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Random walk in random environment in a two-dimensional stratified medium with orientations*

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Abstract

We consider a model of random walk in $\mathbb{Z}^2$ with (fixed or random) orientation of the horizontal lines (layers) and with non constant iid probability to stay on these lines. We prove the transience of the walk for any fixed orientations under general hypotheses. This contrasts with the model of Campanino and Petritis [3], in which probabilities to stay on these lines are all equal. We also establish a result of convergence in distribution for this walk with suitable normalizations under more precise assumptions. In particular, our model proves to be, in many cases, even more superdiffusive than the random walks introduced by Campanino and Petritis.

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1 Introduction

We consider a random walk $(M_n)_n$ starting from 0 on an oriented version of $\mathbb{Z}^2$. Let $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}}$ be a sequence of random variables with values in $\{-1, 1\}$ and joint distribution μ. We assume that the k^{th} horizontal line is entirely oriented to the right if $\varepsilon_k = 1$, and to the left if $\varepsilon_k = -1$. We suppose that the probabilities p_k to stay on the k^{th} horizontal line are given by a sequence of independent identically distributed random variables $\omega = (p_k)_{k\in\mathbb{Z}}$ (with values in $(0, 1)$ and joint distribution κ) and that the probabilities to go up or down are equal. More precisely, given ε and ω, the process $(M_n = (M_n^{(1)}, M_n^{(2)}))_n$ is a Markov chain satisfying $M_0 = (0, 0)$ with transition probabilities given by

$$\mathbb{P}^{\varepsilon,\omega}(M_{n+1} - M_n = (\varepsilon_{M_n^{(2)}}, 0)\,|M_0, ..., M_n) = p_{M_n^{(2)}}$$

$$\text{and} \qquad \forall y \in \{-1, 1\}, \quad \mathbb{P}^{\varepsilon,\omega}(M_{n+1} - M_n = (0, y)\,|M_0, ..., M_n) = \frac{1 - p_{M_n^{(2)}}}{2}.$$

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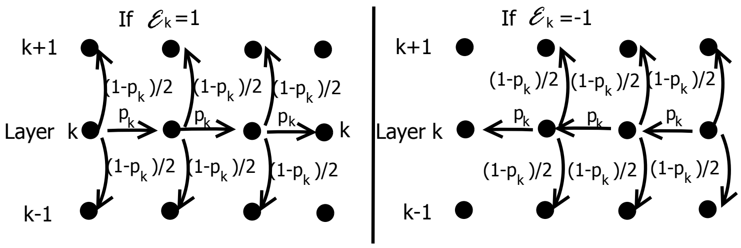

We also define the annealed probability $\mathbb{P}$ as follows:

$$\mathbb{P}(.) := \int \mathbb{P}^{\varepsilon,\omega}(.) d\kappa(\omega)\, d\mu(\varepsilon).$$

We denote by $\mathbb{E}$ and $\mathbb{E}^{\varepsilon,\omega}$ the expectations with regard to $\mathbb{P}$ and $\mathbb{P}^{\varepsilon,\omega}$ respectively.

Our model corresponds to a random walk in a two dimensional stratified medium with oriented horizontal layers and with random probability to stay on each layer.

The model with $p_k = 1/2$ and with the $\varepsilon_k' s$ iid and centered can be seen as a discrete version of a model introduced by G. Matheron and G. de Marsily in [17] to modelize transport in a stratified porus medium. This discrete model appears in [2] to simulate the Matheron and de Marsily model. It has also been introduced, separately, by mathematicians with motivations linked to quantum field theory or propagation on large networks (see respectively [3] and [4] and references therein).

In [3], M. Campanino and D. Petritis proved that, when the p_k's are all equal, the behavior of the walk $(M_n)_n$ depends on the choice of the orientations $(\varepsilon_k)_k$. First, they prove that the walk is recurrent when $\varepsilon_k = (-1)^k$ (i.e. when the horizontal even lines are oriented to the right and the uneven to the left). Second, they prove that the walk is almost surely transient when the $\varepsilon_k' s$ are iid and centered. These results have been recently improved in [4]. Let us mention that extensions of this second model can be found in [8, 19], and that its Martin boundary is computed in [15].

In order to take into account the different nature of the successive layers of a stratified porus medium, it is natural to study the case where the p_k's are random instead of being all equal. In this paper, we prove that taking the p_k's random and i.i.d. can induce very different behaviors for the random walk.

First, we prove that under general hypotheses, the random walk is transient for every deterministic or random orientations, contrarily to the results obtained by Campanino and Petritis in [3] and [4] for their model. Hence, even very small random perturbations of their (constant) p_k's transform their recurrent walks into transient ones.

Second, it was proved in [9] that when the p_k's are all equal, the random walk is superdiffusive, and that the horizontal position at time n is, asymptotically, of order $n^{3/4}$. This was conjectured in [17] and was one main motivation for the introduction of this model. We prove that, depending on the law of p_0, our model can be even more superdiffusive, with horizontal position at time n of order n^δ, where δ can take all the values in $[3/4, 1)$.

More precisely, our results are the following. We start by stating our theorem about transience.

Theorem 1.1 (Transience)**.** *Let $(p_k)_k$ be a sequence of independent identically distributed random variables. Suppose here that p_0 is non-constant and that $\mathbb{E}[(1-p_0)^{-\alpha}] < \infty$ (for some $\alpha > 1$). Then, for every deterministic or random sequence $(\varepsilon_k)_k$, the random walk $(M_n)_n$ is transient for almost every ω.*

We now give a functional theorem under more precise hypotheses. In particular, we will assume that $\frac{p_0}{1-p_0}$ is integrable and that the distribution of $\frac{p_0}{1-p_0} - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]$ belongs to the normal domain of attraction of a strictly stable distribution G_β of index $\beta \in (1,2]$, which means that

$$\mathbb{P}\left(n^{-1/\beta}\sum_{k=1}^{n}\left(\frac{p_k}{1-p_k} - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]\right) \le x\right) \to_{n\to+\infty} G_\beta(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

the characteristic function ζ_β of G_β being of the form

$$\zeta_\beta(\theta) := \exp[-|\theta|^\beta(A_1 + iA_2\mathrm{sgn}(\theta))], \qquad \theta \in \mathbb{R}, \tag{1.2}$$

with $A_1 > 0$ and $|A_1^{-1}A_2| \le |\tan(\pi\beta/2)|$. Notice that this is possible iff $A_2 = A_1\tan(\pi\beta/2)$ (since $\frac{p_0}{1-p_0} \ge 0$ a.s., see e.g. [12, thm 2.6.7]).

If $\beta \in (1,2)$, we consider two independent right continuous stable processes $(Z_x,\ x \ge 0)$ and $(Z_{-x},\ x \ge 0)$, with characteristic functions

$$\mathbb{E}(e^{i\theta Z_t}) = \exp[-A_1|t||\theta|^\beta], \qquad t \in \mathbb{R}, \theta \in \mathbb{R}.$$

If $\beta = 2$, we denote by Z a two-sided standard Brownian motion. We also introduce a standard Brownian motion $(B_t,\ t \ge 0)$, and denote by $(L_t(x),\ x \in \mathbb{R},\ t \ge 0)$ the jointly continuous version of its local time. We assume that Z and B are defined in the same probability space and are independent processes. We now define, as in [14], the continuous process

$$\Delta_t := \int_{\mathbb{R}} L_t(x)\, dZ_x, \qquad t \ge 0.$$

We prove the following result.

Theorem 1.2 (Functional limit theorem)**.** *Let $(p_k)_k$ be a sequence of independent identically distributed random variables with values in $(0,1)$. Suppose here that $\mathbb{E}\left[\frac{p_0}{1-p_0}\right] < \infty$ and that the distribution of $\frac{p_0}{1-p_0} - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]$ belongs to the normal domain of attraction of a strictly stable distribution of index $\beta \in (1,2]$ (i.e. that we have* (1.1) *and* (1.2)*).*

We also assume that $(\varepsilon_k)_k$ satisfies one of the following hypotheses :

(a) for every k, $\varepsilon_k = (-1)^k$,

(b) $(\varepsilon_k)_k$ is a sequence of independent identically distributed centered random variables with values in $\{\pm 1\}$; $(\varepsilon_k)_k$ is independent of $(p_k)_k$.

Then, setting $\delta := \frac{1}{2} + \frac{1}{2\beta}$, the sequence of processes

$$\left(\left(n^{-\delta}M^{(1)}_{\lfloor nt\rfloor}, n^{-1/2}M^{(2)}_{\lfloor nt\rfloor}\right)_{t\ge 0}\right)_n$$

converges in distribution under the annealed probability $\mathbb{P}$ (in the space of Skorokhod $\mathcal{D}([0;+\infty),\mathbb{R}^2)$) to $(\gamma^{-\delta}\sigma\Delta_t, \gamma^{-1/2}B_t)_{t\ge 0}$ with $\gamma := 1 + \mathbb{E}\left[\frac{p_0}{1-p_0}\right]$ and with :

* $\sigma = \left(Var\left(\frac{p_0}{1-p_0}\right)\right)^{1/2}$ *in case (a) with $\beta = 2$,*
* $\sigma = \left(\mathbb{E}\left[\left(\frac{p_0}{1-p_0}\right)^2\right]\right)^{1/2}$ *in case (b) with $\beta = 2$,*
* $\sigma = 1$ *in cases (a) or (b) with $\beta \in (1,2)$.*

We remind that $\frac{p_0}{1-p_0}$ has a finite variance if $\beta = 2$ (see e.g. [12, Thm 2.6.6.]), hence σ is finite in all cases.

The proof of this second result is based on the proof of the functional limit theorem established by N. Guillotin and A. Le Ny [9] for the walk of M. Campanino and D. Petritis (with $(p_k)_k$ constant and $(\varepsilon_k)_k$ centered, independent and identically distributed).

It may be possible that the transience remains true for every non degenerate distribution of the p_k's on $(0, 1)$. Indeed, roughly speaking, taking the p_k's closer to one should make the random walk even more transient; however this is just an intuition and not a mathematical evidence. We prove our Theorem 1.1 under a very general moment condition, which covers all the cases of our Theorem 1.2. In particular, the most superdiffusive cases, with $\delta > 3/4$, are obtained when the support of $1/(1 - p_0)$ is not compact.

The proof of our first result is built from the proof of [3, Thm 1.8] with many adaptations. The idea is to prove that, when $(\varepsilon_k)_{k\in\mathbb{Z}}$ is a fixed sequence of orientations, that is when μ is a Dirac measure,

$$\sum_{k\geq 1} \mathbb{P}(M_k = (0, 0)) < +\infty. \tag{1.3}$$

In the model we consider here, contrarily to the models envisaged in [3], the second coordinate of $(M_n)_n$ is not a random walk but it is a random walk in a random environment, since the probability to stay on a horizontal line depends on the line, which complicates the model. Even if a central limit theorem and a functional limit theorem have been established in [11] and in [10] for $M_n^{(2)}$, the local limit theorem for $M_n^{(2)}$ has not already been proved, to the extent of our knowledge. Moreover, in Theorem 1.1 we do not assume that the distribution of $\frac{p_0}{1-p_0}$ belongs to the domain of attraction of a stable distribution. For these reasons, it does not seem simple to make a precise estimation of $\mathbb{P}(M_n = (0, 0))$ as it has been done in [5]. We also mention that the random walk $(M_n)_n$ is not reversible.

