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Penalization method for a nonlinear Neumann PDE via weak solutions of reflected SDEs *

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Abstract

In this paper we prove an approximation result for the viscosity solution of a system of semi-linear partial differential equations with continuous coefficients and nonlinear Neumann boundary condition. The approximation we use is based on a penalization method and our approach is probabilistic. We prove the weak uniqueness of the solution for the reflected stochastic differential equation and we approximate it (in law) by a sequence of solutions of stochastic differential equations with penalized terms. Using then a suitable generalized backward stochastic differential equation and the uniqueness of the reflected stochastic differential equation, we prove the existence of a continuous function, given by a probabilistic representation, which is a viscosity solution of the considered partial differential equation. In addition, this solution is approximated by solutions of penalized partial differential equations.

Keywords: Reflecting stochastic differential equation; Penalization method; Weak solution; Jakubowski S-topology; Backward stochastic differential equations.
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1 Introduction

Let G be a C^2 convex, open and bounded set from $\mathbb{R}^d$, and for $(t,x)\in[0,T]\times\bar{G}$ we consider the following reflecting stochastic differential equation (SDE for short)

$$X_s+K_s=x+\int_t^s b(X_r)dr+\int_t^s \sigma(X_r)dW_r\,,\ s\in[t,T]\,,$$

with K a bounded variation process such that for any $s\in[t,T]$, $K_s=\int_t^s\nabla\ell(X_r)d\left|K\right|_{[t,r]}$ and $\left|K\right|_{[t,s]}=\int_t^s\mathbb{1}_{\{X_r\in\partial G\}}d\left|K\right|_{[t,r]}$, where the notation $\left|K\right|_{[t,s]}$ stands for the total variation of K on the interval $[t,s]$.

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The coefficients b and σ are supposed to be only bounded continuous on $\mathbb{R}^d$ and $\sigma\sigma^*$ uniformly elliptic. The first main purpose is to prove that the weak solution (X, K) is approximated in law (in the space of continuous functions) by the solutions of the non-reflecting SDE

$$X_s^n = x + \int_t^s \left[b(X_r^n) - n(X_r^n - \pi_{\bar{G}}(X_r^n))\right] dr + \int_t^s \sigma(X_r^n) dW_r, \quad s \in [t, T],$$

where $\pi_{\bar{G}}$ is the projection operator. Since for $n \to \infty$ the term $K_s^n := n \int_t^s (X_r^n - \pi_{\bar{G}}(X_r^n))dr$ forces the solution X^n to remain near the domain, the above equation is called SDE with penalization term.

The case where b and σ are Lipschitz has been considered by Lions, Menaldi and Sznitman in [11] and by Menaldi in [14] where they have proven that $\mathbb{E}(\sup_{s\in[0,\,T]} |X_s^n - X_s|) \longrightarrow 0$, as $n \to \infty$. Note that Lions and Sznitman have shown, using Skorohod problem, the existence of a weak solution for the SDE with normal reflection to a (non-necessarily convex) domain. The case of reflecting SDE with jumps has been treated by Łaukajtys and Słomiński in [8] in the Lipschitz case; the same authors have extended in [9] these results to the case where the coefficient of the reflecting equation is only continuous. In these two papers it is proven that the approximating sequence $(X^n)_n$ is tight with respect to the S-topology, introduced by Jakubowski in [6] on the space $\mathcal{D}\left(\mathbb{R}_+, \mathbb{R}^d\right)$ of càdlàg $\mathbb{R}^d$-valued functions. Assuming the weak (in law) uniqueness of the limiting reflected diffusion X, they prove in [9] that X^n S-converges weakly to X. We mention that $(X^n)_n$ may not be relatively compact with respect to the Skorohod topology J_1.

In contrast to [9], we can not simply assume the uniqueness in law of the limit X, and the weak S-convergence of X^n to X is not sufficient to our goal. In our framework, we need to show the uniqueness in law of the couple (X, K) and that the convergence in law of the sequence (X^n, K^n) to (X, K) holds with respect to uniform topology.

The first main result of our paper will be the weak uniqueness of the solution (X, K), together with the convergence in law (in the space of continuous functions) of the penalized diffusion to the reflected diffusion X and the continuity with respect to the initial data.

Subsequently, using a proper generalized BSDE, we deduce (as a second main result) an approximation result for a continuous viscosity solution of the system of semi-linear partial differential equations (PDEs for short) with a nonlinear Neumann boundary condition

$$\begin{aligned}
&\frac{\partial u_i}{\partial t}(t,x) + Lu_i(t,x) + f_i(t,x,u(t,x)) = 0,\ \forall (t,x) \in [0,T] \times G, \\
&\frac{\partial u_i}{\partial n}(t,x) + h_i(t,x,u(t,x)) = 0,\ \forall (t,x) \in [0,T] \times \partial G, \\
&u_i(T,x) = g_i(x),\ \forall x \in G,\ i = \overline{1,k},
\end{aligned}$$

where L is the infinitesimal generator of the diffusion X, defined by

$$L = \frac{1}{2}\sum_{i,j} (\sigma(\cdot)\,\sigma^*(\cdot))_{ij}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(\cdot) \frac{\partial}{\partial x_i},$$

and $\partial u_i / \partial n$ is the outward normal derivative of u_i on the boundary of the domain.

Boufoussi and Van Casteren have established in [3] a similar result, but in the case where the coefficients b and σ are uniformly Lipschitz.

We mention that the class of BSDEs involving a Stieltjes integral with respect to the continuous increasing process $|K|_{[t,s]}$ was studied first in [16] by Pardoux and Zhang;

the authors provided a probabilistic representation for the viscosity solution of a Neumann boundary partial differential equation. It should be mentioned that the continuity of the viscosity solution is rather hard to prove in our frame. In fact, this property essentially uses the continuity with respect to initial data of the solution of our BSDE. We develop here a more natural method based on the uniqueness in law of the solution (X, K, Y) of the reflected SDE-BSDE and on the continuity property. Similar techniques were developed, in the non reflected case, in [2], but in our situation the proof is more delicate. The difficulty is due to the presence of the reflection process K in the forward component and the generalized part in the backward component.

Throughout this paper we use different types of convergence defined as follows: for the processes $(Y^n)_n$ and Y, by $Y^n \xrightarrow[u]{*} Y$ we denote the convergence in law with respect to the uniform topology, by $Y^n \xrightarrow[J_1]{*} Y^n$ we mean the convergence in law with respect to the Skorohod topology J_1 and by $Y^n \xrightarrow[\mathrm{S}]{*} Y$ we understand the weak convergence considered in S-topology.

The paper is organized as follows: in the next section we give the assumptions, we formulate the problem and we state the two main results. The third section is devoted to the proof of the first main result (proof of the convergence in law of (X^n, K^n) to (X, K) as $n \to \infty$ and the continuous dependence with respect to the initial data). In Section 4 the generalized BSDEs are introduced, the continuity with respect to the initial data is obtained and we prove the approximation result for the PDE introduced above.

2 Formulation of the problem; the main results

Let G be a C^2 convex, open and bounded set from $\mathbb{R}^d$ and we suppose that there exists a function $\ell \in C_b^2(\mathbb{R}^d)$ such that

$$G = \{x \in \mathbb{R}^d : \ell(x) < 0\},\ \partial G = \{x \in \mathbb{R}^d : \ell(x) = 0\},$$

and, for all $x \in \partial G$, $\nabla \ell(x)$ is the unit outward normal to ∂G.

In order to define the approximation procedure we shall introduce the penalization term. Let $p : \mathbb{R}^d \to \mathbb{R}_+$ be given by $p(x) = \mathrm{dist}^2(x, \bar{G})$.

Without restriction of generality we can choose ℓ such that

$$\langle \nabla \ell(x), \delta(x) \rangle \geq 0,\ \forall x \in \mathbb{R}^d,$$

where $\delta(x) := \nabla p(x)$ is called the penalization term.

It can be shown that p is of class C^1 on $\mathbb{R}^d$ with

$$\frac{1}{2}\delta(x) = \frac{1}{2}\nabla(\mathrm{dist}^2(x, \bar{G})) = x - \pi_{\bar{G}}(x),\ \forall x \in \mathbb{R}^d,$$

where $\pi_{\bar{G}}(x)$ is the projection of x on $\bar{G}$. It is clear that δ is a Lipschitz function.

