Clustering and percolation of point processes

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Abstract

We are interested in phase transitions in certain percolation models on point processes and their dependence on clustering properties of the point processes. We show that point processes with smaller void probabilities and factorial moment measures than the stationary Poisson point process exhibit non-trivial phase transition in the percolation of some coverage models based on level-sets of additive functionals of the point process. Examples of such point processes are determinantal point processes, some perturbed lattices, and more generally, negatively associated point processes. Examples of such coverage models are k-coverage in the Boolean model (coverage by at least k grains) and SINR-coverage (coverage if the signal-to-interference-and-noise ratio is large). In particular, we prove the existence of the phase transition in the percolation of a spherical Boolean model depending on the grain radius (and, more generally, k-faces in the Čech simplicial complex, also called clique percolation) on point processes which cluster less than the Poisson process, including determinantal point processes.

We also construct a Cox point process, which is "more clustered" than the Poisson point process and whose Boolean model percolates for arbitrarily small radius. This shows that clustering (at least, as detected by our specific tools) does not always "worsen" percolation.

1 Introduction

Starting with the work of [19], percolation problems on geometric models defined over the Poisson point process have garnered interest among both stochastic geometers and network theorists. Relying and building upon the (on-going) successful study of its discrete counterpart ([20]), continuum percolation has also received considerable attention as evinced in the monographs [27] and [29]. Our work shall deviate from the

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standard approach of studying geometric models based on the Poisson point process, by focusing on percolation models defined over general stationary point processes. In particular, we shall try to *formalize the comparison of "clustering" phenomena in point processes and investigate its impact on percolation models.*

Specifically, we relate percolation properites of Boolean models and, more generally, level-sets of additive functionals of point processes, to the more intrinsic properties of the underlying point processes, such as *moment measures* and *void probabilities*. It has already been observed in [8] that smaller values of these characteristics indicate less clustering. In this paper we show that point processes having voids probabilities and moment measures smaller than a (homogeneous) Poisson point process (we call them *weakly sub-Poisson*), exhibit a *non-trivial phase transition in the percolation* of their level-set coverage models.

Perhaps a most interesting class of point processes which is amenable to our analysis are *determinantal point processes*. Indeed, using the representation of the number of points of a determinantal process as the sum of independent Bernoulli variables and using the Hadamard's theorem on determinants it can be checked that they are weakly sub-Poisson, cf [8]. Hence, in particular, the general results presented in the present paper lead to a proof of the non-triviality of the critical radius for the percolation of the spherical Boolean model on determinantal processes. Moreover, the obtained upper and lower bounds on the critical radius are the same for all determinantal (and all weakly sub-Poisson) processes of a given intensity.

The new approach contributes to both theory and applications. Results regarding percolation on "clustered" or "repulsive" point processes are scarce (we shall say more in Section 1.5). Our methods help in a more systematic study of models over general stationary point processes apart from the ubiquitous Poisson point process. As regards applications, we observe that the Poisson assumption on point processes is not always preferable in many models of spatial networks. In such scenarios, the question of whether clustering or repulsion increases various performance measures (related to some property of the underlying geometric graph) or not arises naturally.

Percolation is one such performance measure naturally related to network connectivity. In this paper we will study also percolation of a more specific coverage model, which arose in the context of wireless communication networks.

We refer the reader to the upcoming survey [6, Section 4] for a more comprehensive overview of applications of our methods for comparison of various other properties of geometric graphs.

With this succinct introduction, we shall now get into our key percolation model. As a sample of our results, we will present here in detail the particular case of k-percolation alone with a brief discussion of other results. The point process notions used in the rest of the section and the paper are described formally in Section 2.1.

1.1 k-percolation

By *percolation of a set* (in the Euclidean space usually), we mean that the set contains an unbounded connected subset. We shall also use *percolation of a graph*, where it means existence of an infinite connected subgraph. By k-percolation in a Boolean model, we understand percolation of the subset of the space covered by at least k grains of the Boolean model and more rigorously, we define as follows.

Definition 1.1 (k-percolation). *Let Φ be a point process in $\mathbb{R}^d$, the d-dimensional Euclidean space. For $r \geq 0$ and $k \geq 1$, we define the coverage number field $V_{\Phi,r}(y) := \sum_{X_i \in \Phi} 1[y \in B_r(X_i)]$, where $B_r(x)$ denotes the Euclidean ball of radius r centred at x.*

The k-covered set is defined as

$$C_k(\Phi, r) := \{y : V_{\Phi,r}(y) \geq k\}.$$

Define the critical radius for k-percolation *as*

$$r_c^k(\Phi) := \inf\{r : \mathsf{P}\left(C_k(\Phi, r) \text{ percolates}\right) > 0\},$$

where, as before, percolation means existence of an unbounded connected subset.

Note that $C(\Phi, r) := C_1(\Phi, r)$ is the standard Boolean model or continuum percolation model mentioned in the first paragraph. It can also be viewed as a graph by taking Φ as the vertex set and the edge-set being $E(\Phi, r) := \{(X, Y) \in \Phi^2 : 0 < |X - Y| \leq 2r\}$. This is the usual random geometric graph, called also the Gilbert's disk graph. The two notions of percolation (graph-theoretical and topological) are the same in this case. The more general set $C_k(\Phi, r)$ can also be viewed as a graph on k-faces of the Čech complex on Φ and this shall be explained later in Remark 3.8. Clearly, $r_c(\Phi) := r_c^1(\Phi)$ is the *critical radius* of the "usual" continuum percolation model on Φ, and we have $r_c(\Phi) \leq r_c^k(\Phi)$. We shall use clustering properties of Φ to get bounds on $r_c^k(\Phi)$.

1.2 Clustering and percolation — heuristics

Before stating our percolation result, let us discuss more about clustering and percolation heuristics. Clustering of Φ roughly means that the points of Φ lie in clusters (groups) with the clusters being well spaced out. When trying to find the minimal r for which the continuum percolation model $C(\Phi, r)$ percolates, we observe that points lying in the same cluster of Φ will be connected by edges for some smaller r but points in different clusters need a relatively higher r for having edges between them. Moreover, percolation cannot be achieved without edges between some points of different clusters. It seems to be evident that spreading points from clusters of Φ "more homogeneously" in the space would result in a decrease of the radius r for which the percolation takes place. This is a heuristic explanation why clustering in a point process Φ should increase the critical radius $r_c(\Phi)$ and a similar reasoning can also be given for $r_c^k(\Phi)$. Our study was motivated by this heuristic.

To make a formal statement of the above heuristic, one needs to adopt a tool to compare clustering properties of point processes[1]. In this regard, our initial choice was *directionally convex* (dcx) order [2] (to be formally defined later). It has its roots in [5], where one shows various results as well as examples indicating that the dcx order on point processes implies ordering of several well-known clustering characteristics in spatial statistics such as Ripley's K-function and second moment densities. Namely, a point process that is larger in the dcx order exhibits more clustering, while having equal mean number of points in any given set.

In view of what has been said above, a possible formalization of the heuristic developed above, that clustering "worsens" percolation, would be $\Phi_1 \leq_{dcx} \Phi_2$ implies $r_c(\Phi_1) \leq r_c(\Phi_2)$. The numerical evidences gathered for a certain class of point processes, called perturbed lattice point processes, were supportive of this conjecture (cf [8]). However, such a statement is not true in full generality and we shall provide a counterexample in Section 4.

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[1]The reader should also keep in mind that we are interested in looking at point processes of same intensities and hence the usual strong order (possibility of coupling as a subset) is not a suitable comparison.

[2]The dcx order of random vectors is an integral order generated by twice differentiable functions with all their second order partial derivatives being non-negative. Its extension to point processes consists in comparison of vectors of number of points in every possible finite collection of bounded Borel subsets of the space.

Trying to make the whole approach more general, one considers some weaker notions of clustering, for which only moment measures or void probabilities need to be compared. Again, smaller values of these characteristics suggest less clustering. When comparing to Poisson point process, this approach lead in [8] to the following definition.

Definition 1.2. *A point process Φ is said to be* weakly sub-Poisson *if the following two conditions are satisfied:*

$$\mathsf{P}\left(\Phi(B) = 0\right) \quad \leq \quad e^{-\mathsf{E}(\Phi(B))} \qquad \textbf{\textit{(ν-weakly sub-Poisson)}} \tag{1.1}$$

$$\mathsf{E}\left(\prod_{i=1}^{k}\Phi(B_i)\right) \quad \leq \quad \prod_{i=1}^{k}\mathsf{E}(\Phi(B_i)) \qquad \textbf{\textit{(α-weakly sub-Poisson)}} \tag{1.2}$$

where $B_i \subset \mathbb{R}^d$ are pairwise disjoint bBs (bounded Borel subsets) and B is any bBs. If only either of the conditions is satisfied, accordingly we call the point process to be ν-weakly sub-Poisson (ν stands for void probabilities) or α-weakly sub-Poisson (α stands for moment measures). Similar notions of super-Poissonianity can be defined by reversing the inequalities.

As already mentioned in the Introduction, determinantal point processes are weakly sub-Poisson. Similarly, permanental processes are weakly super-Poisson. More details on these notions are provided in Section 2 and examples are given in Section 2.4. Let us mention here only that this comparison is weaker than dcx ordering with respect to Poisson process and association as well.