It will be useful to observe that under $\mathbb{P}^{\varepsilon,\omega}$ and $\mathbb{P}$, $(M^{(2)}_{T_n})_n$ is a simple random walk $(S_n)_n$ on $\mathbb{Z}$, where the T_n's are the times of vertical displacement :

$$T_0 := 0;\ \forall n \geq 1,\ \ T_n := \inf\{k > T_{n-1}\ :\ M_k^{(2)} \neq M_{k-1}^{(2)}\}.$$

We will use several times the fact that there exists $M > 0$ such that, for every $n \geq 1$, we have $\mathbb{P}(S_n = 0) \leq Mn^{-\frac{1}{2}}$. Now, let us write X_n the first coordinate of M_{T_n}. We observe that

$$X_{n+1} - X_n = \varepsilon_{S_n}\xi_n,$$

where $\xi_n := T_{n+1} - T_n - 1$ corresponds to the duration of the stay on the horizontal line S_n after the n-th change of line. Moreover, given $\omega = (p_k)_{k\in\mathbb{Z}}$, $\varepsilon = (\varepsilon_k)_{k\in\mathbb{Z}}$ and $S = (S_k)_k$, the ξ_k's are independent and with distribution given by $\mathbb{P}^{\varepsilon,\omega}(\xi_k = m|S) = (1 - p_{S_k})p_{S_k}^m$ for every $k \geq 0$ and $m \geq 0$. With these notations, we have

$$X_n = \sum_{k=0}^{n-1} \varepsilon_{S_k}\xi_k.$$

This representation of $(M_{T_n})_n$ will be very useful in the proof of both the results.

2 Estimate of the variance

To point out the difference between our model and the model with $(p_k)_k$ constant considered by M. Campanino and D. Petritis in [3], we start by estimating the variance of X_{2n} under the probability $\mathbb{P}$ for these two models in the particular case when $\varepsilon_k = (-1)^k$ for every $k \in \mathbb{Z}$ and when $(1 - p_0)^{-1}$ is square integrable.

Proposition 2.1. *Let $\varepsilon_k = (-1)^k$ for every $k \in \mathbb{Z}$.*

1. *If the p_k's do not depend on k, then $Var(X_{2n}) = \mathbb{E}\left[\frac{2p_0}{(1-p_0)^2}\right] n$.*
2. *If the $(1-p_k)^{-1}$'s are iid, square integrable with positive variance, then there exists $C > 0$ such that $Var(X_{2n}) \sim_{n\to+\infty} Cn^{3/2}$.*

Proof of Proposition 2.1. We observe that

$$\mathbb{E}^{\varepsilon,\omega}[\xi_k|S] = \frac{p_{S_k}}{1-p_{S_k}} \text{ and } Var^{\varepsilon,\omega}(\xi_k|S) = \frac{p_{S_k}}{(1-p_{S_k})^2}.$$

Moreover, S is independent of ω under $\mathbb{P}$, hence $\mathbb{E}(X_{2n}) = 0$. We have

$$\begin{aligned}
Var(X_{2n}) &= \sum_{k,\ell=0}^{n-1} \mathbb{E}\left[(\xi_{2k} - \xi_{2k+1})(\xi_{2\ell} - \xi_{2\ell+1})\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\left[(\xi_{2k} - \xi_{2k+1})^2\right] + 2\sum_{0\le k<\ell\le n-1} \mathbb{E}\left[(\xi_{2k} - \xi_{2k+1})(\xi_{2\ell} - \xi_{2\ell+1})\right] \\
&= \sum_{k=0}^{n-1} \mathbb{E}\left[\frac{p_0}{(1-p_0)^2} + \frac{p_1}{(1-p_1)^2}\right. \\
&\qquad \left. + \left(\frac{p_0}{1-p_0} - \frac{p_1}{1-p_1}\right)^2\right] + 2\sum_{k=1}^{n-1}(n-k)\mathbb{E}\left[(\xi_0 - \xi_1)(\xi_{2k} - \xi_{2k+1})\right] \\
&= Cn + 2\sum_{k=1}^{n-1}(n-k)\mathbb{E}\left[\left(\frac{p_{S_0}}{1-p_{S_0}} - \frac{p_{S_1}}{1-p_{S_1}}\right)\left(\frac{p_{S_{2k}}}{1-p_{S_{2k}}} - \frac{p_{S_{2k+1}}}{1-p_{S_{2k+1}}}\right)\right].
\end{aligned}$$

This gives the result in case (1). Now, to prove the result in case (2), we notice that, since p_y and $p_{y'}$ are independent as soon as $y \neq y'$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{p_0}{1-p_0} - \frac{p_{S_1}}{1-p_{S_1}}\right)\left(\frac{p_{S_{2k}}}{1-p_{S_{2k}}} - \frac{p_{S_{2k+1}}}{1-p_{S_{2k+1}}}\right)\right] \\
&\qquad = \mathbb{E}\left[\frac{p_0}{1-p_0}\frac{p_{S_{2k}}}{1-p_{S_{2k}}} + \frac{p_{S_1}}{1-p_{S_1}}\frac{p_{S_{2k+1}}}{1-p_{S_{2k+1}}}\right] - 2\mathbb{E}\left[\frac{p_0}{1-p_0}\right]^2 \\
&\qquad = 2\left(\mathbb{E}\left[\frac{p_0}{1-p_0}\frac{p_{S_{2k}}}{1-p_{S_{2k}}}\right] - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]^2\right) \\
&\qquad = 2\left(\mathbb{E}\left[\frac{p_0^2}{(1-p_0)^2}\right] - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]^2\right)\mathbb{P}(S_{2k}=0) \\
&\qquad = 2Var\left(\frac{p_0}{1-p_0}\right)\mathbb{P}(S_{2k}=0).
\end{aligned}$$

We conclude as H. Kesten and F. Spitzer did in [14, p. 6], using the fact that $\mathbb{P}(S_{2k} = 0) \sim ck^{-1/2}$ (as k goes to infinity) for some $c > 0$. □

3 Proof of Theorem 1.1 (transience)

We come back to the general case. It is enough to prove the result for any fixed $(\varepsilon_k)_k$. Let $(\varepsilon_k)_{k\in\mathbb{Z}}$ be some fixed sequence of orientations. Hence μ is a Dirac measure on $\{-1,1\}^{\mathbb{Z}}$. Without any loss of generality, we assume throughout the proof of Theorem 1.1 that $\varepsilon_0 = 1$ and $\alpha \le 2$. We have

$$\sum_{k\ge1}\mathbb{P}(M_k = (0,0)) = \sum_{n\ge1}\mathbb{P}(S_{2n} = 0 \text{ and } X_{2n} \le 0 \le X_{2n+1}).$$

Hence, to prove the transience, it is enough to prove that

$$\sum_{n\geq 1} \mathbb{P}(S_{2n}=0 \text{ and } X_{2n}\leq 0\leq X_{2n+1})<+\infty. \tag{3.1}$$

This sum is divided into 8 terms which are separately estimated in Lemmas 3.1, 3.2, 3.3, 3.4, 3.5, 3.8, 3.9 and 3.10 provided $\delta_0,\delta_1,\delta_2,\delta_3$ are well chosen. One way to choose these δ_i so that they satisfy simultaneously the hypotheses of all these lemmas is given at the end of this section.

For every $y\in\mathbb{Z}$ and $m\in\mathbb{N}$, we define $N_m(y):=\#\{k=0,...,m-1\ :\ S_k=y\}$. We will use the fact that $X_{2n}=\overline{S}_{2n}+D_{2n}$ with

$$D_{2n}:=\sum_{y\in\mathbb{Z}}\frac{\varepsilon_y p_y}{1-p_y}N_{2n}(y) \text{ and } \overline{S}_{2n}:=\sum_{k=0}^{2n-1}\varepsilon_{S_k}\left(\xi_k-\frac{p_{S_k}}{1-p_{S_k}}\right).$$

Roughly speaking, the idea of the proof is that $X_{2n}\leq 0\leq X_{2n+1}$ implies that X_{2n} cannot be very far away from 0, which means that D_{2n} and $\overline{S}_{2n}$ should be of the same order, but this is false with a large probability. More precisely, we will prove that, with a large probability, we have $|D_{2n}|>n^{\frac{3}{4}-\delta_3}$ and $|\overline{S}_{2n}|<n^{\frac14+\frac{1}{2\alpha}+\upsilon}$ for small $\delta_3>0$ and $\upsilon>0$ (see the definition of B_n and the end of the proof of Lemma 3.3). Now let us carry out carefully this idea.

Let $n\geq 1$. Following [3], we consider $\delta_1>0$ and $\delta_2>0$ and we define :

$$\boxed{A_n:=\left\{\max_{0\leq k\leq 2n}|S_k|\leq n^{\frac12+\delta_1} \text{ and } \max_{y\in\mathbb{Z}}N_{2n}(y)<n^{\frac12+\delta_2}\right\}}.$$

Our first lemma is standard, we give a proof for the sake of completeness.

Lemma 3.1.

$$\sum_{n\geq 0}\mathbb{P}(A_n^c)<+\infty. \tag{3.2}$$

Proof. Let $p>1$. Thanks to Doob's maximal inequality and since $\mathbb{E}(|S_n|^p)=O(n^{p/2})$, we have $\mathbb{E}[\max_{0\leq k\leq 2n}|S_k|^p]=O(n^{\frac{p}{2}})$ and so, by the Chebychev inequality,

$$\mathbb{P}\left(\max_{0\leq k\leq 2n}|S_k|>n^{\frac12+\delta_1}\right)\leq\frac{\mathbb{E}[\max_{0\leq k\leq 2n}|S_k|^p]}{n^{p(\frac12+\delta_1)}}=O(n^{-p\delta_1}).$$

According to [14, Lem. 1], we also have $\max_y\mathbb{E}[N_{2n}(y)^p]=O(n^{\frac{p}{2}})$ and hence

$$\mathbb{P}\left(\max_y N_{2n}(y)>n^{\frac12+\delta_2}\right)\leq\sum_{y=-2n}^{2n}\mathbb{P}(N_{2n}(y)>n^{\frac12+\delta_2})=O(n^{1-p\delta_2}).$$

The result follows by taking p large enough. □

Let $\delta_0>0$ and set

$$\boxed{E_0(n):=\{p_0\leq 1-1/n^{\frac{1}{2\alpha}+\delta_0}\}}.$$

We have

Lemma 3.2.

$$\sum_{n\geq 0}\mathbb{P}(S_{2n}=0,E_0(n)^c)<+\infty. \tag{3.3}$$

Proof. Indeed, since S is independent of $(p_k)_{k\in\mathbb{Z}}$, we have

$$\mathbb{P}(S_{2n}=0, E_0(n)^c) \le \frac{M}{\sqrt{n}}\mathbb{P}\left(\frac{1}{1-p_0} > n^{\frac{1}{2\alpha}+\delta_0}\right) \le \frac{M}{n^{1+\delta_0\alpha}}\mathbb{E}\left[\left(\frac{1}{1-p_0}\right)^\alpha\right]$$

whose sum is finite. □

We also consider the conditional expectation of X_{2n} with respect to $(\omega,(S_p)_p)$ which is equal to $D_{2n}=\sum_{y\in\mathbb{Z}}\frac{\varepsilon_y p_y}{1-p_y}N_{2n}(y)$. We introduce $\delta_3>0$ and

$$\boxed{B_n := \left\{|D_{2n}| > n^{\frac{3}{4}-\delta_3}\right\}}.$$

Let $c_n := n^{\frac{1}{\alpha}(\frac{1}{2}+\delta_1)+\delta_0}$ and

$$\boxed{E_1(n) := \left\{\forall y\in\{-n^{1/2+\delta_1}, n^{1/2+\delta_1}\},\ \frac{1}{1-p_y}\le c_n\right\}}.$$

Since $p_0\in(0,1)$ a.s., there exist $0<a<b<1$ such that $\mathbb{P}(a<p_0<b)=:\gamma_0>0$. Let

$$\Lambda_n := \{k\in\{0,\dots,2n-1\},\ a<p_{S_k}<b\},$$

$P:=\{y\in\mathbb{Z}, a<p_y<b\}$, and $\zeta_y := \mathbf{1}_{\{a<p_y<b\}}$, $y\in\mathbb{Z}$. We have $\#\Lambda_n=\sum_{y\in\mathbb{Z}}\zeta_y N_{2n}(y)=\sum_{y\in P}N_{2n}(y)$. Let

$$\boxed{E_2(n) := \{\#\Lambda_n \ge \gamma_0 n\}}.$$

Define $\overline{V}_{2n} := \left(\sum_{k=0}^{2n-1}\left(\xi_k-\frac{p_{S_k}}{1-p_{S_k}}\right)^2\right)^{1/2}$ and

$$\boxed{E_3(n) := \left\{\overline{V}_{2n}^2 \le n^{d+\delta_0}\right\}},$$

with $d:=\frac{1}{2}+\frac{1}{\alpha}+3\delta_0+\frac{2\delta_1}{\alpha}+\delta_2$ and

$$\boxed{E_4(n) := \left\{\sum_{k=0}^{2n-1}\frac{1}{(1-p_{S_k})^2}\le n^d\right\}}.$$

