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Perturbation analysis of the van den Berg Kesten inequality for determinantal probability measures

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Abstract

This paper describes a second order perturbation analysis of the BK property in the space of Hermitian determinantal probability measures around the subspace of product measures, showing that the second order Taylor approximation of the BK inequality holds for increasing events.

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1 Introduction and results

1.1 Motivation

The van den Berg Kesten (BK) inequality concerns occurrence of two events on disjoint sets. It has numerous applications in percolation theory, see e.g. Grimmett's book on percolation [2]. For increasing events and product measures, the BK inequality was proven by van den Berg and Kesten in [9]; see also variants of it in [6]. Reimer [5] proved the generalization of the BK inequality to arbitrary, not necessarily increasing events and product measures. Quite recently, several variants and generalizations of the BK inequality have been proven; see [8], [7], and [3].

Determinantal probability measures and their continuum analogue, determinantal point processes, have found considerable interest in mathematics and physics, e.g. in the description of quantum systems of fermions and in random matrix theory. For an interesting introduction to the theory of determinantal probability measures, including more details on the history and references, see [4]. One of the facts shown in that paper is that determinantal probability measures are positively associated; compare the remark below (1.12). In §9 of [4], Lyons asks whether determinantal probability measures have the BK property. This question is still not answered, but it motivated the present work.

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1.2 Results

Fix $n \in \mathbb{N}$ with $n \geq 2$, and set $\Omega := \{0,1\}^n$. We imagine any $\omega = (\omega_i)_{1 \leq i \leq n} \in \Omega$ as a particle configuration on the set $[n] := \{1, \ldots, n\}$ of the first n natural numbers, where ω_i denotes the number of particles at location $i \in [n]$. For $\omega, \omega' \in \Omega$, we write $\omega \leq \omega'$ if $\omega_i \leq \omega'_i$ for all $1 \leq i \leq n$. A set $A \subseteq \Omega$ is called increasing if for all $\omega \in A$ and $\omega' \in \Omega$ with $\omega \leq \omega'$ one has $\omega' \in A$. The expressions $\omega \wedge \omega'$ and $\omega \vee \omega'$ denote the componentwise minimum and maximum of ω and ω', respectively. For $i, j \in [n]$ with $i \neq j$, we set

$$\Omega_{i \neq j} := \{\omega \in \Omega : \omega_i \neq \omega_j\}. \tag{1.1}$$

For $x \in \{0,1\}$, we abbreviate $\overline{x} := 1 - x$. This notation is also used componentwise and for sets: For a tuple $\omega \in \Omega$ and a set $B \subseteq \Omega$, we define $\overline{\omega} := (\overline{\omega_i})_{i \in [n]}$ and $\overline{B} := \{\overline{\omega} : \omega \in B\}$. For events $A, B \subseteq \Omega$, the event $A \Box B$ means that A and B occur on disjoint locations. More formally, $A \Box B$ consists of all $\omega \in \Omega$ such that there exist $\tilde{S} \in \Omega$ and $\tilde{T} \in \Omega$ with the following three properties:

1. $\tilde{S} \wedge \tilde{T} = 0$.
2. For all $\eta \in \Omega$ with $\eta \wedge \tilde{S} = \omega \wedge \tilde{S}$ holds $\eta \in A$.
3. Similarly, for all $\eta \in \Omega$ with $\eta \wedge \tilde{T} = \omega \wedge \tilde{T}$ holds $\eta \in B$.

In this paper, we are mostly interested in increasing events A and B. In this case, $A \Box B$ can be characterized as follows.

$$A \Box B = \{\omega \in \Omega : \exists S \in A \;\; \exists T \in B \text{ with } S \wedge T = 0 \text{ and } S \vee T \leq \omega\}. \tag{1.2}$$

Reimer has proven the following theorem; see Theorem 1.2 in [5]. More precisely, we cite here the equivalent version of the theorem given in [8], Proposition 1.3. A detailed review of Reimer's proof can be found in [1], Sections 4 and 5. Reimer's theorem plays the key role in his proof of the van den Berg - Kesten - Reimer inequality.

Fact 1.1 (Reimer's butterfly theorem)**.** *For all events $A, B \subseteq \Omega$, the following holds:*

$$|A \cap \overline{B}| \geq |A \Box B|. \tag{1.3}$$

We deduce the following corollary of this theorem; it plays an important role in this paper.

Corollary 1.2 (A variant of Reimer's theorem)**.** *For all $i, j \in [n]$ with $i \neq j$ and for all increasing events $A, B \subseteq \Omega$, one has*

$$|A \cap \overline{B} \cap \Omega_{i \neq j}| \geq |(A \Box B) \cap \Omega_{i \neq j}|. \tag{1.4}$$

We remark that this variant is not valid for arbitrary events A and B. A counterexample to this and some related counterexamples are given in Remark 2.2, below.

A similar variant of Corollary 1.2 has been proven in [8], Proposition 2.2. It states the following:

Fact 1.3 (cited from Proposition 2.2 in [8])**.** *Let $\hat{\Omega}_m$ be the set of all $\omega \in \{0,1\}^m$ with the property that for all $1 \leq i \leq m/2$, $(\omega_{2i-1}, \omega_{2i})$ is equal to $(1,0)$ or $(0,1)$. Let $\hat{P}_m$ be the probability distribution on $\{0,1\}^m$ which assigns equal probabilities to all $\omega \in \hat{\Omega}_m$, and probability 0 to all other ω.*

Then, for all even m and all increasing $A, B \subset \{0,1\}^m$,

$$\hat{P}_m(A \Box B) \leq \hat{P}_m(A \cap \overline{B}). \tag{1.5}$$

This variant of Reimer's theorem groups *all* indices in pairs, while Corollary 1.2 uses only a *single* pair. Nevertheless, the proofs are quite similar.

For increasing events $A, B \subseteq \Omega$ and measures ν on Ω, we abbreviate

$$\mathrm{Reim}_{A,B}(\nu) := \sum_{\omega \in A \cap \overline{B}} \nu(\omega)\nu(\overline{\omega}) - \sum_{\omega \in A \square B} \nu(\omega)\nu(\overline{\omega}), \tag{1.6}$$

where $\nu(\omega) := \nu(\{\omega\})$.

Corollary 1.4. *Assume that $P = \prod_{k\in[n]}(p_k\delta_1 + (1-p_k)\delta_0)$ is a product probability measure on Ω. Then, for all $i, j \in [n]$ with $i \neq j$ and for all increasing events $A, B \subseteq \Omega$, one has*

$$\mathrm{Reim}_{A,B}(P(\cdot \cap \Omega_{i\neq j})) \geq 0. \tag{1.7}$$

Second order Taylor approximation of the van den Berg-Kesten inequality for determinantal probability measures. Take $k \in \mathbb{N}$. (At this moment, one may think of $k = n$, but later, we use also $k = 2n$.) For $I \subseteq [k]$ and $b \in \{0, 1\}$, we abbreviate

$$\{\omega_I \equiv b\} := \{\omega \in \{0,1\}^k : \omega_i = b \; \forall i \in I\}. \tag{1.8}$$

Thus, $\omega_I \equiv 1$ holds if there are particles at least at I, and $\omega_I \equiv 0$ holds if there are no particles at least at I.

For a matrix G, let $G^* := \overline{G}^t$ denote the Hermitian conjugate of G. Recall that $G \in \mathbb{C}^{k\times k}$ is called Hermitian if $G = G^*$. The identity matrix is denoted by $\mathrm{Id} \in \mathbb{C}^{k\times k}$. For any matrix $M = (M_{ij}) \in \mathbb{C}^{l\times m}$ and $I \subseteq [l]$, $J \subseteq [m]$, we denote by $M_{I,J} = (M_{ij})_{i\in I, j\in J}$ the submatrix with index sets I and J. For convenience of the reader, we have collected some basic facts and notation on positive definite matrices in Appendix A.1.

We introduce the following sets of matrices:

$$\begin{aligned} \mathcal{G}_k :=& \{G \in \mathbb{C}^{k\times k} : G = G^*, 0 < G < \mathrm{Id}\} \quad \text{and} \\ \overline{\mathcal{G}}_k :=& \{G \in \mathbb{C}^{k\times k} : G = G^*, 0 \leq G \leq \mathrm{Id}\}. \end{aligned} \tag{1.9}$$

Fact 1.5. *For every $G \in \overline{\mathcal{G}}_k$, there exists a unique probability measure P_G which satisfies*

$$P_G(\omega_I \equiv 1) = \det G_{I,I} \quad \textit{for all} \quad I \subseteq [k]. \tag{1.10}$$

These probability measures P_G are called Hermitian determinantal probability measures.

Although Fact 1.5 is well-known, we include a proof in Appendix A.2 to make the paper self-contained.