On the other hand, $x \mapsto \mathrm{dist}^2\left(x, \bar{G}\right)$ is a convex function and therefore

$$\langle z - x, \delta(x) \rangle \leq 0,\ \forall x \in \mathbb{R}^d,\ \forall z \in \bar{G}. \tag{2.1}$$

Let $T > 0$ and suppose that:

(A_1) *$b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d'}$ are bounded continuous functions.*

Remark 2.1. *In fact we can assume that the functions b and σ have sublinear growth but, for the simplicity of the calculus, we will work with assumption (A_1).*

(A_2) *the matrix* $\sigma\sigma^*$ *is uniformly elliptic*, i.e. there exists $\alpha_0 > 0$ such that for all $x \in \mathbb{R}^d$, $(\sigma\sigma^*)(x) \geq \alpha_0\, I$.

Moreover, there exist some positive constants C_i, $i = \overline{1,2}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+^*$ and $q \geq 1$ such that

(A_3) $f, h : [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^k$ *and* $g : \mathbb{R}^d \to \mathbb{R}^k$ *are continuous functions and, for all* $x, x' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}^k$, $t, t' \in [0,T]$,

$$\begin{aligned} (i) \quad & \langle y'-y, f(t,x,y') - f(t,x,y)\rangle \leq \alpha\, |y'-y|^2 , \\ (ii) \quad & |h(t',x',y') - h(t,x,y)| \leq \beta\left(|t'-t| + |x'-x| + |y'-y|\right), \\ (iii) \quad & |f(t,x,y)| + |h(t,x,y)| \leq C_1\left(1+|y|\right), \\ (iv) \quad & |g(x)| \leq C_2\left(1+|x|^q\right) . \end{aligned} \tag{2.2}$$

Let us consider the following system of semi-linear PDEs considered on the whole space:

$$\begin{cases} \dfrac{\partial u_i^n}{\partial t}(t,x) + Lu_i^n(t,x) + f_i\left(t,x,u^n(t,x)\right) - \langle \nabla u_i^n(t,x), n\delta(x)\rangle \\ \qquad\qquad\qquad\qquad -\langle \nabla \ell(x), n\delta(x)\rangle\;\; h_i\left(t,x,u^n(t,x)\right) = 0 \\ u_i^n(T,x) = g_i(x),\ \forall\,(t,x) \in [0,T]\times\mathbb{R}^d,\ i = \overline{1,k} , \end{cases} \tag{2.3}$$

and the next semi-linear PDE considered with Neumann boundary conditions:

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,x) + Lu_i(t,x) + f_i\left(t,x,u(t,x)\right) = 0,\ \forall\,(t,x) \in [0,T]\times G, \\ \dfrac{\partial u_i}{\partial n}(t,x) + h_i\left(t,x,u(t,x)\right) = 0,\ \forall\,(t,x) \in [0,T]\times \partial G, \\ u_i(T,x) = g_i(x),\ \forall x \in G,\ i = \overline{1,k}, \end{cases} \tag{2.4}$$

where L is the second order partial differential operator

$$L = \frac{1}{2}\sum_{i,j}(\sigma(\cdot)\,\sigma^*(\cdot))_{ij}(\cdot)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(\cdot)\frac{\partial}{\partial x_i} ,$$

and, for any $x \in \partial G$

$$\frac{\partial u_i}{\partial n}(t,x) = \langle \nabla \ell(x), \nabla u_i(t,x)\rangle$$

is the exterior normal derivative in $x \in \partial G$.

Our goal is to establish a connection between the viscosity solutions for (2.3) and (2.4) respectively. The proof will be given using a probabilistic approach. Therefore we start by studying an SDE with reflecting boundary condition and then we associate a corresponding backward SDE. Since the coefficients of the forward equation are merely continuous, our setting is that of weak formulation of solutions.

For $(t,x) \in [0,T]\times\bar{G}$ we consider the following stochastic differential equation with reflecting boundary condition:

$$\begin{aligned} (i) \quad & X_s^{t,x} + K_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r , \\ (ii) \quad & K_s^{t,x} = \int_t^s \nabla\ell(X_r^{t,x})d\left|K^{t,x}\right|_{[t,r]} , \\ (iii) \quad & \left|K^{t,x}\right|_{[t,s]} = \int_t^s \mathbb{1}_{\{X_r^{t,x}\in\partial G\}}d\left|K^{t,x}\right|_{[t,r]} ,\ \forall s \in [t,T], \end{aligned} \tag{2.5}$$

where $|K^{t,x}|_{[t,s]}$ is the the total variation of $K^{t,x}$ on the interval $[s,t]$ [1].

We denote by $k_s^{t,x}$ the continuous increasing process defined by $k_s^{t,x} := |K^{t,x}|_{[t,s]}$. It follows that

$$k_s^{t,x} = \int_t^s \left\langle \nabla\ell(X_r^{t,x}), dK_r^{t,x} \right\rangle . \tag{2.6}$$

Using the penalization term δ we can define the approximation procedure for the reflected diffusion X.

Under assumption (A_1) we know that (see, e.g., [7, Theorem 5.4.22]), for each $n \in \mathbb{N}^*$, there exists a weak solution of the following penalized SDE

$$X_s^{t,x,n} = x + \int_t^s \left[b(X_r^{t,x,n}) - n\delta(X_r^{t,x,n})\right] dr + \int_t^s \sigma(X_r^{t,x,n}) dW_r \,,\ \forall s \in [t,T] \,. \tag{2.7}$$

Let

$$\begin{aligned} K_s^{t,x,n} &:= \int_t^s n\delta(X_r^{t,x,n}) dr \,, \\ k_s^{t,x,n} &:= \int_t^s \left\langle \nabla\ell(X_r^{t,x,n}), dK_r^{t,x,n} \right\rangle \ , \forall s \in [t,T] \,. \end{aligned} \tag{2.8}$$

We mention that (see, e.g., [7]) the solution process $(X_s^{t,x,n})_{s\in[t,T]}$ is unique in law under the supplementary assumption (A_2).

Here and subsequently, we shall denote by V and V^n:

$$\begin{aligned} V_s^{t,x} &:= x + \int_t^s b(X_r^{t,x}) dr + \int_t^s \sigma(X_r^{t,x}) dW_r \,, \\ V_s^{t,x,n} &:= x + \int_t^s b(X_r^{t,x,n}) dr + \int_t^s \sigma(X_r^{t,x,n}) dW_r \,,\ \forall s \in [t,T] \,. \end{aligned} \tag{2.9}$$

Hence (2.5) and (2.7) become respectively

$$X_s^{t,x} + K_s^{t,x} = V_s^{t,x} \quad \text{and} \quad X_s^{t,x,n} + K_s^{t,x,n} = V_s^{t,x,n} \ ,\ \forall s \in [t,T] \,.$$

Definition 2.2. *We say that $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s\geq t}, W, X, K\right)$ is a weak solution of (2.5) if $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s\geq t}\right)$ is a stochastic basis, W is a d'-dimensional Brownian motion with respect to this basis, X is a continuous adapted process and K is a continuous bounded variation process such that $X_s \in \bar{G}$, $\forall s \in [t,T]$, and system (2.5) is satisfied.*

The main results are the following two theorems. The first one consists in establishing the weak uniqueness (in law) of the solution for (2.5) and the continuous dependence in law with respect to the initial data.

Theorem 2.3. *Under the assumptions $(\mathrm{A}_1 - \mathrm{A}_2)$, there exists a unique weak solution $(X_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ of SDE (2.5). Moreover,*

$$(X^{t,x,n}, K^{t,x,n}) \xrightarrow[u]{*} (X^{t,x}, K^{t,x})$$

and the application

$$[0,T] \times \bar{G} \ni (t,x) \mapsto (X^{t,x}, K^{t,x})$$

is continuous in law.

[1] For $0 \leq s < t \leq T$, the total variation of Y on $[s,t]$ is given by $|Y|_{[s,t]}(\omega) = \sup_{\Delta} \Big\{ \sum_{i=0}^{n-1} |Y_{t_{i+1}}(\omega) - Y_{t_i}(\omega)| \Big\}$, where $\Delta : s = t_0 < t_1 < \cdots < t_n = t$ is a partition of the interval $[s,t]$.

Once this result for the forward part is established we then associate a BSDE involving Stieltjes integral with respect to the increasing process $k^{t,x}$ in order to obtain the probabilistic representation for the viscosity solution of PDE (2.3).

The next result provides the approximation of a viscosity solution for system (2.4) by the solutions sequence of (2.3).

Theorem 2.4. *Under the assumptions* $(\mathrm{A}_1 - \mathrm{A}_3)$*, there exist continuous functions* $u^n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *and* $u : [0,T] \times \bar{G} \to \mathbb{R}^d$ *such that* u^n *is a viscosity solution for system (2.3),* u *is a viscosity solution for system (2.4) with Neumann boundary conditions and, in addition,*

$$\lim_{n\to\infty} u^n(t,x) = u(t,x),\ \forall\,(t,x) \in [0,T] \times \bar{G}.$$

3 Proof of Theorem 2.3

We shall divide the proof of this Theorem into several lemmas. First of all we recall that the existence of a weak solution is given, under assumption (A_1), by [12, Theorem 3.2].