Lemma 3.3. *If $\delta_3+\frac{\delta_1}{\alpha}+\frac{\delta_2}{2}+3\delta_0<\frac{1}{2}-\frac{1}{2\alpha}$, then we have*

$$\sum_{n\in\mathbb{N}}\mathbb{P}\left(S_{2n}=0;\ X_{2n}\le 0\le X_{2n+1},\ A_n, B_n, \cap_{i=0}^3 E_i(n)\right)<\infty. \tag{3.4}$$

Proof. Uniformly on $E_0(n)\cap E_1(n)$, we have

$$\begin{aligned}
&\mathbb{P}^{\varepsilon,\omega}(S_{2n}=0 \text{ and } X_{2n}\le 0\le X_{2n+1}, A_n, B_n, E_2(n), E_3(n))\\
&\le \sum_{k\ge 0}\mathbb{P}^{\varepsilon,\omega}(S_{2n}=0 \text{ and } X_{2n}=-k, A_n, B_n, E_2(n), E_3(n))(1-n^{-1/(2\alpha)-\delta_0})^k\\
&\le \sum_{k=0}^{n^{1/(2\alpha)+2\delta_0}}\mathbb{P}^{\varepsilon,\omega}(S_{2n}=0 \text{ and } X_{2n}=-k, A_n, B_n, E_2(n), E_3(n))+O(n^{-2})\\
&\le \mathbb{P}^{\varepsilon,\omega}(S_{2n}=0 \text{ and } -n^{1/(2\alpha)+2\delta_0}\le X_{2n}\le 0, A_n, B_n, E_2(n), E_3(n))+O(n^{-2})
\end{aligned} \tag{3.5}$$

In order to apply an inequality proved by Nagaev ([18], Thm 1), we define $\overline{X}_k := \varepsilon_{S_k}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)$, recall that $\overline{S}_{2n} = \sum_{k=0}^{2n-1}\overline{X}_k$, and introduce

$$\overline{B}_{2n} := \left(\mathbb{E}^{\varepsilon,\omega}\left[\overline{S}_{2n}^2|S\right]\right)^{1/2} = \left(\sum_{j=0}^{2n-1}\frac{p_{S_j}}{(1-p_{S_j})^2}\right)^{1/2}.$$

We have

$$\overline{B}_{2n}^2 \geq a\sum_{y\,:\,p_y\geq a}\frac{N_{2n}(y)}{(1-p_y)^2}.$$

Let $\overline{C}(2n) := \sum_{k=0}^{2n-1}\mathbb{E}^{\varepsilon,\omega}\left[\left|\overline{X}_k\right|^3|S\right]$. On $A_n\cap E_1(n)\cap E_2(n)$, we have

$$\sum_{y\,:\,p_y<a}\frac{N_{2n}(y)}{(1-p_y)^2} \leq \frac{2n-\#\Lambda_n}{(1-a)^2} \leq \frac{2-\gamma_0}{\gamma_0}\frac{\#\Lambda_n}{(1-a)^2} \leq \frac{2-\gamma_0}{\gamma_0}\sum_{y\,:\,p_y\geq a}\frac{N_{2n}(y)}{(1-p_y)^2} \tag{3.6}$$

and so

$$\overline{C}(2n) \leq \sum_y\frac{16N_{2n}(y)}{(1-p_y)^3} \leq \sum_y\frac{16N_{2n}(y)}{(1-p_y)^2}c_n \leq \frac{32}{\gamma_0}\sum_{y\,:\,p_y\geq a}\frac{N_{2n}(y)}{(1-p_y)^2}c_n.$$

Let $\overline{L}_{2n} := \overline{C}(2n)/\overline{B}_{2n}^3$. On $A_n\cap E_1(n)\cap E_2(n)$, we have

$$\begin{aligned}\overline{L}_{2n} &\leq \frac{32}{\gamma_0 a^{3/2}}\left(\sum_{y\,:\,p_y\geq a}\frac{N_{2n}(y)}{(1-p_y)^2}\right)^{-1/2}c_n \leq \frac{32}{\gamma_0 a^{3/2}}\frac{1-a}{\sqrt{\gamma_0 n}}c_n\\ &\leq \frac{32(1-a)}{(\gamma_0 a)^{3/2}}n^{-\frac{1}{2}+\frac{1}{2\alpha}+\frac{\delta_1}{\alpha}+\delta_0} \leq n^{-2\delta_0}, \end{aligned}\tag{3.7}$$

if n is large enough, since $\frac{\delta_1}{\alpha}+3\delta_0 < \frac{1}{2}-\frac{1}{2\alpha}$.

Let us recall that $\overline{V}_{2n} = \left(\sum_{k=0}^{2n-1}\overline{X}_k^2\right)^{1/2}$. We can now apply Nagaev ([18], Thm 1), which gives uniformly on $A_n\cap E_1(n)\cap E_2(n)$,

$$\begin{aligned}&\mathbb{P}^{\varepsilon,\omega}(|\overline{S}_{2n}| \geq n^{\delta_0}\overline{V}_{2n}|S)\\ \leq\ & 2\left(\frac{n^{2\delta_0}}{4\log 2}+1\right)\exp\left(-\frac{n^{2\delta_0}}{4}(1-c'\overline{L}_{2n}n^{\delta_0})\right)+2\exp\left(-\frac{c''}{\overline{L}_{2n}^2}\right)\\ =\ & O(\exp(-n^{\delta_0}))\end{aligned}\tag{3.8}$$

where $c'>0$ and $c''>0$ are universal constants. We recall that $X_{2n} = \sum_{k=0}^{2n-1}\varepsilon_{S_k}\xi_k = \overline{S}_{2n}+D_{2n}$. We have, for large n, on $A_n\cap B_n\cap E_1(n)\cap E_2(n)$,

$$\begin{aligned}&\mathbb{P}^{\varepsilon,\omega}(-n^{\frac{1}{2\alpha}+2\delta_0} \leq X_{2n} \leq 0, E_3(n)|S)\\ \leq\ & \mathbb{P}^{\varepsilon,\omega}\left(|X_{2n}-D_{2n}| \geq n^{\frac{3}{4}-\delta_3}-n^{\frac{1}{2\alpha}+2\delta_0}, E_3(n)|S\right)\\ \leq\ & \mathbb{P}^{\varepsilon,\omega}(|\overline{S}_{2n}| \geq n^{\delta_0}n^{(d+\delta_0)/2}, E_3(n)|S)\\ \leq\ & \mathbb{P}^{\varepsilon,\omega}(|\overline{S}_{2n}| \geq n^{\delta_0}\overline{V}_{2n}|S),\end{aligned}$$

since $\frac{1}{2\alpha}+2\delta_0 < \frac{3}{4}-\delta_3$ and since $\delta_3+\frac{\delta_1}{\alpha}+\frac{\delta_2}{2}+3\delta_0 < \frac{1}{2}-\frac{1}{2\alpha}$. Integrating this proves the lemma, by (3.5) and (3.8). □

Lemma 3.4.

$$\sum_{n\geq 0}\mathbb{P}(E_2(n)^c) < +\infty. \tag{3.9}$$

Proof. According to [7, Thm 1.3] applied twice with $u = \gamma_0/2$: first with the scenery $(\gamma_0 - \mathbf{1}_{\{a<p_{2y}<b\}})_{y\in\mathbb{Z}}$ and with the strongly aperiodic Markov chain $(S_{2n}/2)_{n\geq 0}$, and second with the scenery $(\gamma_0 - \mathbf{1}_{\{a<p_{2y+S_1}<b\}})_{y\in\mathbb{Z}}$ and with the strongly aperiodic Markov chain $((S_{2n+1} - S_1)/2)_{n\geq 0}$ conditionally on S_1, we get the existence of $c_1 > 0$ such that, for every $n \geq 1$, we have

$$\mathbb{P}(E_2(n)^c) \leq \exp\left(-c_1 n^{\frac{1}{3}}\right).$$

□

Lemma 3.5. *We have*

$$\sum_{n\in\mathbb{N}} \mathbb{P}(E_4(n) \setminus E_3(n)) < \infty. \tag{3.10}$$

Proof. We recall that, taken ω and S, $\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)_{y,k}$ is a sequence of independent, centered random variables. For every integer $\nu \geq 2$, there exists a constant $\tilde{C}_\nu > 0$ such that $\left|\mathbb{E}^{\varepsilon,\omega}\left[(\xi_k - \frac{p_{S_k}}{1-p_{S_k}})^\nu|S\right]\right| \leq \tilde{C}_\nu\left(\frac{1}{1-p_{S_k}}\right)^\nu$ $\mathbb{P}$-almost surely. Consequently, for every $N \geq 1$, there exists a constant $C_N > 0$ such that

$$\forall \nu \in \{2,\dots,2N\}, \qquad \left|\mathbb{E}^{\varepsilon,\omega}\left[(\xi_k - \frac{p_{S_k}}{1-p_{S_k}})^\nu|S\right]\right| \leq \left(\frac{C_N}{1-p_{S_k}}\right)^\nu.$$

Hence, for every $n \geq 1$ and $N \geq 1$, we have on $E_4(n)$:

$$\begin{aligned}
\mathbb{E}^{\varepsilon,\omega}[(\overline{V}_{2n}^2)^N|S] &= \sum_{k_1=0}^{2n-1}\sum_{k_2=0}^{2n-1}\cdots\sum_{k_N=0}^{2n-1} \mathbb{E}^{\varepsilon,\omega}\left[\prod_{i=1}^N \overline{X}_{k_i}^2|S\right] \\
&= \sum_{k_1=0}^{2n-1}\sum_{k_2=0}^{2n-1}\cdots\sum_{k_N=0}^{2n-1} \mathbb{E}^{\varepsilon,\omega}\left[\prod_{j=0}^{2n-1} \overline{X}_j^{2\theta_j(k_1,\dots k_N)}|S\right] \\
&\leq \sum_{k_1=0}^{2n-1}\sum_{k_2=0}^{2n-1}\cdots\sum_{k_N=0}^{2n-1}\prod_{j=0}^{2n-1}\left(\frac{C_N}{1-p_{S_j}}\right)^{2\theta_j(k_1,\dots,k_N)} \\
&= (C_N)^{2N}\left(\sum_{k=0}^{2n-1}\frac{1}{(1-p_{S_k})^2}\right)^N \leq (C_N)^{2N} n^{dN}
\end{aligned}$$

where $\theta_j(k_1,k_2,\dots,k_N) := \#\{1 \leq i \leq N,\ k_i = j\}$. Consequently, on $E_4(n)$,

$$\mathbb{P}^{\varepsilon,\omega}(\overline{V}_{2n}^2 > n^{d+\delta_0}|S) \leq n^{-(d+\delta_0)N}\mathbb{E}^{\varepsilon,\omega}\left[(\overline{V}_{2n}^2)^N|S\right] \leq (C_N)^{2N} n^{-\delta_0 N} = O(n^{-2})$$

by taking N large enough. Integrating this on $E_4(n)$ yields the result. □

Lemma 3.6. *We have on $E_2(n)$, uniformly on ω, S and on $k \in \mathbb{Z}$:*

$$\mathbb{P}^{\varepsilon,\omega}\left(X_{2n} = -k \,|\, S\right) = O\left(\sqrt{\ln(n) n^{-1}}\right). \tag{3.11}$$