Let

$$\mathcal{D} := \{D \in \mathcal{G}_n : D \text{ is diagonal}\} \tag{1.11}$$

denote the set of diagonal matrices in $\mathcal{G}_n$. Note that for any diagonal matrix $D \in \mathcal{D}$, under P_D, the event that there is a particle at position $i \in [n]$ occurs independently of all particles at other locations. For increasing events $A, B \subseteq \Omega$, we define

$$\mathrm{BK}_{A,B} : \overline{\mathcal{G}}_n \to \mathbb{R},\; G \mapsto P_G(A)P_G(B) - P_G(A \square B). \tag{1.12}$$

Whenever A and B are measurable with respect to deterministic disjoint sets of locations, one has $\mathrm{BK}_{A,B} \geq 0$. This is called negative associations of determinantal probability measures and shown in Theorem 6.5 in [4]. The classical BK inequality can be phrased as $\mathrm{BK}_{A,B}(D) \geq 0$ for all $D \in \mathcal{D}$. To our knowledge, it is not known whether $\mathrm{BK}_{A,B}(G) \geq 0$ for all $G \in \overline{\mathcal{G}}_n$. We prove here the following weaker statement:

Theorem 1.6. *For all increasing events $A, B \subseteq \Omega$, the second order Taylor approximation of $\mathrm{BK}_{A,B}$ at $\mathcal{D}$ is non-negative near $\mathcal{D}$. More precisely, let $G : (-1,1) \to \mathcal{G}_n$ be a C^2 path with $G(0) \in \mathcal{D}$. Then, the second order Taylor polynomial of $\mathrm{BK}_{A,B} \circ G$ at 0 is non-negative in a neighborhood of 0.*

The proof of this theorem is based on the following theorem, which also might be interesting on its own.

Theorem 1.7. *For all increasing events $A, B \subseteq \Omega$, the second order Taylor approximation of $\mathcal{G}_n \ni G \mapsto \mathrm{Reim}_{A,B}(P_G)$ at $\mathcal{D}$ is non-negative near $\mathcal{D}$. More precisely, let $G : (-1,1) \to \mathcal{G}_n$ be a C^2 path with $G(0) \in \mathcal{D}$. Then, the second order Taylor polynomial of $(-1,1) \ni t \mapsto \mathrm{Reim}_{A,B}(P_{G(t)})$ at $t_0 = 0$ is non-negative in a neighborhood of 0.*

We remark that $\mathrm{Reim}_{A,B}(P_G)$ may take negative values. This holds even for G arbitrarily close to $\frac{1}{2}\,\mathrm{Id} \in \mathcal{D}$. For a counterexample, see Remark 3.14, below. However, in some numerical and computer algebraic searches, we did not find any counterexample to the conjecture $\mathrm{BK}_{A,B}(G) \geq 0$ for any increasing events A and B and any $G \in \mathcal{G}_n$.

Overview of the proofs and related techniques in the literature. The Corollaries 1.2 and 1.4 of Reimer's butterfly theorem, Fact 1.1, are proven in Section 2. The key idea is to collapse the two locations i and j to a single one.

The Taylor expansions in Theorems 1.6 and 1.7 are proven in Section 3. Reimer's butterfly theorem and Corollary 1.2 are used in these Taylor expansions for the treatment of the 0-th order term and of the second order term, respectively. Parts of the techniques used in Section 3 have also been used by Lyons in [4] and van den Berg and Jonasson in [8], with different goals, perspectives, and notations. More precisely, conditioning of determinantal probability measures is described in §6 of [4], lifting of P_G to $P_{M(G)}$ with a projection $M(G)$ also appears in §8 of [4], and our partitioning of $\Omega \times \Omega \ni (\omega, \eta)$ according to different values of $\xi = \omega + \eta$ has some similarity with the method of proof used in Section 2.2 of [8]. However, we have tried to make the paper as self-contained as possible.

2 Proof of the variant of Reimer's theorem

Throughout this section, we fix $i, j \in [n]$ with $i \neq j$ as in Corollary 1.2. We abbreviate $j^c := [n] \setminus \{j\}$ and $(ij)^c := [n] \setminus \{i, j\}$. The operation $\square$ is adapted to the index set j^c rather than $[n]$; we write $\square_{j^c}$ in this case. We consider the restriction map $' : \Omega_{i \neq j} \to \Omega_{j^c}$, $\omega \mapsto \omega' = (\omega_k)_{k \in j^c}$. Note that this map is a bijection. For an event $A \subseteq \Omega_{i \neq j}$, we write $A' = \{\omega' : \omega \in A\}$. The following lemma allows us to deduce Corollary 1.2 from Reimer's butterfly theorem.

Lemma 2.1. *For increasing events $A, B \subseteq \Omega$, one has:*

$$((A \square B) \cap \Omega_{i \neq j})' \subseteq (A \cap \Omega_{i \neq j})' \square_{j^c} (B \cap \Omega_{i \neq j})'. \tag{2.1}$$

Proof. Let $\zeta \in ((A \square B) \cap \Omega_{i \neq j})'$. We take $\omega \in (A \square B) \cap \Omega_{i \neq j}$ with $\omega' = \zeta$. By the characterization (1.2) of $A \square B$ for increasing events A and B, we can take $S \in A$ and $T \in B$ with $S \wedge T = 0$ and $S \vee T \leq \omega$. In order to show

$$\zeta \in (A \cap \Omega_{i \neq j})' \square_{j^c} (B \cap \Omega_{i \neq j})', \tag{2.2}$$

we define $\tilde{S} \in \Omega_{j^c}$ by $\tilde{S}_k = S_k$ for $k \in (ij)^c$ and $\tilde{S}_i = S_i \vee S_j$, and similarly $\tilde{T}_k = T_k$ for $k \in (ij)^c$ and $\tilde{T}_i = T_i \vee T_j$. By definition of $\square_{j^c}$, it suffices to show the following:

(a) $\tilde{S} \wedge \tilde{T} = 0$.
(b) For all $\eta \in \Omega_{j^c}$ with $\eta \wedge \tilde{S} = \zeta \wedge \tilde{S}$ holds $\eta \in (A \cap \Omega_{i \neq j})'$.
(c) For all $\eta \in \Omega_{j^c}$ with $\eta \wedge \tilde{T} = \zeta \wedge \tilde{T}$ holds $\eta \in (B \cap \Omega_{i \neq j})'$.

To prove claim (a), take $k \in j^c$. In the case $k \in (ij)^c$, we have

$$\tilde{S}_k \wedge \tilde{T}_k = S_k \wedge T_k = 0. \tag{2.3}$$

In the remaining case $k = i$, we get

$$\begin{aligned} \tilde{S}_i \wedge \tilde{T}_i &= (S_i \vee S_j) \wedge (T_i \vee T_j) \\ &= (S_i \wedge T_i) \vee (S_i \wedge T_j) \vee (S_j \wedge T_i) \vee (S_j \wedge T_j) \\ &= (S_i \wedge T_j) \vee (S_j \wedge T_i) \end{aligned} \tag{2.4}$$

because $S \wedge T = 0$. Since $S \leq \omega$ and $T \leq \omega$, we conclude

$$\tilde{S}_i \wedge \tilde{T}_i \leq (\omega_i \wedge \omega_j) \vee (\omega_j \wedge \omega_i) = \omega_i \wedge \omega_j = 0 \tag{2.5}$$

because $\omega \in \Omega_{i \neq j}$.

To prove claim (b), let $\eta \in \Omega_{j^c}$ with $\eta \wedge \tilde{S} = \zeta \wedge \tilde{S}$. Define $\tau \in \Omega_{i \neq j}$ by $\tau_k = \eta_k$ for $k \in j^c$ and $\tau_j = \overline{\eta_i}$. Then, $\tau' = \eta$ holds. We now show that $\tau \geq S$. Indeed, for $k \in j^c$, we have $\tau_k = \eta_k$ and $\tilde{S}_k \geq S_k$. Using $S \leq \omega$ and $\omega' = \zeta$, we conclude $S_k \leq \omega_k = \zeta_k$ and hence

$$\tau_k \geq \tau_k \wedge \tilde{S}_k = \eta_k \wedge \tilde{S}_k = \zeta_k \wedge \tilde{S}_k \geq \zeta_k \wedge S_k = S_k. \tag{2.6}$$

Furthermore, we need to show $\tau_j \geq S_j$. We distinguish two cases.
Case 1: $\tilde{S}_i = 0$: Then, $S_j \leq S_i \vee S_j = \tilde{S}_i = 0$ implies $S_j = 0$ and hence $\tau_j \geq S_j$.
Case 2: $\tilde{S}_i = 1$: Then, $\eta_i = \eta_i \wedge \tilde{S}_i = \zeta_i \wedge \tilde{S}_i = \zeta_i = \omega_i$, $\omega \in \Omega_{i \neq j}$, and $S_j \leq \omega_j$ imply

$$\tau_j = \overline{\eta_i} = \overline{\omega_i} = \omega_j \geq S_j. \tag{2.7}$$

Thus, $\tau \geq S$ is proven.

Using that $S \in A$ and A is increasing, we conclude $\tau \in A$ and hence $\tau \in A \cap \Omega_{i \neq j}$. This yields claim (b) as follows: $\eta = \tau' \in (A \cap \Omega_{i \neq j})'$.

The claim (c) is proven just as claim (b); one only replaces A, S, and $\tilde{S}$ by B, T, and $\tilde{T}$, respectively.

