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Chaos and entropic chaos in Kac's model without high moments

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Abstract

In this paper we present a new local Lévy Central Limit Theorem, showing convergence to stable states that are not necessarily the Gaussian, and use it to find new and intuitive entropically chaotic families with underlying one-particle function that has moments of order 2α, with $1 < \alpha < 2$. We also discuss a lower semi continuity result for the relative entropy with respect to our specific family of functions, and use it to show a form of stability property for entropic chaos in our settings.

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1 Introduction

One of the most influential equations in the kinetic theory of gases, describing the evolution in time of the distribution function of a dilute gas, is the so-called Boltzmann equation. While widely used, the Boltzmann equation poses two fundamental questions in Kinetic Theory, pertaining to the spatially homogeneous case: The validity of the equation and the rate of convergence to equilibrium in it.
In his 1956 paper, [19], Kac attempted to give a partial solution to these two problems. Kac introduced a many-particle model, consisting of N indistinguishable particle with one dimensional velocities, undergoing binary collision and constrained to the energy sphere $\mathbb{S}^{N-1}\left(\sqrt{N}\right)$, which we will call 'the Kac's sphere'. Kac's evolution equation is given by

$$\frac{\partial F_N}{\partial t}\left(v_1, \dots, v_N\right) = -N(I-Q)F_N\left(v_1, \dots, v_N\right), \tag{1.1}$$

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where F_N represents the probability density function of the N particles, and the gain term Q is given by

$$\begin{aligned} QF\left(t, v_1, \dots, v_N\right) = \frac{1}{2\pi} \frac{2}{N(N-1)} \sum_{i<j} \\ \int_0^{2\pi} F\left(t, v_1, \dots, v_i(\theta), \dots, v_j(\theta), \dots, v_N\right) d\theta, \end{aligned} \tag{1.2}$$

with

$$v_i(\theta) = v_i \cos(\theta) + v_j \sin(\theta), \quad v_j(\theta) = -v_i \sin(\theta) + v_j \cos(\theta). \tag{1.3}$$

Motivated by Boltzmann's 'Stosszahlansatz' assumption, Kac defined the concept of *Chaoticity* (what he called 'the Boltzmann property' in his paper) which measures the asymptotic independence of a finite, fixed, number of particles, as the number of total particles goes to infinity. In its modern variant the definition of chaoticity is:

Definition 1.1. *Let X be a Polish space. A family of symmetric probability measures on X^N, $\{\mu_N\}_{N\in\mathbb{N}}$, is called $\mu-$chaotic, where μ is a probability measure on X, if for any $k \in \mathbb{N}$*

$$\lim_{N\to\infty} \Pi_k\left(\mu_N\right) = \mu^{\otimes k}, \tag{1.4}$$

where $\Pi_k(\mu_N)$ is the $k-$th marginal of μ_N and the limit is in the weak topology.

In the context of Kac's work, the symmetric family of measures under investigation is supported on Kac's sphere and is given by $\mu_N = F_N d\sigma^N$, where $d\sigma^N$ is the uniform probability measure on Kac's sphere. The limit measure μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ with a probability density function f. We say that the family $\{F_N\}_{N\in\mathbb{N}}$ is $f-$chaotic in that particular setting. As an interesting remark, we note that it is known, see for instance [27], that it is enough to check the marginals for $k = 1, 2$ in order to conclude chaoticity.
Using a beautiful combinatorial argument, Kac showed that the property of chaoticity propagates with his evolution equation, i.e. if $\{F_N\left(0, v_1, \dots, v_N\right)\}_{N\in\mathbb{N}}$ is f_0-chaotic then the solution to equation (1.1), $\{F_N\left(t, v_1, \dots, v_N\right)\}_{N\in\mathbb{N}}$ is f_t-chaotic, where f_t solves a caricature of the Boltzmann equation.

While Kac's model is not entirely realistic (as it doesn't conserve momentum) and his limit equation wasn't the Boltzmann equation, the ideas presented in his paper were powerful enough that McKean managed to extend them to the $d-$dimensional case (see [24]). Under similar condition to those presented by Kac, McKean construct a similar $N-$particle model from which the *real* spatially homogeneous Boltzmann equation arose as mean field limit for many cases. We will not discuss this model in this work, and refer the interested reader to [8, 13, 24] for more information.
Giving a partial answer to the validation of the Boltzmann equation, Kac set out to try and find a partial solution to the rate of convergence as well. Using his linear model he conjectured that the rate of convergence to equilibrium in the natural L^2 norm would be exponential, with a rate that is independent of the number of particles (the so-called spectral gap problem). While this proved to be true eventually, it is easy to see that for very natural chaotic families the norm of the initial datum depends *very* strongly on N - exponentially so, making the possibility of taking a limit impossible. This effect is due to the multiplicative nature of the L^2 norm and the definition of chaoticity.
A different kind of 'distance' was needed, one that respects the property of chaoticity. To that end the concept of the relative entropy was invoked.

Definition 1.2. *Given two probability measures, μ, ν, on a Polish space X, we define the relative entropy of μ with respect to ν, $H(\mu|\nu)$, as*

$$H(\mu|\nu) = \int_X h \log h d\nu, \tag{1.5}$$

where $h = \frac{d\mu}{d\nu}$, and $H(\mu|\nu) = \infty$ if μ is not absolutely continuous with respect to ν.

Definition 1.3. *Given a probability density function F_N on Kac's sphere we define the* entropy of F_N *to be*

$$H_N(F_N) = H\left(F_N d\sigma^N | d\sigma^N\right) = \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} F_N \log F_N d\sigma^N. \tag{1.6}$$

The reason for this choice of a distance functional lies with the so-called *extensivity* property of the entropy: In a very intuitive way, we'd like to think that 'nice' $f-$chaotic families behave like $F_N \approx f^{\otimes N}$, as such

$$H_N(F_N) \approx N \int_{\mathbb{R}} f(v) \log\left(\frac{f(v)}{\gamma(v)}\right) dv, \tag{1.7}$$

where γ is the standard Gaussian on $\mathbb{R}$, giving a *linear dependence* in N, instead of an exponential one. This intuition was defined formally in [7], where the authors investigated the entropy functional on the Kac's sphere:

Definition 1.4. *A symmetric $\mu-$chaotic family of probability measures on Kac's sphere, $\{\mu_N\}_{N\in\mathbb{N}}$, is called entropically chaotic if*

$$\lim_{N\to\infty} \frac{H_N\left(\mu_N | d\sigma^N\right)}{N} = H\left(\mu|\gamma\right), \tag{1.8}$$

where $d\sigma^N$ is the uniform probability measure on Kac's sphere and $H(\mu|\gamma)$ is the relative entropy of μ and $\gamma(v)dv$.

As before, in the setting of Kac's model we use the measures $\mu_N = F_N d\sigma^N$ and $\mu = f(x)dx$, and say that $\{F_N\}_{N\in\mathbb{N}}$ is $f-$entropically chaotic if (1.8) is satisfied. The concept of entropic chaoticity is much stronger than that of chaoticity as it involves the correlation between arbitrary number of particles. This was shown to be true in [7] and we will verify it in our particular setting as well later on in this paper.
At this point we'd like to mention that the notion of chaoticity, and the use of a rescaled relative entropy between two measures on X^N and its connection to the relative entropy of the limit measures, doesn't solely lie in the realm of Kac's Model. Indeed, in [4] the authors have investigated propagation of chaos for the mean field Gibbs measure with potential F using exactly such ideas and a type of lower semi continuity result for the relative entropy.
Of particular interest to the study of chaoticity, and entropic chaoticity, are special measures that are obtained by a tensorising a measure, μ, on $\mathbb{R}$, and restricting the tensor product to Kac's sphere. In our particular study, much like that of [7], we'll be interested in measures on Kac's sphere with probability density function

$$F_N\left(v_1, \dots, v_N\right) = \frac{f^{\otimes N}\left(v_1, \dots . v_N\right)}{\mathcal{Z}_N\left(f, \sqrt{N}\right)}, \tag{1.9}$$

where f is a probability density function on $\mathbb{R}$ and the so-called *normalisation function*, $\mathcal{Z}_N\left(f, r\right)$, is defined by

$$\mathcal{Z}_N\left(f, r\right) = \int_{\mathbb{S}^{N-1}(r)} f^{\otimes N} d\sigma_r^N, \tag{1.10}$$

with $d\sigma_r^N$ the uniform probability measure on $\mathbb{S}^{N-1}(r)$. In what follows we will call probability density functions F_N of the form (1.9) *conditioned tensorisation of* f. It is worth to mention that Kac himself considered such functions, and have shown that they are chaotic when f has very strong integrability conditions.
The question of whether or not conditioned tensorisation of the function f is well defined rests heavily on the concentration of the tensorised measure $f^{\otimes N}$ on Kac's sphere. The main technical tool that is required is a local central limit theorem that shows exactly how $\mathcal{Z}_N(f,r)$ behaves asymptotically, for any $r > 0$. In [7], the authors have managed to prove that:

Theorem 1.5. *Let f be a probability density on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for some $p > 1$, $\int_{\mathbb{R}} x^2 f(x) = 1$ and $\int_{\mathbb{R}} x^4 f(x)dx < \infty$. Then*

$$\mathcal{Z}_N(f,\sqrt{u}) = \frac{2}{\sqrt{N}\Sigma\left|\mathbb{S}^{N-1}\right| u^{\frac{N-2}{2}}}\left(\frac{e^{-\frac{(u-N)^2}{2N\Sigma^2}}}{\sqrt{2\pi}} + \lambda_N(u)\right), \tag{1.11}$$

where $\Sigma^2 = \int_{\mathbb{R}} v^4 f(v)dv - 1$ and $\sup_u |\lambda_N(u)| \underset{N\to\infty}{\longrightarrow} 0$.

The above approximation yielded more than just an explanation to why our definition is appropriate. Once proven, the above easily proves the following, which can also be found in [7]:

Theorem 1.6. *Let f be a probability density on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for some $p > 1$, $\int_{\mathbb{R}} x^2 f(x) = 1$ and $\int_{\mathbb{R}} x^4 f(x)dx < \infty$. Then the family of conditioned tensorisation of f, given by (1.9), is $f-$chaotic. Moreover, it is $f-$entropically chaotic.*

We'd like to mention at this point that the above theorems were extended to to McKean's model by the first author in [9].

The appearance of the fourth moment of the function f in Theorem 1.5 shouldn't be too surprising: As we're trying to measure fluctuation of the random variable $K_N = \sum_{i=1}^N V_i^2$ from its mean, N, a useful quantity to consider is the variance of K_N, pertaining to the fourth moment of the underlying function f. This, however, is not a necessary condition to be able to obtain the desired concentration result. In this work we will consider families of conditioned tensorisation of a function f, where the underlying generating function f has moment of order 2α, with $1 < \alpha < 2$.
The main approximation theorem of this paper, one that extends Theorem 1.5 and lies at the heart of many subsequent proofs, relies on concepts related to $\alpha-$stable processes. We will introduce them at this point so we'll be able to state our main results.

Definition 1.7. *A random variable U is said to be $\alpha-$stable for $0 < \alpha < 2$, $\alpha \neq 1$ if*

$$\frac{\sum_{i=1}^n X_i}{n^{\frac{1}{\alpha}}}$$

has the same probability distribution function as U, where X_i are independent copies of U. Equivalently, the characteristic function of U is of the form

$$\widehat{\gamma}_{C_S,\alpha,p,q}(\xi) = e^{-C_S|\xi|^\alpha \cdot \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{\pi\alpha}{2}\right)\left(1+isgn(\xi)(p-q)\tan\frac{\alpha\pi}{2}\right)}, \tag{1.12}$$

with $C_S > 0$, $p,q \geq 0$ and $p+q = 1$.

In the above, and what is to follow, we have used the convention

$$\widehat{\varphi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\varphi(x)dx. \tag{1.13}$$

for the characteristic function, $\widehat{\varphi}$, of a probability density φ. It is also worth to mention that some books, including Feller's, refer to above definition as *strict stability*.

Remark 1.8. *Equation (1.12) can be rewritten in the form*

$$\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi) = e^{-\sigma|\xi|^{\alpha}\left(1+i\beta sgn(\xi)\tan\frac{\alpha\pi}{2}\right)}, \tag{1.14}$$

where

$$\sigma = C_S \cdot \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{\pi\alpha}{2}\right) > 0,\ \beta = p - q.$$

We will use both forms in accordance to the situation.

Definition 1.9. *The* Domain of Attraction *(in short, DA) of $\gamma_{\sigma,\alpha,\beta}$ is the set of all real random variables X such that there exist sequences $\{a_n\}_{n\in\mathbb{N}} > 0$ and $\{b_n\}_{n\in\mathbb{N}} \in \mathbb{R}$ such that*

$$\frac{\sum_{i=1}^{n} X_i}{a_n} - nb_n \underset{n\to\infty}{\longrightarrow} U, \tag{1.15}$$

where X_i are independent copies of X, U is the real random variable with characteristic function $\widehat{\gamma}_{\sigma,\alpha,\beta}$ and the limit is to be understood in the weak sense. Equivalently, one can prove that the DA of $\gamma_{\sigma,\alpha,\beta}$ is the set of all real random variables X, whose characteristic function $\widehat{\psi}$ satisfies

$$n\left(\widehat{\psi}\left(\frac{\xi}{a_n}\right)e^{-ib_n\xi} - 1\right) \underset{n\to\infty}{\longrightarrow} -\sigma|\xi|^{\alpha}\left(1 + i\beta sgn(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right), \tag{1.16}$$

where $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\mathbb{N}}$ are sequences as in (1.15) (See [15]).