For the simplicity of presentation we suppress from now on the explicit dependence on (t,x) in the notation of the solution of (2.5) and (2.7).

We first prove an estimation result for the solutions of the penalized SDE (2.7).

Lemma 3.1. *Under assumption* (A_1)*, for any* $q \geq 1$*, there exists a constant* $C > 0$*, depending only on* d, T *and* q*, such that*

$$\mathbb{E}\left(\sup\nolimits_{s\in[t,T]} |X_s^n|^{2q}\right) + \mathbb{E}\left(\sup\nolimits_{s\in[t,T]} |K_s^n|^{2q}\right) + \mathbb{E}|K^n|_{[t,T]}^q \leq C,\ \forall n \in \mathbb{N}\,. \tag{3.1}$$

Proof. Without loss of generality we can assume that $0 \in G$. From Itô's formula applied for $|X_s^n|^2$ it can be deduced that

$$|X_s^n|^2 + 2\int_t^s \langle X_r^n, dK_r^n\rangle = |x|^2 + 2\int_t^s \langle X_r^n, b\,(X_r^n)\rangle\, dr + 2\int_t^s \langle X_r^n, \sigma\,(X_r^n)\, dW_r\rangle \\ + \int_t^s |\sigma(X_r^n)|^2 dr,\ s \in [t,T]\,.$$

Write $\tau_m := \inf\{s \in [t,T] : |X_s^n| \geq m\} \wedge T$, $m \in \mathbb{N}^*$, and by the above,

$$|X_{s\wedge\tau_m}^n|^2 + 2\int_t^{s\wedge\tau_m} \langle X_r^n, dK_r^n\rangle \leq C + |x|^2 + C\int_t^{s\wedge\tau_m} |X_r^n|\, dr + 2\int_t^{s\wedge\tau_m} \langle X_r^n, \sigma\,(X_r^n)\, dW_r\rangle\,, \\ s \in [t,T]\,.$$

Here and in what follows $C > 0$ will denote a generic constant which is allowed to vary from line to line.

Therefore

$$\left(|X_{s\wedge\tau_m}^n|^2 + \int_t^{s\wedge\tau_m} \langle X_r^n, dK_r^n\rangle\right)^q \leq C\left(1 + |x|^{2q}\right) + C\left(\int_t^{s\wedge\tau_m} |X_r^n|^2\, dr\right)^q \\ + C\left|\int_t^{s\wedge\tau_m} \langle X_r^n, \sigma\,(X_r^n)\, dW_r\rangle\right|^q,$$

and

$$\mathbb{E}\sup\nolimits_{r\in[t,s]}\left(|X_{r\wedge\tau_m}^n|^2 + \int_t^{r\wedge\tau_m} \langle X_u^n, dK_u^n\rangle\right)^q \leq C\,\left(1 + |x|^{2q}\right) \\ + C\,\mathbb{E}\int_t^{s\wedge\tau_m} \sup\nolimits_{u\in[t,r]} |X_u^n|^{2q}\, dr + C\,\mathbb{E}\sup\nolimits_{r\in[t,s]}\left|\int_t^{r\wedge\tau_m} \langle X_u^n, \sigma\,(X_u^n)\, dW_u\rangle\right|^q. \tag{3.2}$$

By Burkholder-Davis-Gundy inequality we deduce

$$\mathbb{E}\sup_{r\in[t,s]}\left|\int_t^{r\wedge\tau_m}\langle X_u^n,\sigma(X_u^n)\,dW_u\rangle\right|^q\le C\,\mathbb{E}\left|\int_t^{s\wedge\tau_m}|X_u^n|^2\,|\sigma(X_u^n)|^2\,du\right|^{q/2}$$
$$\le C\,\mathbb{E}\left|\int_t^{s\wedge\tau_m}|X_u^n|^2\,du\right|^{q/2}\le C\,\left(1+\mathbb{E}\int_t^{s\wedge\tau_m}\sup_{u\in[t,r]}|X_u^n|^{2q}\,dr\right),$$

and (3.2) yields

$$\mathbb{E}\sup_{r\in[t,s\wedge\tau_m]}|X_r^n|^{2q}\le C\,\left(1+|x|^{2q}+\mathbb{E}\int_t^s\sup_{u\in[t,r\wedge\tau_m]}|X_u^n|^{2q}\,dr\right),\ \forall s\in[t,T],$$

since from (2.1) applied for $z=0\in G$, we have

$$\int_t^s\langle X_r^n,dK_r^n\rangle=n\int_t^s\langle X_r^n,\delta(X_r^n)\rangle\,dr\ge 0.$$

From the Gronwall lemma,

$$\mathbb{E}\sup_{r\in[t,s\wedge\tau_m]}|X_r^n|^{2q}\le C\,\left(1+|x|^{2q}\right),\ \forall\,n\in\mathbb{N}\,.$$

Taking $m\to\infty$ it follows that

$$\mathbb{E}\sup_{r\in[t,T]}|X_r^n|^{2q}\le C,\ \forall\,n\in\mathbb{N}\,. \tag{3.3}$$

Once again from (3.2) and (3.3) we obtain

$$\mathbb{E}\left(\int_t^T\langle X_r^n,dK_r^n\rangle\right)^q\le C\,\left(1+|x|^{2q}\right).$$

We have that there exists $\varepsilon>0$ such that the ball $\bar{B}(0,\varepsilon)\subset G$, and, for $z=\varepsilon\frac{K_s^n-K_t^n}{|K_s^n-K_t^n|}\in G$, inequality (2.1) becomes

$$\varepsilon\,|K_v^n-K_u^n|\le\int_u^v\langle X_r^n,dK_r^n\rangle\,,\ \forall t\le u\le v\le T,$$

and by the definition of total variation of K^n, it follows that

$$\varepsilon^q\mathbb{E}\left(|K^n|_{[t,T]}^q\right)\le\mathbb{E}\Big(\int_t^T\langle X_r^n,dK_r^n\rangle\Big)^q\le C\ .$$

□

Lemma 3.2. *Under assumption* (A_1) *the sequence* $(X_s^n,K_s^n,k_s^n)_{s\in[t,T]}$ *is tight with respect to the* S*-topology.*

Proof. In order to obtain the S-tightness of a sequence of integrable càdlàg processes U^n, $n\ge 1$, we shall use the sufficient condition given e.g. in [10, Appendix A] which consists in proving the uniform boundedness for:

$$\mathrm{CV}_T(U^n)+\mathbb{E}\Big(\sup_{s\in[t,T]}|U_s^n|\Big),$$

where

$$\mathrm{CV}_T(U^n):=\sup_\pi\sum_{i=0}^{m-1}\mathbb{E}\Big[\big|\mathbb{E}[U_{t_{i+1}}^n-U_{t_i}^n/\mathcal{F}_{t_i}]\big|\Big] \tag{3.4}$$

defines the conditional variation of U^n, with the supremum taken over all finite partitions $\pi : t = t_0 < t_1 < \cdots < t_m = T$.

Using Lemma 3.1, we deduce that there exists a constant $C > 0$ such that for every $n \in \mathbb{N}^*$

$$\mathrm{CV}_T\left(K^n\right) + \mathbb{E}\Big(\sup_{s\in[t,T]} |K^n_s|\Big) \le \mathbb{E}\left(|K^n|_{[t,T]}\right) + \mathbb{E}\Big(\sup_{s\in[t,T]} |K^n_s|\Big) \le C.$$

Since k^n is increasing and $l \in C^2_b\left(\mathbb{R}^d\right)$, then there exist constants $M, C > 0$ such that for every $n \in \mathbb{N}^*$

$$\begin{aligned}
\mathrm{CV}_T\left(k^n\right) + \mathbb{E}\Big(\sup_{s\in[t,T]} |k^n_s|\Big) &\le 2\mathbb{E}\left(k^n_T\right) = 2\mathbb{E}\Big(\int_t^T \langle \nabla\ell(X^n_r), dK^n_r\rangle\Big) \\
&\le 2\mathbb{E}\Big(\sup_{s\in[t,T]} |\nabla\ell(X^n_s)| \cdot |K^n|_{[t,T]}\Big) \\
&\le 2M\,\mathbb{E}\left(|K^n|_{[t,T]}\right) \\
&\le 2M\,C.
\end{aligned}$$