Summarizing, we have proven the claim (2.2). □

Proof of Corollary 1.2. In the following estimate, we use in the first and last step that $' : \Omega_{i \neq j} \to \Omega_{j^c}$ is a bijection. Furthermore, in the first inequality we use Reimer's butterfly theorem Fact 1.1. Finally, in the second inequality, we apply Lemma 2.1. This yields the claim as follows:

$$\begin{aligned} |A \cap \overline{B} \cap \Omega_{i \neq j}| &= |(A \cap \overline{B} \cap \Omega_{i \neq j})'| = |(A \cap \Omega_{i \neq j})' \cap \overline{(B \cap \Omega_{i \neq j})'}| \\ &\geq |(A \cap \Omega_{i \neq j})' \Box_{j^c} (B \cap \Omega_{i \neq j})'| \geq |((A \Box B) \cap \Omega_{i \neq j})'| \\ &= |(A \Box B) \cap \Omega_{i \neq j}|. \end{aligned} \tag{2.8}$$

□

Remark 2.2. *We remark that the claim* (1.4) *of Corollary 1.2 is not valid for arbitrary events A and B. Here is a simple counterexample. Take $n = 2$, $i = 1$, $j = 2$ and consider $A = \{\omega \in \Omega : \omega_1 = 0\}$ and $B = \{\omega \in \Omega : \omega_2 = 1\}$. Then* (1.4) *is false.*

We remark also that the inequality

$$|A \cap \overline{B} \cap \{\omega \in \Omega : \omega_i = 0, \omega_j = 1\}| \geq |(A \Box B) \cap \{\omega \in \Omega : \omega_i = 0, \omega_j = 1\}| \tag{2.9}$$

need not be true for all increasing events $A, B \subseteq \Omega$*. Take for instance* $n = 2$*,* $i = 1$*,* $j = 2$*,* $A = \Omega$*,* $B = \{\omega \in \Omega : \omega_2 = 1\}$*. Then,* $A \cap \overline{B} = \{\omega \in \Omega : \omega_2 = 0\}$*,* $A \Box B = B$*, and hence,*

$$\begin{aligned} &|A \cap \overline{B} \cap \{\omega \in \Omega : \omega_1 = 0, \omega_2 = 1\}| = 0 \\ &< 1 = |(A \Box B) \cap \{\omega \in \Omega : \omega_1 = 0, \omega_2 = 1\}|. \end{aligned} \tag{2.10}$$

Finally, in (1.4) *one cannot replace* $\Omega_{i \neq j}$ *by* $\Omega_{i=j} := \{\omega \in \Omega : \omega_i = \omega_j\}$*. A counterexample is given by* $n = 2$*,* $i = 1$*,* $j = 2$*,* $A = \{\omega_1 = 1\}$*, and* $B = \{\omega_2 = 1\}$*.*

Proof of Corollary 1.4. Let P, i, j, and A, B be as in the assumption of the corollary. Then,

$$P(\omega)P(\overline{\omega}) = \prod_{k \in [n]} p_k(1 - p_k) \tag{2.11}$$

is the same for all $\omega \in \Omega$. We conclude

$$\begin{aligned} \operatorname{Reim}_{A,B}(P(\cdot \cap \Omega_{i\neq j})) &= \sum_{\omega \in \Omega_{i \neq j}} (1_A(\omega) 1_B(\overline{\omega}) - 1_{A \Box B}(\omega)) P(\omega) P(\overline{\omega}) \\ &= (|A \cap \overline{B} \cap \Omega_{i\neq j}| - |(A \Box B) \cap \Omega_{i \neq j}|) \prod_{k \in [n]} p_k(1-p_k) \geq 0 \end{aligned} \tag{2.12}$$

by Corollary 1.2. □

3 Proof of second order Taylor approximations

In Subsection 3.1, we derive a representation of P_G in terms of a triangular matrix $W(G)$, defined in (3.24), below. This representation allows a simpler second order Taylor expansion than the original form. This Taylor expansion is derived in Subsection 3.2 and then applied in Subsection 3.3 to derive Theorem 1.6.

3.1 A representation for determinantal probability measures

Let us first explain what is done in this subsection and why. In the definition (1.6) of $\operatorname{Reim}_{A,B}(P_G)$, the probability $P_G(\omega)$ of individual outcomes $\omega \in \Omega$ plays an essential role. However, these probabilities are difficult to compute directly using the defining property (1.10) of P_G, which is about events $\{\omega_I \equiv 1\}$. Events consisting of a single outcome can be written in the form $\Lambda_{I,n} = \{\omega_I \equiv 1, \omega_{[n]\setminus I} \equiv 0\}$. If P_G is supported on configurations consisting of precisely $|I|$ particles, the events $\{\omega_I \equiv 1\}$ and $\Lambda_{I,n}$ differ only by a null set. Consequently, the probability $P_G(\omega)$ of $\omega \in \Omega$ is a simple determinant in this case. In particular, this holds when G is an orthogonal projector of rank $|I|$. However, $\mathcal{G}_n$ does not contain any orthogonal projector. But one can write P_G as a marginal of another determinantal probability measure $P_{M(G)}$ on the set of configurations $\{0,1\}^{2n}$ with twice the number of locations and an orthogonal projector $M(G)$ of rank n; this is the meaning of Lemma 3.3 in combination with Lemma 3.1. Instead of working with the projector $M(G)$, one can also work with an orthonormal basis of the space it projects to, encoded as columns in a matrix $\Psi(G)^*$. This yields a description of $P_{M(G)}$ in terms of a measure $\mu_{\Psi(G)}$, described in Definition 3.2(a); see also Lemma 3.3. Choosing another basis (not necessarily orthonormal) of the same space only changes a normalizing constant in this measure, as is shown in Lemma 3.5. A convenient choice of such a basis, encoded in a matrix, is of the form $\Sigma^* = (\sigma, \mathrm{Id})^*$, where the identity matrix corresponds to the "second half" of locations which are dropped by taking the marginal P_G of $P_{M(G)}$. Details are given in Lemma 3.4. As a marginal of μ_Σ, one obtains another finite measure ν_σ on $\{0,1\}^n$. For quadratic matrices $\sigma \in \mathbb{C}^{n\times n}$, it is introduced in Definition 3.2(b), below. Unlike P_G, the measure ν_σ is defined in terms of probabilities of individual outcomes. By construction, for an appropriate choice of σ, it turns out to be a multiple of P_G; see Lemma 3.6 below. The second order perturbation analysis of $\mathrm{Reim}_{A,B}(\nu_\sigma)$ gets more elementary for triangular matrices σ with ones in the diagonal. For this reason, we reduce the general case to this special case using a QR-decomposition; this yields Lemma 3.10 below.

To make all this precise, we proceed as follows. Recall the definitions of $\mathcal{G}_n$, $\overline{\mathcal{G}}_n$, and $\mathcal{D}$ from (1.9) and (1.11). For any positive semidefinite Hermitian matrix $A \geq 0$, $\sqrt{A} \geq 0$ denotes its unique positive semidefinite square root. We define

$$\Psi : \overline{\mathcal{G}}_n \to \mathbb{C}^{n\times 2n}, \quad \Psi(G) = (\psi(G), \phi(G)) = (\sqrt{G}, \sqrt{\mathrm{Id} - G}), \text{ and} \tag{3.1}$$

$$M : \overline{\mathcal{G}}_n \to \mathbb{C}^{2n\times 2n}, \quad M(G) = \Psi^*(G)\Psi(G). \tag{3.2}$$

Note that the restriction of ψ and ϕ to $\mathcal{G}_n$ are real analytic functions taking values in the set of positive definite $n \times n$ matrices; see also Appendix A.1. By definition, one has $\psi(G)^* = \psi(G)$, $\phi(G)^* = \phi(G)$, $\psi(G)\psi(G)^* = G$, and $\phi(G)\phi(G)^* = \mathrm{Id} - G$ for $G \in \overline{\mathcal{G}}_n$. Note further that $\phi(G)$ and $\psi(G)$ are diagonal matrices whenever G is a diagonal matrix.

Lemma 3.1. *For $G \in \overline{\mathcal{G}}_n$, the matrix $\Psi(G)$ has orthonormal rows. $M(G)$ is the orthogonal projector to the space spanned by the columns of $\Psi(G)^*$. In particular, $0 \leq M(G) \leq \mathrm{Id}$ and $\mathrm{rank}\, M(G) = n = \mathrm{rank}(\mathrm{Id} - M(G))$ hold.*

Proof. It follows from $\Psi(G)\Psi(G)^* = \psi(G)\psi(G)^* + \phi(G)\phi(G)^* = \mathrm{Id}$ that $\Psi(G)$ has orthonormal rows. As a consequence, the second claim follows. In particular, $\Psi(G)$, $M(G) = \Psi(G)^*\Psi(G)$, and $\mathrm{Id} - M(G)$ have rank n, and we get $0 \leq M(G) \leq \mathrm{Id}$. ☐

For $k \in \mathbb{N}$ and $I \subseteq [k]$, let

$$\Lambda_{I,k} := \{\omega \in \{0,1\}^k : \ \omega_I \equiv 1, \omega_{[k]\setminus I} \equiv 0\} \tag{3.3}$$

denote the event that there are particles precisely at locations in I.

We introduce now two measures with a matrix as a parameter. They are both closely related to P_G as is shown in Lemmas 3.3 and 3.6 below.

Definition 3.2. *(a) For $\Sigma \in \mathbb{C}^{n\times 2n}$, we define a finite measure μ_Σ on $\{0,1\}^{2n}$ by*

$$\mu_\Sigma(\Lambda_{I,2n}) := \begin{cases} \det(\Sigma^*\Sigma)_{I,I} & \text{for } I \subseteq [2n] \text{ with } |I| = n, \\ 0 & \text{for } I \subseteq [2n] \text{ with } |I| \neq n. \end{cases} \tag{3.4}$$

Thus, μ_Σ is supported on configurations with precisely n particles at $2n$ locations.

(b) For $\sigma \in \mathbb{C}^{n\times n}$, we define another finite measure ν_σ on $\{0,1\}^n$ by

$$\nu_\sigma(\Lambda_{I,n}) := \det(\sigma^*\sigma)_{I,I} \text{ for } I \subseteq [n]. \tag{3.5}$$

Thus, ν_σ is supported on particle configurations at n locations with an arbitrary number of particles.