Definition 1.10. *The* Natural Domain of Attraction *(in short, NDA) of $\gamma_{\sigma,\alpha,\beta}$ is the subset of the DA of $\widehat{\gamma}_{\sigma,\alpha,\beta}$ for which $a_n = n^{\frac{1}{\alpha}}$ and $b_n = 0$ are applicable as a sequences in (1.15).*

Definition 1.11. *The* Fourier Domain of Attraction *(in short, FDA) of $\gamma_{\sigma,\alpha,\beta}$ is the set of all real random variables X whose characteristic function $\widehat{\psi}$ satisfies*

$$\widehat{\psi}(\xi) = 1 - \sigma\,|\xi|^{\alpha}\left(1 + i\beta sgn(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right) + \eta_{\psi}(\xi), \tag{1.17}$$

where $\frac{\eta_\psi(\xi)}{|\xi|^\alpha} \in L^\infty$ and $\frac{\eta_\psi(\xi)}{|\xi|^\alpha} \underset{\xi\to 0}{\longrightarrow} 0$. The function η_ψ is called the reminder function *of $\widehat{\psi}$.*

The general local Lévy central limit theorem we prove in this paper the following:

Theorem 1.12. *Let g be the probability density function of a random real variable X. Assume that $g \in L^p(\mathbb{R})$ for some $p > 1$ and g is in the NDA of $\gamma_{\sigma,\alpha,\beta}$ for some $\sigma > 0$, β and $1 < \alpha < 2$. Assume in addition that g has finite moment of some order. Define*

$$g_N(x) = N^{\frac{1}{\alpha}} g^{*N}\left(N^{\frac{1}{\alpha}}x\right),$$

and

$$\gamma_{\sigma,\alpha,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)e^{i\xi x}d\xi. \tag{1.18}$$

Then, for any positive sequence $\{\beta_N\}_{N\to\infty}$ that converges to zero as N goes to infinity, any $\tau > 0$ and N large enough we have that

$$\|g_N - \gamma_{\sigma,\alpha,\beta}\|_\infty \leq C_{g,\alpha}\Bigg(N^{\frac{1}{\alpha}}(1-\beta_N^{2+\tau}+\phi_\tau(\beta_N))^{N-q} + e^{-\frac{\sigma N \beta_N^\alpha}{2}} + \omega_g(\beta_N) + 2\sigma\beta_N^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg) = \epsilon_\tau(N), \tag{1.19}$$

where

(i) $C_{g,\alpha} > 0$ *is a constant depending only on* g*, its moments and* α*.*

(ii) q *can be chosen to be the Hölder conjugate of* $\min(2,p)$*.*

(iii) ϕ_τ *satisfies*

$$\lim_{x\to 0}\frac{\phi_\tau(x)}{|x|^{2+\tau}} = 0,$$

(iv) η_g *is the reminder function of* $\widehat{g}$*, defined in Definition 1.11, and* $\omega_g(\beta) = \sup_{|x|\leq\beta}\frac{|\eta_g(x)|}{|x|^\alpha}$*.*

Section 3, where we prove the above theorem, also provides simple condition to check when a probability density function is in the NDA of some $\gamma_{\sigma,\alpha,\beta}$ as well as a simplified case of the general theorem, one we will use in most of our applications. Theorem 1.12 will allow us to show that:

Theorem 1.13. *Let* f *be a probability density such that* $f \in L^p$ *for some* $p > 1$ *and* $\int x^2 f(x)dx = 1$*. Let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy \tag{1.20}$$

and assume that $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ *for some* $C_S > 0$ *and* $1 < \alpha < 2$*. Then the family of conditioned tensorisation of* f*, given by (1.9), is* $f-$*chaotic. Moreover, it is* $f-$*entropically chaotic.*

As a special case, one has that

Theorem 1.14. *Let* f *be a probability density such that* $f \in L^p$ *for some* $p > 1$ *and* $\int x^2 f(x)dx = 1$*. Assume in addition that*

$$f(x) \underset{x\to\infty}{\sim} \frac{D}{|x|^{1+2\alpha}}, \tag{1.21}$$

for some $1 < \alpha < 2$ *and* $D > 0$*. Then the family of conditioned tensorisation of* f*, given by (1.9), is* $f-$*chaotic. Moreover, it is* $f-$*entropically chaotic.*

The family of conditioned tensorisation of a function f,

$$\nu_N = F_N d\sigma^N, \tag{1.22}$$

where F_N is given by (1.9), plays an important role on Kac's sphere as an 'attractor of chaoticity'. The first clue to this is the following distorted lower semi continuity property:

Theorem 1.15. *Let f be a probability density such that $f \in L^p$ for some $p > 1$ and $\int x^2 f(x)dx = 1$. Let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy$$

and assume that $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S > 0$ and $1 < \alpha < 2$. Let μ_N be a symmetric probability measure on Kac's sphere such that for some $k \in \mathbb{N}$

$$\Pi_k(\mu_N) \underset{N\to\infty}{\rightharpoonup} \mu_k, \tag{1.23}$$

where μ_k is a probability measure on $\mathbb{R}^k$. Then, we find that

(i) $\Pi_1(\mu_N) \underset{N\to\infty}{\rightharpoonup} \Pi_1(\mu_k) = \mu$ *and*

$$H(\mu|f) \leq \liminf_{N\to\infty} \frac{H_N(\mu_N|\nu_N)}{N}, \tag{1.24}$$

where $H(\mu|f)$ is the relative entropy between μ and the measure $f(v)dv$.

(ii) For any $\delta > 0$ we have that

$$\begin{aligned}\liminf_{N\to\infty} \frac{H(\mu_N|\nu_N)}{N} \geq \frac{H(\mu_k|f^{\otimes k})}{k} - \limsup_{N\to\infty} \int_{\mathbb{R}} \log\left(f(v)+\delta\right) d\Pi_1\left(\mu_N\right)(v) \\ + \int \log\left(f(v)\right) d\mu(v) - \frac{1-\int |v|^2 d\mu(v)}{2},\end{aligned} \tag{1.25}$$

where ν_N is given by (1.22).

Theorem 1.15 is the key to proving the following stability property of entropic chaoticity:

Theorem 1.16. *Let f be a probability density such that $f \in L^p$ for some $p > 1$ and $\int x^2 f(x)dx = 1$. Let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy$$

and assume that $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S > 0$ and $1 < \alpha < 2$. Assume in addition that $f \in L^\infty(\mathbb{R})$. Then, if $\{\mu_N\}_{N\in\mathbb{N}}$ is a family of symmetric probability measures on Kac's sphere and

$$\lim_{N\to\infty} \frac{H(\mu_N|\nu_N)}{N} = 0, \tag{1.26}$$

where ν_N is given by (1.22), we have that μ_N is $f-$chaotic. Moreover, μ_N is $f-$entropically chaotic.

A different approach to the stability problem involves the concept of the relative Fisher information functional on $\mathbb{R}$, I, and on Kac's sphere, I_N:

Definition 1.17. *Given two probability measures, μ, ν on $\mathbb{R}$, we define the relative Fisher information of μ with respect to ν, $I(\mu|\nu)$, as*

$$I(\mu|\nu) = \int_{\mathbb{R}} \frac{|h'(x)|^2}{h(x)} d\nu(x) = 4\int_{\mathbb{R}} \left|\frac{d}{dx}\sqrt{h(x)}\right|^2 d\nu(x), \tag{1.27}$$

where $h=\frac{d\mu}{d\nu}$, and $I(\mu|\nu)=\infty$ if μ is not absolutely continuous with respect to ν. Given two probability measures, μ_N,ν_N on Kac's sphere, we define the relative Fisher information of μ_N with respect to ν_N, $I_N(\mu_N|\nu_N)$, as

$$I_N(\mu_N|\nu_N)=\int_{\mathbb{S}^{N-1}(\sqrt{N})}\frac{|\nabla_S h|^2}{h}d\nu, \tag{1.28}$$

where $h=\frac{d\mu_N}{d\nu_N}$, and $I_N(\mu_N|\nu_N)=\infty$ if μ_N is not absolutely continuous with respect to ν_N. Here ∇_S denotes the components of the usual gradient on $\mathbb{R}^N$ that is tangential to Kac's sphere.

Theorem 1.18. *Let $\{\mu_N\}_{N\in\mathbb{N}}$ be a family of symmetric probability measures on Kac's sphere that is $f-$chaotic. Assume that there exists $C_S>0$ and $1<\alpha<2$ such that*

$$\int_{-\sqrt{x}}^{\sqrt{x}} v_1^4 d\Pi_1(\mu_N)(v_1) \underset{x\to\infty}{\sim} C_S x^{2-\alpha} \tag{1.29}$$

uniformly in N, and that

$$\frac{H_N(\mu_N|\sigma^N)}{N}\leq C, \quad \frac{I_N(\mu_N|\sigma^N)}{N}\leq C \tag{1.30}$$

for all N. Then μ_N is $f-$entropically chaotic.

The presented work is structured as follows: In Section 2 we will present some preliminaries to the work, including known results on the normalisation function and marginals of probability measures on Kac's sphere. Section 3 will be focused on proving the newly found local Lévy Central Limit Theorem, described in Theorem 1.12, as well as giving a particular version of it (with additional conditions). While we will use it in Section 4 where we will prove Theorems 1.13 and 1.14, Section 3 is interesting in its own right. As such, we present it in a self contained way (referring to definitions presented in the introduction) in hope for it to be accessible to people who are not familiar with Kac's model. In Section 5 we will discuss the lower semi continuity property of processes of our type (Theorem 1.15) and prove the stability theorems, Theorems 1.16 and 1.18. Once all the proofs are done, Section 6 will give more details about the spectral gap problem, the entropy method, Cercignani's many body conjecture and explain the connection between it and the presented work. Section 7 will see closing remarks for our work, including some connection between the current work and Cercignani's many body conjecture, while the Appendix will discuss a quantitative Lévy type approximation theorem, and include some additional computation that would otherwise encumber the presentation of our paper.

Lastly, we'd like to mention some references for topics that we've touched here. For more information about the Boltzmann equation we refer the interested reader to [10, 25, 29, 28]. For more information about Kac's and McKean Model, the spectral gap problem and the related Entropy method and Cercignani's many body conjecture we refer the interested reader to [3, 5, 6, 7, 9, 12, 13, 14, 17, 18, 19, 20, 22, 24, 25, 29]. For more information about propagation of chaos, from view points of Analysis, Probability and PDE, we refer the interested reader to [1, 2, 4, 7, 9, 11, 14, 17, 25, 27], and for more information about the roles of central limit theorem in the study of Kac-like equation we refer the interested reader to [2, 16, 23].

2 Preliminaries.

The Normalisation Function.

As discussed in the introduction, the normalisation function, $\mathcal{Z}_N\left(f,\sqrt{r}\right)$, plays an important role in the proofs of chaoticity and entropic chaoticity of distribution families of the form

$$F_N = \frac{f^{\otimes N}}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}.$$

In this short subsection we will give a probabilistic interpretation to it, as well as explain why it is well defined under simple conditions on f.

Lemma 2.1. *Let f be a probability density function for the real random variable V. Then*

$$\mathcal{Z}_N\left(f,\sqrt{r}\right) = \frac{2h^{*N}(r)}{\left|\mathbb{S}^{N-1}\right| r^{\frac{N-2}{2}}} \tag{2.1}$$

where h be the associated probability density function for the real random variable V^2 and h^{*N} is the $N-$th iterated convolution of h.

Proof for the above lemma can be found in [7, 12], yet we present it here for completion.

Proof. Denote by $S_N = \sum_{i=1}^N V_i^2$ the sum of independent copies of the real random variable V^2. For any function $\varphi \in C_b\left(\mathbb{R}^N\right)$, depending only on $r = \sqrt{\sum_{i=1}^N v_i^2}$ we find that

$$\mathbb{E}\varphi = \int_{\mathbb{R}^N} \varphi\left(\sqrt{\sum_{i=1}^N v_i^2}\right) \Pi_{i=1}^N f(v_i) dv_1 \dots dv_N =$$

$$\left|\mathbb{S}^{N-1}\right| \int_0^\infty \varphi(r) r^{N-1}\left(\int_{\mathbb{S}^{N-1}(r)} \Pi_{i=1}^N f(v_i) d\sigma_r^N\right) dr = \left|\mathbb{S}^{N-1}\right| \int_0^\infty \varphi(r) r^{N-1} \mathcal{Z}_N\left(f,r\right) dr$$

On the other hand

$$\mathbb{E}\varphi = \int_0^\infty \varphi\left(\sqrt{r}\right) s_N(r) dr = 2\int_0^\infty r\varphi(r) s_N\left(r^2\right) dr.$$

Since the above is valid for any φ we conclude that

$$\mathcal{Z}_N\left(f,\sqrt{r}\right) = \frac{2s_N(r)}{\left|\mathbb{S}^{N-1}\right| r^{\frac{N-2}{2}}}.$$

A known fact from probability theory states that the density function for S_N, s_N, is given by

$$s_N(u) = h^{*N}(u)$$

where h^{*N} is the $N-$th iterated convolution of h. This completes the proof. □

Remark 2.2. *It is easy to see that probability density function h, associated to the probability density function f as described in the above lemma, is given by*

$$h(u) = \begin{cases} \frac{f\left(\sqrt{u}\right)+f\left(-\sqrt{u}\right)}{2\sqrt{u}} & u > 0 \\ 0 & u \leq 0 \end{cases} \tag{2.2}$$

As such, using the convexity of $t \to t^q$ for any $q > 1$, we find that if in addition $f \in L^p(\mathbb{R})$ then

$$\int h(u)^{p'} du \leq \frac{1}{2}\int_0^\infty \frac{f(\sqrt{u})^{p'} + f(-\sqrt{u})^{p'}}{u^{\frac{p'}{2}}} du = \int_{\mathbb{R}} \frac{f(x)^{p'}}{x^{p'-1}}$$
$$\leq \int_{[-1,1]} \frac{f(x)^{p'}}{x^{p'-1}} + \int_{\mathbb{R}} f(x)^{p'} dx \tag{2.3}$$
$$\leq \left(\int_{[-1,1]} f(x)^p dx\right)^{\frac{p'}{p}} \left(\int_{[-1,1]} \frac{dx}{x^{\frac{p(p'-1)}{p-p'}}}\right)^{\frac{p-p'}{p'}} + \int_{f>1} f(x)^p dx + \int_{f<1} f(x) dx,$$

where $p' < p$. Choosing $1 < p' < \frac{2p}{1+p}$ we find $h \in L^{p'}(\mathbb{R})$, showing that h itself gains extra integrability properties in this case. This will serve us later on in Section 4.

Marginals on Kac's Sphere.

By its definition, chaoticity depends strongly on understanding how finite marginal on Kac's sphere behave. In particular, in our presented cases, we'll be interested to find a simple formula for the $k-$th marginal of probability measures of the form $F_N d\sigma^N$. To do that we state the following simple lemma, whose proof we'll omit, but can be found in [12]:

Lemma 2.3. *Let F_N be an integrable function on $\mathbb{S}^{N-1}(r)$, then*

$$\int_{\mathbb{S}^{N-1}(r)} F_N d\sigma_r^N = \frac{\left|\mathbb{S}^{N-j-1}\right|}{\left|\mathbb{S}^{N-1}\right|} \frac{1}{r^{N-2}} \int \left(r^2 - \sum_{i=1}^j v_i^2\right)_+^{\frac{N-j-2}{2}}$$
$$\left(\int_{\mathbb{S}^{N-j-1}\left(\sqrt{r^2-\sum_{i=1}^j v_i^2}\right)} F_N d\sigma^{N-j}_{\sqrt{r^2-\sum_{i=1}^j v_i^2}}\right) dv_1 \dots dv_j,$$

where $g_+ = \max(g, 0)$ for a function g.

Using the above lemma, one can easily show the following:

Lemma 2.4. *Given a distribution function F_N on Kac's sphere, then the probability density function of the $k-$th marginal of the probability measure $F_N d\sigma^N$ is given by*

$$\Pi_k(F_N)(v_1, \dots, v_k) = \frac{\left|\mathbb{S}^{N-k-1}\right|}{\left|\mathbb{S}^{N-1}\right|} \frac{1}{N^{\frac{N-2}{2}}} \left(N - \sum_{i=1}^k v_i^2\right)_+^{\frac{N-k-2}{2}}$$
$$\left(\int_{\mathbb{S}^{N-k-1}\left(\sqrt{r^2-\sum_{i=1}^k v_i^2}\right)} F_N d\sigma^{N-k}_{\sqrt{r^2-\sum_{i=1}^j v_k^2}}\right). \tag{2.4}$$

Next we show a simple condition for chaoticity, one we will use later on in Section 4:

Lemma 2.5. *Let $\{F_N\}_{N\in\mathbb{N}}$ be a family of distribution functions on Kac's sphere. Assume that there exists a distribution function f, on $\mathbb{R}$, such that*

$$\lim_{N\to\infty} \Pi_k(F_N)(v_1, \dots, v_k) = f^{\otimes k}(v_1, \dots, v_k) \tag{2.5}$$

pointwise for all $k \in \mathbb{N}$. Then

$$\lim_{N\to\infty} \left\|\Pi_k(F_N)(v_1, \dots, v_k) - f^{\otimes k}(v_1, \dots, v_k)\right\|_{L^1(\mathbb{R}^k)} = 0, \tag{2.6}$$

for all $k \in \mathbb{N}$, and n particular $\{F_N\}_{N\in\mathbb{N}}$ is $f-$chaotic.

The proof for this (and a more general statement) can be found in [14]. Since the proof is very simple we will add it here, for completion.