Proof. On $E_2(n)$, we have :

$$\begin{aligned}
\mathbb{P}^{\varepsilon,\omega}\left(X_{2n} = -k|\, S\right) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbb{E}^{\varepsilon,\omega}\left[e^{itX_{2n}}|\, S\right] e^{ikt}\, dt \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathbb{E}^{\varepsilon,\omega}\left[e^{itX_{2n}}|\, S\right]\right| dt \\
&\leq \frac{1}{\pi}\int_0^{\pi}\prod_{y\in P}(\chi_{p_y}(\varepsilon_y t))^{N_{2n}(y)}
\end{aligned}$$

with

$$\chi_p(t) := \left|\mathbb{E}^{\varepsilon,\omega}[e^{it\xi_0}|p_0 = p]\right| = \left|\frac{1-p}{1-pe^{it}}\right| = \frac{1-p}{(1+p^2-2p\cos(t))^{\frac{1}{2}}}.$$

Since $\chi_p(t)$ is decreasing in p and since $0 < a < p_y < b < 1$ for $y \in P$, there exist $0 < \beta < \pi/2$ and $c > 0$ such that

$$\text{for a.e. } \omega,\ \ \forall y \in P,\ \ \forall t \in [0,\beta],\ \ \chi_{p_y}(t) \le \frac{1-a}{(1+a^2-2a\cos(t))^{\frac{1}{2}}} \le \exp(-ct^2).$$

Let us define $a_n := \sqrt{2\ln(n)/(c\gamma_0 n)}$. Since $\#\Lambda_n = \sum_{y\in P} N_{2n}(y) \ge \gamma_0 n$ on $E_2(n)$, we have

$$\int_{a_n}^{\beta} \prod_{y\in P} (\chi_{p_y}(t))^{N_{2n}(y)}\, dt \le \int_{a_n}^{\beta} \exp(-ct^2\#\Lambda_n)\, dt \le \int_{a_n}^{\beta} \exp(-ct^2\gamma_0 n)\, dt \le n^{-1}$$

on $E_2(n)$. Moreover,

$$\int_0^{a_n} \prod_{y\in P} (\chi_{p_y}(t))^{N_{2n}(y)}\, dt \le a_n$$

and

$$\int_{\beta}^{\pi} \prod_{y\in P} (\chi_{p_y}(t))^{N_{2n}(y)}\, dt \le \int_{\beta}^{\pi} \prod_{y\in P} \left(\frac{1-p_y}{1-p_y\cos(\beta)}\right)^{N_{2n}(y)} dt \le \pi \left(\frac{1-a}{1-a\cos(\beta)}\right)^{\gamma_0 n/2},$$

since $p > a > 0$ for $p \in P$. □

Lemma 3.7. *Suppose that* $\delta' := \delta_3 - \frac{\delta_2}{2} - \delta_1 > 0$ *and* $\delta_3 + \delta_2 < \frac{1}{4}$*. Then, uniformly on* p_0 *and* $(S_k)_k$*,*

$$\mathbb{P}(A_n \setminus B_n | S, p_0) = O\left(n^{-\delta'}\right).$$

Proof. Up to an enlargement of the probability space, we consider a centered gaussian random variable G with variance $n^{\frac{3}{2}-2\delta_3}$ independent of (ω, S). We have

$$\mathbb{P}(|D_{2n}| \le n^{\frac{3}{4}-\delta_3}|S,p_0)\mathbb{P}(|G| \le n^{\frac{3}{4}-\delta_3}) \le \mathbb{P}\left(|D_{2n}+G| \le 2n^{\frac{3}{4}-\delta_3}|S,p_0\right)$$

and so

$$\mathbb{P}(|D_{2n}| \le n^{\frac{3}{4}-\delta_3}|S,p_0) \le \mathbb{P}\left(|D_{2n}+G| \le 2n^{\frac{3}{4}-\delta_3}|S,p_0\right)/0,6.$$

Let $\tilde\chi$ be the characteristic function of $\frac{p_0}{1-p_0}$. Since p_0 is non-constant, there exist $\tilde\beta > 0$ and $\tilde c > 0$ such that [1]

$$\forall u \in [-\tilde\beta;\tilde\beta],\ \ |\tilde\chi(u)| \le e^{-\tilde c u^2}.$$

Consequently,

$$\mathbb{P}\left(|D_{2n}+G| \le 2n^{\frac{3}{4}-\delta_3}|S,p_0\right) = \frac{2n^{\frac{3}{4}-\delta_3}}{\pi}\int_{\mathbb{R}} \frac{\sin(2tn^{\frac{3}{4}-\delta_3})}{2tn^{\frac{3}{4}-\delta_3}} \mathbb{E}[e^{itD_{2n}}|S,p_0]\mathbb{E}[e^{itG}]\, dt$$

$$= \frac{2n^{\frac{3}{4}-\delta_3}}{\pi}\int_{\mathbb{R}} \frac{\sin(2tn^{\frac{3}{4}-\delta_3})}{2tn^{\frac{3}{4}-\delta_3}} e^{it\frac{p_0}{1-p_0}N_{2n}(0)} \prod_{y\ne 0} \tilde\chi\left(\varepsilon_y N_{2n}(y) t\right) e^{-\frac{t^2}{2}n^{\frac{3}{2}-2\delta_3}}\, dt$$

$$\le \frac{2n^{\frac{3}{4}-\delta_3}}{\pi}\int_{\mathbb{R}} \prod_{y\ne 0} \left|\tilde\chi\left(\varepsilon_y N_{2n}(y) t\right)\right| e^{-\frac{t^2}{2}n^{\frac{3}{2}-2\delta_3}}\, dt.$$

[1] Applying [16, Lemma 3.7.5, p. 58] to the random variable $Y := \frac{p_0}{1-p_0} - \frac{p_1}{1-p_1}$ which is not identically equal to 0 and whose characteristic function is $|\tilde\chi|^2$, we get that for every $r > 0$ and every $t \in \left[-\frac{1}{r};\frac{1}{r}\right]$, $|1-|\tilde\chi(t)|^2| \ge \frac{t^2}{3}\mathbb{E}[Y^2\mathbf{1}_{\{|Y|\le r\}}]$. We take $\tilde\beta$ such that $\tilde c := \frac{1}{6}\mathbb{E}[Y^2\mathbf{1}_{\{|Y|\le\tilde\beta^{-1}\}}] > 0$. For every $u \in [-\tilde\beta;\tilde\beta]$, we have $|1-|\tilde\chi(u)|| \ge \frac{1}{2}|1-|\tilde\chi(u)|^2| \ge \tilde c u^2$ and so $|\tilde\chi(u)| \le 1 - \tilde c u^2$.

Let $\delta_4 > 0$ be such that $\delta_5 := \frac{1}{4} - \delta_3 - \delta_2 - \delta_4 > 0$ and let $b_n := n^{\delta_4+\delta_3-\frac{3}{4}}$. On the one hand, we have

$$\begin{aligned} I_1 := \int_{\{|t|>\tilde\beta b_n\}} \prod_{y\neq 0} |\tilde\chi(\varepsilon_y N_{2n}(y)t)|\, e^{-\frac{t^2}{2} n^{\frac{3}{2}-2\delta_3}}\, dt &\leq \int_{\{|t|>\tilde\beta b_n\}} e^{-\frac{t^2}{2} n^{\frac{3}{2}-2\delta_3}}\, dt \\ &\leq n^{\delta_3-\frac{3}{4}} \int_{\{|s|>\tilde\beta n^{\delta_4}\}} e^{-s^2/2}\, ds \\ &\leq 2n^{\delta_3-\frac{3}{4}} e^{-\tilde\beta^2 n^{2\delta_4}/2}. \end{aligned}$$

On the other hand, we will estimate the following quantity on A_n :

$$I_2 := \int_{\{|t|\leq\tilde\beta b_n\}} \prod_{y\neq 0} |\tilde\chi(\varepsilon_y N_{2n}(y)t)|\, e^{-\frac{t^2}{2} n^{\frac{3}{2}-2\delta_3}}\, dt.$$

Let us define $F_n := \{y \neq 0 \,:\, N_{2n}(y) \geq n^{1/2-\delta_1}/2\}$ and $\rho_n := \#F_n$. On A_n, we have $2n - n^{1/2+\delta_2} \leq \sum_{y\neq 0} N_{2n}(y) \leq \rho_n n^{1/2+\delta_2} + (2n^{1/2+\delta_1} - \rho_n)\frac{n^{1/2-\delta_1}}{2}$ and hence $\rho_n \geq n^{1/2-\delta_2}/2$ (if n is large enough). Therefore, on A_n, we have $\alpha_n := \sum_{y\in F_n} N_{2n}(y) \geq n^{1-\delta_2-\delta_1}/4$. Now, using the Hölder inequality, we have

$$\begin{aligned} I_2 &\leq \prod_{y\in F_n} \left(\int_{\{|t|\leq\tilde\beta b_n\}} |\tilde\chi(\varepsilon_y N_{2n}(y)t)|^{\frac{\alpha_n}{N_{2n}(y)}}\, dt \right)^{\frac{N_{2n}(y)}{\alpha_n}} \\ &\leq \sup_{y\in F_n} \left(\int_{\{|t|\leq\tilde\beta b_n\}} |\tilde\chi(\varepsilon_y N_{2n}(y)t)|^{\frac{\alpha_n}{N_{2n}(y)}}\, dt \right) \\ &\leq b_n \sup_{y\in F_n} \left(\int_{|v|\leq\tilde\beta} |\tilde\chi(\varepsilon_y N_{2n}(y) v b_n)|^{\frac{\alpha_n}{N_{2n}(y)}}\, dv \right). \end{aligned}$$

Let us notice that, if $|v| \leq \tilde\beta$, we have on A_n,

$$|\varepsilon_y N_{2n}(y) v b_n| \leq \tilde\beta n^{1/2+\delta_2} n^{\delta_4+\delta_3-\frac{3}{4}} = \tilde\beta n^{-\delta_5} \leq \tilde\beta,$$

since $\delta_5 > 0$. Hence, on A_n, we have

$$\begin{aligned} I_2 &\leq b_n \sup_{y\in F_n} \left(\int_{\{|v|\leq\tilde\beta\}} e^{-\tilde c (N_{2n}(y))^2 v^2 n^{2\delta_4+2\delta_3-\frac{3}{2}} \frac{\alpha_n}{N_{2n}(y)}}\, dv \right) \\ &\leq b_n \sup_{y\in F_n} \left(\int_{\{|v|\leq\tilde\beta\}} e^{-\tilde c N_{2n}(y) v^2 n^{2\delta_4+2\delta_3-\frac{1}{2}-\delta_2-\delta_1}/4}\, dv \right) \\ &\leq \sup_{y\in F_n} \frac{b_n n^{-\delta_3-\delta_4+\frac{\delta_2+\delta_1}{2}+\frac{1}{4}}}{\sqrt{N_{2n}(y)}} \left(\int_{\mathbb{R}} e^{-\tilde c s^2/4}\, ds \right) \\ &\leq \sqrt{2} n^{-\frac{3}{4}+\delta_1+\frac{\delta_2}{2}} \int_{\mathbb{R}} e^{-\tilde c s^2/4}\, ds. \end{aligned}$$

Hence, uniformly on A_n and on p_0, we have

$$\mathbb{P}(A_n \setminus B_n | (S_k)_k, p_0) = O(n^{\delta_1+\frac{\delta_2}{2}-\delta_3}).$$

□

Lemma 3.8. *Under the same hypotheses, we have*

$$\sum_n \mathbb{P}(S_{2n} = 0,\, X_{2n} \leq 0 \leq X_{2n+1}; A_n \cap E_2(n) \setminus B_n) < \infty.$$