If $\mathrm{rank}\, \Sigma < n$, μ_Σ is the zero measure. Although the definitions of μ_Σ and ν_σ look somehow similar, these measures are quite different and should not be confused with each other. In the special case $\Sigma = \Psi(G)$, the following lemma establishes a connection between μ_Σ and P_G.

Let $\iota : \{0,1\}^{2n} \to \{0,1\}^n$ denote the projection to the first n coordinates. We denote by $\iota[\mu]$ the image measure of any measure μ on $\{0,1\}^{2n}$ with respect to ι.

Lemma 3.3. *For $G \in \mathcal{G}_n$, one has $\mu_{\Psi(G)} = P_{M(G)}$ and $P_G = \iota[\mu_{\Psi(G)}]$. Consequently, $\mu_{\Psi(G)}$ is a probability measure.*

Proof. By definition, $\mu_{\Psi(G)}$ is supported on configurations with precisely n particles. The same is true for $P_{M(G)}$ by Lemma A.3 in the appendix; the assumptions of this lemma are fulfilled by Lemma 3.1. In particular, for all $I \subseteq [2n]$ with $|I| = n$, the events $\Lambda_{I,2n}$ and $\{\omega_I \equiv 1\}$ coincide up to null events with respect to both measures $\mu_{\Psi(G)}$ and $P_{M(G)}$. Since $\mu_{\Psi(G)}(\Lambda_{I,2n}) = \det(\Psi(G)^*\Psi(G))_{I,I} = P_{M(G)}(\omega_I \equiv 1)$ holds for these sets I, the first claim $\mu_{\Psi(G)} = P_{M(G)}$ follows. The second claim $P_G = \iota[\mu_{\Psi(G)}]$ follows then from Lemma A.4. □

For $\omega \in \Omega$, we set

$$I(\omega) := \{i \in [n] : \omega_i = 1\}. \tag{3.6}$$

The measures μ_Σ and ν_σ are related as follows.

Lemma 3.4. *For $\sigma \in \mathbb{C}^{n\times n}$ and $\Sigma = (\sigma, \mathrm{Id}) \in \mathbb{C}^{n\times 2n}$, one has $\iota[\mu_\Sigma] = \nu_\sigma$. In addition, for $\omega \in \Omega$, the following holds:*

$$\nu_\sigma(\omega) = \sum_{K\subseteq[n]:|K|=|I(\omega)|} |\det \sigma_{K,I(\omega)}|^2. \tag{3.7}$$

Proof. Let $I \subseteq [n]$. If at the locations in $[n]$ there are precisely particles in I and the total number of particles in $[2n]$ is n, then there must be $n - |I|$ particles in $[2n] \setminus [n]$. Since μ_Σ is supported on configurations with precisely n particles, we get

$$(\iota[\mu_\Sigma])(\Lambda_{I,n}) = \sum_{J\subseteq[2n]\setminus[n]:|J|=n-|I|} \mu_\Sigma(\Lambda_{I\cup J,2n}). \tag{3.8}$$

For any $J' \subseteq [n]$ with $|J'| = n - |I|$ and $J := J' + n$, one has

$$\begin{aligned}\mu_\Sigma(\Lambda_{I\cup J,2n}) &= \det(\Sigma^*\Sigma)_{I\cup J,I\cup J}\\ &= |\det \Sigma_{[n],I\cup J}|^2 = |\det(\sigma_{[n],I}, \mathrm{Id}_{[n],J'})|^2 = |\det \sigma_{K,I}|^2\end{aligned} \tag{3.9}$$

with $K = [n] \setminus J'$, where in the last equation, we have expanded the determinant with respect to the columns coming from Id.

For matrices $E \in \mathbb{C}^{n\times i}$ and $F \in \mathbb{C}^{i\times n}$ with natural numbers $i \le n$, the well-known Cauchy-Binet formula states the following:

$$\sum_{K\subseteq[n],|K|=i} \det(EF)_{K,K} = \det(FE). \tag{3.10}$$

We use it in the special case $E = \sigma_{[n],I}$ and $F = \sigma^*_{I,[n]}$ to obtain

$$\begin{aligned}(\iota[\mu_\Sigma])(\Lambda_{I,n}) &= \sum_{K\subseteq[n]:|K|=|I|} |\det \sigma_{K,I}|^2 = \sum_{K\subseteq[n]:|K|=|I|} \det(\sigma_{K,I}\sigma^*_{I,K})\\ &= \det(\sigma^*_{I,[n]}\sigma_{[n],I}) = \det(\sigma^*\sigma)_{I,I} = \nu_\sigma(\Lambda_{I,n}).\end{aligned} \tag{3.11}$$

Since $\{\omega\} = \Lambda_{I(\omega),n}$ for $\omega \in \Omega$, the claims follow. □

Lemma 3.5. *For all $\Sigma \in \mathbb{C}^{n\times 2n}$ and all $C \in \mathbb{C}^{n\times n}$, one has $\mu_{C\Sigma} = |\det C|^2\mu_\Sigma$.*

Proof. For $I \subseteq [2n]$ with $|I| = n$, the matrices $\Sigma_{[n],I}$ and ${\Sigma^*}_{I,[n]}$ are square matrices. Hence, the defining equation (3.4) of μ_Σ implies

$$\mu_{C\Sigma}(\Lambda_{I,2n}) = \det(\Sigma^*C^*C\Sigma)_{I,I} = |\det C|^2 \det(\Sigma^*\Sigma)_{I,I} = |\det C|^2\mu_\Sigma(\Lambda_{I,2n}). \tag{3.12}$$

Because the measures $\mu_{C\Sigma}$ and μ_Σ are both supported on configurations with precisely n particles, the claim follows. □

We define the real-analytic map

$$\chi : \mathcal{G}_n \to \mathbb{C}^{n\times n}, \quad \chi(G) = \phi(G)^{-1}\psi(G).$$

Note that for $G \in \mathcal{G}_n$, the matrix $\chi(G) = (\mathrm{Id} - G)^{-1/2}G^{1/2}$ is positive definite; in particular all its diagonal entries are positive. Note further that for diagonal matrices $G \in \mathcal{D}$, the matrix $\chi(G)$ is diagonal.

Lemma 3.6. *For all $G \in \mathcal{G}_n$ one has $P_G = |\det \phi(G)|^2 \nu_{\chi(G)}$.*

Proof. By definition, the formula $\phi(G)^{-1}\Psi(G) = (\chi(G), \mathrm{Id})$ holds for $G \in \mathcal{G}_n$. The following calculation uses this fact in the third equality, Lemma 3.3 in the first equality, Lemma 3.5 in the second equality, and Lemma 3.4 in the last equality.

$$\begin{aligned} P_G &= \iota[\mu_{\Psi(G)}] = |\det \phi(G)|^2 \iota[\mu_{\phi^{-1}(G)\Psi(G)}] \\ &= |\det \phi(G)|^2 \iota[\mu_{(\chi(G),\mathrm{Id})}] = |\det \phi(G)|^2 \nu_{\chi(G)}. \end{aligned} \tag{3.13}$$

This proves the claim. □

Lemma 3.7. *For all $R \in \mathbb{C}^{n\times n}$ and all unitary matrices $Q \in \mathrm{U}(n)$, one has $\nu_{QR} = \nu_R$.*

Proof. By definition (3.5), one has for $I \subseteq [n]$:

$$\nu_{QR}(\Lambda_{I,n}) = \det(R^*Q^*QR)_{I,I} = \det(R^*R)_{I,I} = \nu_R(\Lambda_{I,n}). \tag{3.14}$$

This implies the claim. □

Now let

$$\chi(G) = Q(G)R(G) \quad \text{for } G \in \mathcal{G}_n \tag{3.15}$$

denote the QR-decomposition of $\chi(G)$, where

$$Q : \mathcal{G}_n \to \mathrm{U}(n), \tag{3.16}$$

$$R : \mathcal{G}_n \to \mathcal{T} := \{T \in \mathbb{C}^{n\times n} : \ T \text{ is upper triangular with } T_{ii} > 0 \text{ for all } i \in [n]\}. \tag{3.17}$$

Note that the maps Q and R are uniquely determined by the Gram-Schmidt-algorithm in terms of real-analytic operations. As a consequence, these two maps are real-analytic. Note also that for diagonal matrices $G \in \mathcal{D}$, the matrix $R(G)$ is diagonal. We get

Lemma 3.8. *For all $G \in \mathcal{G}_n$, we have $P_G = |\det \phi(G)|^2 \nu_{R(G)}$.*

Proof. Combining Lemmas 3.6 and 3.7, we get the claim as follows:

$$P_G = |\det \phi(G)|^2 \nu_{\chi(G)} = |\det \phi(G)|^2 \nu_{Q(G)R(G)} = |\det \phi(G)|^2 \nu_{R(G)}. \tag{3.18}$$

□

It is convenient to work with triangular matrices having all diagonal entries equal to 1 rather than using arbitrary positive diagonal entries. To describe the corresponding normalization, we introduce the following notation: For any diagonal matrix $D \in \mathbb{C}^{n\times n}$ with positive entries, we define

$$\kappa(D) : \Omega \to \mathbb{R}_+, \quad \kappa(D)(\omega) = |\det D_{I(\omega),I(\omega)}|^2 = \prod_{i\in I(\omega)} D_{ii}^2. \tag{3.19}$$

