt{R}} + \frac{C_R}{\sqrt{\eta}} + C\delta e^{\eta/2} \\ &\quad + \frac{C_R}{\sqrt{\delta\eta}}\left(\int_0^1 \Big(\|b_s^n - b_s^m\|_{L^q(B_R)} + \|\sigma_s^n - \sigma_s^m\|^2_{L^{2q}(B_R)}\Big)\,\mathrm{d}s\right)^{\frac{1}{2}}.\end{aligned}$$

For fixed $R, \eta, \delta > 0$, letting $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \mathbb{E}\int_{\mathbb{R}^d} \Phi^{n,m}(x)\mu(\mathrm{d}x) \leqslant \frac{C}{\sqrt{R}} + \frac{C_R}{\sqrt{\eta}} + C\delta e^{\eta/2}.$$

Then letting $\delta \to 0$ then $\eta \to \infty$ and $R \to \infty$, we arrive at

$$\lim_{n,m\to\infty} \mathbb{E}\int_{\mathbb{R}^d}\left(\sup_{t\in[0,1]} |X_t^n(x) - X_t^m(x)|\right)\mu(\mathrm{d}x) = \lim_{n,m\to\infty} \mathbb{E}\int_{\mathbb{R}^d} \Phi^{n,m}(x)\mu(\mathrm{d}x) = 0.$$

This means that $\{X_\cdot^n(\cdot)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Banach space

$$L^1(\Omega\times\mathbb{R}^d, P\times\mu; C([0,1];\mathbb{R}^d)).$$

Hence, there exists an adapted càdlàg process $X_t(x)$ such that

$$\lim_{n\to\infty} \mathbb{E}\int_{\mathbb{R}^d}\left(\sup_{t\in[0,1]} |X_t^n(x) - X_t(x)|\right)\mu(\mathrm{d}x) = 0.$$

By Lemma 2.4, it is standard to check that $X_t(x)$ solves SDE (3.1) in the sense of Definition 4.1.

For the uniqueness, let $X_t^i(x), i = 1, 2$ be two almost everywhere stochastic flows of SDE (3.1). As in the proof of Lemma 4.8, we have

$$\mathbb{E}\int_{G_R} \sup_{t\in[0,1]} \log\left(\frac{|X_t^1(x) - X_t^2(x)|^2}{\delta^2} + 1\right)\mu(\mathrm{d}x) \leqslant C,$$

where $G_R(\omega) := \Big\{x \in \mathbb{R}^d : \sup_{t\in[0,1)} |X_t^1(\omega, x)| \vee |X_t^2(\omega, x)| \leqslant R\Big\}$ and C is independent of δ. Letting $\delta \to 0$ and $R \to \infty$, we obtain that $X_t^1(\omega, x) = X_t^2(\omega, x)$ for all $t \in [0, 1]$ and $P \times \mu$-almost all (ω, x). □

5 Probabilistic representation for the solutions of PIDEs

In this section we work in the canonical space $\Omega = \mathbb{D}^d_{[0,1]}$: the set of all right continuous functions with left limits. The generic element in Ω is denoted by w. The space Ω can be endowed with two complete metrics: uniform metric and Skorohod metric. We remark that only under Skorohod metric, Ω is separable. For $t \in [0, 1]$, let $\mathscr{F}_t := \sigma\{w_s : s \in [0, t]\}$ and set $\mathscr{F} = \mathscr{F}_1$. Then $\mathscr{F}$ coincides with the σ-algebra generated by Skorohod's topology. For a Polish space E, by $\mathcal{P}(E)$ we denote the space of all Borel probability measures over E.