It is explained in [8], that weak sub-Poissonianity allows for a slightly different conclusion than the heuristic described before the definition. Suppose that Φ_2 is a stationary Poisson point process and Φ_1 is a stationary weakly sub-Poisson point process on d-dimensional Euclidean space $\mathbb{R}^d$, with $d \geq 2$. Then, Φ_1 exhibits the usual phase transition $0 < r_c(\Phi_1) < \infty$, provided Φ_2 exhibits a (potentially) stronger, "double phase transition": $0 < \underline{r}_c(\Phi_2)$ and $\overline{r}_c(\Phi_2) < \infty$, where $\underline{r}_c(\Phi_2)$ and $\overline{r}_c(\Phi_2)$ are some "nonstandard" critical radii [3] sandwiching $r_c(\Phi_2)$, and exhibiting opposite monotonicity with respect to clustering:

$$\underline{r}_c(\Phi_2) \leq \underline{r}_c(\Phi_1) \leq r_c(\Phi_1) \leq \overline{r}_c(\Phi_1) \leq \overline{r}_c(\Phi_2) \, .$$

Conjecturing that the above "double phase transition" holds for the Poisson point process Φ_2, one obtains the result on the "usual" phase transition for $r(\Phi_1)$ for all weakly sub-Poisson point processes Φ_1. From [8, Proposition 6.1], we know that $\underline{r}_c(\Phi_2) \geq (\lambda\theta_d)^{-\frac{1}{d}}$, where λ is the intensity of the point process and θ_d is the volume of the unit ball in $\mathbb{R}^d$. However, finiteness of $\overline{r}_c(\Phi_2)$ is not clear and hence, in this paper we will prove the results in a slightly different way. The generality of this method shall be explained below and will be obvious from the results in Section 3.

1.3 Results

We can finally state now one of the main results on phase transition in k-percolation which follows as a corollary from Theorem 3.7 (to be stated later). Recall the definition of weak sub-Poissonianity from Definition 1.2.

Corollary 1.3. *Let Φ be a simple, stationary, weakly sub-Poisson point process of intensity λ on $\mathbb{R}^d$, with $d \geq 2$. For $k \geq 1$, $\lambda > 0$, there exist constants $c(\lambda)$ and $c(\lambda, k)$ (not depending on the distribution of Φ) such that*

$$0 < c(\lambda) \leq r_c^1(\Phi) \leq r_c^k(\Phi) \leq c(\lambda, k) < \infty.$$

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[3]$\underline{r}_c(\Phi_2)$ and $\overline{r}_c(\Phi_2)$ are critical radii related, respectively, to the finiteness of asymptotic of the expected number of long occupied paths from the origin and the expected number of void circuits around the origin in suitable discrete approximations of the continuum model.

Remark 1.4. *The above result not only shows non-triviality of critical radius for stationary weakly sub-Poisson processes but also provides uniform bounds. Examples of particular point processes for which this non-triviality result holds are* determinantal point processes with trace-class integral kernels *and* perturbed latices with convexly sub-Poisson replication kernels *(see Section 2.4 and [8]).*

Random field $V_{\Phi,r}(\cdot)$ introduced in Definition 1.1 is but one example of *an additive shot-noise field* (to be defined in Section 3.1). Replacing the indicator function in the definition of $V_{\Phi,r}(\cdot)$ by more general functions of X_i, one can get more general additive shot-noise fields. $C_k(\Phi, r)$ is an example of an excursion set or a level-set (i.e, sets of the form $\{V_{\Phi,r}(\cdot) \geq h\}$ or its complement for some $h \in \mathbb{R}$) that can be associated with the random field $V_{\Phi,r}(\cdot)$. In this paper we shall develop more general methods suitable for the study of percolation of level sets of additive shot-noise fields. Besides $C_k(\Phi, r)$, we shall apply these methods to study percolation of SINR coverage model (coverage by a signal-to-interference-and-noise ratio; [1, 13, 14]) in non-Poisson setting. Though, we do not discuss it here, our methods can also be used to show non-trivial phase transition in the continuum analogue of word percolation (cf.[24]). The details can be found in [31, Section 6.3.3].

1.4 Paper organization

The necessary notions, notations as well as some preliminary results are introduced and recalled in Section 2. In Section 3 we state and prove our main results regarding the existence of the phase transition for percolation models driven by sub-Poisson point processes. A Cox point process, which is *dcx* larger than the Poisson point process (clusters more) and whose Boolean model percolates for arbitrarily small radius ($r_c = 0$) is provided in the Section 4.

1.5 Related work

Let us first remark on studies in continuum percolation which are comparisons of different models driven by the same (usually Poisson) point process. In [21], it was shown that the critical intensity for percolation of the Poisson Boolean model on the plane is minimized when the shape of the typical grain is a triangle and maximized when it is a centrally symmetric set. Similar result was proved in [30] using more probabilistic arguments for the case when the shapes are taken over the set of all polygons and the idea was also used for three dimensional Poisson Boolean models. It is known for many discrete graphs that bond percolation is strictly easier than site percolation. A similar result as well as strict inequalities for spread-out connections in the Poisson random connection model has been proved in [15, 16].

For determinantal point processes, [17, Cor. 3.5] shows non-existence of percolation for small enough integral kernels (or equivalently for small enough radii) via coupling with a Poisson point process. This shows non-zero critical radius ($r_c > 0$) for percolation of determinantal point processes. Our result (Corollary 1.3) goes beyond this result and proves non-degeneracy (non-zero and finite) of critical radius for k-percolation in determinantal point processes.

Non-trivial critical radius for continuum percolation on point processes representing zeros of Gaussian analytic functions, is shown in [18]. These processes are reputed to cluster less than Poisson processes, however currently we are not able to make this comparison formal using our tools. Also, [18] shows uniqueness of infinite clusters for both zeros of Gaussian analytic functions and the Ginibre point process (a special case of determinantal process).

Critical radius of the continuum percolation model on the hexagonal lattice per-

turbed by the Brownian motion is studied in a recent pre-print [4].This is an example of our perturbed lattice and as such it is a dcx sub-Poisson point process.[4] It is shown that for a short enough time of the evolution of the Brownian motion the critical radius is not larger than that of the non-perturbed lattice. This result is shown by some coupling in the sense of set inclusion of point processes.

Many other inequalities in percolation theory depend on such coupling arguments (cf. e.g. [26]), which for obvious reasons are not suited to comparison of point processes with the same mean measures.

2 Notions, notation and basic observations

2.1 Point processes

Let B^d be the Borel σ-algebra and B_b^d be the σ-ring of *bounded (i.e., of compact closure) Borel subsets* (bBs) in the d-dimensional Euclidean space $\mathbb{R}^d$. Let $\mathbf{N}^d = \mathbf{N}(\mathbb{R}^d)$ be the space of non-negative Radon (i.e., finite on bounded sets) counting measures on $\mathbb{R}^d$. The Borel σ-algebra $\mathcal{N}^d$ is generated by the mappings $\mu \mapsto \mu(B)$ for all B bBs. A point process Φ on $\mathbb{R}^d$ is a random element in $(\mathbf{N}^d, \mathcal{N}^d)$ i.e, a measurable map from a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ to $(\mathbf{N}^d, \mathcal{N}^d)$. Further, we shall say that a point process Φ is simple if a.s. $\Phi(\{x\}) \leq 1$ for all $x \in \mathbb{R}^d$. As always, a point process on $\mathbb{R}^d$ is said to be *stationary* if its distribution is invariant with respect to translation by vectors in $\mathbb{R}^d$.

Donote by $\nu(\cdot)$ the void probabilities of Φ; $\nu(B) = \mathsf{P}\left(\Phi(B) = 0\right)$ for bBs B. The measure $\alpha^k(\cdot)$ defined by $\alpha^k(B_1 \times \ldots \times B_k) = \mathsf{E}\left(\prod_{i=1}^{k} \Phi(B_i)\right)$ for all (not necessarily disjoint) bBs B_i $(i = 1, \ldots, k)$ is called k *th order moment measure* of Φ. For simple point processes, the truncation of the measure $\alpha^k(\cdot)$ to the subset $\{(x_1, \ldots, x_k) \in (\mathbb{R}^d)^k : x_i \neq x_j, \text{ for } i \neq j\}$ is equal to the k *th order factorial moment measure* $\alpha^{(k)}(\cdot)$. To explicitly denote the dependence on Φ, we shall sometimes write as α_Φ^k for the moment measures and similarly for factorial moment measures. This is the standard framework for point processes and more generally, random measures (see [23]).

We shall be very brief in our introduction to stochastic ordering of point processes necessary for our purposes. We shall start with a discussion on weakly sub-Poisson point processes defined in Definition 1.2.