Lemma 3.9. *For all $R \in \mathbb{C}^{n\times n}$, all diagonal matrices $D \in \mathbb{C}^{n\times n}$ with positive diagonal entries, and all $\omega \in \Omega$, one has*

$$\nu_R(\omega) = \kappa(D)(\omega)\nu_{RD^{-1}}(\omega). \tag{3.20}$$

In short notation, this means

$$d\nu_R = \kappa(D)\, d\nu_{RD^{-1}}. \tag{3.21}$$

Proof. Using the defining formula (3.5) of ν_R and abbreviating $J = I(\omega)$, the claim is proven as follows:

$$\begin{aligned}\nu_R(\omega) &= \nu_R(\Lambda_{J,n}) = \det(R^*R)_{J,J} = |\det D_{J,J}|^2 \det((D^{-1})^*R^*RD^{-1})_{J,J}\\ &= |\det D_{J,J}|^2\nu_{RD^{-1}}(\Lambda_{J,n}) = |\det D_{J,J}|^2\nu_{RD^{-1}}(\omega).\end{aligned} \tag{3.22}$$

□

We apply this lemma to the real-analytic maps

$$\begin{aligned}&D : \mathcal{G}_n \to \{\Delta \in \mathbb{C}^{n\times n} :\ \Delta \text{ is a diagonal matrix with } \Delta_{ii} > 0 \text{ for all } i \in [n]\},\\ &\qquad D(G) = \operatorname{diag}(R(G)_{ii}, i \in [n]),\end{aligned} \tag{3.23}$$

$$\begin{aligned}&W : \mathcal{G}_n \to \mathcal{T}_1 = \{T \in \mathcal{T} :\ T_{ii} = 1 \text{ for all } i \in [n]\},\\ &\qquad W(G) = R(G)D(G)^{-1}.\end{aligned} \tag{3.24}$$

Note that

$$W(G) = \mathrm{Id} \quad \text{holds for all } G \in \mathcal{D}. \tag{3.25}$$

We get

Lemma 3.10. *For all $G \in \mathcal{G}_n$, one has $dP_G = |\det \phi(G)|^2\kappa(D(G))\, d\nu_{W(G)}$.*

Proof. This follows from Lemmas 3.8 and 3.9. □

We introduce the following real analytic function, which plays the role of a normalizing constant:

$$c : \mathcal{G}_n \to \mathbb{R}_+, \quad c(G) = |\det \phi(G)|^4|\det D(G)|^2. \tag{3.26}$$

Recall Definition (1.6) of $\mathrm{Reim}_{A,B}$.

Lemma 3.11. *For all increasing events $A, B \subseteq \Omega$ and all $G \in \mathcal{G}_n$, one has*

$$\mathrm{Reim}_{A,B}(P_G) = c(G)\,\mathrm{Reim}_{A,B}(\nu_{W(G)}). \tag{3.27}$$

Proof. For $\omega \in \Omega$, we have the following, using the definition (3.19) of $\kappa(D(G))$:

$$\kappa(D(G))(\omega)\kappa(D(G))(\overline{\omega}) = |\det D(G)_{I(\omega),I(\omega)}|^2|\det D(G)_{I(\overline{\omega}),I(\overline{\omega})}|^2 = |\det D(G)|^2. \tag{3.28}$$

Using this together with Lemma 3.10 yields

$$\begin{aligned}P_G(\omega)P_G(\overline{\omega}) &= |\det \phi(G)|^4\kappa(D(G))(\omega)\kappa(D(G))(\overline{\omega})\nu_{W(G)}(\omega)\nu_{W(G)}(\overline{\omega})\\ &= c(G)\nu_{W(G)}(\omega)\nu_{W(G)}(\overline{\omega})\end{aligned}$$

Summing this over $\omega \in A \cap \overline{B}$ and over $\omega \in A\Box B$ and taking the difference, the claim follows. □

3.2 Perturbation analysis around Reimer's butterfly theorem

We now take any matrix norm $\|\cdot\|$ on $\mathbb{C}^{n\times n}$. Recall that $\mathcal{T}_1$ denotes the set of all upper triangular complex $n\times n$ matrices with all diagonal entries equal to 1. Consequently, for $\sigma\in\mathcal{T}_1$, $\|\sigma-\mathrm{Id}\,\|$ measures the size of the off-diagonals in σ. In the following, we use $\sum_{\substack{i,j\in[n]\\ i\neq j}}$ as a short notation for $\sum_{i\in[n]}\sum_{j\in[n]\setminus\{i\}}$.

Lemma 3.12. *For all events $C\subseteq\Omega$ we have the following for $\sigma\in\mathcal{T}_1$ in the limit as $\sigma\to\mathrm{Id}$:*

$$\sum_{\omega\in C}\nu_\sigma(\omega)\nu_\sigma(\overline{\omega})=|C|+\sum_{\substack{i,j\in[n]\\ i\neq j}}|C\cap\Omega_{i\neq j}|\cdot|\sigma_{ji}|^2+O(\|\sigma-\mathrm{Id}\,\|^3) \tag{3.29}$$

Proof. The error terms in this proof are always understood in the limit $\mathcal{T}_1\ni\sigma\to\mathrm{Id}$. We prove the following for $\omega\in\Omega$:

$$\nu_\sigma(\omega)\nu_\sigma(\overline{\omega})=1+\sum_{\substack{i,j\in[n]\\ i\neq j}}1_{\Omega_{i\neq j}}(\omega)|\sigma_{ji}|^2+O(\|\sigma-\mathrm{Id}\,\|^3). \tag{3.30}$$

Summing over $\omega\in C$ then yields the claim.

Recall the definition (3.6) of $I(\omega)$ and formula (3.7):

$$\nu_\sigma(\omega)=\sum_{K\subseteq[n]:|K|=|I(\omega)|}|\det\sigma_{K,I(\omega)}|^2. \tag{3.31}$$

Consider $K\subseteq[n]$ with $|K|=|I(\omega)|$. The following expansion of the determinant in (3.31) is used below.

$$\det\sigma_{K,I(\omega)}=\sum_{\substack{\tau:K\to I(\omega)\\ \text{bijective}}}\operatorname{sgn}\tau\prod_{k\in K}\sigma_{k,\tau_k}. \tag{3.32}$$

We distinguish several cases:

Case 1: $K=I(\omega)$. In this case, we have

$$|\det\sigma_{K,I(\omega)}|^2=|\det\sigma_{I(\omega),I(\omega)}|^2=1 \tag{3.33}$$

because σ is a triangular matrix with ones on the diagonal.

Case 2: $K\setminus I(\omega)$ consists of a single element $j\in K$. Then $I(\omega)\setminus K$ consists also of a single element $i\in I(\omega)$, $i\neq j$. Consider an index τ in the sum (3.32).

Case 2a: $\tau_j=i$ and $\tau_k=k$ for all $k\in K\setminus\{j\}$. Here, we get

$$\prod_{k\in K}\sigma_{k,\tau_k}=\sigma_{ji} \tag{3.34}$$

because σ has ones on the diagonal.

Case 2b: $\tau_k\neq k$ for more than one $k\in K$. In this case, we have the bound $\prod_{k\in K}\sigma_{k,\tau_k}=O(\|\sigma-\mathrm{Id}\,\|^2)$, because the product contains at least two non-diagonal factors, which are bounded by $O(\|\sigma-\mathrm{Id}\,\|)$.

Consequently in case 2, we obtain

$$\det\sigma_{K,I(\omega)}=\pm\sigma_{ji}+O(\|\sigma-\mathrm{Id}\,\|^2) \tag{3.35}$$

and hence

$$|\det\sigma_{K,I(\omega)}|^2=|\sigma_{ji}|^2+O(\|\sigma-\mathrm{Id}\,\|^3) \tag{3.36}$$

Case 3: $K \setminus I(\omega)$ consists of at least two elements. Consider again an index τ in the sum (3.32). Just as in case 2b, we have $\tau_k \neq k$ for more than one $k \in K$. The same argument as in case 2b yields again $\prod_{k\in K} \sigma_{k,\tau_k} = O(\|\sigma - \mathrm{Id}\|^2)$. We conclude in this case:

$$\det \sigma_{K,I(\omega)} = O(\|\sigma - \mathrm{Id}\|^2) \tag{3.37}$$

and hence

$$|\det \sigma_{K,I(\omega)}|^2 = O(\|\sigma - \mathrm{Id}\|^4). \tag{3.38}$$

Summing over K in all three cases, formula (3.31) becomes

$$\nu_\sigma(\omega) = 1 + \sum_{i\in I(\omega)} \sum_{j\in I(\overline{\omega})} |\sigma_{ji}|^2 + O(\|\sigma - \mathrm{Id}\|^3); \tag{3.39}$$

note that $I(\overline{\omega}) = [n] \setminus I(\omega)$. Replacing ω by $\overline{\omega}$, we conclude

$$\nu_\sigma(\overline{\omega}) = 1 + \sum_{i\in I(\overline{\omega})} \sum_{j\in I(\omega)} |\sigma_{ji}|^2 + O(\|\sigma - \mathrm{Id}\|^3); \tag{3.40}$$

Taking the product of (3.39) and (3.40) yields the claim (3.30) as follows:

$$\begin{aligned}
\nu_\sigma(\omega)\nu_\sigma(\overline{\omega}) &= 1 + \sum_{i\in I(\omega)} \sum_{j\in I(\overline{\omega})} |\sigma_{ji}|^2 + \sum_{i\in I(\overline{\omega})} \sum_{j\in I(\omega)} |\sigma_{ji}|^2 + O(\|\sigma - \mathrm{Id}\|^3) \\
&= 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} (1_{\{\omega_i=1,\omega_j=0\}} + 1_{\{\omega_i=0,\omega_j=1\}}) |\sigma_{ji}|^2 + O(\|\sigma - \mathrm{Id}\|^3) \\
&= 1 + \sum_{i=1}^{n} \sum_{j=1}^{n} 1_{\{i\neq j,\omega_i\neq\omega_j\}} |\sigma_{ji}|^2 + O(\|\sigma - \mathrm{Id}\|^3) \\
&= 1 + \sum_{\substack{i,j\in[n] \\ i\neq j}} 1_{\Omega_{i\neq j}}(\omega) |\sigma_{ji}|^2 + O(\|\sigma - \mathrm{Id}\|^3).
\end{aligned} \tag{3.41}$$