Below we consider the more general Lévy generator:

$$\begin{aligned}\mathscr{L}_t\varphi(x) &:= \frac{1}{2}a_t^{ij}(x)\partial_i\partial_j\varphi(x) + b_t^i(x)\partial_i\varphi(x) \\ &\quad + \int_{\mathbb{R}^d\setminus\{0\}}\left[\varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle_{\mathbb{R}^d}}{1+|y|^2}\right]\nu_t(\mathrm{d}y),\end{aligned}$$

where $a_t^{ij}(x) := \sum_k \sigma_t^{ik}(x)\sigma_t^{jk}(x)$ and ν satisfies that

$$\int_0^1\int_{\mathbb{R}^d\setminus\{0\}} \frac{|y|^2}{1+|y|^2}\nu_t(\mathrm{d}y)\mathrm{d}t < +\infty.$$

We recall the following notion of Stroock and Varadhan's martingale solutions (cf. [30, 31]).

Definition 5.1. *(Martingale Solutions) Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. A probability measure P on $(\Omega, \mathscr{F})$ is called a martingale solution corresponding to the operator $\mathscr{L}$ and initial law μ_0 if $\mu_0 = P \circ w_0^{-1}$ and for all $\varphi \in C_0^\infty(\mathbb{R}^d)$,*

$$\varphi(w_t) - \varphi(w_0) - \int_0^t (\mathscr{L}_s\varphi)(w_s)\mathrm{d}s$$

is a P-martingale with respect to $(\mathscr{F}_t)$, which is equivalent that for all $\theta \in \mathbb{R}^d$,

$$\exp\left[\mathrm{i}\left\langle \theta, w_t - w_0 - \int_0^t b_s(w_s)\mathrm{d}s\right\rangle_{\mathbb{R}^d} - \frac{1}{2}\int_0^t a_s^{ij}(w_s)\theta^i\theta^j\mathrm{d}s \right.$$
$$\left. - \int_0^t\int_{\mathbb{R}^d\setminus\{0\}} \left(e^{\mathrm{i}\langle\theta,y\rangle_{\mathbb{R}^d}} - 1 - \frac{\mathrm{i}\langle\theta,y\rangle_{\mathbb{R}^d}}{1+|y|^2}\right)\nu_s(\mathrm{d}y)\mathrm{d}s\right]$$

is a P-martingale with respect to $(\mathscr{F}_t)$.

For any $w \in \Omega$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, we define

$$\eta(t, w, \Gamma) := \sum_{0<s\leqslant t} 1_\Gamma(w(s) - w(s-))$$

and

$$\tilde{\eta}(t, w, \Gamma) := \eta(t, w, \Gamma) - \int_0^t \nu_s(\Gamma)\mathrm{d}s.$$

The following result is from [30, Corollaries 1.3.1 and 1.3.2] (see also [21, Theorem 5]).

Theorem 5.2. *Let $P \in \mathcal{P}(\Omega)$ be a martingale solution corresponding to $(\mathscr{L}, \mu_0)$. Define*

$$\gamma_t(w) := w_t - \int_{|y|<1} y\tilde{\eta}(t, w, \mathrm{d}y) - \int_{|y|\geqslant 1} y\eta(t, w, \mathrm{d}y)$$

and

$$\hat{b}_t(x) := b_t(x) + \int_{|y|<1} \frac{y|y|^2}{1+|y|^2}\nu_t(\mathrm{d}y) - \int_{|y|\geqslant 1} \frac{y}{1+|y|^2}\nu_t(\mathrm{d}y). \tag{5.1}$$

Then $M(t, w) := \gamma_t(w) - \int_0^t \hat{b}_s(w_s)\mathrm{d}s$ is $(\mathscr{F}_t)$-adapted. Moreover, for any $\theta \in \mathbb{R}^d$ and $|g(y)|^2 \leqslant \frac{C|y|^2}{1+|y|^2}$,

$$t \mapsto \exp\left[\mathrm{i}\langle\theta, M(t) - M(0)\rangle_{\mathbb{R}^d} + \frac{1}{2}\int_0^t a_s^{ij}(w_s)\theta^i\theta^j\mathrm{d}s\right]$$

and

$$t \mapsto \int_{\mathbb{R}^d\setminus\{0\}} g(y)\tilde{\eta}(t, w, \mathrm{d}y)$$

are P-martingales with respect to $(\mathscr{F}_t)$.