Proof. Let $k \in \mathbb{N}$ be fixed. Define $g_N = \Pi_k(F_N) + f^{\otimes k}$. By assumption (2.5) we know that

$$\lim_{N\to\infty} g_N = 2f^{\otimes k} = g,$$

pointwise and since $\left|\Pi_k(F_N) - f^{\otimes k}\right| \leq g_N$, and

$$\int_{\mathbb{R}^k} g_N\left(v_1,\dots,v_k\right) dv_1\dots dv_k = \int_{\mathbb{R}^k} g\left(v_1,\dots,v_k\right) dv_1\dots dv_k$$

for all N, we can use the generalised dominated convergence theorem to conclude (2.6). □

3 Lévy Type Local Central Limit Theorem.

This section's purpose is the introduce a new local Lévy type central limit theorem, one that will help us in our investigation of families of conditioned tensorisation of a function f, extending results presented in [7]. The bulk of the work is inspired from [7] and [16] though there are some significant changes on which we will remark.
We start this section with an important technical theorem, taken from [16], which plays a crucial role in the proof of our local central limit theorem. The fact that it only works in $\mathbb{R}$ will affect the lower semi-continuity property, discussed in Section 5. At this point we advice the reader to review the definition of $\alpha-$stability, DA, NDA, FDA of $\gamma_{\sigma,\alpha,\beta}$ given in Definition 1.7, 1.9, 1.10, 1.11, as well as Remark 1.8 and equation (1.18).

Theorem 3.1. *For any $\widehat{\gamma}_{\sigma,\alpha,\beta}$ we have that the NDA equals the FDA.*

Due to its importance, we will present a full proof for this theorem. The proof relies on the following technical lemma (again, taken from [16]):

Lemma 3.2. *Let $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a continuous function that satisfies $\lim_{n\to\infty} g\left(\frac{x}{n}\right) = 0$ for any $x \in \mathbb{R}\setminus\{0\}$. Then $\lim_{x\to 0} g(x) = 0$.*

We leave the proof to the Appendix, and show how one can prove Theorem 3.1 using it.

Proof of Theorem 3.1. We start with the easy direction. Assume that $\widehat{\psi}$ is in the FDA of $\gamma_{\sigma,\alpha,\beta}$. We have that

$$n\left(\widehat{\psi}\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right) - 1\right) = -n\cdot\frac{\sigma|\xi|^\alpha}{n}\left(1 + i\beta\text{sgn}\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)\tan\left(\frac{\pi\alpha}{2}\right)\right) + n\eta\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right).$$

$$= -\sigma\left|\xi\right|^\alpha\left(1 + i\beta\text{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right) + |\xi|^\alpha\cdot\frac{\eta\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)}{\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)^\alpha},$$

concluding the desired result.
Conversely, assume that $\widehat{\psi}$ is in the NDA of $\gamma_{\sigma,\alpha,\beta}$ and define

$$\eta(\xi) = \widehat{\psi}(\xi) - 1 + \sigma|\xi|^\alpha\left(1 + i\beta\text{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right).$$

We have that for any $\xi \neq 0$

$$\frac{\eta\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)}{\left|\frac{\xi}{n^{\frac{1}{\alpha}}}\right|^\alpha} = \frac{1}{|\xi|^\alpha}\left(n\left(\widehat{\psi}\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right) - 1\right) + \sigma|\xi|^\alpha\left(1 + i\beta\text{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right)\right)\underset{n\to\infty}{\longrightarrow} 0.$$

Defining $g(\xi)=\frac{\eta(\xi)}{|\xi|^\alpha}$ we find that g is continuous on $\mathbb{R}\setminus\{0\}$ and

$$g\left(\frac{\xi}{n^{\frac{1}{\alpha}}}\right)\underset{n\to\infty}{\longrightarrow}0$$

for any $\xi\neq 0$. A simple modification of Lemma 3.2 proves that $\lim_{\xi\to 0}g(\xi)=0$. This also shows, since η is continuous, that $\frac{\eta(\xi)}{|\xi|^\alpha}$ is bounded around $\xi=0$. For $|\xi|>\delta$ we have that

$$\frac{|\eta(\xi)|}{|\xi|^\alpha}\leq\frac{2}{\delta^\alpha}+\sigma\left(1+|\beta|\left|\tan\left(\frac{\pi\alpha}{2}\right)\right|\right),$$

proving that $\frac{\eta(\xi)}{|\xi|^\alpha}\in L^\infty$, and the result follows. □

Theorem 3.1 gives us a very convenient approximation for the characteristic function of any real random variable in the NDA of $\gamma_{\sigma,\beta,\alpha}$, one we will use quite strongly.
Next, we're like to find some simple conditions for when a real random variable belongs to the NDA of $\gamma_{\sigma,\beta,\alpha}$. This is given by a theorem from Feller's book, [15]:

Theorem 3.3. *Let F be a probability distribution function of a real random variable, X, that has zero mean, and let $1<\alpha<2$. Denote by*

$$\mu(x)=\int_{-x}^{x}y^2F(dy).\tag{3.1}$$

If

(i)

$$\mu(x)\underset{x\longrightarrow\infty}{\sim}x^{2-\alpha}L(x),\tag{3.2}$$

where L is slowly varying (i.e. $\frac{L(tx)}{L(x)}\underset{x\to\infty}{\longrightarrow}1$ for any $t>0$).

(ii)

$$\begin{aligned}\frac{1-F(x)}{1-F(x)+F(-x)}&\underset{x\to\infty}{\longrightarrow}p,\\ \frac{F(-x)}{1-F(x)+F(-x)}&\underset{x\to\infty}{\longrightarrow}q.\end{aligned}\tag{3.3}$$

(iii) There exists a sequence $\{a_n\}_{n\in\mathbb{N}}>0$ such that

$$\frac{n\mu(a_n)}{a_n^2}\underset{n\to\infty}{\longrightarrow}C_S.\tag{3.4}$$

Then X is in the DA of $\gamma_{C_S,\alpha,p,q}$ with $\{a_n\}_{n\in\mathbb{N}}$ found in (iii) and $b_n=0$.

Remark 3.4. *It is worth mentioning that a similar, less restrictive theorem, holds in the case $0<\alpha<1$. Since we will not use it in this work, we decided to exclude it from this section. For more information we refer the interested reader to [15].*

Remark 3.5. *Of particular interest to us are the following cases:*

- *If in condition (i) of Theorem 3.3 one has that $L(x)\underset{x\to\infty}{\sim}C_S$ then the sequence*

  $$a_n=n^{\frac{1}{\alpha}}$$

  will be suitable for condition (iii) of the same theorem.

- *If the probability distribution function, $F(x)$, is supported in $[\kappa,\infty)$ for some $\kappa\in\mathbb{R}$ then condition (ii) of Theorem 3.3 is immediately satisfied with $p=1$ and $q=0$.*

We now turn our attention to the proof of Theorem 1.12. The main idea of the proof is to evaluate the supremum of the difference between the probability density functions using inversion formula and their characteristic functions. An integral will emerge, one we will have to divide into two domains: frequencies that are close to zero, and frequencies that are 'far' from zero. The domain of frequencies that are almost zero will be taken care of by requiring that the characteristic function would be in the NDA of some stable distribution. The other frequencies will be dealt with presently:

Theorem 3.6. *Let g be a probability density function on $\mathbb{R}$ such that*

$$E_\lambda=\int_{\mathbb{R}}|x|^\lambda g(x)dx<\infty, \tag{3.5}$$

for some $\lambda>0$, and

$$H(g)=\int_{\mathbb{R}}g(x)\log g(x)dx<\infty. \tag{3.6}$$

Then for any $\beta>0$, there exists $\eta=\eta(\beta,H(g),E_\lambda)>0$ such that if $|\xi|>\beta$ then $|\widehat{g}(\xi)|\leq 1-\eta$. Moreover, given $\tau>0$ one can get the estimation

$$|\widehat{g}(\xi)|\leq 1-\beta^{2+\tau}+\phi_\tau(\beta), \tag{3.7}$$

for $\beta<\beta_0$ small enough, where $\frac{\phi_\delta(\tau)}{\beta^{2+\tau}}\underset{\beta\to 0}{\longrightarrow}0$.

Remark 3.7. *The proof of the first part of the above theorem, to be presented shortly, is very similar to the proof found in [7]. The novelty of our approach manifests itself in (3.7), where an explicit distance from 1 is given. The surprising part is that to show this estimation no new machinery is required, only an intermediate approximation.*

Proof. For a given $\xi\in\mathbb{R}$ we can find a $z\in\mathbb{R}$ such that

$$|\widehat{g}(\xi)|=\widehat{g}(\xi)e^{-i\xi z}.$$

By the definition of the Fourier transform, and the fact that $\widehat{g}(0)=1$, we have that

$$|\widehat{g}(\xi)|=\int_{\mathbb{R}}g(x)e^{-i(x+z)\xi}dx=1-\int_{\mathbb{R}}g(x)\left(1-e^{-i(x+z)\xi}\right)dx.$$

Since $|\widehat{g}|$ is real we find that

$$\begin{aligned}|\widehat{g}(\xi)|&=1-\int_{\mathbb{R}}g(x)\left(1-\cos\left((x+z)\xi\right)\right)dx\\&\leq 1-\int_{B}g(x)\left(1-\cos\left((x+z)\xi\right)\right)dx\end{aligned} \tag{3.8}$$

for any measurable set B.
Define:

$$B_{\delta,R}=\left\{x\in[-R,R]\mid 1-\cos\left((z+x)\xi\right)\leq\delta\right\},$$

where δ and R are to be specified later. From its definition, and (3.8), we conclude that

$$\begin{aligned}|\widehat{g}(\xi)|&\leq 1-\int_{[-R,R]\setminus B_{\delta,R}}g(x)\left(1-\cos\left((x+z)\xi\right)\right)dx\\&\leq 1-\delta\int_{[-R,R]\setminus B_{\delta,R}}g(x)dx.\end{aligned} \tag{3.9}$$

Next we notice that $x \in B_{\delta,R}$ if and only if $x \in [-R, R]$ and

$$|(z+x)\xi + 2\pi k| \leq \arccos(1-\delta)$$

for some $k \in \mathbb{Z}$. Since $\arccos(1-\delta) \leq \sqrt{2\delta}$ we conclude that if $x \in B_{\delta,R}$ then, for some $k \in \mathbb{Z}$,

$$\left| x - \left(\frac{2\pi k}{\xi} - z \right) \right| \leq \frac{\sqrt{2\delta}}{|\xi|}. \tag{3.10}$$

We denote by I_k the closed intervals centred in $\frac{2\pi k}{\xi} - z$, with radius $\frac{\sqrt{2\delta}}{|\xi|}$. Since the distance between the centres of any two I_k-s is at least $\frac{2\pi}{|\xi|}$, while the length of each interval is at most $\frac{2}{|\xi|}$, if we pick $\delta < \frac{1}{2}$, we conclude that the intervals I_k-s are mutually disjoint.

From (3.10) we see that the set $B_{\delta,R}$ is contained in a union of I_k-s.

Let n be the number of $k \in \mathbb{Z}$ such that $\frac{2\pi k}{\xi} - z \in [-R, R]$. All such $k-$s, but possibly the biggest and smallest k, satisfy that $\left[\frac{2\pi(k-1)}{|\xi|}, \frac{2\pi k}{|\xi|}\right] \subset [-R, R]$. Thus,

$$(n-2) \cdot \frac{2\pi}{|\xi|} = \sum_k \left| \left[\frac{2\pi(k-1)}{|\xi|}, \frac{2\pi k}{|\xi|} \right] \right| \leq 2R.$$

With $|\cdot|$ denoting the Lebesgue measure, we conclude that

$$|B_{\delta,R}| \leq n \cdot \frac{\sqrt{2\delta}}{|\xi|} \leq \left(\frac{R}{\pi} + \frac{2}{|\xi|} \right) \sqrt{2\delta} \leq \frac{R}{\pi} \left(1 + \frac{2\pi}{R\beta} \right) \sqrt{2\delta}. \tag{3.11}$$

At this point we will use the entropy and moment conditions on g to connect between the known value $|B_{\delta,R}|$ and the desired value $\int_{B_{\delta,R}} g(x)dx$. To do that we will use the relative entropy (see Definition 1.5) and the following known inequality:

$$\mu(B) \leq \frac{2H(\mu|\nu)}{\log\left(1 + \frac{H(\mu|\nu)}{\nu(B)}\right)}, \tag{3.12}$$

where μ and ν are regular probability measure on $\mathbb{R}$ and B is a measurable set.

Define

$$d\mu(x) = \frac{\chi_{[-R,R]}(x)g(x)}{\int_{[-R,R]} g(x)dx} dx, \quad d\nu(x) = \frac{\chi_{[-R,R]}(x)}{2R} dx. \tag{3.13}$$

We have that $\frac{d\mu}{d\nu}(x) = \frac{2R\chi_{[-R,R]}(x)g(x)}{\int_{[-R,R]} g(x)dx}$ and

$$\begin{aligned} H(\mu|\nu) &= \int_{[-R,R]} \log\left(\frac{2Rg(x)}{\int_{[-R,R]} g(x)dx} \right) \frac{g(x)}{\int_{[-R,R]} g(x)dx} dx \\ &= \log(2R) - \log\left(\int_{[-R,R]} g(x)dx \right) + \frac{1}{\int_{[-R,R]} g(x)dx} \int_{[-R,R]} g(x) \log g(x) dx \\ &\leq \log(2R) - \log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{1}{1 - \frac{E_\lambda}{R^\lambda}} \int g(x) |\log g(x)| dx. \end{aligned} \tag{3.14}$$

We have used the fact that

$$\int_{[-R,R]} g(x)dx = 1 - \int_{|x|>R} g(x)dx \geq 1 - \frac{1}{R^\lambda} \int_{|x|>R} |x|^\lambda g(x)dx \geq 1 - \frac{E_\lambda}{R^\lambda}. \tag{3.15}$$

We will now turn our attention to the term $\int g(x) |\log(g(x))| dx$. For *any* positive function $\psi(x)$, we have that

$$\psi(x) \left(\frac{g(x)}{\psi(x)} \log\left(\frac{g(x)}{\psi(x)} \right) - \frac{g(x)}{\psi(x)} + 1 \right) \geq 0.$$

Thus, for any measurable set A we have that

$$\int_A g(x)\log g(x)dx \geq \int_A g(x)\log\psi(x)dx + \int_A g(x) - \int_A \psi(x)dx,$$

when the right hand side is finite. Choosing $\psi(x) = e^{-|x|^\lambda}$ and $A = \{g < 1\}$ we find that

$$\begin{aligned}\left|\int_{g<1} g(x)\log g(x)dx\right| &= -\int_{g<1} g(x)\log g(x) \\ \leq \int_{g<1}|x|^\lambda g(x)dx - \int_{g<1} g(x)dx &+ \int_{g<1}\psi(x)dx < E_\lambda + C_\lambda.\end{aligned} \tag{3.16}$$

where $C_\lambda = \int \psi(x)dx$. Since

$$\int g(x)|\log(g(x)| = H(g) - 2\int_{g<1} g(x)\log g(x)dx.$$

we conclude that

$$H(\mu|\nu) \leq \log(2R) - \log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{H(g) + 2E_\lambda + 2C_\lambda}{1 - \frac{E_\lambda}{R^\lambda}}. \tag{3.17}$$

Together with (3.11) and (3.12) we find that

$$\mu(B_{\delta,R}) \leq \frac{2\log(2R) - 2\log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{2H(g)+4E_\lambda+4C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\log\left(1 + \frac{\log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\frac{1}{2\pi}\left(1+\frac{2\pi}{R\beta}\right)\sqrt{2\delta}}\right)}. \tag{3.18}$$

Next, we notice that

$$\int_{[-R,R]\setminus B_{\delta,R}} g(x)dx = \left(\int_{[-R,R]} g(x)dx\right)\mu\left([-R,R]\setminus B_{\delta,R}\right) \geq \left(1 - \frac{E_\lambda}{R^\lambda}\right)(1 - \mu(B_{\delta,R}))$$

which, along with (3.9) and (3.18) gives us the following control:

$$|\widehat{g}(\xi)| \leq 1 - \delta\cdot\left(1 - \frac{E_\lambda}{R^\lambda}\right)\left(1 - \frac{2\log(2R) - 2\log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{2H(g)+4E_\lambda+4C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\log\left(1 + \frac{\log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\frac{1}{2\pi}\left(1+\frac{2\pi}{R\beta}\right)\sqrt{2\delta}}\right)}\right) \tag{3.19}$$