□

We combine the results obtained so far to obtain a Taylor-expansion of the function $\mathrm{Reim}_{A,B}$, which was defined in formula (1.6):

Corollary 3.13. *For all increasing events $A, B \subseteq \Omega$, one has the following for $\sigma \in \mathcal{T}_1$ in the limit as $\sigma \to \mathrm{Id}$:*

$$\begin{aligned}
\mathrm{Reim}_{A,B}(\nu_\sigma) =& |A \cap \overline{B}| - |A \square B| + \sum_{\substack{i,j\in[n] \\ i\neq j}} (|A \cap \overline{B} \cap \Omega_{i\neq j}| - |(A \square B) \cap \Omega_{i\neq j}|) |\sigma_{ji}|^2 \\
&+ O(\|\sigma - \mathrm{Id}\|^3).
\end{aligned} \tag{3.42}$$

As a consequence, the second order Taylor polynomial T of $\mathcal{T}_1 \ni \sigma \mapsto \mathrm{Reim}_{A,B}(\nu_\sigma)$ at $\sigma_0 = \mathrm{Id}$ is non-negative.

Proof. The first claim (3.42) follows immediately from Lemma 3.12. Now Reimer's theorem, cited as Fact 1.1, shows that the 0th order term $|A \cap \overline{B}| - |A \square B|$ in the Taylor polynomial T is non-negative. The first order terms in T vanish, and Corollary 1.2 shows that the second order terms $(|A \cap \overline{B} \cap \Omega_{i\neq j}| - |(A \square B) \cap \Omega_{i\neq j}|)|\sigma_{ji}|^2$, $i \neq j$, in T are also nonnegative. This proves the second claim. □

Proof of Theorem 1.7. Let $G : (-1,1) \to \mathcal{G}_n$ be a C^2 path with $G(0) \in \mathcal{D}$. From Lemma 3.11 we know

$$\mathrm{Reim}_{A,B}(P_{G(t)}) = c(G(t))\,\mathrm{Reim}_{A,B}(\nu_{W(G(t))}). \tag{3.43}$$

Note that $W(G(0)) = \mathrm{Id}$ by (3.25) and recall that c and W are real-analytic functions. The second order Taylor polynomial of $t \mapsto \mathrm{Reim}_{A,B}(\nu_{W(G(t))})$ at $t_0 = 0$ is non-negative near $t_0 = 0$ as a consequence of Corollary 3.13. Now $c(G(t)) > 0$ for t in a neighborhood of 0. Combining these facts yields the claim of the theorem. □

Remark 3.14. *The following counterexample shows that* $\mathrm{Reim}_{A,B}(\nu_\sigma)$ *may take negative values for* $\sigma \in \mathcal{T}_1$ *arbitrarily close to* Id*. Take* $n = 3$*,* $A = \{\omega_1 = 1\}$*, and* $B = \{\omega_2 = 1\}$*. Then, for* $\sigma \in \mathcal{T}_1 \cap \mathbb{R}^{3\times 3}$*, one has*

$$\mathrm{Reim}_{A,B}(\nu_\sigma) = 2\sigma_{12}^2 - 2\sigma_{12}\sigma_{13}\sigma_{23} + 2\sigma_{12}^2\sigma_{23}^2, \tag{3.44}$$

which is negative for $0 < \sigma_{12} \ll \sigma_{13}\sigma_{23} \ll \sigma_{23} \ll \sigma_{13} \ll 1$*. However, in this example, we get for complex* $\sigma \in \mathcal{T}_1$*,*

$$\nu_\sigma(A)\nu_\sigma(B) - \nu_\sigma(A \square B)\nu_\sigma(\Omega) = \left|\sigma_{13}\sigma_{23}^* - 2\sigma_{12} - \sigma_{12}|\sigma_{23}|^2\right|^2 \geq 0. \tag{3.45}$$

One can show that the function $W : \mathcal{G}_n \to \mathcal{T}_1$ *maps small neighborhoods in* $\mathcal{G}_n$ *of* $\frac{1}{2}\mathrm{Id} \in \mathcal{G}_n$ *onto small neighborhoods in* $\mathcal{T}_1$ *of* $\mathrm{Id} \in \mathcal{T}_1$*. In view of Lemma 3.11, this implies that* $\mathrm{Reim}_{A,B}(P_G)$ *may take negative values for* $G \in \mathcal{G}_n$ *arbitrarily close to* $\frac{1}{2}\mathrm{Id}$*.*

Note that the third order Taylor polynomial at $\sigma_0 = \mathrm{Id}$ *of expression* (3.45)*, which equals* $4(|\sigma_{12}|^2 - \mathrm{Re}(\sigma_{12}\sigma_{23}\sigma_{13}^*))$*, may take negative values for* σ *arbitrarily close to* Id*. This illustrates the following fact: if the BK inequality holds for a family of matrices, the corresponding third order Taylor approximation may violate it even close to the point of expansion. This implies also that our method of proof cannot be generalized to third order Taylor polynomials in a straightforward way.*

3.3 From the variant of Reimer's theorem to the BK inequality

In this subsection, we work with arbitrary finite sets $K \subset \mathbb{N}$ of locations rather than only with $[k]$. For this reason, we adapt the notations (1.8/1.9/1.10) as follows:

$$\{\omega_I \equiv j\} = \{\omega \in \{0,1\}^K : \omega_i = j \text{ for all } i \in I\} \quad \text{for } I \subseteq K \text{ and } j = 0,1, \tag{3.46}$$

$$\mathcal{G}_K = \{G \in \mathbb{C}^{K\times K} : G = G^*, 0 < G < \mathrm{Id}\}, \quad \text{and} \tag{3.47}$$

$$P_G(\omega_I \equiv 1) = \det G_{I,I} \quad \text{for all} \quad G \in \mathcal{G}_K, I \subseteq K. \tag{3.48}$$

The notations $\mathrm{Reim}_{A,B}$ and $\square$ are adapted to arbitrary finite index sets $K \subset \mathbb{N}$ in the obvious way; we use the notations $\mathrm{Reim}^K_{A,B}$ and $\square_K$, respectively. The restriction of a configuration $\omega \in \{0,1\}^K$ to $\{0,1\}^I$, $I \subseteq K$, is denoted by $\omega_I = (\omega_i)_{i\in I}$. For $I \subseteq K$, we define $I^c = K \setminus I$.

We now introduce two functions $C^1_{I,K}$ and $C^0_{I,K}$. The function $C^1_{I,K}$ corresponds to conditioning on having *particles* on I^c, whereas $C^0_{I,K}$ corresponds to conditioning on having *holes* on I^c. A precise formulation of this fact is given in Lemma 3.16, below.

$$C^1_{I,K} : \mathcal{G}_K \to \mathcal{G}_I, \quad C^1_{I,K}(G) = G_{I,I} - G_{I,I^c}(G_{I^c,I^c})^{-1}G_{I^c,I}, \tag{3.49}$$

$$C^0_{I,K} : \mathcal{G}_K \to \mathcal{G}_I, \quad C^0_{I,K}(G) = G_{I,I} + G_{I,I^c}(\mathrm{Id} - G_{I^c,I^c})^{-1}G_{I^c,I}. \tag{3.50}$$

Note that $C^j_{I,K}$, $j = 0,1$, are real analytic functions. They map diagonal matrices to diagonal matrices.

Lemma 3.15. *The maps* $C^j_{I,K}$, $j = 0, 1$, *are well-defined. For all* $G \in \mathcal{G}_K$ *and* $J \subseteq I \subseteq K$, *one has*

$$\det G_{J\cup I^c, J\cup I^c} = \det(C^1_{I,K}(G))_{J,J} \det G_{I^c,I^c}. \tag{3.51}$$

In addition, the following relation holds for all $G \in \mathcal{G}_K$:

$$C^1_{I,K}(\mathrm{Id} - G) = \mathrm{Id} - C^0_{I,K}(G) \tag{3.52}$$

Identity (3.52) means intuitively that $C^1_{I,K}$ and $C^0_{I,K}$ are exchanged when exchanging particles with holes.

Proof of Lemma 3.15. $G \in \mathcal{G}_K$ implies $G_{I^c,I^c} \in \mathcal{G}_{I^c}$ and $G_{I,I} \in \mathcal{G}_I$. In particular, G_{I^c,I^c} and $\mathrm{Id} - G_{I^c,I^c}$ are invertible. For the matrix

$$T = \begin{pmatrix} \mathrm{Id}_{I,I} & -G_{I,I^c}(G_{I^c,I^c})^{-1} \\ 0 & \mathrm{Id}_{I^c,I^c} \end{pmatrix}, \tag{3.53}$$

the following holds:

$$0 < TGT^* = T \begin{pmatrix} G_{I,I} & G_{I,I^c} \\ G_{I^c,I} & G_{I^c,I^c} \end{pmatrix} T^* = \begin{pmatrix} C^1_{I,K}(G) & 0 \\ 0 & G_{I^c,I^c} \end{pmatrix}. \tag{3.54}$$

As a consequence, we get $C^1_{I,K}(G) > 0$; in particular, $C^1_{I,K}(G)$ is Hermitian. Now, $G_{I,I^c}(G_{I^c,I^c})^{-1}G_{I^c,I} = G_{I,I^c}(G_{I^c,I^c})^{-1}(G_{I,I^c})^* \geq 0$ implies $C^1_{I,K}(G) \leq G_{I,I} < \mathrm{Id}$. Hence, $C^1_{I,K}(G) \in \mathcal{G}_I$.