Let us now consider the following integro-differential equation of Fokker-Planck type:

$$\partial_t \mu_t = \mathscr{L}_t^* \mu_t, \tag{5.2}$$

where $\mathscr{L}_t^*$ is the formal adjoint operator of $\mathscr{L}_t$ given by

$$\mathscr{L}_t^* \mu := \frac{1}{2}\partial_i\partial_j(a_t^{ij}(x)\mu) - \partial_i(b_t^i(x)\mu) + \int_{\mathbb{R}^d\setminus\{0\}} \left[\tau_y\mu - \mu + \frac{y^i\partial_i\mu}{1+|y|^2}\right]\nu_t(\mathrm{d}y).$$

Here, PIDE (5.2) is understood in the distributional sense, i.e., for any $\varphi \in C_b^\infty(\mathbb{R}^d)$,

$$\partial_t\langle\mu_t, \varphi\rangle = \langle\mu_t, \mathscr{L}_t\varphi\rangle. \tag{5.3}$$

If $\mu_t(\mathrm{d}x) = u_t(x)\mathrm{d}x$, then (5.2) reads as

$$\partial_t u_t = \mathscr{L}_t^* u_t. \tag{5.4}$$

The following result gives the uniqueness of measure-valued solutions for (5.2) in the case of smooth coefficients.

Theorem 5.3. *Assume that a and b are smooth and satisfies that for all $k \in \{0\}\cup\mathbb{N}$,*

$$\sup_{t\in[0,1]}\|\nabla^k a_t^{ij}\|_\infty + \sup_{t\in[0,1]}\|\nabla^k b_t^i\|_\infty < +\infty.$$

Then for any $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, PIDE (5.2) admits a unique measure-valued solution $\mu_t \in \mathcal{P}(\mathbb{R}^d)$.

Proof. The existence follows by Itô's formula. Let us now prove the uniqueness. For $0 \leqslant s < t \leqslant 1$ and $x \in \mathbb{R}^d$, let $X_{s,t}(x)$ solve the following SDE:

$$\begin{aligned} X_{s,t}(x) = x &+ \int_s^t \hat{b}_r(X_{s,r}(x))\mathrm{d}r + \int_s^t \sqrt{a_r}(X_{s,r}(x))\mathrm{d}W_r \\ &+ \int_{B_1^0} y\tilde{N}((s,t],\mathrm{d}y) + \int_{B_1^c} yN((s,t],\mathrm{d}y), \end{aligned}$$

where $\hat{b}_r(x)$ is defined by (5.1), $\sqrt{a_r}$ denotes the square root of symmetric nonnegative matrix a_r and $N(\mathrm{d}t,\mathrm{d}y)$ is a Poisson random point measure with intensity measure $\nu_t(\mathrm{d}y)\mathrm{d}t$, $B_1^0 := B_1\setminus\{0\}$ and $B_1^c = \mathbb{R}^d\setminus B_1$. For any $\varphi \in C_b^\infty(\mathbb{R}^d)$, define

$$\mathcal{T}_{s,t}\varphi(x) := \mathbb{E}(\varphi(X_{s,t}(x))).$$

Then $\mathcal{T}_{s,t}\varphi(x) \in C_b^\infty(\mathbb{R}^d)$ and for all $0 \leqslant s < r < t \leqslant 1$,

$$\mathcal{T}_{s,r}\mathcal{T}_{r,t}\varphi(x) = \mathcal{T}_{s,t}\varphi(x).$$

It is easy to verify that

$$\partial_s\mathcal{T}_{s,t}\varphi + \mathscr{L}_s\mathcal{T}_{s,t}\varphi = 0.$$