By the definition of V^n, assumption (A_1) and the fact the conditional variation of a martingale is 0, we obtain for each $n \in \mathbb{N}^*$,

$$\begin{aligned}
\mathrm{CV}_T\left(V^n\right) &\le \mathrm{CV}_T\left(\int_t^{\cdot} b(X^{t,x,n}_r)dr\right) + \mathrm{CV}_T\left(\int_t^{\cdot} \sigma(X^{t,x,n}_r)dW_r\right) \\
&= \mathrm{CV}_T\left(\int_t^{\cdot} b(X^{t,x,n}_r)dr\right) \le \mathbb{E}\left(\int_t^T \left|b(X^{t,x,n}_r)\right| dr\right) \\
&\le (T-t)\ M \le C
\end{aligned}$$

Therefore (see also Lemma 3.1), there exists $C > 0$ such that for every $n \in \mathbb{N}^*$

$$\mathrm{CV}_T\left(X^n\right) + \mathbb{E}\Big(\sup_{s\in[t,T]} |X^n_s|\Big) \le \mathrm{CV}_T\left(V^n\right) + \mathrm{CV}_T\left(K^n\right) + \mathbb{E}\Big(\sup_{s\in[t,T]} |X^n_s|\Big) \le C.$$

□

Lemma 3.3. *Under the assumptions* $(\mathrm{A}_1 - \mathrm{A}_2)$*, the uniqueness in law of the stochastic process* $(X_s)_{s\in[t,T]}$ *holds.*

Proof. Let $\left(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge 0}, W, X, K\right)$ be a weak solution of (2.5) and $f \in C^{1,2}\left([0,T]\times\bar{G}\right)$. We apply Itô's formula to $f\left(s, X_s\right)$:

$$\begin{aligned}
f\left(s, X_s\right) = f\left(t, x\right) &+ \int_t^s \left(\frac{\partial f}{\partial r} + Lf\right)\left(r, X_r\right)dr - \int_t^s \left\langle \nabla_x f\left(r, X_r\right), \nabla\ell\left(X_r\right)\right\rangle dk_r \\
&+ \int_t^s \left\langle \nabla_x f\left(r, X_r\right), \sigma\left(X_r\right)dW_r\right\rangle.
\end{aligned} \tag{3.5}$$

Since $\sigma\sigma^*$ is supposed to be invertible, we deduce, using Krylov's inequality for the reflecting diffusions (see [18, Theorem 5.1]), that for any $s \in [t, T]$,

$$\begin{aligned}
&\mathbb{E}\int_t^s \left|\left(\frac{\partial f}{\partial r} + Lf\right)\left(r, X_r\right)\right| \mathbb{1}_{\{X_r\in\partial G\}} dr \\
&\quad\le C\ \left(\int_t^s \int_G \det\left(\sigma\sigma^*\right)^{-1}\left(\frac{\partial f}{\partial r} + Lf\right)^{d+1} \mathbb{1}_{\partial G}\ dr dx\right)^{\frac{1}{d+1}} = 0\ .
\end{aligned}$$

Thus, equality (3.5) becomes

$$f(s,X_s) = f(t,x) + \int_t^s \left(\tfrac{\partial f}{\partial r} + Lf\right)(r,X_r)\,\mathbb{1}_{\{X_r\in G\}}dr - \int_t^s \langle \nabla_x f(r,X_r), \nabla\ell(X_r)\rangle\, dk_r$$
$$+\int_t^s \langle \nabla_x f(r,X_r), \sigma(X_r)\,dW_r\rangle \ , \ \mathbb{P}\text{-a.s.}$$

Therefore

$$f(s,X_s) - f(t,x) - \int_t^s \left(\frac{\partial f}{\partial r} + Lf\right)(r,X_r)\,\mathbb{1}_{\{X_r\in G\}}dr$$

is a $\mathbb{P}$-supermartingale whenever $f\in C^{1,2}\left([0,T]\times\bar{G}\right)$ satisfies

$$\langle \nabla_x f(s,x), \nabla\ell(x)\rangle \ge 0, \forall x\in\partial G.$$

From [21, Theorem 5.7] (applied with $\phi = -\ell$, $\gamma := \nabla\phi$ and $\rho := 0$) we have that the solution to the supermartingale problem is unique for each starting point (t,x), therefore our solution process $(X_s)_{s\in[t,T]}$ is unique in law. □

Remark 3.4. *Following the remark of El Karoui [5, Theorem 6] we obtain the uniqueness in law of the couple (X,K), since the increasing process k depends only on the solution X (and not on the Brownian motion). The uniqueness is essential in order to formulate the issue of the continuity with respect to the initial data.*

Lemma 3.5. *We suppose that the assumptions $(\mathrm{A}_1 - \mathrm{A}_2)$ are satisfied. Then*

$$(i) \quad (X^n,K^n) \xrightarrow[u]{*} (X,K),$$
$$(ii) \quad k^n \xrightarrow[u]{*} k.$$

Proof. (i) First we will prove the convergence:

$$(X^n,K^n) \xrightarrow[\mathrm{S}]{*} (X,K). \tag{3.6}$$

We shall apply [9, Theorem 4.3 (iii)]. We recall that we have the uniqueness of the weak solution. For any $n\in\mathbb{N}$, $s\in[t,T]$, let $H_s^n := x\in\bar{G}$ and the processes $Z_s^n := (s,W_s)$. Our equation can be written as

$$X_s^n = H_s^n + \int_t^s \left\langle (b,\sigma)(X_r^{t,x,n}), dZ_s^n\right\rangle - K_s^n\ ,\ \forall s\in[t,T]\,.$$

The processes Z^n satisfy the **(UT)** condition (introduced in [20]), since for any discrete predictable processes $U^n,\bar{U}^n$ of the form $U_s^n = U_0^n + \sum_{i=0}^k U_i^n$, respectively $\bar{U}_s^n = \bar{U}_0^n + \sum_{i=0}^k \bar{U}_i^n$ with $|U_i^n|, |\bar{U}_i^n| \le 1$,

$$\mathbb{E}\left|\int_0^q U_s^n ds + \int_0^q \bar{U}_s^n dW_s\right|^2 \le 2\mathbb{E}\left|\int_0^q U_s^n ds\right|^2 + 2\mathbb{E}\left|\int_0^q \bar{U}_s^n dW_s\right|^2$$
$$\le 2q^2 + 2\mathbb{E}\int_0^q \left|\bar{U}_s^n\right|^2 ds \le 2q\,(q+1)\,.$$

Therefore the assumptions of [9, Theorem 4.3] are satisfied and thus we obtain that

$$X^n \xrightarrow[\mathrm{S}]{*} X.$$

Using once again [9, Theorem 4.3 (ii)] and definition (2.9) we deduce that

$$\left(X_{t_1}^n, X_{t_2}^n, \dots, X_{t_m}^n, V^n\right) \xrightarrow{*} \left(X_{t_1}, X_{t_2}, \dots, X_{t_m}, V\right),$$

for any partition $t = t_0 < t_1 < \cdots < t_m = T$. The above convergence is considered in law, on the space $\left(\mathbb{R}^d\right)^m \times \mathcal{D}([0,T]\,,\mathbb{R}^d)$ endowed with the product between the usual topology on $\left(\mathbb{R}^d\right)^m$ and the Skorohod topology J_1.

Hence

$$(X^n, V^n) \xrightarrow[\mathrm{S}]{*} (X, V)\,,$$

since $(X^n, V^n)_n$ is tight.