Proof. According to Lemma 3.6, Lemma 3.7 and since $\mathbb{P}(S_{2n}=0)=O(n^{-1/2})$ and $\mathbb{E}[1/(1-p_0)]<\infty$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{\infty}\mathbb{P}^{\varepsilon,\omega}(S_{2n}=0,X_{2n}=-k,A_n\cap E_2(n)\setminus B_n)p_0^k\right]\\
&\quad= \mathbb{E}\left[\sum_{k=0}^{\infty}p_0^k\mathbf{1}_{\{S_{2n}=0\}}\mathbf{1}_{(A_n\setminus B_n)\cap E_2(n)}\mathbb{P}^{\varepsilon,\omega}(X_{2n}=-k|S)\right]\\
&\quad\le C\sqrt{(\ln n)n^{-1}}\,\mathbb{E}\left[\frac{1}{1-p_0}\mathbf{1}_{\{S_{2n}=0\}}\mathbb{P}(A_n\setminus B_n|S,p_0)\right]\\
&\quad= O(n^{-1-\delta'}\sqrt{\ln n}). \qquad (3.12)
\end{aligned}$$

□

Lemma 3.9. *If $\delta_0\alpha<\delta_1$, we have*

$$\sum_n \mathbb{P}\left(S_{2n}=0,X_{2n}\le 0\le X_{2n+1},E_4(n)^c,A_n,E_2(n),E_0(n)\right)<+\infty. \qquad (3.13)$$

Proof. We notice that on $E_0(n)\cap A_n$,

$$\begin{aligned}
&\mathbb{P}(E_4(n)^c|S,p_0) \qquad (3.14)\\
&\le n^{-d\alpha/2}\mathbb{E}\left[\left(\sum_{y=-n^{1/2+\delta_1}}^{n^{1/2+\delta_1}}\frac{1}{(1-p_y)^2}N_{2n}(y)\right)^{\alpha/2}\Big|S,p_0\right]\\
&\le n^{-d\alpha/2}\mathbb{E}\left[\sum_{|y|\le n^{1/2+\delta_1},y\neq 0}\frac{1}{(1-p_y)^\alpha}N_{2n}^{\alpha/2}(y)+\frac{1}{(1-p_0)^\alpha}N_{2n}^{\alpha/2}(0)\Big|S,p_0\right] \qquad (3.15)\\
&\le n^{-d\alpha/2}\left(2n^{1/2+\delta_1}\mathbb{E}\left[\frac{1}{(1-p_0)^\alpha}\right]+n^{\frac{1}{2}+\delta_0\alpha}\right)n^{(1/2+\delta_2)\alpha/2}=O(n^{-3\delta_0\alpha/2}), \qquad (3.16)
\end{aligned}$$

since $\alpha\le 2$, $\delta_0\alpha<\delta_1$ and $d=\frac{1}{2}+\frac{1}{\alpha}+\frac{2\delta_1}{\alpha}+3\delta_0+\delta_2$. Similarly as in (3.12), this yields

$$\mathbb{E}\left[\sum_{k=0}^{\infty}\mathbb{P}^{\varepsilon,\omega}(S_{2n}=0,X_{2n}=-k,E_4(n)^c\cap A_n\cap E_2(n)\cap E_0(n))p_0^k\right]=O(n^{-1-3\delta_0\alpha/2}\sqrt{\ln n}).$$

Hence we have

$$\mathbb{P}\left(S_{2n}=0,X_{2n}\le 0\le X_{2n+1},E_4(n)^c,A_n,E_2(n),E_0(n)\right)=O(n^{-1-\delta_0\alpha/2}\sqrt{\ln n}).$$

□

Lemma 3.10. *If $\delta_0<\frac{1}{2}(1-\frac{1}{\alpha})$, we have*

$$\sum_n \mathbb{P}\left(S_{2n}=0,\ X_{2n}\le 0\le X_{2n+1},\ E_2(n)\setminus E_1(n)\right)<\infty.$$

Proof. Notice that on $\{\frac{1}{1-p_0}\le c_n\}$, we have

$$\mathbb{P}(E_1(n)^c|p_0)\le 2n^{1/2+\delta_1}\mathbb{P}\left(\frac{1}{1-p_0}>c_n\right)\le\frac{2n^{1/2+\delta_1}}{c_n^\alpha}\mathbb{E}\left[\left(\frac{1}{1-p_0}\right)^\alpha\right]=O(n^{-\delta_0\alpha}).$$

Similarly as in (3.12), since $\mathbb{E}[1/(1-p_0)]<\infty$, for δ_0 small enough, we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{k=0}^{\infty}\mathbb{P}^{\varepsilon,\omega}(S_{2n}=0, X_{2n}=-k, E_2(n)\setminus E_1(n))p_0^k\right]\\
&\qquad = \mathbb{E}\left[\sum_{k=0}^{\infty}p_0^k\mathbf{1}_{\{S_{2n}=0\}}\mathbf{1}_{E_2(n)\setminus E_1(n)}\mathbb{P}^{\varepsilon,\omega}(X_{2n}=-k|S)\right]\\
&\qquad \le C\sqrt{(\ln n)n^{-1}}n^{-1/2}\left(\mathbb{E}\left[\sum_{k=0}^{\infty}\mathbb{P}(E_1(n)^c|p_0)\mathbf{1}_{\{(1-p_0)^{-1}\le c_n\}}p_0^k\right]\right.\\
&\qquad\qquad \left.+\,\mathbb{E}\left[\frac{1}{1-p_0}\mathbf{1}_{\{(1-p_0)^{-1}>c_n\}}\right]\right)\\
&\qquad = O(n^{-1-c\delta_0}\sqrt{\ln n}),
\end{aligned}$$

where we can use Hölder's inequality, to deal with the second term of the third line, since $\alpha > 1$ and $\delta_0 < \frac{1}{2}(1-\frac{1}{\alpha})$. □

We take $\delta_3 \in \left(0, \frac{1}{2}-\frac{1}{2\alpha}\right)$ (since $\alpha > 1$) and then $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\delta_1 < \frac{1}{6},\ \delta_2 < \frac{1}{6},\ \delta_2 < \frac{1}{4}-\delta_3,\ \frac{\delta_1}{\alpha}+\frac{\delta_2}{2} < \frac{1}{2}-\frac{1}{2\alpha}-\delta_3,\ \delta_1+\frac{\delta_2}{2} < \delta_3$$

and finally δ_0 such that

$$\delta_0 < \frac{1}{8},\ \delta_0\alpha < \delta_1 \text{ and } \frac{\delta_1}{\alpha}+\frac{\delta_2}{2}+3\delta_0 < \frac{1}{2}-\frac{1}{2\alpha}-\delta_3.$$

Combining all the previous lemmas with these choices for $\delta_0, \delta_1, \delta_2, \delta_3$, we get (3.1), which proves Theorem 1.1.

4 Proof of Theorem 1.2 (functional limit theorem)

We assume that $(p_k)_k$ satisfies the conditions of Theorem 1.2.

Lemma 4.1. *Let $(\varepsilon_k)_k$ be a (fixed or random) sequence with values in $\{-1;1\}$. Let $(p_k)_k$ be as in Theorem 1.2. Then, under $\mathbb{P}$, the sequence of random variables*

$$\left(\left(n^{-\delta}\sum_{k=0}^{\lfloor nt\rfloor-1}\varepsilon_{S_k}\left(\xi_k-\frac{p_{S_k}}{1-p_{S_k}}\right),0\right)_{t\ge 0}\right)_n$$

converges in distribution (in the space of Skorokhod $\mathcal{D}([0;+\infty),\mathbb{R}^2)$) to $(0,0)_{t\ge0}$.

Proof. We first notice that it is enough to prove that

$$N^{-\delta}\sup_{0\le n\le N}\left|\sum_{k=0}^{n-1}\varepsilon_{S_k}\left(\xi_k-\frac{p_{S_k}}{1-p_{S_k}}\right)\right| \to_{N\to+\infty} 0$$

in probability.

Let us define

$$\tilde{E}_4(N,v) := \left\{\sum_{k=0}^{N-1}\frac{1}{(1-p_{S_k})^2} \le N^{\frac{1}{2}+\frac{1}{\beta}+v}\right\}.$$

We proceed as in formula (3.15) (with a conditioning with respect to S only, and $\alpha < \beta$ but close enough to β) to prove that $\mathbb{P}[(\tilde{E}_4(N,v))^c|S] \le N^{-cv}$ on A_N for $c > 0$ and N large enough. Moreover, $\mathbb{P}(A_N^c) \to_{N\to+\infty} 0$ by Lemma 3.1, which gives

$$\lim_{N\to+\infty}\mathbb{P}\left(\tilde{E}_4(N,v)\right) = 1.$$

Now, taken (ε, S, ω), $\left(\sum_{k=0}^{n-1} \varepsilon_{S_k}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)\right)_n$ is a martingale. Hence, according to the maximal inequality for martingales we have, for every $\theta > 0$,

$$\mathbb{P}^{\varepsilon,\omega}\left(\sup_{n\le N}\left|\left(\sum_{k=0}^{n-1}\varepsilon_{S_k}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)\right)^2\right| \ge \theta^2 N^{2\delta}\middle| S\right)$$

$$\le \frac{2\sup_{n\le N}\mathbb{E}^{\varepsilon,\omega}\left[\sum_{k=0}^{n-1}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)^2 |S\right]}{\theta^2 N^{2\delta}}$$

$$\le \frac{2\sum_{k=0}^{N-1}\frac{1}{(1-p_{S_k})^2}}{\theta^2 N^{2\delta}}$$

$$\le \frac{2N^{\frac12+\frac1\beta+v}}{\theta^2 N^{2\delta}} = 2N^{-\frac12+v}\theta^{-2},$$

on $\tilde{E}_4(n,v)$, since $\delta = \frac12 + \frac{1}{2\beta}$. Hence, we get

$$\mathbb{P}\left(\sup_{n\le N}\left|\sum_{k=0}^{n-1}\varepsilon_{S_k}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)\right| \ge \theta N^\delta\right) \le 1 - \mathbb{P}(\tilde{E}_4(N,v)) + 2N^{-\frac12+v}\theta^{-2}.$$

From this we conclude that $\lim_{n\to+\infty}\mathbb{P}\left(\sup_{n\le N}\left|\sum_{k=0}^{n-1}\varepsilon_{S_k}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right)\right| \ge \theta N^\delta\right) = 0.$ □

The next lemma follows from the proof of [9, Thm 4] when $\beta = 2$. The proof of the general case $\beta \in (1,2]$ is postponed to Section 5.

Lemma 4.2. *Let $\beta \in (1,2]$. Let $S = (S_n)_{n\ge0}$ be a random walk on $\mathbb{Z}$ starting from $S_0 = 0$, with iid centered square integrable and non-constant increments and such that $gcd\{k : \mathbb{P}(S_1 = k) > 0\} = 1$. Let $(\tilde\varepsilon_y)_{y\in\mathbb{Z}}$ be a sequence of iid random variables independent of S with symmetric distribution and such that $(n^{-\frac1\beta}\sum_{k=1}^n \tilde\varepsilon_k)_n$ converges in distribution to a random variable Y with stable distribution of index β. Then, the following convergence holds in distribution in $\mathcal{D}([0,+\infty),\mathbb{R}^2)$*

$$\left(n^{-\delta}\sum_{k=0}^{\lfloor nt\rfloor-1}\tilde\varepsilon_{S_k}, n^{-\frac12}S_{\lfloor nt\rfloor}\right)_{t\ge0} \longrightarrow_{n\to+\infty} (\tilde\Delta_t, \tilde B_t)_{t\ge0},$$

with $\delta = \frac12 + \frac{1}{2\beta}$, where $(\tilde B_t)_t$ is a Brownian motion such that $Var(\tilde B_1) = Var(S_1)$ and with $(\tilde L_t(x))_{t,x}$ the jointly continuous version of its local time and where

$$\tilde\Delta_t := \int_{\mathbb{R}} \tilde L_t(x)\, d\tilde Z_x,$$

with $\tilde Z$ independent of $\tilde B$ given by two independent right continuous stable processes $(\tilde Z_x)_{x\ge0}$ and $(\tilde Z_{-x})_{x\ge0}$ with stationary independent increments such that $\tilde Z_1$, $\tilde Z_{-1}$ have the same distribution as Y.