At this point we can choose R and $\delta < \frac{1}{2}$ appropriately. For any $\tau > 0$ we choose

$\delta = \beta^{2+\tau}$ and $R = -\log\beta$ we find that for β going to zero

$$\begin{aligned}
&\frac{2\log(2R) - 2\log\left(1 - \frac{E_\lambda}{R^\lambda}\right) + \frac{2H(g)+4E_\lambda+4C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\log\left(1 + \frac{\log(2R)-\log\left(1-\frac{E_\lambda}{R^\lambda}\right)+\frac{H(g)+2E_\lambda+2C_\lambda}{1-\frac{E_\lambda}{R^\lambda}}}{\frac{1}{2\pi}\left(1+\frac{2\pi}{R\beta}\right)\sqrt{2\delta}}\right)} \\
&\approx \frac{2\log\left(-\log(\beta)\right)\left(1 + O\left(\frac{\log\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}{\log(-\log(\beta))}\right) + O\left(\frac{1}{\log(-\log(\beta))\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}\right)\right)}{\log\left(1 + \frac{2\pi\log(-\log(\beta))\left(1+O\left(\frac{\log\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}{\log(-\log(\beta))}\right)+O\left(\frac{1}{\log(-\log(\beta))\left(1-\frac{1}{(-\log(\beta))^\lambda}\right)}\right)\right)}{\sqrt{2}\beta^{1+\frac{\tau}{2}} - \frac{2\pi\sqrt{2}\beta^{\frac{\tau}{2}}}{\log(\beta)}}\right)} \\
&\approx \frac{2\log\left(-\log(\beta)\right)}{\log\left(1 - \frac{\log(-\log(\beta))\log(\beta)}{\sqrt{2}\beta^{\frac{\tau}{2}}(1+O(\beta\log(\beta)))}\right)} \approx \frac{2\log\left(-\log(\beta)\right)}{\log\left(-\frac{\log(-\log(\beta))\log(\beta)}{\sqrt{2}\beta^{\frac{\tau}{2}}}\right)} \\
&\approx -\frac{4\log\left(-\log(\beta)\right)}{\tau\log(\beta) - 2\log\left(-\log(\beta)\log\left(-\log(\beta)\right)\right)} \underset{\beta\to 0}{\longrightarrow} 0.
\end{aligned}$$

Thus,

$$|\widehat{g}(\xi)| \leq 1 - \beta^{2+\tau} + \phi_\tau(\beta),$$

where $\frac{\phi_\tau(\beta)}{\beta^{2+\tau}} \underset{\beta\to 0}{\longrightarrow} 0$. □

Before we present the proof for Theorem 1.12, we state the next simple lemma, whose proof we leave to the appendix. A similar argument can be found in [16].

Lemma 3.8. *Let $\widehat{g}$ be the characteristic function of a random real variable X that is in the NDA of $\gamma_{\sigma,\alpha,\beta}$. Then there exists $\beta_0 > 0$ such that for all $|\xi| < \beta_0$ we have that*

$$|\widehat{g}(\xi)| \leq e^{-\frac{\sigma|\xi|^\alpha}{2}}. \tag{3.20}$$

Proof of Theorem 1.12. We start by noticing that

$$\widehat{g_N}(\xi) = \widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right),$$

and from the inversion formula for characteristic functions (see [15]) we have that $\widehat{\gamma}_{\sigma,\alpha,\beta}$ is the characteristic function of $\gamma_{\sigma,\alpha,\beta}$.
Since $g \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ we conclude that $g \in L^{p'}(\mathbb{R})$ for any $1 \leq p' \leq p$. Thus, its characteristic function belongs to some $L^q(\mathbb{R})$ for some $q > 1$. One can choose q to be the Hölder conjugate of $\min(2,p)$. For any $N > q$ we have that

$$\int_{\mathbb{R}} |\widehat{g_N}(\xi)|\, d\xi \leq \|\widehat{g}\|_\infty^{N-q} \int_{\mathbb{R}} \left|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right|^q d\xi \leq N^{\frac{1}{\alpha}} \|\widehat{g}\|_{L^q}^q < \infty.$$

This implies that we can use the inversion formula for g: For any $x \in \mathbb{R}$:

$$\begin{aligned}
|g_N(x) - \gamma_{\sigma,\alpha,\beta}(x)| &\leq \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right) - \widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)\right| d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right) - \widehat{\gamma}^N_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right| d\xi \\
&\leq \frac{1}{2\pi}\int_{|\xi|<\beta_N N^{\frac{1}{\alpha}}} \left|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right) - \widehat{\gamma}^N_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right| d\xi \\
&+\frac{1}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}} \left|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right| d\xi + \frac{1}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}} \left|\widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)\right| d\xi \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.21}$$

The partition in (3.21) corresponds to the near to/far from zero discussed earlier. We wil start with estimating I_1.

Since $\widehat{g}$ is in the NDA of $\gamma_{\sigma,\alpha,\beta}$, Theorem 3.1 assures us that $\widehat{g}$ is in the FDA of $\gamma_{\sigma,\alpha,\beta}$ and there exists a reminder function, η_g, such that

$$|\widehat{g}(\xi) - \widehat{\gamma}_{\sigma,\alpha,\beta}(\xi)| = |\eta_g(\xi)| + |\eta_\gamma(\xi)|, \tag{3.22}$$

with

$$|\eta_\gamma(\xi)| \leq 2\sigma^2 |\xi|^{2\alpha}\left(1 + \beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right) \tag{3.23}$$

when $|\xi| < \beta_1$ for some small $\beta_1 > 0$. Thus,

$$\sup_{|\zeta|<\beta_N} \frac{|\widehat{g}(\zeta) - \widehat{\gamma}_{\sigma,\alpha,\beta}(\zeta)|}{|\zeta|^\alpha} \leq \omega_g(\beta_N) + 2\sigma\beta_N^\alpha\left(1 + \beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right) \tag{3.24}$$

for N large enough such that $\beta_N < \beta_1$.

Next, we see that

$$\begin{aligned}
&\left|\widehat{g}^N\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right) - \widehat{\gamma}^N_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right| \\
&\leq \left|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right) - \widehat{\gamma}_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right| \sum_{k=0}^{N-1}\left|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right|^k \left|\widehat{\gamma}_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right|^{N-1-k}.
\end{aligned} \tag{3.25}$$

Picking N such that $\frac{|\xi|}{N^{\frac{1}{\alpha}}} < \beta_N < \beta_0$ from Lemma 3.8 we find that

$$\begin{aligned}
\sum_{k=0}^{N-1}\left|\widehat{g}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right|^k \left|\widehat{\gamma}_{\sigma,\alpha,\beta}\left(\frac{\xi}{N^{\frac{1}{\alpha}}}\right)\right|^{N-1-k} &\leq \sum_{k=0}^{N-1} e^{-\frac{\sigma k|\xi|^\alpha}{2N}} \cdot e^{-\frac{\sigma(N-k-1)|\xi|^\alpha}{N}} \\
&\leq Ne^{-\frac{\sigma(N-1)|\xi|^\alpha}{2N}} \leq Ne^{-\frac{\sigma|\xi|^\alpha}{4}},
\end{aligned} \tag{3.26}$$

when $N \geq 2$. Combining (3.24), (3.25) and (3.26) we see that

$$\begin{aligned}
I_1 &\leq \frac{\omega_g(\beta_N) + 2\sigma\beta_N^\alpha\left(1 + \beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)}{2\pi}\int_{|\xi|<\beta_N N^{\frac{1}{\alpha}}} \frac{|\xi|^\alpha}{N} \cdot Ne^{-\frac{\sigma|\xi|^\alpha}{4}} d\xi \\
&\leq C\left(\omega_g(\beta_N) + 2\sigma\beta_N^\alpha\left(1 + \beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\right),
\end{aligned} \tag{3.27}$$

where $C = \int_{\mathbb{R}} |\xi|^\alpha e^{-\frac{\sigma|\xi|^\alpha}{4}} d\xi$. Next, we estimate I_2.

The expression I_2 is connected to Theorem 3.6, and as such we need to check that its conditions are satisfied. From the conditions given in the statement of our theorem, we know that there exists $\lambda > 0$ such that $E_\lambda < \infty$, following the notations of Theorem 3.6.

We only need to show that $H(g)<\infty$. Indeed, since $g\in L^p(\mathbb{R})$ for some $p>1$ we have that

$$\int_{\mathbb{R}} g(x)\left|\log g(x)\right| dx = -\int_{g<1} g(x)\log g(x)dx + \int_{g\geq 1} g(x)\log g(x)dx.$$

We already showed in the proof of Theorem 3.6 that $-\int_{g<1} g(x)\log g(x)dx<\infty$, and since we can always find $C_p>0$ such that $\log x\leq C_p x^{p-1}$ for $x\geq 1$ we conclude that

$$\int_{g\geq 1} g(x)\log g(x)dx \leq C_p \left\|g\right\|^p_{L^p(\mathbb{R})}<\infty,$$

showing that $H(g)<\infty$. Thus, for any $\tau>0$ and for β small enough we have that

$$|\widehat{g}(\xi)|\leq 1-\beta^{2+\tau}+\phi_\tau(\beta),$$

with $\frac{\phi_\delta(\tau)}{\beta^{2+\tau}} \underset{\beta\to 0}{\longrightarrow} 0$.

Using the above, we conclude that

$$I_2 = \frac{N^{\frac{1}{\alpha}}}{2\pi}\int_{|\xi|>\beta_N} |\widehat{g}(\xi)|^N d\xi \leq \frac{N^{\frac{1}{\alpha}}}{2\pi}\left(1-\beta_N^{2+\tau}+\phi_\tau(\beta_N)\right)^{N-q}\left\|\widehat{g}\right\|^q_{L^q(\mathbb{R})}. \tag{3.28}$$

Lastly, we need to estimate I_3, which is the simplest of the three integrals. Indeed

$$\begin{aligned} I_3 = \frac{1}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}} e^{-\sigma|\xi|^\alpha}d\xi &\leq \frac{e^{-\frac{\sigma N\beta_N^\alpha}{2}}}{2\pi}\int_{|\xi|>\beta_N N^{\frac{1}{\alpha}}} e^{-\frac{\sigma|\xi|^\alpha}{2}}d\xi \\ &\leq De^{-\frac{\sigma N\beta_N^\alpha}{2}}, \end{aligned} \tag{3.29}$$

where $D=\frac{1}{2\pi}\int_{\mathbb{R}} e^{-\frac{\sigma|\xi|^\alpha}{2}}d\xi$. Combining (3.27), (3.28) and (3.29) yields the desired result. □

Remark 3.9. *It is clear that if $\{\beta_N\}_{N\in\mathbb{N}}$ is chosen such that it goes to zero and*

$$\beta_N^{2+\tau}N \underset{N\to\infty}{\longrightarrow} \infty$$

then $\epsilon_\tau(N)$, defined in the above theorem, goes to zero as N goes to infinity, and we have an explicit rate to how fast it does it. A different method to undertake here is to pick β_0 small enough that all the steps of the proof the theorem work, and get that

$$\begin{aligned} \left\|g_N-\gamma_{\sigma,\alpha,\beta}\right\|_\infty \leq C_{g,\alpha}\Bigg(& N^{\frac{1}{\alpha}}(1-\beta_0^{2+\tau}+\phi_\tau(\beta_0))^{N-q}+e^{-\frac{\sigma N\beta_0^\alpha}{2}} \\ & +\omega_g(\beta_0)+2\sigma\beta_0^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg). \end{aligned}$$

Thus

$$\limsup_{N\to\infty}\left\|g_N-\gamma_{\sigma,\alpha,\beta}\right\|_\infty \leq \lim_{\beta_0\to 0}\left(\omega_g(\beta_0)+2\sigma\beta_0^\alpha\left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\right)=0,$$

proving the desired convergence, but losing the explicit N dependency!

An immediate corollary of Theorem 1.12 is the following:

Theorem 3.10. *Let g be the probability density function of a random real variable X. Assume that $g\in L^{p'}(\mathbb{R})$ for some $p'>1$ and*

(1) $\int |x|g(x)dx < \infty$.

(2) $\mu_g(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ *for some* $C_S > 0$ *and* $1 < \alpha < 2$ *where*

$$\mu_g(x) = \int_{-x}^{x} y^2 g(y)dy.$$

(3)

$$\frac{1-G(x)}{1-G(x)+G(-x)} \underset{x\to\infty}{\longrightarrow} p$$
$$\frac{G(-x)}{1-G(x)+G(-x)} \underset{x\to\infty}{\longrightarrow} q,$$

where $G(x) = \int_{-\infty}^{x} g(y)dy$.

Then, for any positive sequence $\{\beta_N\}_{N\in\mathbb{N}}$ *that converges to zero as* N *goes to infinity and satisfies*

$$\beta_N^{2+\tau} N \underset{N\to\infty}{\longrightarrow} \infty, \tag{3.30}$$

for some $\tau > 0$ *and for* N *large enough, we have that*

$$\begin{aligned}\sup_x \left| g^{*N}(x) - \frac{\gamma_{\sigma,\alpha,\beta}\left(\frac{x-NE}{N^{\frac{1}{\alpha}}}\right)}{N^{\frac{1}{\alpha}}} \right| \leq \frac{C_{g,\alpha}}{N^{\frac{1}{\alpha}}} \Bigg(N^{\frac{1}{\alpha}} (1-\beta_N^{2+\tau} + \phi_\tau(\beta_N))^{N-q'} \\ + e^{-\frac{\sigma N \beta_N^\alpha}{2}} + \omega_\eta(\beta_N) + 2\sigma\beta_N^\alpha \left(1+\beta^2\tan^2\left(\frac{\pi\alpha}{2}\right)\right)\Bigg) = \frac{\epsilon_\tau(N)}{N^{\frac{1}{\alpha}}},\end{aligned} \tag{3.31}$$

where

(i) $\sigma = C_S \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)} \cos\left(\frac{\pi\alpha}{2}\right)$, $\beta = p - q$.

(ii) $E = \int_{\mathbb{R}} xg(x)dx$.

(iii) $C_{g,\alpha} > 0$ *is a constant depending only on* g*, its moments and* α*.*

(iv) q' *can be chosen to be the Hölder conjugate of* $\min(2, p')$*.*

(v) ϕ_τ *satisfies*

$$\lim_{x\to 0} \frac{\phi_\tau(x)}{|x|^{2+\tau}} = 0,$$

(vi) $\eta(\xi)$ *is the reminder function of* $e^{-i\xi E}\widehat{g}(\xi)$*, defined in Definition 1.11, and* $\omega_\eta(\beta) = \sup_{|x|\leq\beta} \frac{|\eta(x)|}{|x|^\alpha}$.

Under the condition (3.30) and the conclusions $(i)-(vi)$ *one finds that*

$$\lim_{N\to\infty} \epsilon_\tau(N) = 0.$$

Proof. We start by defining $g_0(x) = g(x+E)$. Clearly $g_0 \in L^{p'}(\mathbb{R})$ and $\int_{\mathbb{R}} |x|g_0(x)dx < \infty$. If we will be able to show that g_0 is in the NDA of $\gamma_{\sigma,\alpha,\beta}$, then, using Theorem 1.12, we can conclude that

$$\sup_x \left| g^{*N}\left(N^{\frac{1}{\alpha}}x + NE\right) - \frac{\gamma_{\sigma,\alpha,\beta}(x)}{N^{\frac{1}{\alpha}}} \right| \leq \frac{\epsilon_\tau(N)}{N^{\frac{1}{\alpha}}},$$

as $g_0^{*N}(x) = g^{*N}(x+NE)$, and the desired result follows.
We only have to prove that g_0 is in the appropriate NDA. To do that we will use Theorem 3.3. From its definition we know that g_0 has zero mean. Clearly

$$\frac{1-G_0(x)}{1-G_0(x)+G_0(-x)} \underset{x\to\infty}{\longrightarrow} p$$
$$\frac{G_0(-x)}{1-G_0(x)+G_0(-x)} \underset{x\to\infty}{\longrightarrow} q,$$

with $G_0(x) = \int_{-\infty}^{x} g_0(y)dy$, as $G_0(x) = G(x+E)$.
Next, we see that

$$\mu_{g_0}(x) = \int_{-x}^{x} y^2 g_0(y)dy = \int_{-x+E}^{x+E} y^2 g(y)dy - 2E\int_{-x+E}^{x+E} yg(y)dy + E^2\int_{-x+E}^{x+E} g(y)dy.$$

The first term is bounded between $\mu_g(x-E)$ and $\mu_g(x+E)$ and as such behaves like $C_S x^{2-\alpha}$ as x goes to infinity. The rest of the terms have a limit as x goes to infinity, implying that

$$\mu_{g_0}(x) \sim C_S x^{2-\alpha}.$$

All the conditions of Theorem 3.3 are satisfied (see Remark 3.5), with σ and β given by (i), and the proof is complete. □

Before we end this section we'd like to mention that with additional conditions on g, the estimation on ϵ_τ, defined in Theorem 3.10, can become more explicit. This will be done via an explicit estimation for $\omega_\eta(\xi)$. Such estimation can be found in [16], yet the additional conditions are very restrictive. As it is still of interest we will provide some information on the matter in the Appendix.