Let $\mu_t^i, i = 1,2$ be two solutions of PIDE (5.2) with the same initial values. Then by (5.3), we have

$$\partial_s\langle\mu_s^i, \mathcal{T}_{s,t}\varphi\rangle = \langle\mu_s^i, \partial_s\mathcal{T}_{s,t}\varphi + \mathscr{L}_s\mathcal{T}_{s,t}\varphi\rangle = 0, \quad i = 1,2.$$

Since $\mu_0^1 = \mu_0^2$, we have

$$\langle\mu_s^1, \mathcal{T}_{s,t}\varphi\rangle = \langle\mu_s^2, \mathcal{T}_{s,t}\varphi\rangle, \quad s \in [0,t].$$

In particular,

$$\langle\mu_t^1, \varphi\rangle = \langle\mu_t^2, \varphi\rangle,$$

which implies that $\mu_t^1 = \mu_t^2$ for any $t \in [0,1]$. □

We now prove the following extension of Figalli's result [11, p.116, Theorem 2.6], which is originally due to Ambrosio [1].

Theorem 5.4. *Assume that b and a are bounded and measurable functions. Let $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ be a measure-valued solution of PIDE (5.2) with initial value $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Then there exists a martingale solution $P \in \mathcal{P}(\Omega)$ corresponding to $(\mathscr{L}, \mu_0)$ such that for all $t \in [0,1]$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$,*

$$\langle \mu_t, \varphi \rangle = \mathbb{E}^P(\varphi(w_t)). \tag{5.5}$$

Proof. Let $\rho : \mathbb{R}^d \to (0, +\infty)$ be a convolution kernel such that $|\nabla^k \rho(x)| \leqslant C_k \rho(x)$ for any $k \in \mathbb{N}$ (for instance $\rho(x) = e^{-|x|^2/2}/(2\pi)^{d/2}$). Let $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$, $\varepsilon > 0$, and define

$$\mu_t^\varepsilon := \mu_t * \rho_\varepsilon, \quad b_t^\varepsilon := \frac{(b_t \mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}, \quad a_t^\varepsilon := \frac{(a_t \mu_t) * \rho_\varepsilon}{\mu_t^\varepsilon}.$$

It is easy to see that for any $k \in \{0\} \cup \mathbb{N}$,

$$\|\nabla^k b_t^\varepsilon\|_\infty \leqslant C_k \|\nabla^k b_t\|_\infty, \quad \|\nabla^k a_t^\varepsilon\|_\infty \leqslant C_k \|\nabla^k a_t\|_\infty.$$

With a little abuse of notation, we are denoting the measure μ_t^ε and its density with respect to the Lebesgue measure by the same symbol. If we take the convolutions with ρ_ε for both sides of PIDE (5.2), then

$$\partial_t \mu_t^\varepsilon = \frac{1}{2}\partial_i \partial_j (a_t^{\varepsilon,ij} \mu_t^\varepsilon) - \partial_i (b_t^{\varepsilon,i} \mu_t^\varepsilon) + \int_{\mathbb{R}^d \setminus \{0\}} \left[\tau_y \mu_t^\varepsilon - \mu_t^\varepsilon + \frac{\langle y, \nabla \mu_t^\varepsilon \rangle_{\mathbb{R}^d}}{1 + |y|^2} \right] \nu_t(\mathrm{d}y),$$

subject to $\mu_0^\varepsilon = \mu_0 * \rho_\varepsilon$. By Theorem 5.3, the unique solution to this PIDE can be represented by

$$\mu_t^\varepsilon = \text{Law of } X_t^\varepsilon,$$

i.e., for any $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\langle \mu_t^\varepsilon, \varphi \rangle = \mathbb{E}\varphi(X_t^\varepsilon), \tag{5.6}$$

where X_t^ε solves the following SDE with jump

$$X_t^\varepsilon = X_0^\varepsilon + \int_0^t \hat{b}_s^\varepsilon(X_s^\varepsilon)\mathrm{d}s + \int_0^t \sqrt{a_s^\varepsilon}(X_s^\varepsilon)\mathrm{d}W_s + \int_{B_1^0} y\tilde{N}((0,t],\mathrm{d}y) + \int_{B_1^c} yN((0,t],\mathrm{d}y),$$

and the law of X_0^ε is μ_0^ε. Here, $\hat{b}_s^\varepsilon(x)$ is defined by (5.1) with replacing b by b^ε.