2.2 Weak sub- and super-Poisson point processes

Note that, **α-weakly sub-Poisson (super-Poisson)** point process have factorial moment measures $\alpha^{(k)}(\cdot)$ smaller (respectively larger) than those of the Poisson point process of the same mean measure; inequalities hold everywhere provided the point processes is simple and "off the diagonals" otherwise. Recall also that moment measures $\alpha^k(\cdot)$ of a general point process can be expressed as non-negative combinations of products of its (lower-dimensional) factorial moment measures (cf [11, Ex. 5.4.5, p. 143]) i.e.,

$$\alpha^k(\mathrm{d}x_1 \times \ldots \times \mathrm{d}x_k) = \sum_{j=1}^{k} \sum_{\mathcal{V}} \alpha^{(j)}(\mathrm{d}y_1(\mathcal{V}) \times \ldots \times \mathrm{d}y_j(\mathcal{V}))\delta(\mathcal{V}), \qquad (2.1)$$

where the inner sum is taken over all partitions of k coordinates into j non-empty subsets, the $y_i(\mathcal{V})$ constitute one coordinate chosen arbitrarily among the coordinates in each of the j non-empty subsets and $\delta(\mathcal{V})$ is a δ-function that equals zero unless there is equality among all the coordinatees in the each of the j non-empty subsets of the $\mathcal{V}$.

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[4]More precisely, at any time t of the evolution of the Brownian motion, it is dcx smaller than a non-homogeneous Poisson point process of some intensity which depends on t, and converges to the homogeneous one for $t \to \infty$.

Consequently, simple, *α-weakly sub- (super-)Poisson point processes have also moment measures $\alpha^k(\cdot)$ smaller (larger) than those of Poisson point process.*

The reason for using the adjective "weak" will be clear once we introduce the stronger notion of *directionally convex ordering*. This will be very much needed for the example presented in Section 4 but not for the results of Section 3. So, the reader may skip the following subsection now and return back to it later when needed.

2.3 Directionally convex ordering

Let us quickly introduce the theory of directionally convex ordering. We refer the reader to [28, Section 3.12] for a more detailed introduction.

For a function $f : \mathbb{R}^k \to \mathbb{R}$, define the discrete differential operators as $\Delta_\epsilon^i f(x) := f(x + \epsilon e_i) - f(x), 1 \leq i \leq k$ where $\epsilon > 0$ and $\{e_i\}_{1 \leq i \leq k}$ are the canonical basis vectors for $\mathbb{R}^k$. Now, one introduces the following families of *Lebesgue-measurable* functions on $\mathbb{R}^k$: A function $f : \mathbb{R}^k \to \mathbb{R}$ is said to be *directionally convex* (*dcx*) if for every $x \in \mathbb{R}^k, \epsilon, \delta > 0, i, j \in \{1, \ldots, k\}$, we have that $\Delta_\epsilon^i \Delta_\delta^j f(x) \geq 0$. We abbreviate *increasing* (i.e. increasing in each co-ordinate) and *dcx* by *idcx* and *decreasing* and *dcx* by *ddcx*. There are various equivalent definitions of these and other multivariate functions suitable for dependence ordering (see [28, Chapter 3]).

Unless mentioned, when we state $\mathsf{E}(f(X))$ for a function f and a random vector X, we assume that the expectation exists. Assume that X and Y are real-valued random vectors of the same dimension. Then X *is said to be less than Y in dcx order* if $\mathsf{E}(f(X)) \leq \mathsf{E}(f(Y))$ for all f *dcx* such that both the expectations are finite. We shall denote it as $X \leq_{dcx} Y$. This property clearly regards only the distributions of X and Y, and hence sometimes we will say that the law of X is less in *dcx* order than that of Y.

A point process Φ on $\mathbb{R}^d$ can be viewed as the random field $\{\Phi(B)\}_{B \in \mathsf{B}_b^d}$. As the *dcx* ordering for random fields is defined via comparison of their finite dimensional marginals, for two point processes on $\mathbb{R}^d$, one says that $\Phi_1(\cdot) \leq_{dcx} \Phi_2(\cdot)$, if for any $B_1, \ldots, B_k$ bBs in $\mathbb{R}^d$,

$$(\Phi_1(B_1), \ldots, \Phi_1(B_k)) \leq_{dcx} (\Phi_2(B_1), \ldots, \Phi_2(B_k)). \tag{2.2}$$

The definition is similar for other orders, i.e., those defined by *idcx, ddcx* functions. It was shown in [5] that it is enough to verify the above condition for B_i pairwise disjoint.

In order to avoid technical difficulties, we will consider here only point processes whose *mean measures* $\mathsf{E}(\Phi(\cdot))$ are Radon (finite on bounded sets). For such point processes, *dcx* order is a transitive order. Note also that $\Phi_1(\cdot) \leq_{dcx} \Phi_2(\cdot)$ implies the *equality of their mean measures*: $\mathsf{E}(\Phi_1(\cdot)) = \mathsf{E}(\Phi_2(\cdot))$ as both x and $-x$ are *dcx* functions on $\mathbb{R}$. For more details on *dcx* ordering of point processes and random measures, see [5].

2.4 Relations and examples

We now concentrate on comparison of point processes to the Poisson point process of same mean measure. Following [8] we will call a point process *dcx sub-Poisson* (respectively *dcx super-Poisson*) if it is smaller (larger) in *dcx* order than the Poisson point process (necessarily of the same mean measure). For simplicity, we will just refer to them as sub-Poisson or super-Poisson point process omitting the word *dcx*.

From [8, Proposition 3.1 and Fact 3.2], we can see that weak sub- and super-Poissonianity are actually weaker than that of *dcx* sub- and super-Poissonianity respectively. Interestingly, they are also weaker than the notion of association (see [8, Section 2]). More precisely, it is shown in [8, Cor. 3.1] that under very mild regularity conditions, positively associated point processes are weakly super-Poisson, while negatively associated point processes are weakly sub-Poisson.

We list here briefly some examples of dcx and weak sub-Poisson and super-Poisson point processes. It was observed in [5] that some doubly-stochastic Poisson (Cox) point processes, such as Poisson-Poisson cluster point processes and, more generally, Lévy based Cox point processes are super-Poisson. Examples of positively associated Cox point processes, namely those driven by a positively associated random measure, are provided in [10].

A rich class of point processes called the perturbed lattices, including both sub- and super-Poisson point processes, is provided in [8] (see Section 4 for one of the simpler perturbed lattices). These point processes can be seen as toy models for determinantal and permanental point processes; cf. [3]. Regarding these latter point processes, it is shown in [8] that determinantal and permanental point processes are weakly sub-Poisson and weakly super-Poisson respectively.

3 Non-trivial phase transition for percolation models on sub-Poisson point processes

We will be particularly interested in percolation models on level-sets of additive shot-noise fields. The rough idea is as follows: level-crossing probabilities for these models can be bounded using Laplace transform of the underlying point process. For weakly sub-Poisson point process, this can further be bounded by the Laplace transform of the corresponding Poisson point process, which has a closed-form expression. For "nice" response functions of the shot-noise, these expressions are amenable enough to deduce the asymptotic bounds that are good enough to use the standard arguments of percolation theory. Hence, we can deduce percolation or non-percolation of a suitable discrete approximation of the model.

3.1 Bounds on Shot-Noise fields

Denote by

$$V_\Phi(y) := \sum_{X \in \Phi} \ell(X, y) \tag{3.1}$$

the (additive) shot-noise field generated by a point process Φ where $\ell(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is called the response function. The response function is assumed to be Lebesgue measurable in its first co-ordinate. We shall start with two complimentary results about ordering of Laplace transforms of weakly sub-Poisson (super-Poisson) point processes which are needed for the Proposition 3.3 that follows them. This key proposition will provide us with bounds on level-crossing probabilities of shot-noise fields which drive all the proofs that follow later in the section. We shall prove the two lemmas only for the case of weakly sub-Poisson point processes and the analagous results for weakly super-Poisson point processes follow via similar arguments.

Lemma 3.1. *Assume that Φ is a simple point process of a Radon mean measure α. Let the response function $\ell(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be Lebesgue measurable in its first co-ordinate and the shot-noise field $V_\Phi(.)$ be defined as in (3.1). If Φ is α-weakly sub-Poisson then for all $y \in \mathbb{R}^d$,*

$$\mathsf{E}\left(e^{V_\Phi(y)}\right) \leq \exp\left[\int_{\mathbb{R}^d} (e^{\ell(x,y)} - 1) \, \alpha(dx)\right]. \tag{3.2}$$

If Φ is α-weakly super-Poisson then the above inequality is reversed.

Proof. From the known representation of the Laplace transform of a functional of Poisson point process Φ_α of intensity measure α (cf [12, eqn. 9.4.17 p. 60]), we observe

that the RHS of (3.2) is the same as $\mathsf{E}\left(e^{V_{\Phi_\alpha}(y)}\right)$. So, the rest of the proof will be only concerned with proving that for any $y \in \mathbb{R}^d$,

$$\mathsf{E}\left(e^{V_\Phi(y)}\right) \leq \mathsf{E}\left(e^{V_{\Phi_\alpha}(y)}\right). \tag{3.3}$$

From Taylor's series expansion for the exponential function and the positivity of summands, we get that

$$\mathsf{E}\left(e^{V_\Phi(y)}\right) = 1 + \sum_{k=1}^\infty \frac{1}{k!} \int_{\mathbb{R}^{dk}} \ell(x_1, y) \ldots \ell(x_k, y) \, \alpha_\Phi^k(\mathrm{d}(x_1, \ldots, x_k)),$$

where α_Φ^k are moment measures of Φ. By the assumption that Φ is simple and $\boldsymbol{\alpha}$-weakly sub-Poisson $\alpha_\Phi^k \leq \alpha_{\Phi_\alpha}^k$, which completes the proof. □

Lemma 3.2. *Assume that Φ is a simple point process of a Radon mean measure α. Let the response function $\ell(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ be Lebesgue measurable in its first coordinate and the shot-noise field $V_\Phi(.)$ be defined as in (3.1). Φ is $\boldsymbol{\nu}$-weakly sub-Poisson if and only if for all $y \in \mathbb{R}^d$,*

$$\mathsf{E}\left(e^{-V_\Phi(y)}\right) \leq \exp\left[\int_{\mathbb{R}^d} (e^{-\ell(x,y)} - 1) \, \alpha(dx)\right]. \tag{3.4}$$

Φ is $\boldsymbol{\nu}$-weakly super-Poisson if and only if the above inequality is reversed.