4 Chaoticity and Entropic Chaoticity for Families with Unbounded Fourth Moment.

The study of the chaoticity and entropic chaoticity of probability density functions, $\{F_N\}_{N\in\mathbb{N}}$, on Kac's sphere that are obtained by conditioning a tensorisation of a one particle function, f (equation (1.9)), is intimately connected to the asymptotic behaviour of the normalisation function $\mathcal{Z}_N(f,r)$ at all r, and not only its value at $r=\sqrt{N}$. Formula (2.1) for the normalisation function, presented in Section 2, and the local central limit theorem we proved in Section 3 provide us with the necessary tools to find the desired behaviour.

Theorem 4.1. *Let f be the probability density function of a random real variable V such that $f\in L^p(\mathbb{R})$ for some $p>1$ and let*

$$\nu_f(x) = \int_{-\sqrt{x}}^{\sqrt{x}} y^4 f(y)dy.$$

Assume that

$$\int_{\mathbb{R}} x^2 f(x)dx = E < \infty.$$

and $\nu_f(x) \underset{x\to\infty}{\sim} C_S x^{2-\alpha}$ for some $C_S>0$ and $1<\alpha<2$. Then

$$\sup_x \left| h^{*N}(x) - \frac{\gamma_{\sigma,\alpha,1}\left(\frac{x-NE}{N^{\frac{1}{\alpha}}}\right)}{N^{\frac{1}{\alpha}}} \right| \leq \frac{\epsilon(N)}{N^{\frac{1}{\alpha}}}, \tag{4.1}$$

where $\lim_{N\to\infty}\epsilon(N)=0$*,* $\sigma=C_S\frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{\pi\alpha}{2}\right)$ *and* h *is the probability density function of the random variable* V^2*. Moreover,* $\epsilon(N)$ *can be bound by* $\epsilon_\tau(N)$*, given in Theorem 3.10, with* η *the reminder function of* $e^{-i\xi}\widehat{h}$*.*
In addition,

$$\mathcal{Z}_N\left(f,\sqrt{r}\right)=\frac{2}{\left|\mathbb{S}^{N-1}\right|r^{\frac{N-2}{2}}}\frac{1}{N^{\frac{1}{\alpha}}}\left(\gamma_{\sigma,\alpha,1}\left(\frac{r-NE}{N^{\frac{1}{\alpha}}}\right)+\lambda_N(r)\right), \tag{4.2}$$

where $\sup_u|\lambda_N(u)|\underset{N\to\infty}{\longrightarrow}0$*.*

Proof. We start by noticing that (4.2) follows immediately from (2.1) and (4.1). Next, we will show that the conditions of Theorem 3.10 are satisfied by h, concluding inequality (4.1), and the estimation for $\epsilon(N)$.
As was mentioned before, the function h is given by

$$h(x)=\begin{cases}\frac{f(\sqrt{x})+f(-\sqrt{x})}{2\sqrt{x}} & x>0\\ 0 & x\leq 0\end{cases}$$

and $h\in L^{p'}(\mathbb{R})$ for some $p'>1$ when $f\in L^p(\mathbb{R})$ with $p>1$ (see Remark 2.2). Moreover, for any $\kappa>0$

$$\int_{\mathbb{R}}|x|^\kappa h(x)dx=\int_{\mathbb{R}}|x|^{2\kappa}f(x)dx,$$

from which we conclude that

$$\int_{\mathbb{R}}|x|h(x)dx=\int_{\mathbb{R}}xh(x)=E<\infty.$$

By its definition

$$\mu_h(x)=\int_{-x}^{x}y^2h(y)dy=\nu_f(x)\underset{x\to\infty}{\sim}C_Sx^{2-\alpha},$$

and recalling Remark 3.5, we conclude that if H is the probability distribution function of V^2 then for any $x>0$

$$\frac{1-H(x)}{1-H(x)+H(-x)}=1$$
$$\frac{H(-x)}{1-H(x)+H(-x)}=0.$$

Thus, all the condition of Theorem 3.10 are satisfied by h with the appropriate σ,α and $\beta=1$, and the proof is complete. ☐

Remark 4.2. *A couple of remarks:*

- *As was discussed in the introduction: the finiteness of the fourth moment of* f *guarantees a* normal *local central limit theorem. When* f *lacks that condition, a thing that manifests itself via the function* $\nu_f(x)$ *in the above theorem, there is still something that can be said and our local central limit theorem comes into play by replacing the Gaussian with the stable laws.*
- *The parameter* β *represents the skewness of the stable distribution. In general* $\beta\in[-1,1]$ *and the closer it is to* 1*, the more right skewed the distribution is. The closer it gets to* -1*, the more left skewed the distribution is. Since our probability density function* h *is supported on the positive real line, it is not surprising that we got that* β *must be* 1*!*

We are now ready to prove Theorems 1.13 and 1.14.

Proof of Theorem 1.13. Due to the given information on f, we see that it satisfies all the conditions of Theorem 4.1, and as such for any finite $k \in \mathbb{R}$

$$\begin{aligned} &\left|\mathbb{S}^{N-k-1}\right| r^{\frac{N-k-2}{2}} \mathcal{Z}_{N-k}\left(f,\sqrt{r}\right) \\ &= \frac{2}{(N-k)^{\frac{1}{\alpha}}}\left(\gamma_{\sigma,\alpha,1}\left(\frac{r-(N-k)}{(N-k)^{\frac{1}{\alpha}}}\right)+\lambda_{N-k}(r)\right), \end{aligned} \tag{4.3}$$

for some $\sigma = C_S \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\cos\left(\frac{\pi\alpha}{2}\right)$ and λ_{N-k} such that

$$\epsilon_{N-k} = \sup_r |\lambda_{N-k}(r)| \underset{N\to\infty}{\longrightarrow} 0.$$

Using Lemma 2.4 with $F_N = \frac{f^{\otimes N}}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}$ we find that

$$\Pi_k(F_N)\left(v_1,\dots,v_k\right) = \frac{\left|\mathbb{S}^{N-k-1}\right|\left(N-\sum_{i=1}^k v_i^2\right)_+^{\frac{N-k-2}{2}} \mathcal{Z}_{N-k}\left(f,\sqrt{N-\sum_{i=1}^k v_i^2}\right)}{\left|\mathbb{S}^{N-1}\right| N^{\frac{N-2}{2}} \mathcal{Z}_N\left(f,\sqrt{N}\right)} \cdot f^{\otimes k}\left(v_1,\dots,v_k\right).$$

Combining this with (4.3) yields

$$\begin{aligned} \Pi_k(F_N)\left(v_1,\dots,v_k\right) = \left(\frac{N}{N-k}\right)^{\frac{1}{\alpha}} &\frac{\gamma_{\sigma,\alpha,1}\left(\frac{k-\sum_{i=1}^k v_i^2}{(N-k)^{\frac{1}{\alpha}}}\right)+\lambda_{N-k}\left(N-\sum_{i=1}^k v_i^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)} \\ &\cdot f^{\otimes k}\left(v_1,\dots,v_k\right)\chi_{\sum_{i=1}^k v_i^2 \leq N}\left(v_1,\dots,v_k\right), \end{aligned} \tag{4.4}$$

where χ_A is the characteristic function of the set A. By its definition, given in (1.18), and the properties of $\widehat{\gamma}_{\sigma,\alpha,\beta}$, we know that $\gamma_{\sigma,\alpha,1}$ is bounded and continuous on $\mathbb{R}$. As such, along with the conditions on λ_{N-k} and λ_N, we conclude that

$$\Pi_k(F_N)\left(v_1,\dots,v_k\right) \underset{N\to\infty}{\longrightarrow} f^{\otimes k}\left(v_1,\dots,v_k\right),$$

pointwise. Using Lemma 2.5 we obtain that $\{F_N\}_{N\in\mathbb{N}}$ is $f-$chaotic.
Next we turn our attention to the entropic chaos. Using symmetry, (4.3) and (4.4) we find that

$$\begin{aligned} H_N(F_N) &= \frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}\int_{\mathbb{S}^{N-1}(\sqrt{N})} f^{\otimes N}\log\left(f^{\otimes N}\right)d\sigma^N - \log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right) \\ &= N\int_{\mathbb{R}} \Pi_1(F_N)(v_1)\log\left(f(v_1)\right)dv_1 - \log\left(\frac{2\left(\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)\right)}{|\mathbb{S}|^{N-1} N^{\frac{N-2}{2}+\frac{1}{\alpha}}}\right) \\ &= N\left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}}\int_{-\sqrt{N}}^{\sqrt{N}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{(N-1)^{\frac{1}{\alpha}}}\right)+\lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)} f(v_1)\log f(v_1)dv_1 \\ &- \log\left(2\sqrt{\pi}\left(\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)\right)\left(1+O\left(\frac{1}{N}\right)\right)\right) + \left(\frac{1}{\alpha}-\frac{1}{2}\right)\log N + \frac{N}{2}\log\left(2\pi e\right). \end{aligned}$$

where we have used the fact that $\left|\mathbb{S}^{N-1}\right| = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}$, and an asymptotic approximation for the Gamma function.

We have that

$$\left|\frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{(N-1)^{\frac{1}{\alpha}}}\right)+\lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)}f(v_1)\log f(v_1)\right|$$
$$\leq \frac{\|\gamma_{\sigma,\alpha,1}\|_\infty+\epsilon_{N-1}}{\gamma_{\sigma,\alpha,1}(0)-\epsilon_N}f(v_1)\left|\log f(v_1)\right|$$
$$\leq \frac{2\left(\|\gamma_{\sigma,\alpha,1}\|_\infty+1\right)}{\gamma_{\sigma,\alpha,1}(0)}f(v_1)\left|\log f(v_1)\right|\in L^1(\mathbb{R}),$$

for N large enough. Combining this with the fact that $\{\Pi_1(F_N)\}_{N\in\mathbb{N}}$ converges to f pointwise, we can use the dominated convergence theorem to conclude that

$$\lim_{N\to\infty}\frac{H_N(F_N)}{N}=\int_{\mathbb{R}}f(v_1)\log f(v_1)dv_1+\frac{\log 2\pi+1}{2}=H(f|\gamma), \tag{4.5}$$

and the proof is complete. □

Proof of Theorem 1.14. It is easy to see that the condition $f(x)\underset{x\to\infty}{\sim}\frac{D}{|x|^{1+2\alpha}}$ for some $1<\alpha<2$ and $D>0$ implies that

$$\nu_f(x)\underset{x\to\infty}{\sim}\frac{D}{2-\alpha}x^{2-\alpha}.$$

Thus, with the added information given in the theorem we know that f satisfies the conditions of Theorem 1.13, and we conclude the desired result. □

Remark 4.3. *Theorem 1.14 gives rise to many, previously unknown, entropically chaotic families, determined mainly by a simple growth condition. An explicit example to such family is the one generated by the function*

$$f(x)=\frac{\sqrt{2}}{\pi\left(1+x^4\right)}.$$

5 Lower Semi Continuity and Stability Property.

As discussed in Section 1, the concept of entropic chaoticity is much stronger than that of normal chaoticity. This is due to the inclusion of all correlation information and an appropriate rescaling of the relative entropy. In this section we will show that the rescaled entropy is a good form of distance, one that is stable under certain conditions. The first step we must make, inspired by [7], is a form of lower semi continuity property for the relative entropy on Kac's sphere, expressed in Theorem 1.15. To begin with, we mention that in [7], the authors proved the following:

Theorem 5.1. *Let f be a probability density function on $\mathbb{R}$ such that $f\in L^p(\mathbb{R})$ for some $p>1$. Assume in addition that*

$$\int_{\mathbb{R}}x^2f(x)dx=1,\quad \int_{\mathbb{R}}x^4f(x)dx<\infty.$$

Denote by $d\nu_N=F_Nd\sigma^N$, where $F_N=\frac{f^{\otimes N}}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}$, restricted to Kac's sphere and let $\{\mu_N\}_{N\in\mathbb{N}}$ be a family of symmetric probability measures on Kac's sphere such that for some $k\in\mathbb{N}$ we have that

$$\Pi_k(\mu_N)\underset{N\to\infty}{\rightharpoonup}\mu_k.$$

Then

$$\frac{H\left(\mu_k|f^{\otimes k}\right)}{k}\leq\liminf_{N\to\infty}\frac{H_N(\mu_N|\nu_N)}{N}. \tag{5.1}$$

Note that due to the so-called Csiszar-Kullback-Leibler-Pinsker inequality ([26]) one has that

$$\|\mu-\nu\|_{TV} \leq \sqrt{2H(\mu|\nu)}, \tag{5.2}$$

showing that (5.1) gives a stronger result than an L^1 convergence. We will use this theorem as a motivation for our lower semi continuity property, as well as in the particular case of

$$f(x)=\gamma(x), \quad d\nu_N = F_N d\sigma^N = d\sigma^N,$$

where $\gamma(x)$ is the standard Gaussian.
Before we begin the proof of Theorem 1.15 we point out the obvious difference between the $k=1$ and $k>1$ cases. This is due to the fact that the proof relies heavily on our approximation theorem, Theorem 4.1, which is valid *only* in one dimension. The higher dimension case needs to be tackled differently, unlike the proof of Theorem 5.1, where the higher dimension case is proven in a very similar way.
The proof of Theorem 1.15 follows ideas presented in [7], with some modification to our current discussion.