Let P_ε be the law of $t \mapsto X_t^\varepsilon$ in Ω. Since

$$P_\varepsilon(|w_0| \geqslant R) = \mu_0^\varepsilon(B_R^c) \to 0 \text{ uniformly in } \varepsilon \text{ as } R \to \infty,$$

by [30, p.237, Theorem A.1], $(P_\varepsilon)_{\varepsilon \in (0,1)}$ is tight in $\mathcal{P}(\Omega)$. By Prohorov's theorem (cf. [8, page 104, Theorem 2.2]), there exist a subsequence of $P_{\varepsilon_n} \in \mathcal{P}(\Omega)$ and $P \in \mathcal{P}(\Omega)$ such that P_{ε_n} weakly converges to P as $n \to \infty$. Fix t and let $t_n \downarrow t$. By [8, page 131, Theorem 7.8], one has

$$\mathbb{E}^{P_{\varepsilon_n}}(\varphi(w_{t_n})) \to \mathbb{E}^P(\varphi(w_t)). \tag{5.7}$$

On the other hand, by (5.6), we have

$$\langle \mu_{t_n}^{\varepsilon_n}, \varphi \rangle = \mathbb{E}^{P_{\varepsilon_n}}(\varphi(w_{t_n})).$$

Since $t \mapsto \langle \mu_t, \varphi \rangle$ is continuous, using the property of convolutions, by taking limits for the above identity, one finds that (5.5) holds.

For completing the proof, it remains to show that P is a martingale solution corresponding to $(\mathscr{L}, \mu_0)$. That is, we need to prove that for any $0 \leqslant s < t \leqslant 1$ and bounded continuous and $\mathscr{F}_s$-measurable function Φ^s on Ω, $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\mathbb{E}^P\left[\left(\varphi(w_t) - \varphi(w_s) - \int_s^t (\mathscr{L}_r\varphi)(w_r)\mathrm{d}r\right)\Phi^s(w)\right] = 0.$$

As above, let $(s_n, t_n) \downarrow (s,t)$. The above identity will follow by (5.7) and taking limits for

$$\mathbb{E}^{P_{\varepsilon_n}}\left[\left(\varphi(w_{t_n}) - \varphi(w_{s_n}) - \int_{s_n}^{t_n} (\mathscr{L}_r^{\varepsilon_n}\varphi)(w_r)\mathrm{d}r\right)\Phi^s(w)\right] = 0.$$

The more details can be found in [11, p.118, Step 3]. □

Definition 5.5. *(Weak solution) If there exist a probability space $(\Omega, \mathscr{F}, P)$ with filtration $(\mathscr{F}_t)_{t\in[0,1]}$ and an $(\mathscr{F}_t)$-adapted Brownian motion W_t, an $(\mathscr{F}_t)$-adapted Poisson random measure $N(\mathrm{d}t, \mathrm{d}y)$ with intensity measure $\nu_t(\mathrm{d}y)\mathrm{d}t$ and an $(\mathscr{F}_t)$-adapted process X_t on $(\Omega, \mathscr{F}, P)$ such that for all $t \in [0,1]$,*

$$X_t = X_0 + \int_0^t \hat{b}_s(X_s)\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s + \int_{B_1^0} y\tilde{N}((0,t], \mathrm{d}y) + \int_{B_1^c} yN((0,t], \mathrm{d}y), \quad (5.8)$$

where $\hat{b}_s(x)$ is defined by (5.1), then we say $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]})$ together with (W, N, X) a weak solution. By weak uniqueness, we means that any two weak solutions with the same initial law have the same law in Ω.