Proof. Firstly, let us prove the easy implication by assuming that (3.4) holds. Let B be a bBS and set $\ell(x, y) = t \mathbf{1}[x \in B]$ for $t > 0$. Then $V_\Phi(y) = t\Phi(B)$ and so we get the required inequality (1.1) to prove that Φ is $\boldsymbol{\nu}$-weakly sub-Poisson:

$$\nu(B) = \lim_{t \to \infty} \mathsf{E}\left(e^{-t\Phi(B)}\right) \leq \lim_{t \to \infty} \exp\left[(e^{-t} - 1)\alpha(B)\right] = e^{-\alpha(B)},$$

where the inequality is due to (3.4) for our specific choice of ℓ.

Now for the reverse implication, assume that Φ is $\boldsymbol{\nu}$-weakly sub-Poisson. As with many other proofs, we shall only prove the inequality in the case of simple functions i.e, $\ell(\cdot, y) = \sum_{i=1}^k t_i \mathbf{1}[x \in B_i]$ for disjoint bBs $B_i, i = 1, \ldots, k$ and appeal to standard-measure theoretic arguments for extension to the general case. Thus for a simple function $\ell(\cdot, y)$, we need to prove the following:

$$\mathsf{E}\left(e^{-\sum_i t_i \Phi(B_i)}\right) \leq \prod_i \exp\left[\alpha(B_i)(e^{-t_i} - 1)\right]. \tag{3.5}$$

Setting $s_i = e^{-t_i}$, let Φ' be the thinned point process obtained from Φ by deleting points independently with probability $p(x)$, where $p(x) = s_i$ for points $x \in B_i$ and $p(x) \equiv 1$ outside $\bigcup_i B_i$. Similary, we define Φ'_α for the Poisson point process Φ_α of intensity measure α. Thus, we have that

$$\mathsf{E}\left(\prod_i s_i^{\Phi(B_i)}\right) = \sum_{n_1, \ldots, n_k \geq 0} \prod_i s_i^{n_i} \mathsf{P}\left(\Phi(B_i) = n_i, 1 \leq i \leq k\right)$$

$$= \mathsf{P}\left(\Phi'(\bigcup_i B_i) = 0\right).$$

Now to prove (3.5), it suffices to prove that for any bBs B,

$$\mathsf{P}\left(\Phi'(B) = 0\right) \leq \mathsf{P}\left(\Phi'_\alpha(B) = 0\right).$$

This follows from a more general observation that independent thinning preserves ordering of void probabilities of simple point processes, which we show in the remaining part of the proof using a coupling argument.

Consider a null-array of partitions $\{B_{n,j}\}_{n \geq 1, j \geq 1}$ of $\mathbb{R}^d$ [5]. Further, we assume that either $B_{n,j} \subset B_i$ for some $i \in \{1, \ldots, k\}$ or $B_{n,j} \cap (\cup_i B_i) = \emptyset$. Such a choice of partition can be always made by refining any given partition $\{B'_{n,j}\}_{n \geq 1, j \geq 1}$ to $\{B'_{n,j} \cap B_i\}_{i \in \{1, \ldots, k\}, n \geq 1, j \geq 1}$. For every $x \in \mathbb{R}^d$, let $j(n, x)$ be the unique index such that $x \in B_{n,j(n,x)}$. For every n, j, define $s(n, j) = s_i$ if $B_{n,j} \subset B_i$ else $s(n, j) \equiv 1$. Thus by the choice of partition, we get that $s(n, j(n, x)) = s_i$ if $x \in B_i$ for some $i \in \{1, \ldots, k\}$ or else $s(n, j(n, x)) = 1$. Let $\{\xi\} = \{\xi_{n,j}\}_{n \geq 1, j \geq 1}$ be a family of independent Bernoulli random variables $\mathrm{Ber}(1 - s(n, j))$ defined on a common probability space with Φ and independent of it. Define the family of point processes $\Phi_n = \sum_{X_i \in \Phi} \xi_{n,j(n,X_i)} \delta_{X_i}$. For given $n \geq 1$, Φ_n is a (possibly dependent) thinning of Φ. Moreover, because Φ is simple, $\Phi_n(B)$ converges in distribution to $\Phi'(B)$ (recall, Φ' is an independent thinning of Φ with retention probability $1 - s_i$ in B_i and 0 outside $\bigcup_i B_i$). The result follows by conditioning on $\{\xi\}$:

$$\nu'(B) = \lim_{n \to \infty} \mathsf{P}\left(\Phi_n(B) = 0\right) = \lim_{n \to \infty} \mathsf{E}\left(\mathsf{P}\left(\Phi(B_n^1) = 0 \,|\, (\xi)\right)\right) = \lim_{n \to \infty} \mathsf{E}\left(\nu(B_n^1)\right),$$

with $B_n^1 = \bigcup_{j : \xi_{n,j} = 1} B_{n,j}$. This completes the proof as for all realizations of ξ, we have that

$$\nu(B_n^1) \leq \mathsf{P}\left(\Phi_\alpha(B_n^1) = 0 \,|\, (\xi)\right).$$

$\square$

Proposition 3.3. *Let Φ be a simple, stationary point process of intensity λ.*

1. *If Φ is $\boldsymbol{\alpha}$-weakly sub-Poisson then we have that for any $y_1, \ldots, y_m \in \mathbb{R}^d$ and $s, h > 0$,*

$$\mathsf{P}\left(V_\Phi(y_i) \geq h, 1 \leq i \leq m\right) \leq e^{-smh} \exp\left\{\lambda \int_{\mathbb{R}^d} (e^{s \sum_{i=1}^m \ell(x, y_i)} - 1) dx\right\}. \tag{3.6}$$

2. *If Φ is $\boldsymbol{\nu}$-weakly sub-Poisson, then we have that for any $y_1, \ldots, y_m \in \mathbb{R}^d$ and $s, h > 0$,*

$$\mathsf{P}\left(V_\Phi(y_i) \leq h, 1 \leq i \leq m\right) \leq e^{smh} \exp\left\{\lambda \int_{\mathbb{R}^d} (e^{-s \sum_{i=1}^m \ell(x, y_i)} - 1) dx\right\}. \tag{3.7}$$

Proof. Suppose that inequalities (3.6) and (3.7) are true for $m = 1$. Observe that $\sum_{i=1}^m V_\Phi(y_i) = \sum_{X \in \Phi} \sum_{i=1}^m \ell(X, y_i)$ is itself a shot-noise field driven by the response function $\sum_{i=1}^m \ell(\cdot, y_i)$. Thus if (3.6) and (3.7) are true for $m = 1$, we can get the general case from the following easy inequalities:

$$\mathsf{P}\left(V_\Phi(y_i) \geq h, 1 \leq i \leq m\right) \quad \leq \quad \mathsf{P}\left(\sum_{i=1}^m V_\Phi(y_i) \geq mh\right)$$

$$\mathsf{P}\left(V_\Phi(y_i) \leq h, 1 \leq i \leq m\right) \quad \leq \quad \mathsf{P}\left(\sum_{i=1}^m V_\Phi(y_i) \leq mh\right).$$

Hence, we shall now prove only the case of $m = 1$ for both the inequalities. Setting $y_1 = y$ and using Chernoff's bound, we have that

$$\mathsf{P}\left(V_\Phi(y) \geq h\right) \quad \leq \quad e^{-sh} \mathsf{E}\left(e^{sV_\Phi(y)}\right),$$

$$\mathsf{P}\left(V_\Phi(y) \leq h\right) \quad \leq \quad e^{sh} \mathsf{E}\left(e^{-sV_\Phi(y)}\right).$$

Using Lemmas 3.1 and 3.2, we can upper bound the RHS of the both the equations and thus we have shown the inequalities in the case $m = 1$ as required. $\square$

---

[5]i.e., for every n, $\{B_{n,j}\}_{j \geq 1}$ form a partition of $\mathbb{R}^d$ with finitely many elements, and $\max_{j \geq 1}\{|B_{n,j}|\} \to 0$ as $n \to \infty$ where $|\cdot|$ denotes the diameter in any fixed metric; see [23, page 11])

Remark 3.4. *Assuming in Proposition 3.3 that Φ is stationary and weakly sub-Poisson, we compare Φ to the Poisson point process of the same intensity (λ). We see from the proof that this assumption can be weakened in the following way:*

1. *If Φ is simple and its factorial moment measures $\alpha_\Phi^{(k)}$ can be bounded by those of a homogeneous Poisson point process of some intensity $\lambda' > 0$ then (3.6) holds true for any $y_1, \ldots, y_m \in \mathbb{R}^d$ and $s, h > 0$ with λ replaced by λ'.*
2. *If Φ is simple and its void probabilities $\nu(\cdot)$ can be bounded by those of a homogeneous Poisson point process of some intensity $\lambda'' > 0$ then (3.7) holds true for any $y_1, \ldots, y_m \in \mathbb{R}^d$ and $s, h > 0$ with with λ replaced by λ''.*

The two bounds can be obtained by comparison to Poisson processes of two different intensities, with $\lambda'' \le \lambda'$.