Proof of Theorem 1.15. We start by noticing that since $C_b\left(\mathbb{R}^{k_0}\right)$ can be considered a subspace of $C_b\left(\mathbb{R}^{k}\right)$ whenever $k_0 \leq k$, the weak convergence condition on $\Pi_k(\mu_N)$ implies that

$$\Pi_{k_0}(\mu_N) \underset{N\to\infty}{\rightharpoonup} \mu_{k_0} = \Pi_{k_0}(\mu_k).$$

In particular we find that $\Pi_1(\mu_N)$ converges weakly to $\mu=\Pi_1(\mu_k)$.
Next, we recall a duality formula for the relative entropy (see [21] for instance, for the compact case):

$$H(\mu|\nu) = \sup_{\varphi\in C_b}\left\{\int \varphi d\mu - \log\left(\int e^{\varphi} d\nu\right)\right\}. \tag{5.3}$$

Given $\epsilon>0$ we can find $\varphi_\epsilon \in C_b(\mathbb{R})$ such that

$$\int_{\mathbb{R}} e^{\varphi_\epsilon(v)} f(v)dv = 1$$

and

$$H(\mu|f) \leq \int_{\mathbb{R}} \varphi_\epsilon(v) d\mu(v) + \frac{\epsilon}{2}. \tag{5.4}$$

We can find a compact set $K_\epsilon \subset \mathbb{R}$ such that

$$\mu\left(K_\epsilon^c\right) \leq \frac{\epsilon}{4\left\|\varphi_\epsilon\right\|_\infty}, \quad \int_{K_\epsilon^c} f(v)dv \leq \frac{\epsilon}{2e^{\|\varphi_\epsilon\|_\infty}}.$$

Let $\eta_\epsilon \in C_c(\mathbb{R})$ be such that

$$0 \leq \eta_\epsilon \leq 1, \quad \eta_\epsilon|_{K_\epsilon} = 1,$$

and define $\varphi(v)=\eta_\epsilon(v)\varphi_\epsilon(v)$. Clearly $\varphi\in C_c(\mathbb{R})$, $|\varphi|\leq|\varphi_\epsilon|$ and

$$H(\mu|f) \leq \int_{\mathbb{R}} \varphi(v)d\mu(v) + 2\left\|\varphi_\epsilon\right\|_\infty \mu\left(K_\epsilon^c\right) + \frac{\epsilon}{2} < \int_{\mathbb{R}} \varphi(v)d\mu(v) + \epsilon. \tag{5.5}$$

Also,

$$\left|\int_{\mathbb{R}} e^{\varphi(v)} f(v)dv - \int_{\mathbb{R}} e^{\varphi_\epsilon(v)} f(v)dv\right| \leq 2e^{\|\varphi_\epsilon\|_\infty} \int_{K_\epsilon^c} f(v)dv < \epsilon. \tag{5.6}$$

For any $N \in \mathbb{N}$, define $\phi_N(v_1,\dots,v_N) = \sum_{i=1}^N \varphi(v_i) \in C_b\left(\mathbb{R}^N\right)$. Plugging ϕ_N as a candidate in (5.3), in the setting of Kac's sphere, and using symmetry we find that

$$H_N(\mu_N|\nu_N) \geq N\int_{\mathbb{R}} \varphi(v_1)d\Pi_1(\mu_N)(v_1) - \log\left(\frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}\int_{\mathbb{S}^{N-1}(\sqrt{N})} \Pi_{i=1}^N\left(e^{\varphi(v_i)}f(v_i)\right)d\sigma^N\right)$$
$$= N\int_{\mathbb{R}} \varphi(v_1)d\Pi_1(\mu_N)(v_1) - \log\left(\frac{\mathcal{Z}_N\left(\frac{e^{\varphi}f}{a},\sqrt{N}\right)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}\right) - N\log a,$$

where $a = \int_{\mathbb{R}} e^{\varphi(v)}f(v)dv$. Since f satisfies the conditions of Theorem 4.1, so does the probability density function $\frac{e^{\varphi}}{a}f$. Denoting by $E = \frac{1}{a}\int_{\mathbb{R}} v^2 e^{\varphi(v)}f(v)dv$ we find that

$$\frac{\mathcal{Z}_N\left(\frac{e^{\varphi}f}{a},\sqrt{N}\right)}{\mathcal{Z}_N(f,\sqrt{N})} = \frac{\gamma_{\sigma_1,\alpha,1}\left(\frac{N-NE}{N^{\frac{1}{\alpha}}}\right)+\epsilon_1(N)}{\gamma_{\sigma,\alpha,1}(0)+\epsilon_2(N)}, \tag{5.7}$$

for some σ,σ_1, and $\{\epsilon_i(N)\}_{i=1,2}$ that go to zero as N goes to infinity. Since $\gamma_{\sigma_1,\alpha,1}$ is the defined as the inverse Fourier transform of an L^1 function we know that

$$\lim_{|x|\to\infty} \gamma_{\sigma_1,\alpha,1}(x) = 0.$$

Thus,

$$\liminf_{N\to\infty}\left(-\frac{\log\left(\gamma_{\sigma_1,\alpha,1}\left(\frac{N-NE}{N^{\frac{1}{\alpha}}}\right)+\epsilon_1(N)\right)}{N}\right) \geq 0. \tag{5.8}$$

Together with the fact that

$$\lim_{N\to\infty}\left(-\frac{\log\left(\gamma_{\sigma,\alpha,1}(0)+\epsilon_2(N)\right)}{N}\right) = 0,$$

the weak convergence of $\Pi_1(\mu_N)$ and (5.5), we find that

$$\begin{aligned}\liminf_{N\to\infty}\frac{H_N(\mu_N|\nu_N)}{N} &\geq \int_{\mathbb{R}}\varphi(v)d\mu(v) - \log(1+\epsilon)\\ &\geq H(\mu|f) - \epsilon - \log(1+\epsilon),\end{aligned} \tag{5.9}$$

where we have used (5.6) to conclude that $|a-1|<\epsilon$. Since ϵ was arbitrary, (i) is proved.

In order to show (ii), we notice that

$$H_N(\mu_N|\nu_N) = \int_{\mathbb{S}^{N-1}(\sqrt{N})}\log\left(\frac{d\mu_N}{F_N d\sigma^N}\right)d\mu_N = H_N(\mu_N|\sigma^N) - \int_{\mathbb{S}^{N-1}(\sqrt{N})}\log\left(F_N\right)d\mu_N$$
$$= H_N(\mu_N|\sigma^N) - N\int_{\mathbb{R}}\log\left(f(v_1)\right)d\Pi_1(\mu_N) + \log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right).$$

Thus, for any $\delta>0$,

$$\begin{aligned}\liminf_{N\to\infty}\frac{H_N(\mu_N|\nu_N)}{N} &+ \limsup_{N\to\infty}\int_{\mathbb{R}}\log\left(f(v_1)+\delta\right)d\Pi_1(\mu_N)\\ &\geq \liminf_{N\to\infty}\frac{H_N(\mu_N|\sigma^N)}{N} - \frac{\log(2\pi)+1}{2},\end{aligned} \tag{5.10}$$

where we have used the fact that $\lim_{N\to\infty}\frac{\log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right)}{N} = -\frac{\log(2\pi)+1}{2}$, shown in the proof of Theorem 1.13. From Theorem 5.1 we know that

$$\liminf_{N\to\infty}\frac{H_N(\mu_N|\sigma^N)}{N}\geq\frac{H(\mu_k|\gamma^{\otimes k})}{k},$$

and since

$$H(\mu_k|f^{\otimes k})=H(\mu_k|\gamma^{\otimes k})+\int_{\mathbb{R}^k}\log\left(\frac{\gamma^{\otimes k}}{f^{\otimes k}}\right)d\mu_k$$
$$=H(\mu_k|\gamma^{\otimes k})-\frac{k\left(\log(2\pi)+\int_{\mathbb{R}}v^2d\mu(v)\right)}{2}-k\int_{\mathbb{R}}\log\left(f(v)\right)d\mu(v)$$

we get the desired result from (5.10). □

We will now prove our first stability result, Theorem 1.16. Again, the ideas presented here are motivated by [7].

Proof of Theorem 1.16. We start with the simple observation that if $\{\mu_N\}_{N\in\mathbb{N}}$ is a family of symmetric probability measures on Kac's sphere then $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ is a tight family, for any $k\in\mathbb{N}$. Indeed, given $k\in\mathbb{N}$ we can find $m_N, r_N\in\mathbb{N}$ such that

$$N=m_Nk+r_N,$$

where $0\leq r_N<k$. We have that

$$\Pi_k(\mu_N)\left(\left\{\sqrt{\sum_{i=1}^k v_i^2}>R\right\}\right)\leq\frac{1}{R^2}\int_{\sum_{i=1}^k v_i^2>R^2}\left(\sum_{i=1}^k v_i^2\right)d\Pi_k(\mu_N)$$
$$\leq\frac{1}{m_NR^2}\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)}\left(\sum_{i=1}^{m_Nk}v_i^2\right)d\mu_N\leq\frac{N}{m_NR^2}<\frac{2k}{R^2},$$

proving the tightness.
Since $\{\Pi_1\mu_N\}_{N\in\mathbb{N}}$ is tight, we can find a subsequence, $\left\{\Pi_1\left(\mu_{N_{k_j}}\right)\right\}_{j\in\mathbb{N}}$, to any subsequence $\{\Pi_1\left(\mu_{N_k}\right)\}_{k\in\mathbb{N}}$, that converges to a limit. Denote by κ the weak limit of such one subsequence. Using (1.24) we conclude that

$$H(\kappa|f)\leq\liminf_{j\to\infty}\frac{H_{N_{k_j}}\left(\mu_{N_{k_j}}|\nu_{N_{k_j}}\right)}{N_{k_j}}=0, \tag{5.11}$$

due to condition (1.26). Thus, $\kappa=f(v)dv$, and since κ was an arbitrary weak limit, we conclude that all possible weak limit points must be $f(v)dv$. Since the weak topology on $P(\mathbb{R})$ is metrisable we conclude that

$$\Pi_1(\mu_N)\underset{N\to\infty}{\rightharpoonup}f(v)dv=\mu.$$

We will show that the convergence is actually in L^1 with the weak topology.
As an intermediate step in the proof of Theorem 1.15 we have shown that

$$\begin{aligned}H(\mu_N|\nu_N)=H(\mu_N|\sigma^N)-N\int_{\mathbb{R}}\log\left(f(v_1)\right)d\Pi_1(\mu_N)(v_1)\\+\log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right).\end{aligned} \tag{5.12}$$

Using condition (1.26), the fact that $\lim_{N\to\infty} \frac{\mathcal{Z}_N\left(f,\sqrt{N}\right)}{N} = -\frac{\log(2\pi)+1}{2}$, and the fact that $f \in L^\infty(\mathbb{R})$ we conclude that there exists $C>0$, independent of N, such that for any $\delta > 0$

$$\frac{H(\mu_N|\sigma_N)}{N} \leq C + \log\left(\|f\|_\infty + \delta\right). \tag{5.13}$$

The inequality

$$\frac{H\left(\Pi_k(\mu_N)|\Pi_k(\sigma^N)\right)}{k} \leq 2\frac{H_N(\mu_N|\sigma^N)}{N}$$

proven in [3] and valid for any $k \geq 1$ and $N \geq k$, implies that

$$H\left(\Pi_k(\mu_N)|\Pi_k(\sigma^N)\right) \leq 2k\left(C + \log\left(\|f\|_\infty\right) + \delta\right), \tag{5.14}$$

for all $k \in \mathbb{N}$, $N \geq k$ and $\delta > 0$.
Similar to the proof of Theorem 1.15, one can easily see that

$$H\left(\Pi_k(\mu_N)|\gamma^{\otimes k}\right) = H\left(\Pi_k(\mu_N)|\Pi_k(\sigma^N)\right) + \int_{\mathbb{R}^k} \log\left(\frac{\Pi_k(\sigma^N)}{\gamma^{\otimes k}}\right) d\Pi_k(\mu_N) \tag{5.15}$$

where γ is the standard Gaussian. Since $d\sigma^N = \frac{\gamma^{\otimes N}}{\mathcal{Z}_N\left(\gamma,\sqrt{N}\right)} d\sigma^N$, and γ is a probability density with finite fourth moment, one can employ similar theorems to those presented here and find that

$$\frac{\Pi_k(\sigma^N)\left(v_1,\dots,v_k\right)}{\gamma^{\otimes k}\left(v_1,\dots,v_k\right)} = \sqrt{\frac{N}{N-k}} \cdot \frac{\gamma\left(\frac{k-\sum_{i=1}^k v_i^2}{\sqrt{2N}}\right) + \lambda_{N-k}\left(N-k-\sum_{i=1}^k v_I^2\right)}{1+\lambda_N(N)} \chi_{\sum_{i=1}^k v_i^2 \leq N},$$

where $\sup_u |\lambda_{N-k}(u)| \underset{N\to\infty}{\longrightarrow} 0$ and $\lambda_N(N) \underset{N\to\infty}{\longrightarrow} 0$ (see [7] for more details). As such,

$$\int_{\mathbb{R}^k} \log\left(\frac{\Pi_k(\sigma^N)}{\gamma^{\otimes k}}\right) d\Pi_k(\mu_N) \leq \log\left(\max_{N>k} \sqrt{\frac{N}{N-k}} \frac{\|\gamma\|_\infty + \sup_N \sup_u |\lambda_{N-k}(u)|}{1+\inf_N \lambda_N(N)}\right),$$

which, together with (5.14) and (5.15) shows that

$$H\left(\Pi_k(\mu_N)|\gamma^{\otimes k}\right) \leq 2k\left(C + \log\left(\|f\|_\infty\right) + \delta\right) + D,$$

for some $C, D > 0$ independent of N, and $\delta > 0$. Thus, $\{\Pi_k \mu_N\}_{N\in\mathbb{N}}$ has bounded relative entropy with respect to $\gamma^{\otimes k}$ and we can apply the Dunford-Pettis compactness theorem and conclude that the densities of $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ form a relatively compact set in $L^1(\mathbb{R}^k)$ with the weak topology. Since this is true for all k, and we know that $\{\Pi_1(\mu_N)\}_{N\in\mathbb{N}}$ converge weakly (in the measure sense) to μ, with density function $f(v)$, we conclude that for any $\phi \in L^\infty(\mathbb{R})$ we have that

$$\int_{\mathbb{R}} \phi(v) d\Pi_1(\mu_N)(v) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} \phi(v) f(v) dv. \tag{5.16}$$

In particular, since $f \in L^\infty(\mathbb{R})$ and $f \geq 0$ we have that for any $\delta > 0$

$$\int_{\mathbb{R}} \log\left(f(v)+\delta\right) d\Pi_1(\mu_N)(v) \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}} \log\left(f(v)+\delta\right) f(v) dv. \tag{5.17}$$

Combining (5.17), (1.26) with the fact that $\Pi_1(\mu_N)$ converges to $f(v)dv$, we find that if $\left\{\Pi_k\left(\mu_{N_j}\right)\right\}_{j\in\mathbb{N}}$ converges weakly to κ_k, then by (1.25)

$$\frac{H(\kappa_k|f^{\otimes k})}{k} \leq \int_{\mathbb{R}} \log\left(f(v)+\delta\right) f(v) dv - \int_{\mathbb{R}} \log\left(f(v)\right) f(v) dv \tag{5.18}$$

where we have used the fact that $\int_{\mathbb{R}} v^2 d\mu(v) = \int_{\mathbb{R}} v^2 f(v)dv = 1$. Using the dominated convergence theorem to take δ to zero shows that $H(\kappa_k|f^{\otimes k}) = 0$, and so

$$\kappa_k = f^{\otimes k}(v_1, \dots, v_k)\, dv_1 \dots dv_k.$$

Much like $\{\Pi_1(\mu_N)\}_{N\in\mathbb{N}}$, since $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ is tight we can always find weak limits for some subsequences of it. We have just proved that all possible weak limits of subsequences of $\{\Pi_k(\mu_N)\}_{N\in\mathbb{N}}$ are $f^{\otimes k}$, from which we conclude that

$$\Pi_k(\mu_N) \underset{N\to\infty}{\rightharpoonup} f^{\otimes k},$$

showing the chaoticity. It is worth to note that we actually proved more than the above: we have proved convergence in $L^1(\mathbb{R}^k)$ with the weak topology.

Going back to (5.12), and using (1.26), (5.17) and the known limit of $\frac{\log\left(\mathcal{Z}_N\left(f,\sqrt{N}\right)\right)}{N}$ we find that

$$\limsup_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \leq \int_{\mathbb{R}} \log\left(f(v)+\delta\right) f(v)dv + \frac{\log(2\pi)+1}{2}. \tag{5.19}$$

Taking δ to zero we conclude that

$$\limsup_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \leq H(f|\gamma). \tag{5.19}$$

Since the inequality

$$\liminf_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \geq H(f|\gamma)$$

follows from Theorem 5.1, we see that

$$\lim_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} = H(f|\gamma), \tag{5.18}$$

proving the entropic chaoticity and completing the proof. □

The last proof of this section will involve the second 'closeness' criteria, associated with the Fisher information functional, and given by Theorem 1.18. The proof is similar to those appearing in [17] and [9] with appropriate modifications. The proof will rely heavily on tools from the field of Optimal Transportation.