The following result gives the equivalence between weak solutions and martingale solutions.

Theorem 5.6. *The existence of martingale solutions implies the existence of weak solutions. In particular, the uniqueness of weak solutions implies the uniqueness of martingale solutions.*

Proof. Let $P \in \mathcal{P}(\Omega)$ be a martingale solution. By Theorem 5.2, one knows that under P, η is a Poisson random point measure with intensity measure $\nu_t(\mathrm{d}y)\mathrm{d}t$ and M is a continuous martingale with covariation process

$$\langle M^i, M^j\rangle_t = \frac{1}{2}\sum_k \int_0^t (\sigma_s^{ik}\sigma_s^{jk})(w_s)\mathrm{d}s.$$

Let $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}; (\hat{\mathscr{F}}_t)_{t\in[0,1]})$ be another filtered probability space supporting a Brownian motion $\hat{W}_t$. Let $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}; (\tilde{\mathscr{F}}_t)_{t\in[0,1]})$ be product probability space of $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]})$ and $(\hat{\Omega}, \hat{\mathscr{F}}, \hat{P}; (\hat{\mathscr{F}}_t)_{t\in[0,1]})$. Let $\pi : \tilde{\Omega} \to \Omega$ be the canonical projection. Define

$$\tilde{M}_t(\tilde{\omega}) := M_t(\pi(\tilde{\omega})), \quad \tilde{\sigma}_t(\tilde{\omega}) := \sigma_t(\pi(\tilde{\omega})_t)$$

and

$$\tilde{N}_t(\tilde{\omega}, \mathrm{d}y) := \eta(t, \pi(\tilde{\omega}), \mathrm{d}y), \quad \tilde{X}_t(\tilde{\omega}) := \pi(\tilde{\omega}).$$

Then by the proof of [30, p.108, Theorem 4.5.2], there exists another Brownian motion $(\tilde{W}_t)_{t\in[0,1]}$ defined on $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}; (\tilde{\mathscr{F}}_t)_{t\in[0,1]})$ such that

$$\tilde{M}_t = \int_0^t \tilde{\sigma}_s \mathrm{d}\tilde{W}_s, \quad \tilde{P} - a.s.$$

Hence, $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}; (\tilde{\mathscr{F}}_t)_{t\in[0,1]})$ together with $(\tilde{W}, \tilde{N}, \tilde{X})$ is a weak solution. □

The main result of this section is:

Theorem 5.7. *Assume that for some* $q > 1$*,*

$$|\nabla b| \in L^\infty([0,1]; L^q_{loc}(\mathbb{R}^d;\mathbb{R}^d)), \quad [\mathrm{div} b]^-, |b|, |\sigma|, |\nabla\sigma| \in L^\infty([0,1]\times\mathbb{R}^d),$$

and for any $p \geqslant 1$*,*

$$\int_0^1\int_{\mathbb{R}^d\setminus\{0\}} |y|^2(1+|y|)^p \nu_t(\mathrm{d}y)\mathrm{d}t < +\infty.$$

Let $r > \frac{q}{q-1} = q^*$*. Then for any probability density function* ϕ *with*

$$\int_{\mathbb{R}^d} \phi(x)^r(1+|x|^2)^{(r-1)d}\mathrm{d}x < +\infty,$$

there exists a unique solution u_t *to PIDE (5.4) in the class of*

$$\mathscr{M}_{q^*} := \Bigg\{ u_t \in L^{q^*}(\mathbb{R}^d) : u_t(x) \geqslant 0, \int_{\mathbb{R}^d} u_t(x)\mathrm{d}x = 1,$$
$$\sup_{t\in[0,1]} \int_{\mathbb{R}^d} u_t(x)^{q^*}(1+|x|^2)^{(q^*-1)d}\mathrm{d}x < +\infty \Bigg\}.$$