3.2 Auxiliary discrete models

Though we focus on the percolation of Boolean models (continuum percolation models), but as is the wont in the subject we shall extensively use discrete percolation models as approximations. For $r > 0, x \in \mathbb{R}^d$, define the following subsets of $\mathbb{R}^d$. Let $Q_r := (-r, r]^d$ and $Q_r(x) := x + Q_r$. We will consider the following discrete graph : $\mathbb{L}^{*d}(r) = (r\mathbb{Z}^d, \mathbb{E}^{*d}(r))$ is a close-packed graph on the scaled-up lattice $r\mathbb{Z}^d$; the edge-set is $\mathbb{E}^{*d}(r) := \{(z_i, z_j) \in (r\mathbb{Z}^d)^2 : Q_r(z_i) \cap Q_r(z_j) \ne \emptyset\}$.

In what follows, we will define auxiliary site percolation models on the above graph by randomly declaring some of its vertices (called also sites) open. As usual, we will say that a given discrete site percolation model percolates if the corresponding sub-graph consisting of all open sites contains an infinite component.

Define the corresponding lower and upper level sets of the shot-noise field $V_\Phi(., .)$ on the lattice $r\mathbb{Z}^d$ by $\mathbb{Z}_r^d(V_\Phi, \le h) := \{z \in r\mathbb{Z}^d : V_\Phi(z) \le h\}$ and $\mathbb{Z}_r^d(V_\Phi, \ge h) := \{z \in r\mathbb{Z}^d : V_\Phi(z) \ge h\}$. The percolation of these two discrete models (i.e, $\mathbb{Z}_r^d(V_\Phi, \le h)$ and $\mathbb{Z}_r^d(V_\Phi, \ge h)$) understood in the sense of site-percolation of the close-packed lattice $\mathbb{L}^{*d}(r)$ will be of interest to us.

There are two standard arguments used in percolation theory to show non-percolation and percolation in discrete $(d \ge 2)$-dimensional models. We shall describe them here below as we use one or the other of these two arguments in our proofs for Theorems 3.7, 3.10, 3.12 and 3.13. In the remaining part of this section we assume $d \ge 2$. In sections 3.4 and 4 we will consider two-dimensional models (i.e, $d = 2$).

Remark 3.5 (Standard argument for non-percolation)**.** *Since the number of paths on $\mathbb{L}^{*d}(r)$ of length n starting from the origin is at most $(3^d - 1)^n$, in order to show non-percolation of a given model it is enough to show that the corresponding probability of having n distinct sites simultaneously open is smaller than ρ^n for some $0 \le \rho < (3^d - 1)^{-1}$ for n large enough. From this we can get that the expected number of open paths of length n starting from the origin (which is at most $(\rho(3^d - 1))^n$) tends to 0 and hence by Markov's inequality, we get that almost surely there is no infinite path from the origin. Since the same observation holds true for any node of the lattice, we can also conclude that almost surely there is no percolation.*

Remark 3.6 (Peierls' argument for percolation)**.** *Recall that the number of contours surrounding the origin in $\mathbb{L}^{*d}(r)$ [6] is at most $n(3^d - 2)^{n-1}$ [7]. Hence, in order to prove percolation of a given model using Peierls argument (cf. [20, pp. 17–18]), it is enough to*

---

[6] A contour surrounding the origin in $\mathbb{L}^{*d}(r)$ is a minimal collection of vertices of $\mathbb{L}^{*d}(r)$ such that any infinite path on this graph from the origin has to contain one of these vertices.

[7] The bounds $n(3^d - 2)^{n-1}$ and $(3^d - 1)^n$ (in Remark 3.5) are not tight; we use them for simplicity of exposition. For more about the former bound, refer [25, 2].

show that the corresponding probability of having n distinct sites simultaneously closed is smaller than ρ^n for some $0 \leq \rho < (3^d - 2)^{-1}$ for n large enough. Thus the expected the number of closed contours around the origin (which is at most $\sum_{n \geq 1} n(3^d - 2)^{n-1} \rho^n$) is finite and hence by a duality argument, we can infer that almost surely there will be at least one infinite path i.e, percolation.

3.3 k-percolation in Boolean model

Recall that k-percolation has already been introduced rigorously in the introduction itself; see Definition 1.1. The aim of this section is to show that for weakly sub-Poisson point processes, the critical intensity for k-percolation of the Boolean model is non-degenerate.

Theorem 3.7. *Let Φ be a simple, stationary point process of intensity λ on $\mathbb{R}^d$, with $d \geq 2$. For $k \geq 1$, there exist constants $c(\lambda)$ and $c(\lambda, k)$ (not depending on the distribution of Φ) such that $0 < c(\lambda) \leq r_c^1(\Phi)$ provided Φ is $\boldsymbol{\alpha}$-weakly sub-Poisson and $r_c^k(\Phi) \leq c(\lambda, k) < \infty$ provided Φ is $\boldsymbol{\nu}$-weakly sub-Poisson.*

More simply, Theorem 3.7 gives an upper and lower bound for the critical radius of a stationary weakly sub-Poisson point process dependent only on its intensity and not on the finer structure. This is the content of Corollary 1.3 stated in the introduction. Recall that examples of such point processes, including *determinantal point processes with trace-class integral kernels*, have been already mentioned in Section 2.4.

Remark 3.8 (Clique percolation). *The Čech simplicial complex on a point process Φ is defined as the simplicial complex whose k-dimensional faces are subsets $\{X_0, \ldots, X_k\} \subset \Phi$ such that $\bigcap_{i=0}^{k} B_{X_i}(r) \neq \emptyset$. Define a graph on the k-dimensional faces by placing edges between two k-dimensional faces if they are contained within the same $(k + 1)$-dimensional face. The case $k = 0$ corresponds to the random geometric graph or the Boolean model $C(\Phi, r)$. Is there a non-trivial phase transition for percolation in this graph for all $k \geq 1$? This question was posed in [22, Section 4] for the case of the Poisson point process. The question was motivated by positive answer to the discrete analogue of this question for Erdős-Rényi random graphs in [9] where it was called as clique percolation. Note that when Φ is the Poisson point process and $k = 0$, existence of non-trivial phase transition for percolation follows from the classical results of [27]. For $k \geq 1$, once we observe that percolation of k-faces in the Čech simplicial complex is equivalent to $(k + 1)$-percolation in the Boolean model, Corollary 1.3 answers the question in affirmative not only for the Poisson point process but also for weakly sub-Poisson point processeses.*

Proof of Theorem 3.7. As explained before, we shall use the standard arguments as described in Remarks 3.5 and 3.6 to show the lower and upper bounds respectively.

In order to prove the first statement, let Φ be $\boldsymbol{\alpha}$-weakly sub-Poisson and $r > 0$. Consider the close packed lattice $\mathbb{L}^{*d}(3r)$. Define the response function $l_r(x, y) := \mathbf{1}\left[x \in Q_{\frac{3r}{2}}(y)\right]$ and the corresponding shot-noise field $V_\Phi^r(z)$ on $\mathbb{L}^{*d}(3r)$. Note that if $C(\Phi, r)$ percolates then $\mathbb{Z}_{3r}^d(V_\Phi^r, \geq 1)$ percolates as well. To see why, consider an infinite path $(X_0, X_1, \ldots)$ of the graph $C(\Phi, r)$ and let $w_i \in 3r\mathbb{Z}^d$ be defined such that $X_i \in Q_{\frac{3r}{2}}(w_i)$. Since $\forall i \geq 0$, $|X_i - X_{i+1}| \leq 2r$, we have that

$$\emptyset \neq B_{\frac{3r}{2}}(X_i) \cap B_{\frac{3r}{2}}(X_{i+1}) \subset Q_{3r}(w_i) \cap Q_{3r}(w_{i+1})$$

Thus, $w_i = w_{i+1}$ or $(w_i, w_{i+1}) \in \mathbb{E}^{*d}(3r)$. This shows that the subset $\{w_i, i \geq 1\}$ forms a connected subgraph of $\mathbb{L}^{*d}(3r)$ and since the infinite set of points $\{X_i, i \geq 1\}$ is contained in $\Phi(\bigcup_{i \geq 1} Q_{\frac{3r}{2}}(w_i))$, the set $\{w_i, i \geq 0\}$ is also infinite. Thus, we have shown the

existence of an infinite connected subgraph in $\mathbb{Z}_{3r}^d(V_\Phi^r, \geq 1)$ or equivalently, $\mathbb{Z}_{3r}^d(V_\Phi^r, \geq 1)$ percolates whenever $C(\Phi, r)$ percolates.