Proof of Theorem 1.18. The first step of the proof will be to show that conditions (1.29) and (1.30) imply that the marginal limit, f, satisfies the conditions of Theorem 1.13.

We start by showing that $f \in L^p(\mathbb{R})$ for some $p > 1$. In [17] the authors have presented a lower semi continuity result for the relative Fisher Information, from which we conclude that

$$I(f|\gamma) \leq \liminf_{N\to\infty} \frac{I_N(\mu_N|\sigma^N)}{N} \leq C. \tag{5.19}$$

Denoting by

$$I(f) = \int_{\mathbb{R}} \frac{|f'(x)|}{f(x)} dx = 4\int_{\mathbb{R}} \left|\frac{d}{dx}\sqrt{f(x)}\right|^2 dx$$

we see that

$$I(f) = I(f|\gamma) + 2 - \int_{\mathbb{R}} v^2 f(v)dv < C + 2 - \int_{\mathbb{R}} v^2 f(v)dv < \infty,$$

as f is a weak limit of $\Pi_1(\mu_N)$, implying that

$$\int_{\mathbb{R}^2} v^2 f(v)dv \leq \liminf_{N\to\infty} \int_{\mathbb{R}} v^2 d\Pi_1(\mu_N)(v) = 1.$$

We conclude that $\sqrt{f} \in H^1(\mathbb{R})$ and using a Sobolev embedding theorem we find that $\sqrt{f} \in L^\infty(\mathbb{R})$. Thus, since f is also in $L^1(\mathbb{R})$, we have that $f \in L^p(\mathbb{R})$ for all $p \geq 1$.
The next step will be to show that condition (1.29) implies a uniform bound for the $1+\alpha$ moment of $\Pi_1(\mu_N)$, i.e.

$$\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\mu_N)(v_1) \leq C, \tag{5.20}$$

for some $C > 0$, independent of N. This will show that

$$\int_{\mathbb{R}} v^2 f(v) dv = \lim_{N\to\infty} \int_{\mathbb{R}} v^2 d\Pi_1(\mu_N)(v) = 1, \tag{5.21}$$

as well as

$$\int_{\mathbb{R}} |v|^{1+\alpha} f(v) dv \leq \liminf_{N\to\infty} \int_{\mathbb{R}} |v|^{1+\alpha} d\Pi_1(\mu_N)(v) \leq C. \tag{5.22}$$

To prove (5.20) we notice that

$$\begin{aligned} &\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi(\mu_N)(v_1) = \frac{3-\alpha}{2^{3-\alpha}-1} \int_{\mathbb{R}} \int_{\frac{|v_1|}{2}}^{|v_1|} x^{\alpha-4} v_1^4 d\Pi_1(\mu_N)(v_1) dx \\ &= \frac{3-\alpha}{2^{3-\alpha}-1} \int_0^\infty x^{\alpha-4} \left(\int_{-2x}^{-x} v_1^4 d\Pi(\mu_N)(v_1) + \int_x^{2x} v_1^4 d\Pi(\mu_N)(v_1) \right) dx \\ &= \frac{3-\alpha}{2^{3-\alpha}-1} \int_0^\infty x^{\alpha-4} \left(\int_{-2x}^{2x} v_1^4 d\Pi(\mu_N)(v_1) - \int_{-x}^{x} v_1^4 d\Pi(\mu_N)(v_1) \right) dx \end{aligned} \tag{5.23}$$

Using condition (1.29) we know that for any $\epsilon > 0$ we can find $R > 0$, such that for any $|x| > R$ and any $N \in \mathbb{N}$

$$(1-\epsilon) C_S x^{2-\alpha} \leq \int_{-\sqrt{x}}^{\sqrt{x}} v_1^4 d\Pi_1(\mu_N)(v_1) \leq (1+\epsilon) C_S x^{2-\alpha} \tag{5.24}$$

In addition, for any probability measure μ on $\mathbb{R}$ we have that

$$\int_{-x}^{x} v^4 d\mu(v) \leq 2x^4. \tag{5.25}$$

Combining (5.23), (5.24) and (5.25) we conclude that

$$\begin{aligned} &\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi(\mu_N)(v_1) \leq \frac{3-\alpha}{2^{3-\alpha}-1} \Bigg(32 R^{\frac{\alpha+1}{2}} \\ &+ C_S \left((1+\epsilon) 2^{4-2\alpha} - (1-\epsilon) \right) \int_{\sqrt{R}}^\infty \frac{dx}{x^\alpha} \Bigg) = C \end{aligned} \tag{5.26}$$

for a choice of $0 < \epsilon < 1$.
Lastly, we want to show that ν_f, defined in Theorem 1.13, satisfies the appropriate growth condition.
Since $\Pi_1(\mu_N)$ converges to f weakly, we have that for any lower semi continuous function, ϕ, that is bounded from below,

$$\int_{\mathbb{R}} \phi(v) f(v) dv \leq \liminf_{N\to\infty} \int_{\mathbb{R}} \phi(v_1) d\Pi_1(\mu_N)(v_1). \tag{5.27}$$

Similarly, if ϕ is upper semi continuous and bounded from above then

$$\int_{\mathbb{R}} \phi(v) f(v) dv \geq \limsup_{N\to\infty} \int_{\mathbb{R}} \phi(v_1) d\Pi_1(\mu_N)(v_1). \tag{5.28}$$

Choosing $\phi(v)=v^4\chi_{(-\sqrt{x},\sqrt{x})}(v)$ and $\phi(v)=v^4\chi_{[-\sqrt{x},\sqrt{x}]}(v)$ respectively, and using condition (1.29) proves that

$$\nu_f(x)=\int_{-\sqrt{x}}^{\sqrt{x}} v^4 f(v)dv \underset{x\to\infty}{\sim} C_S x^{2-\alpha},$$

and we can conclude that f satisfies the conditions of Theorem 1.13. This implies that the function $F_N=\frac{f^{\otimes N}}{\mathcal{Z}_N\left(f,\sqrt{N}\right)}$ is well defined, and as usual we denote $\nu_N=F_N d\sigma^N$.

Next, we will show that $\frac{I_N(\nu_N|\sigma^N)}{N}$ is uniformly bounded in N. Denoting by ∇ the normal gradient on $\mathbb{R}^N$ and by ∇_S its tangential component to Kac's sphere we find that

$$\begin{aligned}
\int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} \frac{\left|\nabla_S F_N\right|^2}{F_N} d\sigma^N &\leq \frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)} \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} \frac{\left|\nabla f^{\otimes N}\right|^2}{f^{\otimes N}} d\sigma^N \\
&= \sum_{i=1}^N \frac{1}{\mathcal{Z}_N\left(f,\sqrt{N}\right)} \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)} \frac{\left|f'(v_i)\right|^2}{f(v_i)} \Pi_{j=1,j\neq i}^N f(v_j) d\sigma^N \qquad (5.29)\\
&= N\int_{\mathbb{R}} \frac{\left|\mathbb{S}^{N-2}\right| \left(N-v_1^2\right)_+^{\frac{N-3}{2}}}{\left|\mathbb{S}^{N-1}\right| N^{\frac{N-2}{2}}} \frac{\mathcal{Z}_{N-1}\left(f,\sqrt{N-v_1^2}\right)}{\mathcal{Z}_N\left(f,\sqrt{N}\right)} \cdot \frac{\left|f'(v_1)\right|^2}{f(v_1)} dv_1,
\end{aligned}$$

where we have used Lemma 2.3, and the definition of the normalisation function. Using the asymptotic behaviour of $\mathcal{Z}_N\left(f,\sqrt{r}\right)$ from Theorem 4.1 we conclude that

$$\begin{aligned}
\frac{I_N(\nu_N|\sigma^N)}{N} &\leq \left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}} \int_{\mathbb{R}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{N^{\frac{1}{\alpha}}}\right)+\lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0)+\lambda_N(N)} \frac{\left|f'(v_1)\right|^2}{f(v_1)} dv_1 \qquad (5.30)\\
&\leq CI(f)\leq C_1,
\end{aligned}$$

for $C_1>0$, independently of N.

At this point we'd like to invoke the HWI inequality, a strategy that was first proved to be successful in this context in [17] and [25]. In our settings we find that

$$\begin{aligned}
H(\mu_N|\sigma^N)-H(\nu_N|\sigma^N) &\leq \frac{\pi}{2}\sqrt{I_N(\mu_N|\sigma_N)}W_2(\mu_N,\nu_N) \\
H(\nu_N|\sigma^N)-H(\mu_N|\sigma^N) &\leq \frac{\pi}{2}\sqrt{I_N(\nu_N|\sigma_N)}W_2(\mu_N,\nu_N),
\end{aligned} \qquad (5.31)$$

where W_2 stands for the quadratic Wasserstein distance with distance function induced from the quadratic distance function on $\mathbb{R}^N$:

$$W_2^2(\mu_N,\nu_N)=\inf_{\pi\in\Pi(\mu_N,\nu_N)} \int_{\mathbb{S}^{N-1}\left(\sqrt{N}\right)\times\mathbb{S}^{N-1}\left(\sqrt{N}\right)} |x-y|^2 d\pi(x,y),$$

where $\Pi(\mu_N,\nu_N)$, the space of pairing, is the space of all probability measures on $\mathbb{S}^{N-1}\left(\sqrt{N}\right)\times\mathbb{S}^{N-1}\left(\sqrt{N}\right)$ with marginal μ_N and ν_N respectively.

The reason we are allowed to use the HWI inequality follows from the fact that Kac's sphere has a positive Ricci curvature. Moreover, in the original statement of the HWI inequality, the quadratic Wasserstein distance is taken with the quadratic *geodesic* distance, yet, fortunately for us, it is equivalent to the normal distance on $\mathbb{R}^N$, hence the factor $\frac{\pi}{2}$ that appears in (5.31). For more information about the Wasserstein distance and the HWI inequality, we refer the interested reader to [30].

Combining (5.31) with the boundness of the rescaled relative Fisher information of μ_N and ν_N with respect to σ^N, we conclude that

$$\left|\frac{H(\mu_N|\sigma^N)}{N}-\frac{H(\nu_N|\sigma^N)}{N}\right| \leq C\frac{W_2(\mu_N,\nu_N)}{\sqrt{N}} \qquad (5.32)$$

for some $C > 0$.
The next step of the proof is to show that the first marginals of μ_N and ν_N have some joint bounded moment of order $l > 2$, uniformly in N. This will help us give a quantitative estimation to the quadratic Wasserstein distance. Indeed, using several results from [17], one can show the following estimation:

$$\frac{W_2(\kappa_N, f^{\otimes N})}{\sqrt{N}} \leq C_1 B_l^{\frac{1}{l}} \left(W_1\left(\Pi_2(\kappa_N), f^{\otimes 2}\right) + \frac{1}{N^{p_1}} \right)^{\frac{1}{2}-\frac{1}{l}} \tag{5.33}$$

where C_1 and p_1 are positive constants that depends only on $l > 2$, κ_N is a probability measure on Kac's sphere, f is a probability measure on $\mathbb{R}$ and

$$B_l = \int_{\mathbb{R}} |v_1|^l d\Pi_1(\kappa_N)(v_1) + \int_{\mathbb{R}} |v_1|^l f(v_1) dv_1 < \infty.$$

We have already shown that $\{\Pi_1(\mu_N)\}_{N\in\mathbb{N}}$ has a uniformly bounded moment of order $1+\alpha$. Using (4.4) from the proof of Theorem 1.13, we find that

$$\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\nu_N)(v_1) = \left(\frac{N}{N-1}\right)^{\frac{1}{\alpha}} \int_{|v_1|\leq\sqrt{N}} \frac{\gamma_{\sigma,\alpha,1}\left(\frac{1-v_1^2}{N^{\frac{1}{\alpha}}}\right) + \lambda_{N-1}\left(N-v_1^2\right)}{\gamma_{\sigma,\alpha,1}(0) + \lambda_N(N)} |v_1|^{1+\alpha} f(v_1) dv_1$$

for some $\sigma > 0$, $1 < \alpha < 2$ and λ_{N-k}, λ_N with

$$\sup_u |\lambda_{N-1}(u)| \underset{N\to\infty}{\longrightarrow} 0, \quad \lambda_N(N) \underset{N\to\infty}{\longrightarrow} 0.$$

Thus, along with (5.22), we conclude that

$$\int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\nu_N)(v_1) \leq C, \tag{5.34}$$

for some $C > 0$.
Defining

$$\begin{aligned} M = \int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\mu_N)(v_1) + \int_{\mathbb{R}} |v_1|^{1+\alpha} d\Pi_1(\nu_N)(v_1) \\ + \int_{\mathbb{R}} |v_1|^{1+\alpha} f(v_1) dv_1 < \infty \end{aligned} \tag{5.35}$$

and combining (5.32), (5.33), and the triangle inequality for the Wasserstein distance, leads us to conclude that

$$\begin{aligned} \left| \frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N} \right| \leq C M^{\frac{1}{1+\alpha}} \Bigg[\left(W_1\left(\Pi_2(\mu_N), f^{\otimes 2}\right) + \frac{1}{N^{p_1}} \right)^{\frac{1}{2}-\frac{1}{1+\alpha}} \\ + \left(W_1\left(\Pi_2(\nu_N), f^{\otimes 2}\right) + \frac{1}{N^{p_1}} \right)^{\frac{1}{2}-\frac{1}{1+\alpha}} \Bigg]. \end{aligned} \tag{5.36}$$

As $\Pi_2(\nu_N), \Pi_2(\nu_N)$ and $f^{\otimes 2}$ all have unit second moment (for any N), the Wasserstein distance is equivalent to weak topology with respect to them. Since $\{\mu_N\}_{N\in\mathbb{N}}$ and $\{\nu_N\}_{N\in\mathbb{N}}$ are $f-$chaotic, we conclude that

$$W_1\left(\Pi_2(\mu_N), f^{\otimes 2}\right) \underset{N\to\infty}{\longrightarrow} 0, \quad W_1\left(\Pi_2(\nu_N), f^{\otimes 2}\right) \underset{N\to\infty}{\longrightarrow} 0,$$

implying that

$$\lim_{N\to\infty} \left| \frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N} \right| = 0. \tag{5.37}$$

We are almost ready to conclude the proof. Before we do, we use the lower semi continuity of the entropy, discussed in Theorem 5.1, to see that

$$H(f|\gamma) \leq \liminf_{N\to\infty} \frac{H_N(\mu_N|\sigma^N)}{N} \leq C < \infty.$$

Thus,

$$\begin{aligned} \left| \frac{H(\mu_N|\sigma^N)}{N} - H(f|\gamma) \right| &\leq \left| \frac{H(\mu_N|\sigma^N)}{N} - \frac{H(\nu_N|\sigma^N)}{N} \right| \\ &+ \left| \frac{H(\nu_N|\sigma^N)}{N} - H(f|\gamma) \right| \underset{N\to\infty}{\longrightarrow} 0, \end{aligned} \tag{5.38}$$

where we have used (5.37) and Theorem 1.13, completing proof. □

Remark 5.2. *We'd like to point out that following the above proof, one can see that condition (1.29), giving us a uniform asymptotic behaviour for the fourth moments of the first marginals of* $\{\mu_N\}_{N\in\mathbb{N}}$*, can be replaced with the conditions that* f *satisfies the conditions of Theorem 1.13, and the first marginals of* $\{\mu_N\}_{N\in\mathbb{N}}$ *have a uniformly bounded* $k-$*th moment, for some* $k > 2$*. This gives us a different approach to the stability problem, expressed with the Fisher information functional, one that assumes less information on the first marginals, but more conditions on the marginal limit.*