Now, we prove a functional limit theorem for $(X_{\lfloor nt\rfloor}, S_{\lfloor nt\rfloor})$ from which we will deduce our theorem 1.2.

Proposition 4.3. *Under the assumptions and with the notations of Theorem 1.2, the sequence of processes*

$$\left(\left(n^{-\delta}X_{\lfloor nt\rfloor}, n^{-1/2}S_{\lfloor nt\rfloor}\right)_{t\ge0}\right)_n$$

converges in distribution under $\mathbb{P}$ (in the space of Skorokhod $\mathcal{D}([0;+\infty),\mathbb{R}^2)$) to the process $(\sigma\Delta_t, B_t)_{t\geq 0}$.

Proof of Proposition 4.3. We observe that X_n can be rewritten

$$X_n = \sum_{k=0}^{n-1} \varepsilon_{S_k}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right) + \sum_{k=0}^{n-1} \varepsilon_{S_k}\frac{p_{S_k}}{1-p_{S_k}}.$$

According to Lemma 4.1, it is enough to prove, under $\mathbb{P}$, the convergence

$$\left(\left(n^{-\delta}\sum_{k=0}^{\lfloor nt\rfloor-1}\varepsilon_{S_k}\frac{p_{S_k}}{1-p_{S_k}}, n^{-1/2}S_{\lfloor nt\rfloor}\right)_{t\geq 0}\right)_n \to_{n\to+\infty} (\sigma\Delta_t, B_t)_{t\geq 0} \tag{4.1}$$

in distribution in $\mathcal{D}([0;+\infty),\mathbb{R}^2)$.

In case *(b)*, $\left(\tilde\varepsilon_y := \varepsilon_y\frac{p_y}{1-p_y}\right)_y$ is a sequence of independent identically distributed random variables with symmetric distribution such that $(n^{-1/\beta}\sum_{y=1}^n \tilde\varepsilon_y)_n$ converges in distribution to a random variable with characteristic function $\theta\mapsto\exp(-A_1|\theta|^\beta)$, where $A_1 := \mathbb{E}(p_0^2/(1-p_0)^2)/2$ if $\beta = 2$. Hence the result follows from Lemma 4.2.

In case *(a)* with $\beta = 2$, we observe that $\sum_{k=0}^{n-1}\varepsilon_{S_k}$ is equal to 0 if n is even and is equal to 1 if n is odd. Hence, $((n^{-3/4}\sum_{k=0}^{\lfloor nt\rfloor-1}\varepsilon_{S_k})_{t\geq 0})_n$ converges to 0 in $\mathcal{D}([0;+\infty),\mathbb{R})$ and it remains to prove the convergence of

$$\left(\left(n^{-3/4}\sum_{k=0}^{\lfloor nt\rfloor-1}\varepsilon_{S_k}\left(\frac{p_{S_k}}{1-p_{S_k}} - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]\right), n^{-1/2}S_{\lfloor nt\rfloor}\right)_{t\geq 0}\right)_n.$$

Let us write λ for the characteristic function of $\frac{p_0}{1-p_0} - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]$. Since $\frac{p_0}{1-p_0}$ has a finite variance and $\lambda(\varepsilon_y\cdot)$ behaves as λ at 0, we can follow the proof of the convergence of the finite distributions of [9, prop 1], which gives the convergence in distribution in $\mathcal{D}([0;+\infty),\mathbb{R}^2)$ thanks to the tightness that can be proved for the first coordinate as in [14].

Now, let us explain how case *(a)* with $\beta\in(1,2)$ will also be deduced from Lemma 4.2. This comes from the following lemma.

Lemma 4.4. *Let $\beta\in(1,2)$. Let $S=(S_n)_n$ be a simple symmetric random walk on $\mathbb{Z}$ starting from $S_0 = 0$. Let $(\tilde a_y)_{y\in\mathbb{Z}}$ be a sequence of iid random variables such that $\mathbb{E}(|\tilde a_0|)<\infty$, independent of S. We have*

$$\left(n^{-\delta}\left(\sum_{k=0}^{\lfloor nt\rfloor-1}(-1)^k\tilde a_{S_k} - \sum_y(\tilde a_{2y}-\tilde a_{2y-1})N_{\lfloor nt\rfloor}(2y)\right), 0\right)_{t\geq 0} \longrightarrow (0,0)$$

in distribution as n goes to infinity (in $\mathcal{D}([0;+\infty),\mathbb{R}^2)$), with $\delta := \frac{1}{2}+\frac{1}{2\delta}$.

Proof of Lemma 4.4. Let us write

$$e_n := \sum_{k=0}^{n-1}(-1)^k\tilde a_{S_k} - \sum_y(\tilde a_{2y}-\tilde a_{2y-1})N_n(2y).$$

We notice that it is enough to prove that

$$n^{-\delta}\sup_{0\leq k\leq n}|e_k| \xrightarrow{\mathbb{P}}_{n\to+\infty} 0.$$

Let $\eta > 0$ be such that $2\eta < \frac{1}{2\beta} - \frac{1}{4}$ (such a η exists since $\beta < 2$). For every $n \geq 1$, we consider the set Ω'_n defined by

$$\Omega'_n := \left\{ \sup_{k \leq n} |S_k| \leq n^{\frac{1}{2}+\eta}, \ \sup_{0 \leq k \leq n} \ \sup_{|y| \leq n^{\frac{1}{2}+\eta}+1} |N_k(y) - N_k(y-1)| \leq n^{\frac{1}{4}+\eta} \right\}.$$

Let us show that $\lim_{n\to+\infty} \mathbb{P}(\Omega'_n) = 1$. As in Lemma 3.1, we have,

$$\lim_{n\to+\infty} \mathbb{P}\left(\sup_{k\leq n} |S_k| \leq n^{\frac{1}{2}+\eta} \right) = 1.$$

Now we recall that for any even integer m,

$$\sup_y \mathbb{E}[|N_n(y) - N_n(y-1)|^m] = O(n^{\frac{m}{4}}),$$

as n goes to infinity (see [14, lem 3] and [13, p. 77]). Hence, using again the Markov inequality and taking m large enough, we get

$$\mathbb{P}(\Omega'_n) \geq 1 - o(1) - 3n^{\frac{3}{2}+\eta} \sup_{n,y} \frac{\mathbb{E}[|N_n(y) - N_n(y-1)|^m]}{n^{\frac{m}{4}+\eta m}} = 1 - o(1).$$

On Ω'_n, using the fact that

$$\sum_{\ell=0}^{k-1} (-1)^\ell \tilde{a}_{S_\ell} = \sum_y \left(\tilde{a}_{2y} N_k(2y) - \tilde{a}_{2y-1} N_k(2y-1) \right),$$

for every $k = 0, ..., n$, we have

$$|e_k| = \left| \sum_y \tilde{a}_{2y-1}(N_k(2y) - N_k(2y-1)) \right| \leq \sum_{|y|\leq n^{\frac{1}{2}+\eta}+1} |\tilde{a}_y| n^{\frac{1}{4}+\eta}.$$

Hence, thanks to the Markov inequality, we get for $\theta > 0$.

$$\begin{aligned} \mathbb{P}\left(n^{-\delta} \sup_{0\leq k\leq n} |e_k| > \theta \right) &\leq (1 - \mathbb{P}(\Omega'_n)) + \mathbb{P}\left(\sum_{|y|\leq n^{\frac{1}{2}+\eta}+1} |\tilde{a}_y| > \theta n^{\frac{1}{2}+\frac{1}{2\beta}-\frac{1}{4}-\eta} \right) \\ &\leq (1 - \mathbb{P}(\Omega'_n)) + \frac{3\mathbb{E}\left(|\tilde{a}_0|\right)}{\theta} n^{-\frac{1}{2\beta}+\frac{1}{4}+2\eta}. \end{aligned}$$

Hence, for every $\theta > 0$, we have $\lim_{n\to+\infty} \mathbb{P}(n^{-\delta} \sup_{0\leq k\leq n} |e_k| > \theta) = 0$. □

Now we observe that the characteristic function of $\tilde{\varepsilon}_y := \frac{p_{2y}}{1-p_{2y}} - \frac{p_{2y-1}}{1-p_{2y-1}}$ is $t \mapsto |\tilde{\chi}(t)|^2$ (where $\tilde{\chi}$ stands for the characteristic function of $\frac{p_0}{1-p_0}$). The distribution of $\tilde{\varepsilon}_0$ is symmetric and $(n^{-\frac{1}{\beta}} \sum_{k=1}^n \tilde{\varepsilon}_k)_n$ converges in distribution to a random variable with characteristic function $\theta \mapsto \exp(-2A_1|\theta|^\beta)$. According to Lemma 4.2 applied with the random walk $\left(\tilde{S}_k := \frac{S_{2k}}{2}\right)_k$, we have

$$\left(n^{-\delta} \sum_{k=0}^{\lfloor nt\rfloor-1} \tilde{\varepsilon}_{\tilde{S}_k}, n^{-\frac{1}{2}} \frac{S_{\lfloor 2nt\rfloor}}{2} \right)_{t\geq 0} \longrightarrow_{n\to+\infty} (\tilde{\Delta}_t, \tilde{B}_t)_{t\geq 0},$$

in distribution in $\mathcal{D}([0;+\infty),\mathbb{R}^2)$, where $(\tilde B_t)_t$ is a Brownian motion such that $Var(\tilde B_1)=\frac12$ and with $(\tilde L_t(x))_{t,x}$ the jointly continuous version of its local time and where

$$\tilde\Delta_t := \int_{\mathbb{R}} \tilde L_t(x)\, d\tilde Z_x,$$

with $\tilde Z$ independent of $\tilde B$ given by two independent right continuous stable processes $(\tilde Z_x)_{x\ge 0}$ and $(\tilde Z_{-x})_{x\ge0}$ such that the characteristic functions of $\tilde Z_1$ and of $\tilde Z_{-1}$ are $\theta\mapsto \exp(-2A_1|\theta|^\beta)$. Hence, we have

$$\left(n^{-\delta}\sum_{k=0}^{\lfloor nt/2\rfloor -1}\tilde\varepsilon_{\tilde S_k}, n^{-\frac12}\frac{S_{\lfloor nt\rfloor}}{2}\right)_{t\ge0} \longrightarrow_{n\to+\infty} (\tilde\Delta_{t/2},\tilde B_{t/2})_{t\ge 0},$$

and so

$$\left(n^{-\delta}\sum_y \tilde\varepsilon_y N_{\lfloor nt\rfloor}(2y), n^{-\frac12}S_{\lfloor nt\rfloor}\right)_{t\ge0} \longrightarrow_{n\to+\infty} (\tilde\Delta_{t/2}, B_t)_{t\ge0},$$

with $B_t := 2\tilde B_{t/2}$. Now we observe that

$$\tilde\Delta_{t/2} = \int_{\mathbb{R}}\tilde L_{t/2}(x)\,d\tilde Z_x = \int_{\mathbb{R}} L_t(2x)\,d\tilde Z_x = \int_{\mathbb{R}} L_t(x)\,dZ_x,$$

where L denotes the local time of B and with $Z_x := \tilde Z_{x/2}$. Now Lemma 4.4 applied to $\left(\frac{p_y}{1-p_y}\right)_{y\in\mathbb{Z}}$ gives (4.1), which proves Proposition 4.3 in the case (a) with $\beta\in(1,2)$. □

Proof of Theorem 1.2. We recall that for every n, we have

$$X_n = M^{(1)}_{T_n} \text{ and } S_n = M^{(2)}_{T_n}.$$

Moreover we observe that we have

$$T_n = \sum_{k=0}^{n-1}(\xi_k+1),$$

that can be rewritten

$$T_n = \sum_{k=0}^{n-1}\left(\xi_k - \frac{p_{S_k}}{1-p_{S_k}}\right) + \sum_{k=0}^{n-1}\left(\frac{p_{S_k}}{1-p_{S_k}} - \mathbb{E}\left[\frac{p_0}{1-p_0}\right]\right) + n\left(1+\mathbb{E}\left[\frac{p_0}{1-p_0}\right]\right).$$