Moreover, if $q > d$*, then the uniqueness holds in the measure-valued space* $\mathcal{P}(\mathbb{R}^d)$*.*

Proof. (Existence). Set $\mu(\mathrm{d}x) := \mathrm{d}x/(1+|x|^2)^d$ and let $X_t(x)$ be the μ-almost everywhere stochastic flow of the following SDE

$$X_t(x) = x + \int_0^t \hat{b}_s(X_s(x))\mathrm{d}s + \int_0^t \sigma_s(X_s(x))\mathrm{d}W_s + \int_{\mathbb{R}^d\setminus\{0\}} y\tilde{N}((0,t],\mathrm{d}y),$$

where $N((0,t],\mathrm{d}y)$ is a Poisson random measure with intensity $\nu_t(\mathrm{d}y)\mathrm{d}t$ and

$$\hat{b}_s(x) := b_s(x) + \int_{\mathbb{R}^d\setminus\{0\}} \frac{y|y|^2}{1+|y|^2}\nu_s(\mathrm{d}y).$$

Since in this case, $L_1 = 0$ in Theorem 4.2, the p in (4.1) can be arbitrarily close to 1. Let X_0 be an $\mathscr{F}_0$-measurable random variable with law $\phi(x)\mathrm{d}x$. Define

$$Y_t := X_t(X_0).$$

It is easy to check that Y_t solves the following SDE:

$$Y_t = X_0 + \int_0^t \hat{b}_s(Y_s)\mathrm{d}s + \int_0^t \sigma_s(Y_s)\mathrm{d}W_s + \int_{\mathbb{R}^d\setminus\{0\}} y\tilde{N}((0,t],\mathrm{d}y). \tag{5.9}$$

Now for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, by Hölder's inequality, we have

$$\begin{aligned}
\mathbb{E}\varphi(Y_t) &= \mathbb{E}(\mathbb{E}\varphi(X_t(x))|x = X_0) = \int_{\mathbb{R}^d} \mathbb{E}\varphi(X_t(x))\phi(x)\mathrm{d}x \\
&\leqslant \left(\int_{\mathbb{R}^d} |\mathbb{E}\varphi(X_t(x))|^{\frac{r}{r-1}}\mu(\mathrm{d}x)\right)^{1-\frac{1}{r}} \left(\int_{\mathbb{R}^d} \left(\phi(x)(1+|x|^2)^d\right)^r \mu(\mathrm{d}x)\right)^{\frac{1}{r}} \\
&= \left(\mathbb{E}\int_{\mathbb{R}^d} |\varphi(X_t(x))|^{\frac{r}{r-1}}\mu(\mathrm{d}x)\right)^{1-\frac{1}{r}} \left(\int_{\mathbb{R}^d} \phi(x)^r(1+|x|^2)^{(r-1)d}\mathrm{d}x\right)^{\frac{1}{r}} \\
&\overset{(4.1)}{\leqslant} C_\phi\|\varphi\|_{L^q_\mu},
\end{aligned}$$

which then implies that Y_t has an absolutely continuous probability density $u_t \in \mathscr{M}_{q^*}$ with

$$\int_{\mathbb{R}^d} u_t(x)\varphi(x)\mathrm{d}x = \mathbb{E}\varphi(Y_t) \leqslant C_\phi\|\varphi\|_{L^q_\mu}, \quad \frac{1}{q^*} + \frac{1}{q} = 1.$$

By Itô's formula, it is immediate that u_t solves PIDE (5.4) in the distributional sense.