It is known that the space $\mathcal{D}([0,T]\,,\mathbb{R}^d)$ of càdlàg functions endowed with S-topology is not a linear topological space, but the sequential continuity of the addition, with respect to the S-topology, is fulfilled (see Jakubowski [6, Remark 3.12]). Therefore

$$K^n = V^n - X^n \xrightarrow[\mathrm{S}]{*} V - X = K.$$

In order to obtain the uniform convergence[4] of the sequence $(X^n, K^n)_n$ we remark that, since V^n, V are continuous and $V^n \xrightarrow[J_1]{*} V$, this convergence is uniform in distribution:

$$V^n \xrightarrow[u]{*} V.$$

Using the Skorohod theorem, there exists a new probability space $\left(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}\right)$ on which we can define random variables $\hat{V}, \hat{V}^n$ such that

$$\hat{V} \overset{law}{=\!=} V, \;\; \hat{V}^n \overset{law}{=\!=} V^n, \; \forall n \in \mathbb{N},$$

and

$$\sup_{s\in[t,T]} |\hat{V}^n_s - \hat{V}_s| \xrightarrow{a.s.} 0.$$

Let $\hat{X}^n$ be the solution of the equation

$$\hat{X}^n_s + \int_t^s n\delta(\hat{X}^n_r)dr = \hat{V}^n_s\,,\; s \in [t,T]\,,$$

$\bar{X}^n$ be the solution of

$$\bar{X}^n_s + \int_t^s n\delta(\bar{X}^n_r)dr = \hat{V}_s\,,\; s \in [t,T]\,,$$

and denote

$$\hat{K}^n_s := \int_t^s n\delta(\hat{X}^n_r)dr,\;\; \bar{K}^n_s := \int_t^s n\delta(\bar{X}^n_r)dr.$$

It is easy to prove (see, e.g., [8, Lemma 2.2] or [22, Lemma 2.2]) that

$$\left|\hat{X}^n_s - \bar{X}^n_s\right|^2 \le \left|\hat{V}^n_s - \hat{V}_s\right|^2 + 2\int_t^s \left\langle (\hat{V}^n_s - \hat{V}_s) - (\hat{V}^n_r - \hat{V}_r), d(\hat{K}^n_r - \bar{K}^n_r)\right\rangle,$$

therefore

$$\sup_{s\in[t,T]} \left|\hat{X}^n_s - \bar{X}^n_s\right|^2 \le \sup_{s\in[t,T]} \left|\hat{V}^n_s - \hat{V}_s\right|^2 + 4\sup_{s\in[t,T]} \left|\hat{V}^n_s - \hat{V}_s\right|\left(|\hat{K}^n|_{[t,T]} + |\bar{K}^n|_{[t,T]}\right). \tag{3.7}$$

Since $(\hat{X}^n, \hat{K}^n) \overset{law}{=\!=} (X^n, K^n)$ and $|K^n|_{[t,T]}$ is bounded in probability by inequality (3.1), $|\hat{K}^n|_{[t,T]}$ is bounded in probability. Applying [8, Theorem 2.7], it follows that $|\bar{K}^n|_{[t,T]}$ is also bounded in probability.

[4] We are thankful to professor L. Słomiński for his useful suggestion in the proof of this part.

But

$$\sup_{s\in[t,T]} \left|\hat{V}_s^n - \hat{V}_s\right| \xrightarrow{prob} 0,$$

therefore, from (3.7),

$$\sup_{s\in[t,T]} \left|\hat{X}_s^n - \bar{X}_s^n\right|^2 \xrightarrow{prob} 0. \tag{3.8}$$

On the other hand, let $\hat{X}$ be the solution of the Skorohod problem

$$\hat{X}_s + \hat{K}_s = \hat{V}_s\,,\ s \in [t,T]\,.$$

It can be shown (see the proof of [12, Theorem 2.1] or the proof of [17, Theorem 4.17]) that

$$\sup_{s\in[t,T]} \left|\bar{X}_s^n - \hat{X}_s\right|^2 \xrightarrow{prob} 0,$$

therefore, from (3.8),

$$\sup_{s\in[t,T]} \left|\hat{X}_s^n - \hat{X}_s\right|^2 \xrightarrow{prob} 0.$$

Since $\hat{K}_s = \hat{V}_s - \hat{X}_s$, $\hat{K}_s^n = \hat{V}_s^n - \hat{X}_s^n$, $s \in [t,T]$,

$$(\hat{X}^n, \hat{K}^n) \xrightarrow[u]{prob} (\hat{X}, \hat{K}),$$

and

$$(\hat{X}^n, \hat{K}^n) \overset{law}{=\!=} (X^n, K^n)\,,$$

the conclusion follows.

(ii) In order to pass to the limit in the integral $\displaystyle\int_t^s \langle \nabla \ell(X_r), dK_r\rangle$, we apply the stochastic version of Helly-Bray theorem given by [23, Proposition 3.4]. For the convenience of the reader we give the statement of that result:

Lemma 3.6. *Let $(X^n, K^n) : (\Omega^n, \mathcal{F}^n, \mathbb{P}^n) \longrightarrow \mathcal{C}\left([0,T]\,,\mathbb{R}^d\right)$ be a sequence of random variables and (X, K) such that*

$$(X^n, K^n) \xrightarrow[u]{*} (X, K)\,.$$

If $(K^n)_n$ has bounded variation a.s. and

$$\sup_{n\in\mathbb{N}^*} \mathbb{P}\left(\left|K^n\right|_{[0,T]} > a\right) \longrightarrow 0,\ \text{as } a \longrightarrow \infty,$$

then K has a.s. bounded variation and

$$\int_0^T \langle X_r^n, dK_r^n\rangle \xrightarrow[u]{*} \int_0^T \langle X_r, dK_r\rangle\,,\ \text{as } n \longrightarrow \infty.$$

Returning to the proof of Lemma 3.5, the conclusion (ii) follows now easily, since k and k^n are defined by (2.6) and (2.8) respectively. □

Remark 3.7. *Let the assumptions $(\mathrm{A}_1{-}\mathrm{A}_2)$ be satisfied. Then the weak solution $(X_s^{t,x})_{s\in[t,T]}$ is a strong Markov process. Indeed, taking into account the equivalence between the existence for the (sub-)martingale problem and the existence of a weak solution for reflected SDE (2.5) (see [5, Theorem 7]), we obtain that the weak solution $(X_s^{t,x})_{s\in[t,T]}$ is a strong Markov process since the uniqueness holds (see [5, Theorem 10]). In our situation, this equivalence can be obtained by using Krylov's inequality for reflecting diffusions.*

The following result will finalize the proof of Theorem 2.3.

We extend the solution process to $[0,T]$ by denoting

$$X_s^{t,x} := x,\ K_s^{t,x} := 0,\ \forall s \in [0,t). \tag{3.9}$$

Lemma 3.8. *We suppose that the assumptions* $(\mathrm{A}_1-\mathrm{A}_2)$ *are satisfied and let* $(X_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ *be the weak solution of (2.5). Then* $(X_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ *is continuous in law with respect to the initial data* (t,x)*.*

Proof. Let $(t,x) \in [0,T] \times \bar{G}$ be fixed and $(t_n, x_n) \to (t,x)$, as $n \to \infty$. We denote

$$(X_s^n, K_s^n) := (X_s^{t_n,x_n}, K_s^{t_n,x_n}).$$

We will prove first that the family (X^n, K^n) is tight as family of $\mathcal{C}([0,T],\mathbb{R}^d \times \mathbb{R}^d)$-valued random variables.

Applying Itô's formula for the process $X_s^n - X_r^n$, where r is fixed and $s \geq r$, we deduce

$$\begin{aligned}
|X_s^n - X_r^n|^2 &= 2\int_r^s \langle X_u^n - X_r^n, b(X_u^n)\rangle\, du - 2\int_r^s \langle X_u^n - X_r^n, dK_u^n\rangle + \int_r^s |\sigma(X_u^n)|^2\, du \\
&\quad + 2\int_r^s \langle X_u^n - X_r^n, \sigma(X_u^n)\, dW_u^n\rangle \\
&\leq 2\int_r^s \langle X_u^n - X_r^n, b(X_u^n)\rangle\, du + \int_r^s |\sigma(X_u^n)|^2\, du + 2\int_r^s \langle X_u^n - X_r^n, \sigma(X_u^n)\, dW_u^n\rangle,
\end{aligned}$$

since $X_u^n, X_r^n \in \bar{G}$ and

$$\int_r^s \langle z - X_u^n, dK_u^n\rangle = \int_r^s \langle z - X_u^n, \nabla\ell(X_u^n)\rangle\, dk_u^n \leq 0,\ \forall\, 0 \leq r \leq s,\ \forall\, z \in \bar{G}.$$

Therefore, using that b, σ are bounded functions and $\bar{G}$ is a bounded domain,

$$\begin{aligned}
\mathbb{E}\Big(|X_s^n - X_r^n|^8\Big) &\leq C|s-r|^4 + C\mathbb{E}\Big(\sup_{v\in[r,s]}\int_r^v \langle X_u^n - X_r^n, \sigma(X_u^n)\, dW_u^n\rangle\Big)^4 \\
&\leq C|s-r|^4 + C\mathbb{E}\Big(\int_r^s |X_u^n - X_r^n|^2 |\sigma(X_u^n)|^2\, du\Big)^2 \\
&\leq C|s-r|^4 + C|s-r|^2 \leq C\max\Big(|s-r|^4, |s-r|^2\Big).
\end{aligned} \tag{3.10}$$

Concerning K, we remark first that

$$K_s^n - K_r^n = \int_r^s b(X_u^n)\, du + \int_r^s \sigma(X_u^n)\, dW_u^n - (X_s^n - X_r^n).$$

Hence

$$\begin{aligned}
\mathbb{E}\Big(|K_s^n - K_r^n|^8\Big) &\leq C\mathbb{E}\Big(|X_s^n - X_r^n|^8\Big) + C\mathbb{E}\Big(\int_r^s b(X_u^n)\, du\Big)^8 \\
&\quad + C\mathbb{E}\Big(\sup_{v\in[r,s]}\int_r^v \sigma(X_u^n)\, dW_u^n\Big)^8 \\
&\leq C\max\Big(|s-r|^4, |s-r|^2\Big) + C|s-r|^8 + C\mathbb{E}\Big(\int_r^s |\sigma(X_u^n)|^2\, du\Big)^4 \\
&\leq C\max\Big(|s-r|^8, |s-r|^2\Big).
\end{aligned} \tag{3.11}$$

Observe that the constants in the right hand of the inequalities (3.10) and (3.11) do not depend on (t,x). Therefore, applying a tightness criterion (see, e.g. [17, Cap. I]) we deduce that the family $(X^{t,x},K^{t,x})$ is tight (viewed as a family of $\mathcal{C}([0,T],\mathbb{R}^d\times\mathbb{R}^d)$-valued random variables) with respect to the initial data (t,x).