We shall now show that there exists a $r > 0$ such that $\mathbb{Z}_{3r}^d(V_\Phi^r, \geq 1)$ does not percolate. For any n and distinct $z_i \in 3r\mathbb{Z}^d, 1 \leq i \leq n$, $\sum_{i=1}^n l_r(x, z_i) = 1$ iff $x \in \bigcup_{i=1}^n Q_{\frac{3r}{2}}(z_i)$ and else 0. Thus, from Proposition 3.3, for $s \geq 0$, we have that

$$
\begin{aligned}
\mathsf{P}\left(V_\Phi^r(z_i) \geq 1, 1 \leq i \leq n\right) &\leq e^{-sn} \exp\left\{\lambda \int_{\mathbb{R}^d}(e^{s\sum_{i=1}^n l_r(x, z_i)} - 1)\mathrm{d}x\right\}, \\
&= e^{-sn} \exp\left\{\lambda\|\bigcup_{i=1}^n Q_{\frac{3r}{2}}(z_i)\|(e^s - 1)\right\}, \\
&= (\exp\{-(s + (1 - e^s)\lambda(3r))^d)\})^n, \quad (3.8)
\end{aligned}
$$

where $\|\cdot\|$ denote the d-dimensional Lebesgue's measure. Choosing s large enough that $e^{-s} < (3^d - 1)^{-1}$ and then by continuity of $s + (1 - e^s)\lambda(3r)^d$ in r, we can choose a $c(\lambda, s) > 0$ such that for all $r < c(\lambda, s)$, $\exp\{-(s+(1-e^s)\lambda(3r)^d))\} < (3^d-1)^{-1}$. Now, using the standard argument involving the expected number of open paths (cf Remark 3.5), we can show non-percolation of $\mathbb{Z}_{3r}^d(V_\Phi^r, \geq 1)$ for $r < c(\lambda) := \sup_{s > \log(3^d - 1)} c(\lambda, s)$. Hence for all $r < c(\lambda)$, $C(\Phi, r)$ does not percolate and so $c(\lambda) \leq r_c(\Phi)$.

For the second statement on the finiteness of $r_c^k(\Phi)$, let Φ be $\boldsymbol{\nu}$-weakly sub-Poisson. Consider the close packed lattice, $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$. Define the response function $l_r(x, y) := \mathbb{1}\left[x \in Q_{\frac{r}{2\sqrt{d}}}(y)\right]$ and the corresponding additive shot-noise field $V_\Phi^r(z)$ on $\mathbb{L}^{*d}(\frac{r}{\sqrt{d}})$.

We will now show that $C_k(\Phi, r)$ percolates if $\mathbb{Z}_{\frac{r}{\sqrt{d}}}^d(V_\Phi^r, \geq k)$ percolates. Let $(w_0, w_1, \ldots)$ be an infinite path in $\mathbb{Z}_{\frac{r}{\sqrt{d}}}^d(V_\Phi^r, \geq k)$. We have that $\forall i \geq 0$, $\overline{Q_{\frac{r}{2\sqrt{d}}}(w_i)} \subset C_k(\Phi, r)$ as $\sup_{x, y \in Q_{\frac{r}{2\sqrt{d}}}(w_i)} |x - y| \leq r$ and where $\overline{A}$ denotes the topological closure of a set A. Since $\bigcup_i \overline{Q_{\frac{r}{2\sqrt{d}}}(w_i)}$ is connected, unbounded and contained in $C_k(\Phi, r)$, we have shown the existence of an unbounded connected subset of $C_k(\Phi, r)$ i.e., $C_k(\Phi, r)$ percolates.

We shall now show that there exists a $r < \infty$ such that $\mathbb{Z}_{\frac{r}{\sqrt{d}}}^d(V_\Phi^r, \geq k)$ percolates and thereby completing the proof. For any n and $z_i, 1 \leq i \leq n$, from Proposition 3.3, for $s \geq 0$, we have that

$$
\mathsf{P}\left(V_\Phi^r(z_i) \leq k - 1, 1 \leq i \leq n\right) \quad (3.9)
$$
$$
\begin{aligned}
&\leq e^{sn(k-1)} \exp\left\{\lambda \int_{\mathbb{R}^d}(e^{-s\sum_{i=1}^n l_r(x, z_i)} - 1)\,\mathrm{d}x\right\} \\
&= e^{sn(k-1)} \exp\left\{\lambda\|\bigcup_{i=1}^n Q_{\frac{r}{2\sqrt{d}}}(z_i)\|(e^{-s} - 1)\right\} \\
&= (\exp\{-((1 - e^{-s})\lambda(\frac{r}{\sqrt{d}})^d - s(k-1))\})^n. \quad (3.10)
\end{aligned}
$$

For any $s > 0$, there exists $c(\lambda, k, s) < \infty$ such that for all $r > c(\lambda, k, s)$, the the expression in (3.10) is strictly less than $(3^d - 2)^{-n}$. Thus one can use the standard argument involving the expected number of closed contours around the origin (cf Remark 3.6) to show that $\mathbb{Z}_{\frac{r}{\sqrt{d}}}^d(V_\Phi^r, \geq k)$ percolates. Further defining $c(\lambda, k) := \inf_{s>0} c(\lambda, k, s)$, we have that $C_k(\Phi, r)$ percolates for all $r > c(\lambda, k)$. Thus $r_c^k(\Phi) \leq c(\lambda, k)$. □

Remark 3.9. *Following Remark 3.4, bounds on the critical radii can be obtained by comparison to Poisson processes of two different intensities: $0 < c(\lambda') \leq r_c^1(\Phi)$ provided Φ is simple and its factorial moment measures are bounded by those of a homogeneous Poisson point process of intensity $\lambda' > 0$ and $r_c^k(\Phi) \leq c(\lambda'', k) < \infty$ provided the void*

probabilities of Φ are bounded by those of the homogeneous Poisson point process of intensity $\lambda'' > 0$.

For $k = 1$; i.e., for the usual percolation in Boolean model, we can avoid the usage of exponential estimates of Proposition 3.3 and work directly with void probabilities and factorial moment measures. The gain is improved bounds on the critical radius.

Theorem 3.10. *Let Φ be a stationary point process of intensity λ on $\mathbb{R}^d$, $d \geq 2$, and $\boldsymbol{\nu}$-weakly sub-Poisson. Then $r_c(\Phi) \leq \tilde{c}(\lambda) := \sqrt{d} \left(\frac{\log(3^d - 2)}{\lambda} \right)^{1/d} \leq c(\lambda, 1) < \infty$.*

Proof. We follow the same arguments as in the second part of the proof of Theorem 3.7 and use different bound for the probabilities in (3.9) with $k = 1$:

$$
\begin{aligned}
\mathsf{P}\left(V_\Phi^r(z_i) = 0, 1 \leq i \leq n \right) &= \mathsf{P}\left(\Phi \cap \bigcup_{i=1}^n Q_{\frac{r}{2\sqrt{d}}}(z_i) = \emptyset \right) \\
&\leq \mathsf{P}\left(\Phi_\lambda \cap \bigcup_{i=1}^n Q_{\frac{r}{2\sqrt{d}}}(z_i) = \emptyset \right) \\
&= \left(\exp\{ -\lambda (\frac{r}{\sqrt{d}})^d \} \right)^n .
\end{aligned}
\tag{3.11}
$$

Clearly, for $r > \tilde{c}(\lambda)$, the expression in (3.11) is less than $(3^d - 2)^{-n}$ and thus $\{z : V_\Phi^r(z) \geq 1\}$ percolates by Peierls argument (cf Remark 3.6). It is easy to see that for any $s > 0$, $\exp\{ -\lambda(\frac{r}{\sqrt{d}})^d \} \leq \exp\{ -(1 - e^{-s})\lambda(\frac{r}{\sqrt{d}})^d \}$ and hence $\tilde{c}(\lambda) \leq c(\lambda, 1)$. $\qquad \square$

Combining the results of Theorem 3.10 and [8, Proposition 6.1], we have the following phase-transition result for usual continuum percolation. Recall that θ_d is the volume of the unit ball in $\mathbb{R}^d$.

Corollary 3.11. *For a stationary weakly sub-Poisson point process Φ on $\mathbb{R}^d$, $d \geq 2$, we have that $0 < \frac{1}{(\lambda \theta_d)^{1/d}} \leq r_c(\Phi) \leq \sqrt{d} \left(\frac{\log(3^d - 2)}{\lambda} \right)^{1/d} < \infty$*

3.4 Percolation in SINR graphs

Study of percolation in the Boolean model $C(\Phi, r)$ was proposed in [19] to address the feasibility of multi-hop communications in large "ad-hoc" networks, where full connectivity is typically hard to maintain. The Signal-to-interference-and-noise ratio (SINR) model (see [1, 13, 14] [8]) is more adequate than the Boolean model in the context of wireless communication networks as it allows one to take into account the *interference* intrinsically related to wireless communications. For more motivation to study SINR model, refer [7] and the references therein.