(Uniqueness). Let $u^i_t \in \mathscr{M}_{q^*}$ be any two solutions of PIDE (5.4) with the same initial value $u_0 = \phi$. Let $P^i \in \mathcal{P}(\Omega)$ be two martingale solutions corresponding to $\mu^i_t(\mathrm{d}x) = u^i_t(x)\mathrm{d}x$ by Theorem 5.4. Since for any $\varphi \in C^\infty_0(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u^i_t(x)\varphi(x)\mathrm{d}x = \mathbb{E}^{P^i}\varphi(w_t), \quad i = 1, 2,$$

we only need to prove that $P^1 = P^2$. By Theorem 5.6 and [29, p.104, Theorem 137], it suffices to prove the pathwise uniqueness of SDE (5.9). Let $Y^i_t, i = 1, 2$ be two solutions of SDE (5.9) defined on the same filtered probability space supporting a Brownian motion W and a Poisson random measure N with intensity measure $\nu_t(\mathrm{d}y)\mathrm{d}t$, where for any $\varphi \in C^\infty_0(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} u^i_t(x)\varphi(x)\mathrm{d}x = \mathbb{E}\varphi(Y^i_t), \quad i = 1, 2.$$

Since $u^i_t \in \mathscr{M}_p$, by suitable approximation, we have for any $\varphi \in L^q_\mu(\mathbb{R}^d)$,

$$\sup_{t\in[0,1]} \mathbb{E}\varphi(Y^i_t) \leqslant C\|\varphi\|_{L^q_\mu}, \quad i = 1, 2. \tag{5.10}$$

Set

$$Z_t := Y^1_t - Y^2_t, \quad \tau_R := \inf\{t \in [0, 1] : |Y^1_t| \vee |Y^2_t| > R\}.$$

Basing on (5.10), as in the proof of Lemma 4.8, we have for any $\delta > 0$,

$$\begin{aligned}
&\mathbb{E}\log\left(\frac{|Z_{t\wedge\tau_R}|^2}{\delta^2} + 1\right)\\
&\leqslant 2\mathbb{E}\int_0^{t\wedge\tau_R} \frac{\langle Z_s, b_s(Y^1_s) - b_s(Y^2_s)\rangle_{\mathbb{R}^d}}{|Z_s|^2 + \delta^2}\mathrm{d}s + \mathbb{E}\int_0^{t\wedge\tau_R} \frac{\|\sigma_s(Y^1_s) - \sigma_s(Y^2_s)\|^2}{|Y^1_s - Y^2_s|^2 + \delta^2}\mathrm{d}s\\
&\overset{(2.9)}{\leqslant} C\mathbb{E}\int_0^{t\wedge\tau_R} \Big(\mathcal{M}_{2R}|\nabla b_s|(Y^1_s) + \mathcal{M}_{2R}|\nabla b_s|(Y^2_s)\Big)\mathrm{d}s + \int_0^1 \|\nabla\sigma_s\|^2_\infty\mathrm{d}s \qquad (5.11)\\
&\leqslant C\int_0^t \|1_{B_R}\cdot\mathcal{M}_{2R}|\nabla b_s|\|_{L^q_\mu}\mathrm{d}s + \int_0^1 \|\nabla\sigma_s\|^2_\infty\mathrm{d}s\\
&\leqslant C\int_0^t \|\mathcal{M}_{2R}|\nabla b_s|\|_{L^q(B_R)}\mathrm{d}s + \int_0^1 \|\nabla\sigma_s\|^2_\infty\mathrm{d}s\\
&\overset{(2.10)}{\leqslant} C\int_0^1 \|\nabla b_s\|_{L^q(B_{3R})}\mathrm{d}s + \int_0^1 \|\nabla\sigma_s\|^2_\infty\mathrm{d}s,
\end{aligned}$$

where C is independent of δ. Letting first $\delta \to 0$ and then $R \to \infty$, we obtain that $Z_t = 0$ a.s., i.e., $Y^1_t = Y^2_t$ a.s.

In the case of $q > d$, let Y^1_t be the solution constructed in the proof of existence and Y^2_t another solution of SDE (5.9) corresponding to any measure-valued solution μ_t with $\mu_0(\mathrm{d}x) = \phi(x)\mathrm{d}x$. In the above proof of (5.11), instead of using (2.9), we use Morrey's inequality (2.8) to deduce that $Y^1_t = Y^2_t$. □

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