Taking into account the above conclusion and the Prokhorov theorem, we have that if $(t_n,x_n)\to(t,x)$, as $n\to\infty$, then there exists a subsequence, still denoted by (t_n,x_n), such that

$$X^n:=X^{t_n,x_n}\xrightarrow[u]{*}X,\quad K^n:=K^{t_n,x_n}\xrightarrow[u]{*}K,\text{ as }n\to\infty.$$

It remains to identify the limits, i.e. $X\overset{law}{=\!=}X^{t,x}$ and $K\overset{law}{=\!=}K^{t,x}$.

By the Skorohod theorem, we can choose a probability space $\big(\hat{\Omega},\hat{\mathcal{F}},\hat{\mathbb{P}}\big)$ (which can be taken in fact as $\big([0,1],\mathcal{B}_{[0,1]},\mu\big)$ where μ is the Lebesgue measure), and $(\hat{X}^n,\hat{K}^n,\hat{W}^n)$, $(\hat{X},\hat{K},\hat{W})$ defined on this probability space, such that

$$(\hat{X}^n,\hat{K}^n,\hat{W}^n)\overset{law}{=\!=}(X^n,K^n,W^n)\,,\qquad (\hat{X},\hat{K},\hat{W})\overset{law}{=\!=}(X,K,W)$$

and

$$(\hat{X}^n,\hat{K}^n,\hat{W}^n)\xrightarrow{a.s.}(\hat{X},\hat{K},\hat{W}),\ \text{ as }n\to\infty.$$

It is not difficult to see that $\big(\hat{W}^n,\mathcal{F}_t^{\hat{W}^n,\hat{X}^n}\big)$ and $\big(\hat{W},\mathcal{F}_t^{\hat{W},\hat{X}}\big)$ are Brownian motions.

We now define,

$$\begin{aligned}\hat{V}_s^n&:=x+\int_t^s b(\hat{X}_r^n)dr+\int_t^s\sigma(\hat{X}_r^n)d\hat{W}_r^n\ \text{ and}\\ \hat{V}_s&:=x+\int_t^s b(\hat{X}_r)dr+\int_t^s\sigma(\hat{X}_r)d\hat{W}_r\,,\ s\in[t,T]\,.\end{aligned}$$

Arguing as in the proof of [1, Proposition 12] (see also [19, Chapter III-3] for more details), it can be shown that $\int_t^s\sigma(\hat{X}_r^n)d\hat{W}_r^n$ and $\int_t^s b(\hat{X}_r^n)dr$ converge in probability to $\int_t^s\sigma(\hat{X}_r)dW_r$ and respectively $\int_t^s b(\hat{X}_r)dr$. Since σ and b are bounded, we deduce using the Lebesgue theorem that this convergence holds in $L^q(\hat{\Omega})$ for each $q\geq1$. Therefore,

$$\mathbb{E}\big(\sup_{s\in[t,T]}\big|\hat{V}_s^n-\hat{V}_s\big|^q\big)\to0,\ \text{as }n\to\infty,$$

If V^n is defined by

$$V_s^n:=x+\int_t^s b(X_r^n)dr+\int_t^s\sigma(X_r^n)dW_r^n$$

then $X_s^n+K_s^n=V_s^n,$ $\mathbb{P}$-a.s. And it is not difficult to see that

$$(X^n,K^n,W^n,V^n)\overset{law}{=\!=}(\hat{X}^n,\hat{K}^n,\hat{W}^n,\hat{V}^n)\ \text{ on }\ \mathcal{C}([0,T]\,,\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^{d'}\times\mathbb{R}^d)$$

and

$$\hat{X}_s^n+\hat{K}_s^n=\hat{V}_s^n,\ \text{ a.s.}$$

which yields, passing to the limit, that

$$\hat{X}_s+\hat{K}_s=\hat{V}_s,\ \text{a.s.}$$

Then the coupled process $(\hat{X}_s,\hat{K}_s)$ is a solution of (2.5) corresponding to the initial data (t,x). Taking into account the uniqueness in law of the solution $(X_s^{t,x},K_s^{t,x})_{s\in[t,T]}$ (see Remark 3.4) we deduce that the whole sequence $(X_s^n,K_s^n)_{s\in[t,T]}$ converges to the process $(X_s^{t,x},K_s^{t,x})_{s\in[t,T]}$, and therefore the continuity with respect to (t,x) follows. □

4 BSDEs and nonlinear Neumann boundary problem

Let us now consider the processes $(X_s^{t,x,n}, k_s^{t,x,n})_{t\le s\le T}$ and $(X_s^{t,x}, k_s^{t,x})_{t\le s\le T}$ given by relations (2.5) - (2.8), for $(t,x)\in[0,T]\times\bar{G}$.

For the proof of Theorem 2.4 we associate the following generalized backward stochastic differential equations (BSDEs for short) on $[t,T]$:

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x})dr - \int_s^T U_r^{t,x} dM_r^{X^{t,x}} - \int_s^T h(r, X_r^{t,x}, Y_r^{t,x}) dk_r^{t,x}\,, \tag{4.1}$$

and respectively the BSDE corresponding to the solution of (2.7)

$$\begin{aligned} Y_s^{t,x,n} = g(X_T^{t,x,n}) + \int_s^T f(r, X_r^{t,x,n}, Y_r^{t,x,n})dr - \int_s^T U_r^{t,x,n} dM_r^{X^{t,x,n}} \\ - \int_s^T h(r, X_r^{t,x,n}, Y_r^{t,x,n}) dk_r^{t,x,n}\,, \end{aligned} \tag{4.2}$$

where

$$M_s^{X^{t,x}} := \int_t^s \sigma(X_r^{t,x})dW_r\,, \quad M_s^{X^{t,x,n}} := \int_t^s \sigma(X_r^{t,x,n})dW_r \tag{4.3}$$

are the martingale part of the reflected diffusion process $X^{t,x}$ and $X^{t,x,n}$ respectively. We assume for simplicity that the processes $(X_s^{t,x,n}, K_s^{t,x,n})_{s\in[t,T]}$ and $(X_s^{t,x}, K_s^{t,x})_{s\in[t,T]}$ are considered on the canonical space.

We recall that the coefficients f, g and h satisfy assumption (A_3). Then, given the processes $(X_s^{t,x,n}, k_s^{t,x,n})_{s\in[t,T]}$ and $(X_s^{t,x}, k_s^{t,x})_{s\in[t,T]}$, this assumption ensures (see [16]) the existence and the uniqueness for the couples $(Y_s^{t,x,n}, U_s^{t,x,n})_{s\in[t,T]}$ and $(Y_s^{t,x}, U_s^{t,x})_{s\in[t,T]}$ respectively. Arguing as in [3], one can establish the following result.