6 Connections to the Trend to Equilibrium in Kac's Model and Cercignani's Conjecture.

The study of the conditioned tensorisation of a function f is closely related to the problem of finding the rate of convergence to equilibrium in Kac's Model. In this section we will outline some of the history, and recent results, dealing with this subject.
Recall that Kac's model have managed to show validity (in some sense) for the spatially homogenous Boltzmann equation. Kac hoped to use his simple model to infer quantitative rate of convergence to equilibrium in Boltzmann equation, as a limit of his 'master' equation. He started by noticing that his evolution equation, (1.1), is ergodic, with an equilibrium state represented by the constant function 1. As such, for any fixed N, one can easily see that

$$\lim_{t\to\infty} F_N(t, v_1, \dots, v_N) = 1,$$

for any solution to Kac's equation, $F_N(t, v_1, \dots, v_N)$. The rate of convergence to equilibrium under the L^2 norm is determined by the spectral gap

$$\Delta_N = \inf \left\{ \frac{\langle \varphi, N(I-Q)\varphi \rangle_{L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right)}}{\|\varphi\|^2_{L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right)}} \mid \varphi \text{ is symmetric }, \varphi \in L^2\left(\mathbb{S}^{N-1}\left(\sqrt{N}\right)\right), \ \varphi \perp 1 \right\}.$$

Kac's conjectured that

$$\Delta = \liminf_{N\to\infty} \Delta_N > 0,$$

resulting in an exponential convergence to equilibrium, for any fixed N, with a rate that is independent with the number of particles.
The spectral gap problem remained open until 2000, when a series of papers by authors such as Janversse, Maslen, Carlen, Carvahlo, Loss and Geronimo gave a satisfactory positive answer to the conjecture, even in McKean's model (see [18, 22, 5, 8] for more details). However, the L^2 norm is catastrophic when dealing with chaotic families and in this setting attempts to pass to the limit in the number of particles is futile. Indeed, one can easily find a chaotic family, $\{F_N\}_{N\in\mathbb{N}}$, such that

$$\|F_N\|_{L^2} \geq C^N,$$

where $C > 1$ (for example, the conditioned tensorisation of a function f we discussed so extensively in this paper!).
As was mentioned in the introduction, the next 'distance' to be considered was the entropy

$$H_N(F_N) = \int_{\mathbb{S}^{N-1}(\sqrt{N})} F_N \log F_N d\sigma^N.$$

In an attempt to imitate the idea behind the spectral gap method, one can define the *entropy production* to be the minus of the formal derivation of the entropy under Kac's evolution equation

$$D_N(F_N) = -\frac{d}{dt}H_N(F_N) = \langle \log F_N, N(I-Q)F_N \rangle_{L^2(\mathbb{S}^{N-1}(\sqrt{N}))}, \tag{6.1}$$

The appropriate 'spectral gap' will be given by

$$\Gamma_N = \inf_{F_N} \frac{D_N(F_N)}{H_N(F_N)}$$

and the appropriate conjecture is: Can one find a positive constant, $C > 0$, such that

$$\Gamma_N \geq C$$

for all N. This problem is called *Cercignani's many body conjecture*, named after a similar conjecture posed for the real Boltzmann equation in [10]. If there exists such a C, we have that

$$H_N(F_N(t)) \leq e^{-Ct} H_N(F_N(0))$$

then, hoping the entropic chaoticity propagates with time, one can divide by N and take a limit as N to find that

$$H(f_t|\gamma) \leq e^{-Ct} H(f_0|\gamma). \tag{6.2}$$

This, along with a known inequality on $H(f|\gamma)$ gives an exponential rate of decay towards the equilibrium.
Unfortunately, in general, Cercignani's many body conjecture is false. The first hint to it was revealed in [29] where Villani managed to prove that

$$\Gamma_N \geq \frac{2}{N-1}.$$

Villani has conjectured that $\Gamma_N = O\left(\frac{1}{N}\right)$, which was proven to be essentially true by the second author. In [12] (and later on in [13] for McKean's model) the second author extended the *normal* local central limit theorem, Theorem 1.5, to the case where the underlying generating function, f, also varies with N. In particular, the second author showed that:

Theorem 6.1. *Let* $0 < \eta < 1$ *and* $\delta_N = \frac{1}{N^\eta}$. *Define*

$$f_N(v) = \delta_N M_{\frac{1}{2\delta_N}}(v) + (1-\delta_N) M_{\frac{1}{2(1-\delta_N)}}(v),$$

where $M_a(v) = \frac{e^{-\frac{v^2}{2a}}}{\sqrt{2\pi a}}$. *Then*

$$\mathcal{Z}_N(f,\sqrt{u}) = \frac{2}{\sqrt{N}\Sigma_N \left|\mathbb{S}^{N-1}\right| u^{\frac{N-2}{2}}} \left(\frac{e^{-\frac{(u-N)^2}{2N\Sigma_N^2}}}{\sqrt{2\pi}} + \lambda_N(u) \right), \tag{6.3}$$

where $\Sigma^2 = \frac{3}{4\delta_N(1-\delta_N)} - 1$ *and* $\sup_u |\lambda_N(u)| \underset{N\to\infty}{\longrightarrow} 0$*. Moreover, using the same notation as (1.9) with* f *replaced by* f_N*, one finds that there exists* $C_{\eta'} > 0$*, depending only on* η' *such that*

$$\Gamma_N \leq \frac{D_N(F_N)}{H_N(F_N)} < \frac{C_{\eta'}}{N^{\eta'}} \tag{6.4}$$

for $0 < \eta' < \eta$*.*

The above theorem poses an interesting insight: The family constructed in Theorem 6.1 has two peculiar properties:

(i)
$$\int_{\mathbb{R}} v^4 f_N(v)dv = \frac{3}{4\delta_N(1-\delta_N)} \underset{N\to\infty}{\longrightarrow} \infty$$
so the fourth moment condition unbounded in some sense in this example.

(ii) F_N is $M_{\frac{1}{2}}-$chaotic yet $\lim_{N\to\infty} \frac{H_N(F_N)}{N}$ exists but doesn't equal $H\left(M_{\frac{1}{2}}|\gamma\right)$!

This insinuates that moments of the limit function, as well as entropic chaoticity, may be very important to the validity of Cercignani's many body conjecture. This is one reason that prompted us to try and investigate the conditioned tensorisation of a function f that has an unbounded fourth moment. While we have attained some answers, we believe that there is much more that can be discovered.

7 Final Remarks.

While Kac's model, chaoticity and entropic chaoticity, and Cercignani's many body conjecture are far from being completely understood and resolved, we hope that our paper has shed some light on the interplay between the moments of a generating function and its associated tensorised measure, restricted to Kac's sphere. As an epilogue, we present here a few remarks about our work, along with associated questions we'll be interested in investigating next.

- One fundamental problem we're very interested in is finding conditions under which Cercignani's many body conjecture is valid. While our work showed that the requirement of a bounded fourth moment is not a major issue for chaoticity and even entropic chaoticity, we still believe that the fourth moment plays an important role in the conjecture. At the very least, due to its probabilistic interpretation as a measurement of deviation from the sphere, we believe that the fourth moment will be needed for an initial positive answer to the conjecture.
- The following was communicated to us by Clément Mouhot: Using a Talagrand inequality, one can show that if the family of functions $\{G_N\}_{N\in\mathbb{N}}$, restricted to the sphere, satisfies a Log-Sobolev inequality that is uniform in N, one has that
$$\lim_{N\to\infty} \frac{H(F_N|G_N)}{N} = 0$$
implies that $\lim_{N\to\infty} (\Pi_k(F_N) - \Pi_k(G_N)) = 0$. Our stability result, Theorem 1.16, gives many examples where the function G_N doesn't satisfy any Log-Sobolev inequality (due to how the underlying function behaves), but we still get equality of marginal. Moreover, we actually get that F_N is entropically chaotic! The connection between the limit of the 'distance'
$$d(F_N, G_N) = \frac{H(F_N|G_N)}{N}$$
and the convergence of marginals is still not understood fully.

- We'll be interested to know if one can find an easy criteria for which we can evaluate quantitatively the convergence of h^{*N} (appearing in Theorem 4.1) without relying on the reminder function. This will allow for possibilities to extend the work done by the second author in [12, 13] and allow the underlying generating function, f, to rely on N as well. While we present such quantitative estimation in the Appendix, we found them to be unusable while trying to deal with concrete examples.

A Additional Proofs.

In this section of the appendix we will present several proofs of technical items we thought would only hinder the flow of the paper.

Proof of Lemma 3.2. Assume that the conclusion is false. We can find a sequence $x_n \underset{n\to\infty}{\longrightarrow} 0$, $x_n \neq 0$, and an $\epsilon_0 > 0$ such that

$$|g(x_n)| \geq \epsilon_0.$$

Due to continuity, we can find $d_1 > 0$ such that for any $x \in [x_1, x_1 + d_1]$ we have

$$|g(x)| \geq \frac{\epsilon_0}{2}.$$

Denote $n_1 = 1$, $x_{k_1} = x_1$ and $\xi_1 = n_1 \cdot x_1 = x_1$.
Since x_n converges to zero and is non zero, we can find x_{k_2} such that $0 < x_{k_2} < \frac{\xi_1}{2}$. Let $n_2 = \left[\frac{\xi_1}{x_{k_2}}\right] + 1 \geq 2$, where $[\cdot]$ is the lower integer part function. We may assume that $x_{k_2} < d_1$ and conclude that

$$\xi_1 \leq n_2 x_{k_2} < \xi_1 + x_{k_2} \leq \xi_1 + n_1 d_1.$$

Next, we can find d_2 such that $n_2(x_{k_2} + d_2) \leq \xi_1 + n_1 d_1$. We may also assume that d_2 is small enough so that $x \in [x_{k_2}, x_{k_2} + d_2]$ implies

$$|g(x)| \geq \frac{\epsilon_0}{2}.$$

Denoting by $\xi_2 = n_2 x_{k_2}$, we notice that $[\xi_2, \xi_2 + n_2 d_2] \subset [\xi_1, \xi_1 + n_1 d_1]$ and the closed intervals are non empty.
We continue by induction. Assume we found $n_i, k_i \in \mathbb{N}$, $n_i \geq i$, and $d_i > 0$ for $i = 1, \dots, j$ such that $\xi_i = n_i x_{k_i}$ satisfies

$$[\xi_i, \xi_i + n_i d_i] \subset [\xi_{i-1}, \xi_{i-1} + n_{i-1} d_{i-1}]$$

and for any $x \in [\xi_i, \xi_i + n_i d_i]$ we have that

$$\left| g\left(\frac{x}{n_i}\right) \right| \geq \frac{\epsilon_0}{2}.$$

We find $x_{k_{j+1}}$ such that $x_{k_{j+1}} < \frac{\xi_j}{j+1}$ and define $n_j = \left[\frac{\xi_j}{x_{k_{j+1}}}\right] + 1 \geq j + 1$. As such, we have that

$$\xi_j \leq n_{j+1} x_{k_{j+1}} < \xi_j + x_{k_{j+1}} < \xi_j + n_j d_j,$$

where the last inequality is valid since we can pick $x_{k_{j+1}} < n_j d_j$. We can find d_{j+1} such that $n_{j+1}(x_{k_{j+1}} + d_{j+1}) < \xi_j + n_j d_j$ and for any $x \in [x_{k_{j+1}}, x_{k_{j+1}} + d_{j+1}]$

$$|g(x)| \geq \frac{\epsilon_0}{2}.$$

Denoting $\xi_{j+1} = n_{j+1}x_{k_{j+1}}$ gives us the interval with the desired properties.
Since we have a nested sequence of non-empty closed intervals in $\mathbb{R}$ we know that the intersection of all of them must be non-empty. Thus, there exists $x \in [\xi_i, \xi_i + n_i d_i]$ for all $i \in \mathbb{N}$. Moreover, by construction

$$\left| g\left(\frac{x}{n_i}\right)\right| \geq \frac{\epsilon_0}{2}$$

which contradicts the assumption that $\lim_{n\to\infty} g\left(\frac{x}{n}\right) = 0$ for any $x \neq 0$. □

The next result we will prove, is Lemma 3.8:

Proof of Lemma 3.8. Since $\widehat{g}$ is in the NDA of $\gamma_{\sigma,\alpha,\beta}$ we conclude that $\widehat{g}$ is actually in the FDA of $\gamma_{\sigma,\alpha,\beta}$, due to Theorem 3.1. Thus, there exists η_1, with $\frac{\eta_1(\xi)}{|\xi|^\alpha} \in L^\infty(\mathbb{R})$ and

$$\frac{\eta_1(\xi)}{|\xi|^\alpha} \underset{\xi\to 0}{\longrightarrow} 0,$$

such that

$$\widehat{g}(\xi) = 1 - \sigma\left|\xi\right|^\alpha \left(1 + i\beta \mathrm{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right) + \eta_1(\xi)$$
$$= e^{-\sigma|\xi|^\alpha\left(1+i\beta\mathrm{sgn}(\xi)\tan\left(\frac{\pi\alpha}{2}\right)\right)} + \eta_2(\xi) + \eta_1(\xi),$$

where $\eta_2(\xi)$ has the same properties as $\eta_1(\xi)$. We conclude that

$$|\widehat{g}(\xi)| \leq e^{-\sigma|\xi|^\alpha} + |\eta_1(\xi) + \eta_2(\xi)| \leq 1 - \sigma\left|\xi\right|^\alpha + |\eta_1(\xi)| + |\eta_2(\xi)| + |\eta_3(\xi)|\,,$$

where $\eta_3(\xi)$ has the same properties as $\eta_1(\xi)$.
Let $\beta_0 > 0$ be such that if $|\xi| < \beta_0$

$$|\eta_1(\xi)| + |\eta_2(\xi)| + |\eta_3(\xi)| \leq \frac{\sigma\left|\xi\right|^\alpha}{2}.$$

For any $|\xi| < \beta_0$ one has that

$$|\widehat{g}(\xi)| \leq 1 - \frac{\sigma\left|\xi\right|^\alpha}{2} \leq e^{-\frac{\sigma|\xi|^\alpha}{2}},$$

completing the proof. □

B Quantitative Approximation Theorem.

An item of great importance in Kinetic Theory, and our problem in particular, is *quantitative* estimation of errors. Our local Lévy Central Limit Theorem involves such an estimation, yet it is dependent on the function

$$\omega(\beta) = \sup_{|\xi|} \frac{|\eta(\xi)|}{|\xi|^\alpha},$$

where η is the reminder function of a probability density function g in the NDA of some $\gamma_{\sigma,\alpha,\beta}$. In some cases one can find explicit estimation for the behaviour of η near zero, and get a better quantitative estimation on the error term $\epsilon(N)$. Such conditions are explored in [16] and we will satisfy ourselves by mentioning them, but providing no proof.

Definition B.1. *Let $\delta > 0$. The Fourier Domain of Attraction* of order δ *of $\gamma_{\sigma,\alpha,\beta}$ is the subset of the FDA of $\gamma_{\sigma,\alpha,\beta}$ such that the reminder function, η, satisfies*

$$\frac{|\eta(\xi)|}{|\xi|^{\alpha}} \leq C\,|\xi|^{\delta},$$

for some $C > 0$.

Clearly the FDAs of order δ are nested sets, all contained in the FDA. Also, if g is in the FDA of order δ of $\gamma_{\sigma,\alpha,\beta}$ then we can replace $\omega(\beta)$, defined in Theorem 1.12 by $C\beta^{\delta}$ and get an explicit estimation to the error term $\epsilon(N)$!.
The following is a variant of a theorem appearing in [16] that gives sufficient conditions to be in the FDA of order δ of some $\gamma_{\sigma,\alpha,\beta}$:

Theorem B.2. *Let g be a probability density on $\mathbb{R}$ that has zero mean. Let $1 < \alpha < 2$ and $0 < \delta < 2 - \alpha$ be given. Then if*

$$\int_{\mathbb{R}} |x|^{\alpha+\delta}\,|g(x) - \gamma_{\sigma,\alpha,\beta}(x)|\,dx < \infty \tag{B.1}$$

for some $\sigma > 0$ and $\beta \in [-1,1]$, g is in the FDA of order δ of $\gamma_{\sigma,\alpha,\beta}$.

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