We recall that $\gamma = 1+\mathbb{E}\left[\frac{p_0}{1-p_0}\right]$ and we define $(U_n)_n$ such that

$$U_n := \max\{k\ge 0\,:\, T_k\le n\}.$$

We notice that the sequences of processes

$$\left(\frac1n\sum_{k=0}^{\lfloor nt\rfloor-1}\left(\xi_k-\frac{p_{S_k}}{1-p_{S_k}}\right), t\ge0\right)_n \text{ and } \left(\frac1n\sum_{k=0}^{\lfloor nt\rfloor -1}\left(\frac{p_{S_k}}{1-p_{S_k}}-\mathbb{E}\left[\frac{p_0}{1-p_0}\right]\right), t\ge 0\right)_n$$

converge in distribution in $\mathcal{D}([0,+\infty),\mathbb{R})$ to 0. The first convergence follows from Lemma 4.1 where we take $\varepsilon_k=1$ for every $k\in\mathbb{Z}$. The second convergence is a consequence of [14, Thm 1.1] since $n^\delta/n\to0$ as $n\to+\infty$. Hence $\left(n^{-1}T_{\lfloor nt\rfloor}, t\ge0\right)_n$ converges in distribution to $(\gamma t)_t$, We conclude that $\left(\left(n^{-1}U_{\lfloor nt\rfloor}\right)_{t\ge0}\right)_n$ converges in distribution (in

$\mathcal{D}([0;+\infty),\mathbb{R})$) to $(t/\gamma)_t$. Therefore, according to Proposition 4.3 and to [1, Lem p. 151, Thm 3.9], the sequence of processes

$$\left(\left(n^{-\delta}X_{U_{\lfloor nt\rfloor}}, n^{-1/2}S_{U_{\lfloor nt\rfloor}}\right)_{t\geq 0}\right)_n$$

converges in distribution (in $\mathcal{D}([0;+\infty),\mathbb{R}^2)$) to $(\sigma\Delta_{\frac{t}{\gamma}}, B_{\frac{t}{\gamma}})_{t\geq 0}$. This means that

$$\left(\left(n^{-\delta}M^{(1)}_{T_{U_{\lfloor nt\rfloor}}}, n^{-1/2}M^{(2)}_{T_{U_{\lfloor nt\rfloor}}}\right)_{t\geq 0}\right)_n$$

converges in distribution (in $\mathcal{D}([0;+\infty),\mathbb{R}^2)$) to $(\sigma\Delta_{\frac{t}{\gamma}}, B_{\frac{t}{\gamma}})_{t\geq 0}$.

Moreover, we have $B_{\frac{t}{\gamma}} = \gamma^{-1/2}B'_t$ and $Z_{\frac{x}{\sqrt{\gamma}}} = \gamma^{-1/(2\beta)}Z'_x$, where $(B'_t)_{t\geq 0}$ is a standard Brownian motion, and $(Z'_x)_{x\in\mathbb{R}}$ has the same distribution as $(Z_x)_{x\in\mathbb{R}}$ and is independent of $(B'_t)_{t\geq 0}$. Furthermore we have

$$L_{\frac{t}{\gamma}}(x) = \gamma^{-1/2}L'_t(\gamma^{1/2}x), \qquad t\geq 0,\ x\in\mathbb{R},$$

where $(L'_t)_{t\geq 0}$ is the local time of $(B'_t)_t$ and so

$$\Delta_{\frac{t}{\gamma}} = \gamma^{-\frac{1}{2}}\int_{\mathbb{R}} L'_t(\gamma^{\frac{1}{2}}x)\,dZ_x = \gamma^{-\delta}\int_{\mathbb{R}} L'_t(y)\,dZ'_y.$$

Hence $(\sigma\Delta_{\frac{t}{\gamma}}, B_{\frac{t}{\gamma}})_{t\geq 0}$ has the same distribution as $(\sigma\gamma^{-\delta}\Delta_t, \gamma^{-\frac{1}{2}}B_t)_{t\geq 0}$.

Now we observe that we have

$$M^{(2)}_{\lfloor nt\rfloor} = M^{(2)}_{T_{U_{\lfloor nt\rfloor}}} \quad\text{and}\quad \left|M^{(1)}_{\lfloor nt\rfloor} - M^{(1)}_{T_{U_{\lfloor nt\rfloor}}}\right| \leq \xi_{U_{\lfloor nt\rfloor}}$$

and that for every $\theta>0$ and $T>0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0;T]} n^{-\delta}\xi_{U_{\lfloor nt\rfloor}} \geq \theta\right) &\leq \sum_{k=0}^{nT}\mathbb{P}\left(\xi_k\geq\theta n^{\delta}\right)\\ &\leq \sum_{k=0}^{nT}\frac{\mathbb{E}[(\xi_0)^{\beta-\eta}]}{(\theta n^{\delta})^{\beta-\eta}} = o(1)\end{aligned} \tag{4.2}$$

for $\eta>0$ small enough, since $\delta\beta>1$ and since [2] (if $\eta<\beta-1$) $\mathbb{E}[(\xi_0)^{\beta-\eta}|p_0]\leq C\frac{1}{(1-p_0)^{\beta-\eta}}$ a.s. and $\mathbb{E}\left[\frac{1}{(1-p_0)^{\beta-\eta}}\right]<\infty$. This completes the proof of Theorem 1.2. □

5 Proof of Lemma 4.2

The proof is very similar to those in [14] and [9], with some adaptations.

We define $\tilde{D}_n := \sum_{y\in\mathbb{Z}}\tilde{\varepsilon}_y N_n(y),\ n\in\mathbb{N}$.

Lemma 5.1. *If $\beta\in(1,2]$, the finite dimensional distributions of $(\tilde{D}_{\lfloor nt\rfloor}/n^{\delta}, S_{\lfloor nt\rfloor}/\sqrt{n})_{t\geq 0}$ converge to those of $(\tilde{\Delta}_t,\tilde{B}_t)_{t\geq 0}$.*

Before proving Lemma 5.1, we first introduce some preliminary results.

We observe that $n^{-1/\beta}\sum_{y=1}^n\tilde{\varepsilon}_y$ converges in distribution to a stable random variable of parameter β, with characteristic function $\widetilde{\zeta}_\beta(\theta) := \exp(-A_0|\theta|^\beta)$ (for some $A_0>0$). We can now compute the characteristic function of the finite dimensional distributions of $(\tilde{\Delta}_t,\tilde{B}_t)_{t\geq 0}$.

[2] This comes from the Hölder inequality since $\mathbb{E}\left[(\xi_0)^2|p_0\right]\leq\frac{2}{(1-p_0)^2}$.

Lemma 5.2. *Let* $k \in \mathbb{N}^*$, $(t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k$ *and* $(\theta_1^{(i)}, \theta_2^{(i)}, \dots, \theta_k^{(i)})_{i=1,2} \in \mathbb{R}^{2k}$. *We have,*

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\left(\theta_j^{(1)}\tilde{\Delta}_{t_j} + \theta_j^{(2)}\tilde{B}_{t_j}\right)\right)\right]$$

$$= \mathbb{E}\left[\exp\left(-A_0\int_{-\infty}^{+\infty}\left|\sum_{j=1}^{k}\theta_j^{(1)}\tilde{L}_{t_j}(x)\right|^{\beta} dx\right)\exp\left(i\sum_{j=1}^{k}\theta_j^{(2)}\tilde{B}_{t_j}\right)\right]. \tag{5.1}$$

Proof. We condition by $\tilde{B}$ and we proceed as in [14, Lem 5]. We get

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\theta_j^{(1)}\tilde{\Delta}_{t_j}\right)\middle|\tilde{B}\right] = \exp\left(-A_0\int_{-\infty}^{+\infty}\left|\sum_{j=1}^{k}\theta_j^{(1)}\tilde{L}_{t_j}(x)\right|^{\beta} dx\right),$$

which gives the result. □

For fixed $k \in \mathbb{N}^*$ and $(t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k$, we define for every $(\theta_1, \theta_2, \dots, \theta_k) \in (\mathbb{R}^2)^k$,

$$\psi_n(\theta_1, \theta_2, \dots, \theta_k) := \mathbb{E}\left[\exp\left(-A_0\sum_{y\in\mathbb{Z}}\left|\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)n^{-\delta}\right|^{\beta}\right)\exp\left(i\sum_{j=1}^{k}\theta_j^{(2)}\frac{S_{\lfloor nt_j\rfloor}}{\sqrt{n}}\right)\right]$$

and

$$\begin{aligned}\phi_n(\theta_1, \theta_2, \dots, \theta_k) &:= \mathbb{E}\left[\exp\left(i\sum_{j=1}^{k}\left(\theta_j^{(1)}n^{-\delta}\tilde{D}_{\lfloor nt_j\rfloor} + \theta_j^{(2)}\frac{S_{\lfloor nt_j\rfloor}}{\sqrt{n}}\right)\right)\right] \\ &= \mathbb{E}\left[\prod_{y\in\mathbb{Z}}\tilde{\lambda}\left(\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)n^{-\delta}\right)\exp\left(i\sum_{j=1}^{k}\theta_j^{(2)}\frac{S_{\lfloor nt_j\rfloor}}{\sqrt{n}}\right)\right]\end{aligned}$$

where $\tilde{\lambda}(\theta) := \mathbb{E}[\exp(i\theta\tilde{\varepsilon}_0)]$ for every $\theta \in \mathbb{R}$ and $\theta_j = (\theta_j^{(1)}, \theta_j^{(2)})$ for every $j \in \{1, \dots, n\}$.

Lemma 5.3. *For every* $k \in \mathbb{N}^*$, $(t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k$ *and* $(\theta_1, \theta_2, \dots, \theta_k) \in (\mathbb{R}^2)^k$,

$$\lim_{n\to+\infty}|\psi_n(\theta_1, \theta_2, \dots, \theta_k) - \phi_n(\theta_1, \theta_2, \dots, \theta_k)| = 0.$$

Proof. As in [14, p. 7], we have $1 - \tilde{\lambda}(\theta) \sim_{\theta\to 0} A_0|\theta|^{\beta}$ since the distribution of $\tilde{\varepsilon}_0$ belongs to the normal domain of attraction of the stable distribution with characteristic function $\widetilde{\zeta}_{\beta}$. The remainder of the proof is the same as in [9, Lem 5] with δ instead of $3/4$ and β instead of 2, since $\mathbb{P}(n^{-\delta}\sup_{y\in\mathbb{Z}}|\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)| > \varepsilon) \to_{n\to+\infty} 0$ for $\varepsilon > 0$ by [14, Lem 4] and since we have $\mathbb{E}(\sum_{y\in\mathbb{Z}}|n^{-\delta}\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)|^{\beta}) \leq C < \infty$ by [6, Lem 3.3]. □

We now prove

Lemma 5.4. *For every* $k \in \mathbb{N}^*$, $(t_1, t_2, \dots, t_k) \in \mathbb{R}_+^k$ *and* $(\theta_1, \theta_2, \dots, \theta_k) \in (\mathbb{R}^2)^k$,

$$\left(n^{-\delta\beta}\sum_{y\in\mathbb{Z}}\left|\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)\right|^{\beta}, \sum_{j=1}^{k}\theta_j^{(2)}S_{\lfloor nt_j\rfloor}/\sqrt{n}\right)_n$$

converges in distribution as $n \to +\infty$ *to*

$$\left(\int_{-\infty}^{+\infty} \left| \sum_{j=1}^{k} \theta_j^{(1)} \tilde{L}_{t_j}(x) \right|^{\beta} dx, \sum_{j=1}^{k} \theta_j^{(2)} \tilde{B}_{t_j} \right).$$