Proposition 4.1. *Let the assumptions* $(\mathrm{A}_1 - \mathrm{A}_3)$ *be satisfied.*
Let $(Y_s^{t,x,n}, U_s^{t,x,n})_{s\in[t,T]}$ *and* $(Y_s^{t,x}, U_s^{t,x})_{s\in[t,T]}$ *be the solutions of the BSDEs (4.2) and (4.1), respectively. Then*

$$\left(Y^{t,x,n}, M^{t,x,n}, H^{t,x,n}\right) \xrightarrow[\mathrm{S}\times\mathrm{S}\times\mathrm{S}]{*} \left(Y^{t,x}, M^{t,x}, H^{t,x}\right),$$

where

$$\begin{aligned} M_s^{t,x,n} &:= \int_t^s U_r^{t,x,n} dM_r^{X^{t,x,n}}\,,\ \ H_s^{t,x,n} := \int_0^s h(r, X_r^{t,x,n}, Y_r^{t,x,n}) dk_r^{t,x,n}, \\ M_s^{t,x} &:= \int_t^s U_r^{t,x} dM_r^{X^{t,x}}\,,\ \ H_s^{t,x} := \int_0^s h(r, X_r^{t,x}, Y_r^{t,x}) dk_r^{t,x} \end{aligned} \tag{4.4}$$

and $M^{X^{t,x,n}}$ *and* $M^{X^{t,x}}$ *are defined by (4.3).*

Moreover, we have that $\lim_{n\to\infty} Y_t^{t,x,n} = Y_t^{t,x}$.

Remark 4.2. *The solution process* $(Y_s^{t,x})_{s\in[t,T]}$ *is unique in law. Indeed, following [4, Theorem 3.4], it can be proven that, since the coefficients* b *and* σ *satisfy the assumptions* $(\mathrm{A}_1 - \mathrm{A}_2)$ *and the solution process has the Markov property, there exists a deterministic measurable function* u *such that the solution* $Y_s^{t,x} = u(s, X_s^{t,x})$, $s\in[t,T]$ $d\mathbb{P}\otimes ds$ *a.s. The conclusion follows by Lemma 3.3 and the uniqueness (as a strong solution) of* Y.

In the following, we extend $X^{t,x}, K^{t,x}$ to $[0,T]$ as in (3.9) and $(Y^{t,x}, U^{t,x})$ by denoting

$$Y_s^{t,x} := Y_t^{t,x},\ U_s^{t,x} := 0 \quad \text{and} \quad M_s^{X^{t,x}} := 0,\ \forall s\in[0,t).$$

Proposition 4.3. *Let $(t_n, x_n) \to (t, x)$, as $n \to \infty$. Then there exists a subsequence $(t_{n_k}, x_{n_k})_{k\in\mathbb{N}}$ such that $Y^{t_{n_k},x_{n_k}} \xrightarrow[\mathrm{S}]{*} Y^{t,x}$.*

Proof. The proof will follow the techniques used in [3, Theorem 3.1] (see also [15, Theorem 6.1]).

It is clear that

$$\begin{aligned} Y_s^{t_n,x_n} = g(X_T^{t_n,x_n}) + \int_s^T \mathbb{1}_{[t_n,T]}(r)\, f(r, X_r^{t_n,x_n}, Y_r^{t_n,x_n})dr - \int_s^T U_r^{t_n,x_n} dM_r^{X^{t_n,x_n}} \\ - \int_s^T h(r, X_r^{t_n,x_n}, Y_r^{t_n,x_n})dk_r^{t_n,x_n}\,,\ s \in [0,T]. \end{aligned} \tag{4.5}$$

For the proof we will adapt the steps from the proof of [3, Theorem 3.1].

Step 1. The solutions satisfy the boundedness conditions (for the proof see, e.g., [16, Proposition 1.1]):

$$\mathbb{E}\big(\sup_{s\in[t,T]} |Y_s^{t_n,x_n}|^2\big) + \mathbb{E}\int_t^T ||U_s^{t_n,x_n}\sigma(X_r^{t_n,x_n})||^2 ds \le C,\ \forall t \in [0,T]\,,\ \forall n \in \mathbb{N}$$

$$\mathbb{E}\big(\sup_{s\in[t,T]} |Y_s^{t,x}|^2\big) + \mathbb{E}\int_t^T ||U_s^{t,x}\sigma(X_r^{t,x})||^2 ds \le C,\ \forall t \in [0,T]\,,$$

where $C > 0$ is a constant not depending on n.

Step 2. To obtain the tightness property with respect to the S-topology it is sufficient to compute the conditional variation CV_T (see definition (3.4)) of the processes Y^{t_n,x_n}, M^{t_n,x_n} and H^{t_n,x_n} respectively; we recall the notation (4.4) for the quantities M^{t_n,x_n} and H^{t_n,x_n}.

As in [3, Theorem 3.1], after some easy computation we deduce that there exists a constant $C > 0$ independent of n, such that

$$\begin{aligned} \mathrm{CV}_T\,(Y^{t_n,x_n}) + \mathbb{E}\Big(\sup_{s\in[0,T]} |Y_s^{t_n,x_n}|\Big) + \mathbb{E}\Big(\sup_{s\in[0,T]} |M_s^{t_n,x_n}|\Big) + \mathrm{CV}_T\,(H^{t_n,x_n}) \\ + \mathbb{E}\Big(\sup_{s\in[0,T]} |H_s^{t_n,x_n}|\Big) \le C,\ \forall n \in \mathbb{N}^*. \end{aligned}$$

Step 3. The above condition ensures (see [10, Appendix A] or [3, Theorem 3.5]) the tightness of the sequence $(Y^{t_n,x_n}, M^{t_n,x_n}, H^{t_n,x_n})$ with respect to the S-topology. Therefore there exists a subsequence, still denoted by $(Y^{t_n,x_n}, M^{t_n,x_n}, H^{t_n,x_n})$, and a process $\big(\bar{Y},\, \bar{M},\, \bar{H}\big) \in \big(\mathcal{D}\left([0,T], \mathbb{R}^k\right)\big)^3$ such that

$$\big(X^{t_n,x_n}, K^{t_n,x_n}, Y^{t_n,x_n}, M^{t_n,x_n}, H^{t_n,x_n}\big) \xrightarrow[\mathrm{U\times U\times S\times S\times S}]{*} \big(X^{t,x}, K^{t,x}, \bar{Y}, \bar{M}, \bar{H}\big)\,, \tag{4.6}$$

weakly on $(\mathcal{C}([0,T], \mathbb{R}^d))^2 \times \big(\mathcal{D}([0,T], \mathbb{R}^k)\big)^3$.

In order to pass to the limit in (4.5) we use the continuity of f, [6, Corollary 2.11], the Lipschitzianity of h,

$$k^{t_n,x_n} \xrightarrow[u]{*} k^{t,x},$$

and we apply [3, Lemma 3.3]; we precise that the conclusion of this lemma is still true in the point T, hence there exists a countable set $Q \subset [0,T)$ such that, for any $s \in [0,T]\backslash Q$,

$$\bar{Y}_s = g(X_T^{t,x}) + \int_s^T \mathbb{1}_{[t,T]}(r)\, f(r, X_r^{t,x}, \bar{Y}_r)dr - (\bar{M}_T - \bar{M}_s) - \int_s^T h(r, X_r^{t,x}, \bar{Y}_r)dk_r\ .$$

Since the processes $\bar{Y}$, $\bar{M}$ and $\bar{H}$ are càdlàg, the above equality holds true for any $s \in [0,T]$.

We mention that $M^{X^{t,x}}$ and $\bar{M}$ are martingales with respect to the same filtration. Indeed, $\bar{M}_s$ is $\mathcal{F}_s^{X^{t,x},\bar{Y},\bar{M}}$-adapted and, moreover, $\bar{M}$ is an $\mathcal{F}^{X^{t,x},\bar{Y},\bar{M}}$-martingale (for the proof see [10, Lemma A.1]).

Let now ψ_s be a bounded continuous mapping from $\mathcal{C}([0,s],\mathbb{R}^d)\times\mathcal{D}([0,s],\mathbb{R}^k)^2$, $\varphi\in C^\infty(\mathbb{R}^d)$ and

$$L=\frac{1}{2}\sum_{i,j}(\sigma(\cdot)\,\sigma^*(\cdot))_{ij}(\cdot)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_i b_i(\cdot)\frac{\partial}{\partial x_i}$$

be the infinitesimal generator of the diffusion process X^{t_n,x_n}.

From Itô's formula we obtain that

$$\varphi(X_s^{t_n,x_n})-\varphi(x_n)-\int_{t_n}^s L\varphi(X_r^{t_n,x_n})dr+\int_{t_n}^s\nabla\varphi(X_r^{t_n,x_n})dK_r^{t_n,x_n}$$

is a martingale.