We begin with a formal introduction of the SINR graph model. In this subsection, we shall work only in $\mathbb{R}^2$. The parameters of the model are non-negative numbers P (signal power), N (environmental noise), γ (interference parameter), T (SINR threshold) and a continuous (power attenuation) function $l : \mathbb{R}_+ \to \mathbb{R}_+$, strictly decreasing on its support, with $l(0) \geq TN/P$, $l(\cdot) \leq 1$, and $\int_0^\infty x l(x) \mathrm{d}x < \infty$. These are exactly the assumptions made in [14] and we refer to this paper for a discussion on their validity.

Given a point process Φ, the *interference* generated by the point process Φ at a location x is defined as the following shot-noise field $I_\Phi(x) := \sum_{X \in \Phi \setminus \{x\}} l(|X - x|)$. Define the SINR value from x to y as follows:

$$
\mathrm{SINR}(x, y, \Phi, \gamma) := \frac{P l(|x - y|)}{N + \gamma P I_{\Phi \setminus \{x\}}(y)}.
\tag{3.12}
$$

---

[8]The name *shot-noise germ-grain process* was also suggested by D. Stoyan in his private communication to BB.

Let Φ_B and Φ_I be two point processes. Let $P, N, T > 0$ and $\gamma \geq 0$. The SINR graph is defined as $G(\Phi_B, \Phi_I, \gamma) := (\Phi_B, E(\Phi_B, \Phi_I, \gamma))$ with vertices Φ_B and edges

$$E(\Phi_B, \Phi_I, \gamma) := \{(X, Y) \in \Phi_B^2 : \mathrm{SINR}(Y, X, \Phi_I, \gamma) > T, \ \mathrm{SINR}(X, Y, \Phi_I, \gamma) > T\}.$$

The SNR graph $G(\Phi_B) := (\Phi_B, E(\Phi_B))$ (i.e, the graph without interference) is defined as above assuming $\gamma = 0$. Note that it does not depend on Φ_I and thus we omit it from the parameters and notation of the SNR graph.

Observe also that the SNR graph $G(\Phi)$ is same as the (Boolean model) graph $C(\Phi, r_l)$ with $2r_l = l^{-1}(\frac{TN}{P})$. Recall that percolation in all the above graphs is existence of an infinite connected component in the graph-theoretic sense.

3.4.1 Poissonian back-bone nodes

Firstly, we consider the case when the backbone nodes (Φ_B) form a Poisson point process and in the next section, we shall relax this assumption. When $\Phi_B = \Phi_\lambda$, the Poisson point process of intensity λ, we shall use $G(\lambda, \Phi_I, \gamma)$ and $G(\lambda)$ to denote the SINR and SNR graphs respectively. Denote by $\lambda_c(r) := \lambda(r_c(\Phi_\lambda)/r)^2$ the *critical intensity* for percolation of the Boolean model $C(\Phi_\lambda, r)$. The following result says that for any sub-Poisson point process $\Phi = \Phi_I$, the SINR graph $G(\lambda, \Phi, \gamma)$ percolates for sufficiently small $\gamma > 0$ if and only if the corresponding SNR graph $G(\lambda)$ (assuming $\gamma = 0$) percolates.

Theorem 3.12. *Let $\lambda > \lambda_c(r_l)$ and Φ be a stationary α-weakly sub-Poisson point process on $\mathbb{R}^2$, with intensity $\mu > 0$. Then there exists a $\gamma_0 > 0$ such that $G(\lambda, \Phi, \gamma)$ percolates for all $\gamma < \gamma_0$. If $\lambda < \lambda_c(r_l)$ then $G(\lambda, \Phi, \gamma)$ does not percolate for any $\gamma \geq 0$.*

Note that we have not assumed the independence of Φ and Φ_λ. In particular, Φ could be $\Phi_\lambda \cup \Phi_0$ where Φ_0 is an independent α-weakly sub-Poisson point process. The case $\Phi_0 = \emptyset$ was proved in [14]. Our proof follows their idea of coupling the continuum model with a discrete model and then using the Peierls argument (see Remark 3.6). As in [14], it is clear that for $N \equiv 0$, the above result holds with $\lambda_c(r_l) = 0$.

Sketch of the proof of Theorem 3.12. The second statement is trivial, as $G(\lambda, \Phi, \gamma)$ is a sub-graph of $G(\lambda)$. Regarding the first statement, since the probability of percolation is a non-increasing function of γ, it suffices to show that there exists $\gamma > 0$ such that $G(\lambda, \Phi, \gamma)$ percolates. Our proof follows the arguments given in [14] and here, we will only give a sketch of the proof. The details can be found in [31, Section 6.3.4].

Assuming $\lambda > \lambda_c(r_l)$, one observes first that the graph $G(\lambda)$ also percolates with any slightly larger constant noise $N' = N + \delta'$, for some $\delta' > 0$. Essential to the proof of the result is to show that the level-set $\{x : I_{\Phi_I}(x) \leq M\}$ of the interference field percolates (contains an infinite connected component) for sufficiently large M. Suppose that it is true. Then taking $\gamma = \delta'/M$ one has percolation of the level-set $\{y : \gamma I_{\Phi_I}(y) \leq \delta'\}$. The main difficulty consists in showing that $G(\lambda)$ with noise $N' = N + \delta'$ percolates *within* an infinite connected component of $\{y : \gamma I_{\Phi_I}(y) \leq \delta'\}$. This was done in [14], by mapping both models $G(\lambda)$ and the level-set of the interference field to a discrete lattice and showing that both discrete approximations not only percolate but actually satisfy a stronger condition, related to the Peierls argument. We follow exactly the same steps and the only fact that we have to prove, regarding the interference, is that there exists a constant $\epsilon < 1$ such that for arbitrary $n \geq 1$ and arbitrary choice of locations $x_1, \ldots, x_n$ one has $\mathsf{P}(I_{\Phi_I}(x_i) > M, i = 1, \ldots, n) \leq \epsilon^n$. In this regard, we use the first statement of Proposition 3.3 to prove, exactly as in [14, Prop. 2], that for sufficiently small s it is not larger than K^n for some constant K which depends on λ but not on M. This completes the proof. □

3.4.2 Non-Poissonian back-bone nodes

We shall now consider the case when the backbone nodes are formed by a $\boldsymbol{\nu}$-weakly sub-Poisson point process. In this case, we can give a weaker result, namely that with an increased signal power P the SINR graph will percolate for small interference parameter $\gamma > 0$.

Theorem 3.13. *Let Φ be a stationary, $\boldsymbol{\nu}$-weakly sub-Poisson point process on $\mathbb{R}^2$, and Φ_I be an $\boldsymbol{\alpha}$-weakly sub-Poisson point process with intensity μ for some $\mu > 0$. Also assume that $l(x) > 0$ for all $x \in \mathbb{R}_+$. Then there exist $P_0 < \infty$ and $\gamma_0 > 0$ such that $G(\Phi, \Phi_I, \gamma)$ percolates for all $P > P_0$ and $\gamma < \gamma_0$.*

As in Theorem 3.12, we have not assumed the independence of Φ_I and Φ. For example, $\Phi_I = \Phi \cup \Phi_0$ where Φ and Φ_0 are independent $\boldsymbol{\nu}$-weakly and $\boldsymbol{\alpha}$-weakly sub-Poisson point processes respectively. Let us also justify the assumption of unbounded support for $l(\cdot)$. Suppose that $r = \sup\{x : l(x) > 0\} < \infty$. Then if $C(\Phi, r)$ does not percolate, then $G(\Phi, \Phi_I, \gamma)$ does not percolate for any Φ_I, P, γ.

Sketch of the proof of Theorem 3.13. Again by noting that percolation probability is a non-decreasing function of P and a non-increasing function of γ, it suffices to show the result for a finite P and a non-zero γ. In this scenario, increased power is equivalent to increased radius in the Boolean model corresponding to SNR model. From this observation, it follows from Theorem 3.10 that with possibly increased power the associated SNR model percolates. Then, we use the approach from the proof of Theorem 3.12 to obtain a $\gamma > 0$ such that the SINR network percolates as well. The details can be found in [31, Section 6.3.4]. □

4 Super-Poisson point process with null critical radius

The objective of this section is to show an example of highly clustered and well percolating point process. More precisely we construct a Poisson-Poisson cluster point process (which is known to be dcx larger than the Poisson point process) for which $r_c = 0$. This invalidates the temptation to conjecture the monotonicity of r_c with respect to the dcx order (and hence with respect to void probabilities and moment measures) of point process, in full generality. Recall that $B_x(r)$ is the ball of radius r centered at $x \in \mathbb{R}^2$.

Example 4.1 (Poisson-Poisson cluster point process on $\mathbb{R}^2$, with annular clusters). *Let Φ_α be the Poisson point process of intensity α on the plane $\mathbb{R}^2$; we call it the process of cluster centers. For any δ, R, μ such that $0 < \delta \leq R < \infty$ and $0 < \mu < \infty$, consider a Poisson-Poisson cluster point process $\Phi_\alpha^{R,\delta,\mu}$; i.e., a Cox point process with the random intensity measure $\Lambda(\cdot) := \mu \sum_{X \in \Phi_\alpha} \mathcal{X}(X, \cdot)$, where $\mathcal{X}(x, \cdot)$ is the uniform distribution on the annulus $B_x(R) \setminus B_x(R - \delta)$ of inner and outer radii $R - \delta$ and R respectively; see Figure 1.*

By [5, Proposition 5.2], it is a super-Poisson point process. More precisely, $\Phi_\lambda \leq_{dcx} \Phi_\alpha^{R,\delta,\mu}$, where Φ_λ is homogeneous Poisson point process of intensity $\lambda = \alpha\mu$.