Next, we show (3.51). We abbreviate $L = J \cup I^c$ and $L^c = K \setminus L = I \setminus J$. Note that $T_{K,I} = \mathrm{Id}_{K,I}$ implies $T_{L,L^c} = 0$. Consequently, we get $(TGT^*)_{L,L} = T_{L,L}G_{L,L}T^*_{L,L}$. Using $\det T_{L,L} = 1$ and (3.54), we get

$$\det G_{L,L} = \det(TGT^*)_{L,L} = \det(C^1_{I,K}(G))_{J,J} \det G_{I^c,I^c}. \tag{3.55}$$

The claim (3.52) follows then from the definitions of $C^1_{I,K}(\mathrm{Id} - G)$ and $C^0_{I,K}(G)$ together with $(\mathrm{Id} - G)_{I,I^c} = -G_{I,I^c}$ and $(\mathrm{Id} - G)_{I^c,I} = -G_{I^c,I}$.

Finally, we conclude $C^0_{I,K}(G) = \mathrm{Id} - C^1_{I,K}(\mathrm{Id} - G) \in \mathrm{Id} - \mathcal{G}_I = \mathcal{G}_I$. □

Lemma 3.16. *For all* $G \in \mathcal{G}_K$, *all* $I \subseteq K$, *and all* $j \in \{0, 1\}$, *one has*

$$P_G(\omega_I \in \cdot | \omega_{I^c} \equiv j) = P_{C^j_{I,K}(G)}. \tag{3.56}$$

Proof. Let $G \in \mathcal{G}_K$ and $I \subseteq K$. First, we show (3.56) in the case $j = 1$. It suffices to prove the claim for the events $\{\omega_J \equiv 1\}$, $J \subseteq I$. Note that $P_G(\omega_{I^c} \equiv 1) = \det G_{I^c,I^c} \neq 0$. We calculate using (3.51) in the second but last inequality:

$$\begin{aligned} P_G(\omega_J \equiv 1 | \omega_{I^c} \equiv 1) &= \frac{P_G(\omega_{J\cup I^c} \equiv 1)}{P_G(\omega_{I^c} \equiv 1)} = \frac{\det G_{J\cup I^c, J\cup I^c}}{\det G_{I^c,I^c}} \\ &= \det(C^1_{I,K}(G))_{J,J} = P_{C^1_{I,K}(G)}(\omega_J \equiv 1). \end{aligned} \tag{3.57}$$

The case $j = 0$ is reduced to the case $j = 1$ by exchanging particles and holes as follows. It suffices to verify the claim for the events $\{\omega_J \equiv 0\}$, $J \subseteq I$. Using Lemma A.2 from the appendix in the first and the third step and (3.52) in the last step, we obtain

$$\begin{aligned} P_G(\omega_J \equiv 0 | \omega_{I^c} \equiv 0) =& P_{\mathrm{Id} - G}(\omega_J \equiv 1 | \omega_{I^c} \equiv 1) = P_{C^1_{I,K}(\mathrm{Id} - G)}(\omega_J \equiv 1) \\ =& P_{\mathrm{Id} - C^1_{I,K}(\mathrm{Id} - G)}(\omega_J \equiv 0) = P_{C^0_{I,K}(G)}(\omega_J \equiv 0). \end{aligned} \tag{3.58}$$

□

Set $\Xi = \{0,1,2\}^n$. For $\xi \in \Xi$ and $j = 0,1,2$, we write $I_j(\xi) = \{i \in [n] : \ \xi_i = j\}$. We introduce the map

$$C_\xi : \mathcal{G}_n \to \mathcal{G}_{I_1(\xi)}, \quad C_\xi(G) = C^1_{I_1(\xi), I_1(\xi)\cup I_2(\xi)}(C^0_{I_1(\xi)\cup I_2(\xi),[n]}(G)). \tag{3.59}$$

Corollary 3.17. *For all $G \in \mathcal{G}_n$ and $\xi \in \Xi$, one has*

$$P_{C_\xi(G)} = P_G(\omega_{I_1(\xi)} \in \cdot | \omega_{I_0(\xi)} \equiv 0, \omega_{I_2(\xi)} \equiv 1). \tag{3.60}$$

Proof. This follows immediately by applying Lemma 3.16 twice. □

For $\omega \in \{0,1\}^{I_1(\xi)}$, we denote by $\omega 0_{I_0(\xi)} 1_{I_2(\xi)}$ the configuration in $\{0,1\}^n$ that agrees with ω on $I_1(\xi)$, equals 0 on $I_0(\xi)$, and equals 1 on $I_2(\xi)$. For an event $A \subseteq \Omega$, we define

$$A_\xi = \{\omega \in \{0,1\}^{I_1(\xi)} : \ \omega 0_{I_0(\xi)} 1_{I_2(\xi)} \in A\}. \tag{3.61}$$

Lemma 3.18. *For all increasing events $A, B \subseteq \Omega$ and all $\xi \in \{0,1,2\}^n$, one has*

$$(A \Box B)_\xi \subseteq A_\xi \Box_{I_1(\xi)} B_\xi. \tag{3.62}$$

Proof. Let $\omega \in (A\Box B)_\xi$, i.e. $\omega 0_{I_0(\xi)} 1_{I_2(\xi)} \in A \Box B$. By the characterization (1.2) of the disjoint occurrence operator $\Box$, there are $S \in A$ and $T \in B$ with $S \wedge T = 0$ and $S \vee T \leq \omega 0_{I_0(\xi)} 1_{I_2(\xi)}$. Let $\tilde{S} = S_{I_1(\xi)}$ and $\tilde{T} = T_{I_1(\xi)}$ denote the restrictions to $I_1(\xi)$ of S and T, respectively. Since $S \leq \omega 0_{I_0(\xi)} 1_{I_2(\xi)}$, we must have $S_{I_0(\xi)} \equiv 0$. Consequently, $S \leq \tilde{S} 0_{I_0(\xi)} 1_{I_2(\xi)}$ and since A is increasing, it follows that $\tilde{S} 0_{I_0(\xi)} 1_{I_2(\xi)} \in A$. This means $\tilde{S} \in A_\xi$. By the same argument, $\tilde{T} \in B_\xi$ holds. Clearly, $\tilde{S} \wedge \tilde{T} = 0$ and $\tilde{S} \vee \tilde{T} \leq \omega$. Consequently, $\omega \in A_\xi \Box_{I_1(\xi)} B_\xi$. □

For $\xi \in \Xi$, we define

$$\Omega^{(\xi)} := \{\omega \in \Omega : \omega_{I_0(\xi)} \equiv 0, \omega_{I_2(\xi)} \equiv 1\}. \tag{3.63}$$

For any probability measure P on Ω, we set

$$\Xi(P) := \{\xi \in \Xi : P(\Omega^{(\xi)}) > 0\}. \tag{3.64}$$

For $\xi \in \Xi(P)$, we introduce the probability measure

$$P^{(\xi)} = P(\omega_{I_1(\xi)} \in \cdot | \Omega^{(\xi)}) \tag{3.65}$$

on $\{0,1\}^{I_1(\xi)}$.

Lemma 3.19. *For any probability measure P on Ω and any increasing events $A, B \subseteq \Omega$, one has*

$$P(A)P(B) - P(A\Box B) \geq \sum_{\xi \in \Xi(P)} P(\Omega^{(\xi)})^2 \operatorname{Reim}^{I_1(\xi)}_{A_\xi, B_\xi}(P^{(\xi)}). \tag{3.66}$$

In particular, for all $G \in \mathcal{G}_n$, this reduces to

$$\mathrm{BK}_{A,B}(G) \geq \sum_{\xi \in \Xi} P_G(\Omega^{(\xi)})^2 \operatorname{Reim}^{I_1(\xi)}_{A_\xi, B_\xi}(P_{C_\xi(G)}). \tag{3.67}$$

Proof. We partition $\Omega \times \Omega$ into the sets $\{(\omega,\eta) \in \Omega^{(\xi)} \times \Omega^{(\xi)} : \ \omega+\eta=\xi\}$, $\xi \in \Xi$. Using this, we calculate

$$
\begin{aligned}
&P(A)P(B) - P(A\Box B) = P(A)P(B) - P(A\Box B)P(\Omega) \\
&= \sum_{\xi\in\Xi} \sum_{\substack{(\omega,\eta)\in\Omega^{(\xi)}\times\Omega^{(\xi)}: \\ \omega+\eta=\xi}} (1_A(\omega)1_B(\eta) - 1_{A\Box B}(\omega))P(\omega)P(\eta) \\
&= \sum_{\xi\in\Xi(P)} P(\Omega^{(\xi)})^2 \sum_{\substack{(\omega,\eta)\in\Omega^{(\xi)}\times\Omega^{(\xi)}: \\ \omega+\eta=\xi}} (1_A(\omega)1_B(\eta) - 1_{A\Box B}(\omega))P^{(\xi)}(\omega_{I_1(\xi)})P^{(\xi)}(\eta_{I_1(\xi)}) \\
&= \sum_{\xi\in\Xi(P)} P(\Omega^{(\xi)})^2 \sum_{\omega\in\{0,1\}^{I_1(\xi)}} (1_{A_\xi}(\omega)1_{B_\xi}(\overline{\omega}) - 1_{(A\Box B)_\xi}(\omega))P^{(\xi)}(\omega)P^{(\xi)}(\overline{\omega});
\end{aligned} \tag{3.68}
$$

for the last step, note that $\omega+\eta=1$ on $I_1(\xi)$ holds if and only if $\eta=\overline{\omega}$ on $I_1(\xi)$. Using Lemma 3.18, this yields the first claim (3.66) as follows:

$$
\begin{aligned}
&P(A)P(B) - P(A\Box B) \\
&\geq \sum_{\xi\in\Xi(P)} P(\Omega^{(\xi)})^2 \sum_{\omega\in\{0,1\}^{I_1(\xi)}} (1_{A_\xi}(\omega)1_{B_\xi}(\overline{\omega}) - 1_{A_\xi\Box_{I_1(\xi)}B_\xi}(\omega))P^{(\xi)}(\omega)P^{(\xi)}(\overline{\omega}) \\
&= \sum_{\xi\in\Xi(P)} P(\Omega^{(\xi)})^2 \operatorname{Reim}^{I_1(\xi)}_{A_\xi,B_\xi}(P^{(\xi)}).
\end{aligned} \tag{3.69}
$$