Therefore, for any $0\le s_1<s_2\le T$,

$$\mathbb{E}\Big[\psi_{s_1}\left(X^{t_n,x_n},Y^{t_n,x_n},M^{t_n,x_n}\right)\Big(\varphi(X_{s_2}^{t_n,x_n})-\varphi(X_{s_1}^{t_n,x_n})-\int_{s_1\vee t_n}^{s_2\vee t_n}L\varphi(X_r^{t_n,x_n})dr$$
$$+\int_{s_1\vee t_n}^{s_2\vee t_n}\nabla\varphi(X_r^{t_n,x_n})dK_r^{t_n,x_n}\Big)\Big]=0,\ \forall n.$$

It can be proved, using (4.6), that

$$\lim_{n\to\infty}\mathbb{E}\Big[\psi_{s_1}\left(X^{t_n,x_n},Y^{t_n,x_n},M^{t_n,x_n}\right)\Big(\varphi(X_{s_2}^{t_n,x_n})-\varphi(X_{s_1}^{t_n,x_n})-\int_{s_1\vee t_n}^{s_2\vee t_n}L\varphi(X_r^{t_n,x_n})dr\Big)\Big]$$
$$=\mathbb{E}\Big[\psi_{s_1}\left(X^{t,x},\bar{Y},\bar{M}\right)\Big(\varphi(X_{s_2}^{t,x})-\varphi(X_{s_1}^{t,x})-\int_{s_1\vee t}^{s_2\vee t}L\varphi(X_r^{t,x})dr\Big)\Big].$$

On the other hand,

$$\lim_{n\to\infty}\mathbb{E}\Big[\psi_{s_1}\left(X^{t_n,x_n},Y^{t_n,x_n},M^{t_n,x_n}\right)\int_{s_1\vee t_n}^{s_2\vee t_n}\nabla\varphi(X_r^{t_n,x_n})dK_r^{t_n,x_n}\Big]$$
$$=\mathbb{E}\Big[\psi_{s_1}\left(X^{t,x},\bar{Y},\bar{M}\right)\int_{s_1\vee t}^{s_2\vee t}\nabla\varphi(X_r^{t,x})dK_r^{t,x}\Big],$$

by [23, Proposition 3.4].

Therefore

$$\mathbb{E}\Big[\psi_{s_1}\left(X^{t,x},\bar{Y},\bar{M}\right)\Big(\varphi(X_{s_2}^{t,x})-\varphi(X_{s_1}^{t,x})-\int_{s_1\vee t}^{s_2\vee t}L\varphi(X_r^{t,x})dr\Big)\Big]$$
$$+\int_{s_1\vee t}^{s_2\vee t}\nabla\varphi(X_r^{t,x})dK_r^{t,x}\Big)\Big]=0.$$

Using Itô's formula we see that

$$\mathbb{E}\left[\psi_{s_1}\left(X^{t,x},\bar{Y},\bar{M}\right)\int_{s_1\vee t}^{s_2\vee t}\nabla\varphi(X_r^{t,x})dM_r^{X^{t,x}}\right]=0$$

and therefore $M^{X^{t,x}}$ is a $\mathcal{F}^{X^{t,x},\bar{Y},\bar{M}}$-martingale.

Now since $Y^{t,x}$ and $U^{t,x}$ are $\mathcal{F}^{X^{t,x}}$-adapted, $M^{t,x}:=\int_t^\cdot U_r^{t,x}dM_r^{X^{t,x}}$ is also $\mathcal{F}^{X^{t,x},\bar{Y},\bar{M}}$-martingale.

Let us take $0 \le s_1 \le s_2 \le T$. Itô's formula yields

$$\begin{aligned}
&|Y_{s_1}^{t,x} - \bar{Y}_{s_1}|^2 + \left([M^{t,x} - \bar{M}]_{s_2} - [M^{t,x} - \bar{M}]_{s_1}\right) = |Y_{s_2}^{t,x} - \bar{Y}_{s_2}|^2 \\
&\quad +2\int_{s_1}^{s_2} \left\langle Y_r^{t,x} - \bar{Y}_r, f(r, X_r^{t,x}, Y_r^{t,x}) - f(r, X_r^{t,x}, \bar{Y}_r)\right\rangle dr \\
&\quad +2\int_{s_1}^{s_2} \left\langle Y_r^{t,x} - \bar{Y}_r, h(r, X_r^{t,x}, Y_r^{t,x}) - h(r, X_r^{t,x}, \bar{Y}_r)\right\rangle dk_r^{t,x} \\
&\quad -2\int_{s_1}^{s_2} \left\langle Y_r^{t,x} - \bar{Y}_r, d(M_r^{t,x} - \bar{M}_r)\right\rangle \\
&\le |Y_{s_2}^{t,x} - \bar{Y}_{s_2}|^2 + 2\alpha \vee \beta \int_{s_1}^{s_2} |Y_r^{t,x} - \bar{Y}_r|^2 d(r + k_r^{t,x}) - 2\int_{s_1}^{s_2} \left\langle Y_r^{t,x} - \bar{Y}_r, d(M_r^{t,x} - \bar{M}_r)\right\rangle .
\end{aligned}$$

since $\int_t^{\cdot} \left\langle Y_r^{t,x} - \bar{Y}_r, d(M_r^{t,x} - \bar{M}_r)\right\rangle$ is a $\mathcal{F}^{X^{t,x},\bar{Y},\bar{M}}$-martingale.

Hence, from a generalized Gronwall lemma (see, e.g., [13, Lemma 12]), by taking $s_2 = T$, we deduce the identification

$$Y^{t,x} = \bar{Y} \ \text{ and } \ M^{t,x} = \bar{M}.$$

□

4.1 Proof of Theorem 2.4

Let us denote

$$u^n(t,x) := Y_t^{t,x,n} \quad \text{and} \quad u(t,x) := Y_t^{t,x}. \tag{4.7}$$

Hence, u^n and u are deterministic functions since $Y^{t,x,n}$ is adapted with respect to the filtration generated by $X^{t,x,n}$ and $Y^{t,x}$ is adapted with respect to the filtration generated by $X^{t,x}$.

First we prove that the functions $u^n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $u : [0,T] \times \bar{G} \to \mathbb{R}^d$ defined by (4.7) are continuous. We will show only that the function u is continuous. Let $(t_n, x_n) \to (t,x) \in [0,T] \times \bar{G}$, as $n \to \infty$. From the proof of Proposition 4.3, we can extract a subsequence still denoted (t_n, x_n), such that

$$\left(X^{t_n,x_n}, K^{t_n,x_n}, Y^{t_n,x_n}, M^{t_n,x_n}\right) \xrightarrow[\mathrm{U}\times\mathrm{U}\times\mathrm{S}\times\mathrm{S}]{*} \left(X^{t,x}, K^{t,x}, Y^{t,x}, M^{t,x}\right).$$

We know from [3, Lemma 3.3] applied for $t = T$, that

$$\int_0^T h(r, X_r^{t_n,x_n}, Y_r^{t_n,x_n}) dk_r^{t_n,x_n} \to \int_0^T h(r, X_r^{t,x}, Y_r^{t,x}) dk_r^{t,x} \text{ in law, as } n \to \infty.$$

Using [6, Remark 2.4], we see that $M_T^{t_n,x_n} \longrightarrow M_T^{t,x}$ in law, since $M^{t_n,x_n} \xrightarrow[\mathrm{S}]{*} M^{t,x}$.

Hence we can pass to the limit in

$$\begin{aligned}
u(t_n, x_n) = Y_{t_n}^{t_n,x_n} = Y_0^{t_n,x_n} = g(X_T^{t_n,x_n}) + \int_0^T \mathbb{1}_{[t_n,T]}(r) f(r, X_r^{t_n,x_n}, Y_r^{t_n,x_n}) dr - M_T^{t_n,x_n} \\
- \int_0^T h(r, X_r^{t_n,x_n}, Y_r^{t_n,x_n}) dk_r^{t_n,x_n}
\end{aligned}$$

and, as in the the proof of Proposition 4.3, we deduce that the limit of $u(t_n, x_n)$ is

$$\begin{aligned}
g(X_T^{t,x}) + \int_0^T \mathbb{1}_{[t,T]}(r) f(r, X_r^{t,x}, Y_r^{t,x}) dr - M_T^{t,x} - \int_0^T h(r, X_r^{t,x}, Y_r^{t,x}) dk_r^{t,x} \\
= Y_0^{t,x} = Y_t^{t,x} = u(t,x).
\end{aligned}$$

It is easy to show that, even if b and σ are only continuous functions, the proof from [16, Theorem 4.3] (see also [15, Theorem 3.2] for nonreflecting case) still works in order to show that the functions u^n and u defined by (4.7) are viscosity solutions of the PDEs (2.3) and (2.4) respectively.

Finally, as a consequence of Proposition 4.1 we deduce the solution u of the deterministic system (2.4) is approximated by the functions u^n, i.e.

$$\lim_{n\to\infty} u^n(t,x) = u(t,x),\ \forall\, (t,x) \in [0,T] \times \bar{G}.$$

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