Proof. The proof is very similar to the one of [9, Lem 6], and to the proof of [14, Lem 6] which deals with the first coordinate. Throughout the proof, C denotes a positive constant, which can vary from line to line, and can depend on $(\theta_j^{(i)}, i = 1, 2; j = 1, \dots, k)$. For $n \in \mathbb{N}$ and real numbers $a < b$ and $t > 0$, we introduce the notation

$$T_t^n(a,b) := \int_0^{\lfloor nt \rfloor / n} 1_{\{a \le S_{\lfloor ns \rfloor}/\sqrt{n} < b\}} ds,$$

which is the occupation time of $[a, b)$ by $S_{\lfloor n. \rfloor}/\sqrt{n}$ up to time $\lfloor nt \rfloor / n$. We consider $\tau > 0$ and two real numbers μ_1 and μ_2. We define for $M > 0$, $\ell \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\begin{aligned} U(\tau, M, n) &:= \mu_1 n^{-\delta\beta} \sum_{\substack{y < -M\tau\sqrt{n} \\ \text{or } y \ge M\tau\sqrt{n}}} \left| \sum_{j=1}^{k} \theta_j^{(1)} N_{\lfloor nt_j \rfloor}(y) \right|^{\beta}, \\ T(\ell, n) &:= \sum_{j=1}^{k} \theta_j^{(1)} T_{t_j}^n(\ell\tau, (\ell+1)\tau) = \frac{1}{n} \sum_{j=1}^{k} \theta_j^{(1)} \sum_{\ell\tau\sqrt{n} \le y < (\ell+1)\tau\sqrt{n}} N_{\lfloor nt_j \rfloor}(y), \\ V(\tau, M, n) &:= \mu_1 \tau^{1-\beta} \sum_{-M \le \ell < M} |T(\ell, n)|^{\beta} + \mu_2 n^{-1/2} \sum_{j=1}^{k} \theta_j^{(2)} S_{\lfloor nt_j \rfloor}. \end{aligned}$$

We are interested in

$$\begin{aligned} A(\tau, M, n) &:= \frac{\mu_1}{n^{\delta\beta}} \sum_{y \in \mathbb{Z}} \left| \sum_{j=1}^{k} \theta_j^{(1)} N_{\lfloor nt_j \rfloor}(y) \right|^{\beta} + \mu_2 \sum_{j=1}^{k} \theta_j^{(2)} \frac{S_{\lfloor nt_j \rfloor}}{\sqrt{n}} - U(\tau, M, n) - V(\tau, M, n) \\ &= \frac{\mu_1}{n^{\delta\beta}} \sum_{-M \le \ell < M} \sum_{\ell\tau\sqrt{n} \le y < (\ell+1)\tau\sqrt{n}} \left| \sum_{j=1}^{k} \theta_j^{(1)} N_{\lfloor nt_j \rfloor}(y) \right|^{\beta} \\ &\quad - \mu_1 \sum_{-M \le \ell < M} \tau^{1-\beta} |T(\ell, n)|^{\beta}. \end{aligned}$$

First step: We define $c(\ell, n) := \#\{y \in \mathbb{Z},\ \ell\tau\sqrt{n} \le y < (\ell+1)\tau\sqrt{n}\}$. As in [14], we have for $\mu_1 \neq 0$,

$$\begin{aligned} &\mu_1^{-1} A(\tau, M, n) \\ &= \sum_{-M \le \ell < M} \sum_{\ell\tau\sqrt{n} \le y < (\ell+1)\tau\sqrt{n}} n^{-\delta\beta} \left[\left| \sum_{j=1}^{k} \theta_j^{(1)} N_{\lfloor nt_j \rfloor}(y) \right|^{\beta} - n^{\beta} (\tau\sqrt{n})^{-\beta} |T(\ell, n)|^{\beta} \right] \quad (5.2) \\ &+ \sum_{-M \le \ell < M} \left[n^{\beta - \delta\beta} (\tau\sqrt{n})^{-\beta} c(\ell, n) - \tau^{1-\beta} \right] |T(\ell, n)|^{\beta}. \quad (5.3) \end{aligned}$$

As in [14, p. 19], the right hand side of (5.3) tends to 0 in probability as $n \to +\infty$. Then we just have to study (5.2). To this aim, we use the inequality suggested by [14], that is

$$\forall (a, b) \in \mathbb{R}_+^2, \qquad |a^{\beta} - b^{\beta}| \le \beta |a - b| (a^{\beta-1} + b^{\beta-1}) \le 2\beta |a - b| (a + b)^{\beta-1}$$

since $\beta > 1$. We define $T'(\ell, n)$ by the same formula as $T(\ell, n)$ where we replace each $\theta_j^{(1)}$ by $|\theta_j^{(1)}|$. We consider, for $\ell\tau\sqrt{n} \le y < (\ell+1)\tau\sqrt{n}$,

$$\mathbb{E}\left(\left|\left|\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)\right|^{\beta} - n^{\beta}(\tau\sqrt{n})^{-\beta}|T(\ell,n)|^{\beta}\right|\right) \tag{5.4}$$

$$\le\ 2\beta\mathbb{E}\left(\left|\left|\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y)\right| - \frac{\sqrt{n}}{\tau}|T(\ell,n)|\right|.\left|\sum_{j=1}^{k}|\theta_j^{(1)}|N_{\lfloor nt_j\rfloor}(y) + \frac{\sqrt{n}}{\tau}T'(\ell,n)\right|^{\beta-1}\right)$$

$$\le\ 2\beta\mathbb{E}\left(\left|\sum_{j=1}^{k}\theta_j^{(1)}N_{\lfloor nt_j\rfloor}(y) - \frac{\sqrt{n}}{\tau}T(\ell,n)\right|^2\right)^{1/2} \tag{5.5}$$

$$\times\mathbb{E}\left(\left|\sum_{j=1}^{k}|\theta_j^{(1)}|N_{\lfloor nt_j\rfloor}(y) + \frac{\sqrt{n}}{\tau}T'(\ell,n)\right|^{2(\beta-1)}\right)^{1/2} \tag{5.6}$$

by the Cauchy-Schwarz inequality and by the second triangular inequality in (5.5). In the following RHS will stand for right hand side. We have by [14] equations (3.9) and (2.26),

$$\text{RHS of (5.5)} \le (C_1\tau n)^{1/2}, \tag{5.7}$$

where C_1 is a constant, which is finite since $gcd\{k\ :\ \mathbb{P}(S_1 = k) > 0\} = 1$. Moreover, setting $a(\ell, n) := \tau\ell\sqrt{n}$, by the Hölder inequality and [9, p. 346], we have

$$[\text{RHS of (5.6)}]^{\frac{2}{\beta-1}} \ \le\ \mathbb{E}\left(\left|\sum_{j=1}^{k}|\theta_j^{(1)}|N_{\lfloor nt_j\rfloor}(y) + \frac{\sqrt{n}}{\tau}T'(\ell,n)\right|^2\right) \tag{5.8}$$

$$\le\ C\sum_{j=1}^{k}\max_{a(\ell,n)\le x<a(\ell+1,n)}(\mathbb{E}(N_{\lfloor nt_j\rfloor}(x)^3)^{2/3} + \mathbb{E}(N_{\lfloor nt_j\rfloor}(y)^3)^{2/3})$$

$$\le\ C\mathbb{E}(N_{\lfloor n\max(t_1,\dots,t_k)\rfloor}(0)^3)^{2/3} \le Cn \tag{5.9}$$

by [14, Lem 1]. Combining (5.7) and (5.9), we get

$$\text{RHS of (5.4)} \le C\tau^{\frac{1}{2}}n^{\frac{\beta}{2}}.$$

Hence,

$$\mathbb{E}(|\text{RHS of (5.2)}|) \le C(2M+1)\tau^{\frac{3}{2}}.$$

As in [14, p. 20], for each $\eta > 0$ we can take $M\tau$ so large that

$$\mathbb{P}(U(M,n,\tau) \neq 0) \le \eta \tag{5.10}$$

and then τ so small that

$$\mathbb{E}(|\text{RHS of (5.2)}|) \le \eta^2/|\mu_1|. \tag{5.11}$$

Hence, by (5.10) and (5.11), and since the right hand side of (5.3) tends to 0 in probability as $n \to +\infty$, we get for n large enough (even when $\mu_1 = 0$),

$$\mathbb{P}(|A(\tau,M,n) + U(\tau,M,n)| > 3\eta) \le \mathbb{P}(|A(\tau,M,n)| > 3\eta) + \mathbb{P}(U(M,n,\tau) \neq 0) \le 3\eta.$$

Second step: As in [9, Lem 2], we have

$$(T_{t_j}^{(n)}(\ell\tau,(\ell+1)\tau), S_{\lfloor nt_j\rfloor}/\sqrt{n})_{j=1,\dots,k,\ell=-M,\dots,M} \to (\Lambda_{t_j}(\ell\tau,(\ell+1)\tau), \tilde{B}_{t_j})_{j=1,\dots,k,\ell=-M,\dots,M}$$

in distribution, as $n \to +\infty$, where $\Lambda_t(a,b) := \int_a^b \tilde{L}_t(x)dx$ for $t > 0$ and $a < b$. Consequently, $(V(\tau, M, n))_n$ converges in distribution as $n \to +\infty$ to

$$\overline{V}(\tau, M) := \mu_1 \tau^{1-\beta} \sum_{-M \leq \ell < M} \left| \sum_{j=1}^k \theta_j^{(1)} \Lambda_{t_j}(\ell\tau, (\ell+1)\tau) \right|^\beta + \mu_2 \sum_{j=1}^k \theta_j^{(2)} \tilde{B}_{t_j}.$$

Since $\tilde{L}_t(.)$ is continuous with a compact support, we get almost surely

$$\overline{V}(\tau, M) \to_{M\tau \to +\infty, \tau \to 0} \mu_1 \int_{-\infty}^{+\infty} \left| \sum_{j=1}^k \theta_j^{(1)} \tilde{L}_{t_j}(x) \right|^\beta dx + \mu_2 \sum_{j=1}^k \theta_j^{(2)} \tilde{B}_{t_j} =: \widehat{V}.$$

Hence by choosing adequate M and τ we get for n large enough

$$\left| \mathbb{E}\left[\exp\left(i \frac{\mu_1}{n^{\delta\beta}} \sum_{y \in \mathbb{Z}} \left| \sum_{j=1}^k \theta_j^{(1)} N_{\lfloor nt_j \rfloor}(y) \right|^\beta + i\mu_2 \sum_{j=1}^k \theta_j^{(2)} \frac{S_{\lfloor nt_j \rfloor}}{\sqrt{n}} \right) \right] - \mathbb{E}\exp(i\widehat{V}) \right| \leq 11\eta.$$

Since this is true for every $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$ and $\eta > 0$, this proves Lemma 5.4. □

Proof of Lemma 5.1. Applying Lemma 5.4, we get the convergence of $\psi_n(\theta_1, \ldots, \theta_k)$ to the right hand side of (5.1) as $n \to +\infty$. This combined with Lemma 5.2 and Lemma 5.3 proves Lemma 5.1. □

Proof of Lemma 4.2. We now turn to the tightness. We know that $(\tilde{D}_{\lfloor nt \rfloor}/n^\delta,\ t \geq 0)_n$ and $(S_{\lfloor nt \rfloor}/\sqrt{n},\ t \geq 0)_n$ both converge in distribution in $\mathcal{D}([0, +\infty), \mathbb{R})$ to continuous processes (respectively by [14] and by the theorem of Donsker), and the finite dimensional distributions of $(\tilde{D}_{\lfloor nt \rfloor}/n^\delta, S_{\lfloor nt \rfloor}/\sqrt{n})_{t \geq 0}$ converge to those of $(\tilde{\Delta}_t, \tilde{B}_t)_{t \geq 0}$ by Lemma 5.1, hence the distributions of $(\tilde{D}_{\lfloor nt \rfloor}/n^\delta, S_{\lfloor nt \rfloor}/\sqrt{n})_{t \geq 0}$ are tight in $\mathcal{D}([0, +\infty), \mathbb{R}^2)$ (this is a consequence of [1] Theorems 13.2 and 13.4, Corollary p.142 and inequalities (12.7) and (12.9)). This proves Lemma 4.2. □

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