For a given arbitrarily large intensity $\lambda < \infty$, taking sufficiently small α, R, $\delta = R$ and sufficiently large μ, it is straightforward to construct a Poisson-Poisson cluster point process $\Phi_\alpha^{R,R,\mu}$ with spherical clusters, which has an arbitrarily large critical radius r_c for percolation. It is less evident that one can construct a Poisson-Poisson cluster point process that always percolates, i.e., with degenerate critical radius $r_c = 0$.

Proposition 4.2. *Let $\Phi_\alpha^{R,\delta,\mu}$ be a Poisson-Poisson cluster point process with annular clusters on the plane $\mathbb{R}^2$ as in Example 4.1. Given arbitrarily small $a, r > 0$, there exist*
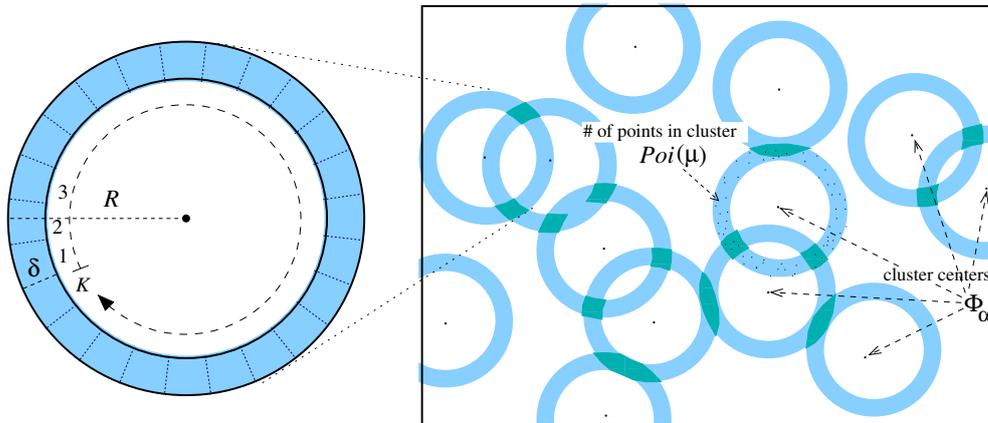
Figure 1: Poisson-Poisson cluster process of annular cluster; cf. Example 4.1.

constants α, μ, δ, R such that $0 < \alpha, \mu, \delta, R < \infty$, the intensity $\alpha\mu$ of $\Phi_\alpha^{R,\delta,\mu}$ is equal to a and the critical radius for percolation $r_c(\Phi_\alpha^{R,\delta,\mu}) \leq r$. Moreover, for any $a > 0$ there exists point process Φ of intensity a, which is dcx-larger than the Poisson point process of intensity a, and which percolates for any $r > 0$; i.e., $r_c(\Phi) = 0$.

Proof. Let $a, r > 0$ be given. Assume $\delta = r/2$. We will show that there exist sufficiently large μ, R such that $r_c(\Phi_\alpha^{R,\delta,\mu}) \leq r$ where $\alpha = a/\mu$. In this regard, denote $K := 2\pi R/r$ and assume that R is chosen such that K is an integer. For a $\alpha > 0$ and any point (cluster center) $X_i \in \Phi_\alpha$, let us partition the annular support $A_{X_i}(R, \delta) := B_{X_i}(R) \setminus B_{X_i}(R - \delta)$ of the Poisson point process constituting the cluster centered at X_i into K cells as shown in Figure 1. Recall that all (individual) clusters have finite, Poisson $Poi(\mu)$, total number of points. We will call X_i "open" if in each of the K cells of $A_{X_i}(R, \delta)$, there exists at least one point of the cluster centered at X_i. Note that given Φ_α, each point $X_i \in \Phi_\alpha$ is open with probability $p(R, \mu) := (1 - e^{-\mu/K})^K$, independently of other points of Φ_α. Consequently, open points of Φ_α form a Poisson point process of intensity $\alpha p(R, \mu)$; call it Φ_{open}. Note that the maximal distance between any two points in two neighbouring cells of the same cluster is not larger than $2(\delta + 2\pi R/K) = 2r$. Similarly, the maximal distance between any two points in two non-disjoint cells of two different clusters is not larger than $2(\delta + 2\pi R/K) = 2r$. Consequently, if the Boolean model $C(\Phi_{open}, A_O(R, \delta)) := \bigcup_{X_i \in \Phi_{open}} A_{X_i}(R, \delta)$ with annular grains $A_O(R, \delta)$ centered at the origin O percolates then the Boolean model $C(\Phi_\alpha^{R,\delta,\mu}, r)$ with spherical grains of radius r percolates as well. The former Boolean model percolates if and only if $C(\Phi_{open}, R)$ percolates.

Since Φ_{open} is a Poisson point process, the Boolean model $C(\Phi_{open}, R)$ percolates if its volume fraction $\mathsf{P}\,(O \in C(\Phi_{open}, R)) = 1 - e^{-\alpha p(R,\mu)\pi R^2}$ is large enough. Hence, in order to guarantee $r_c(\Phi_\alpha^{R,\delta,\mu}) \leq r$, it is enough to chose R, μ such that the volume fraction $1 - e^{-\alpha p(R,\mu)\pi R^2} = 1 - e^{-ap(R,\mu)\pi R^2/\mu}$ is larger than the critical volume fraction for the percolation of the Poisson Boolean model on the plane. In what follows, we will show that by choosing appropriate R, μ one can make $p(R, \mu)R^2/\mu$ arbitrarily large. Indeed, take

$$\mu := \mu(R) = \frac{2\pi R}{r} \log \frac{R}{\sqrt{\log R}} = \frac{2\pi R}{r}\left(\log R - \frac{1}{2}\log\log R\right).$$

Then, as $R \to \infty$

$$
\begin{aligned}
p(R,\mu)R^2/\mu &= \frac{R^2}{\mu}(1-e^{-\mu r/(2\pi R)})^{2\pi R/r} \\
&= \frac{Rr}{2\pi(\log R - \frac{1}{2}\log\log R)}\left(1-\frac{\sqrt{\log R}}{R}\right)^{2\pi R/r} \\
&= e^{O(1)+\log R - \log(2\pi(\log R - \frac{1}{2}\log\log R))-O(1)\sqrt{\log R}} \to \infty \, .
\end{aligned}
$$

This completes the proof of the first statement.

In order to prove the second statement, for a given $a > 0$, denote $a_n := a/2^n$ and let $r_n = 1/n$. Consider a sequence of independent (super-Poisson) Poisson-Poisson cluster point processes $\Phi_n = \Phi_{\alpha_n}^{R_n,\delta_n,\mu_n}$ with intensities $\lambda_n := \alpha_n\mu_n = a_n$, satisfying $r_c(\Phi_n) \leq r_n$. The existence of such point processes was shown in the first part of the proof. By the fact that Φ_n are super-Poisson for all $n \geq 0$ and by [5, Proposition 3.2(4)] the superposition $\Phi = \bigcup_{n=1}^{\infty}\Phi_n$ is dcx-larger than Poisson point process of intensity a. Obviously $r_c(\Phi) = 0$. This completes the proof of the second statement. $\qquad\square$

Remark 4.3. *By Proposition 4.2, we know that there exists point process Φ with intensity $a > 0$ such that $r_c(\Phi) = 0$ and $\Phi_a \leq_{dcx} \Phi$, where Φ_a is homogeneous Poisson point process. Since one knows that $r_c(\Phi_a) > 0$ so Φ is a counterexample to the monotonicity of r_c in dcx ordering of point processes.*

5   Concluding remarks

We come back to the initial heuristic discussed in the Introduction — clustering in a point process should increase the critical radius for the percolation of the corresponding continuum percolation model. As we have seen, even a relatively strong tool such as the dcx order falls short, when it comes to making a formal statement of this heuristic.

The two natural questions are what would be a more suitable comparison of clustering that can be used to affirm the heuristic and whether dcx order can satisfy a weaker version of the conjecture.

As regards the first question, one might start by looking at other dependence orders such as super-modular, component-wise convex or convex order but it has been already shown that the first two are not suited to comparison of clustering in point processes (cf. [31, Section 4.4]). Properties of convex order on point processes are yet to be investigated fully and this research direction is interesting in its own right, apart from its relation to the above conjecture. In a similar vein, it is of potential interest to study other stochastic orders on point processes.

On the second question, it is pertinent to note that sub-Poisson point processes surprisingly exhibited non-trivial phase transitions for percolation. Such well-behavedness of the sub-Poisson point processes makes us wonder if it is possible to prove a rephrased conjecture saying that any homogeneous sub-Poisson point process has a smaller critical radius for percolation than the Poisson point process of the same intensity. Such a conjecture matches well with [4, Conjecture 4.6].

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