In the special case $P = P_G$, we have $\Xi = \Xi(P_G)$. In this case, $P^{(\xi)} = P_{C_\xi(G)}$ holds for $\xi \in \Xi$ by Corollary 3.17. This proves the second claim (3.67). □

Proof of Theorem 1.6. Note that for all $\xi \in \Xi$, the events A_ξ and B_ξ are increasing. The maps $\mathcal{G}_n \ni G \mapsto \operatorname{Reim}^{I_1(\xi)}_{A_\xi,B_\xi}(P_{C_\xi(G)})$ and $\mathcal{G}_n \ni G \mapsto P_G(\Omega^{(\xi)})^2$ are real-analytic. Note further that $C_\xi(G(0))$ is a diagonal matrix because $G(0)$ is a diagonal matrix. Now $P_G(\Omega^{(\xi)})^2 \geq 0$ holds, and the second order Taylor expansion of $t \mapsto \operatorname{Reim}^{I_1(\xi)}_{A_\xi,B_\xi}(P_{C_\xi(G(t))})$ at $t_0 = 0$ is nonnegative for t near $t_0 = 0$ by Theorem 1.7. Combining these facts with Lemma 3.19 yields the claim of Theorem 1.6. □

A Appendix

A.1 Positive definite matrices

A matrix $G \in \mathbb{C}^{n\times n}$, $n \in \mathbb{N}$, is called positive semidefinite if it is Hermitian, i.e. $G^* = G$, and fulfills $x^*Gx \geq 0$ for all column vectors $x \in \mathbb{C}^n$. It is called positive definite, if in addition $x^*Gx = 0$ implies $x = 0$. Equivalently, a Hermitian matrix $G \in \mathbb{C}^{n\times n}$ is positive semidefinite if and only if all its eigenvalues are positive or zero, and it is positive definite if and only if all its eigenvalues are positive. As a consequence, the determinant of any positive semidefinite matrix is positive or zero, and the determinant of any positive definite matrix is positive.

The relation $M \leq N$ between Hermitian matrices $M, N \in \mathbb{C}^{n\times n}$ means that $N - M$ is positive semidefinite. Similarly, $M < N$ means that $N - M$ is positive definite. These two relations are transitive because the sum of two positive (semi)definite matrices is positive (semi)definite. In particular, $G \geq 0$ means that G is positive semidefinite and $G > 0$ means that G is positive definite.

A Hermitian matrix $M \in \mathbb{C}^{n\times n}$ is called an orthogonal projector if $M^2 = M$. For orthogonal projectors $M \in \mathbb{C}^{n\times n}$, one has $0 \leq M \leq \mathrm{Id}$ because $x^*Mx = \|Mx\|^2$ and $x^*(\mathrm{Id}-M)x = \|(\mathrm{Id}-M)x\|^2$ holds for all column vectors $x \in \mathbb{C}^n$.

For all positive semidefinite matrices $G \in \mathbb{C}^{n\times n}$, there is a unique positive semidefinite matrix $\sqrt{G} \in \mathbb{C}^{n\times n}$ such that $(\sqrt{G})^2 = G$. It is given by

$$\sqrt{G} = T \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})T^*$$

when $G = T \operatorname{diag}(\lambda_1, \dots, \lambda_n)T^*$ is a diagonalization of G with a unitary matrix $T \in \mathbb{C}^{n\times n}$. This square root is a real analytic function on the set of positive definite matrices. We need this fact only on the set $\mathcal{G}_n$ defined in (1.9). On $\mathcal{G}_n$, convergence of the binomial power series $\sqrt{G} = \sum_{k=0}^{\infty} \binom{1/2}{k}(G - \mathrm{Id})^k$ proves real analyticity.

More generally, let $M \in \mathbb{C}^{n\times n}$ be any Hermitian matrix. Let $\lambda_1, \dots, \lambda_k$ be its different eigenvalues, listed without multiplicities. Let $\Pi_j \in \mathbb{C}^{n\times n}$ be the orthogonal projection onto the eigenspace of M corresponding to the eigenvalue λ_j, $j = 1, \dots, k$. For any real-valued function f defined at least on the eigenvalues of M, the Hermitian matrix $f(M) \in \mathbb{C}^{n\times n}$ is defined by

$$f(M) := \sum_{j=1}^{k} f(\lambda_j)\Pi_j. \tag{A.1}$$

If g is another real-valued function defined at least on $\{\lambda_1, \dots, \lambda_k\}$ one has $f(M)g(M) = (fg)(M) = g(M)f(M)$.

If $\Sigma \in \mathbb{C}^{m\times n}$, $m, n \in \mathbb{N}$, is any rectangular complex matrix, then $\Sigma^*\Sigma$ is positive semidefinite because $x^*\Sigma^*\Sigma x = \|\Sigma x\|^2 \geq 0$ for all column vectors $x \in \mathbb{C}^n$.

Let $G \in \mathbb{C}^{n\times n}$ be positive (semi)definite. Then, for any $I \subseteq [n]$, the submatrix $G_{I,I}$ is positive (semi)definite as well. In particular, $\det G_{I,I} \geq 0$ when G is positive semidefinite because $\det G_{I,I}$ is the product of all eigenvalues of $G_{I,I}$, counted with multiplicity, and all these eigenvalues are positive or zero.

A.2 Hermitian determinantal probability measures

We include here a proof of the well-known existence of Hermitian determinantal probability measures, Fact 1.5. The proof constructs first P_G as a *signed* measure. Then, it shows that P_G is a *positive* measure. Some of the steps in the proof are also useful in the rest of the paper.

Lemma A.1. *For all $k \in \mathbb{N}$ and $G \in \overline{\mathcal{G}}_k$, there exists a unique* signed *measure P_G satisfying* (1.10).

Proof. Uniqueness. The collection of events $\{\omega_I \equiv 1\}$, $I \subseteq [k]$, is stable under intersections and generates the power set of $\{0,1\}^k$. Consequently, there exists at most one signed measure satisfying (1.10).

Existence. Let $\mathcal{M}_s$ denote the set of signed measures on $(\{0,1\}^k, \mathcal{P}(\{0,1\}^k))$. Then, $\mathcal{M}_s$ is a vector space with dimension $\dim \mathcal{M}_s = |\{0,1\}^k| = 2^k$. Consider the linear map

$$\mathcal{M}_s \to \mathbb{R}^{\mathcal{P}([k])}, \quad \mu \mapsto (\mu(\omega_I \equiv 1))_{I\subseteq[k]}. \tag{A.2}$$

By the uniqueness statement, it is one-to-one. Since

$$\dim \mathbb{R}^{\mathcal{P}([k])} = 2^k = \dim \mathcal{M}_s, \tag{A.3}$$

it is a bijection. Thus, there exists a signed measure P_G satisfying (1.10). □

Lemma A.2. *For $l \in \mathbb{N}$, for all $G \in \overline{\mathcal{G}}_l$ and all $\omega \in \{0,1\}^l$, one has $P_G(\overline{\omega}) = P_{\mathrm{Id}-G}(\omega)$.*

Proof. In the following, we abbreviate $L = [l]$. Let $H = \mathrm{Id} - G$. Let I, J, K be a partition of L. In this proof, $(G_{L,I}, H_{L,J}, \mathrm{Id}_{L,K}) \in \mathbb{C}^{l\times l}$ denotes the matrix with (i,j)-entry G_{ij} for $j \in I$, H_{ij} for $j \in J$, and δ_{ij} for $j \in K$. We prove the following by induction over J:

$$P_G(\omega_I \equiv 1, \omega_J \equiv 0) = \det(G_{L,I}, H_{L,J}, \mathrm{Id}_{L,K}) \tag{A.4}$$

For $J = \emptyset$, the claim (A.4) reduces to the fact

$$P_G(\omega_I \equiv 1) = \det G_{I,I} = \det(G_{L,I}, \mathrm{Id}_{L,I^c}) \tag{A.5}$$

with the abbreviation $I^c = L \setminus I$. As induction hypothesis, assume that the claim (A.4) holds for all partitions I, J, K of L with some given $J \subsetneq L$. Take $J' = J \cup \{j\}$ with $j \in L \setminus J$ and a partition I, J', K of L. We abbreviate $I' = I \cup \{j\}$ and $K' = K \cup \{j\}$. By the induction hypothesis, one has

$$P_G(\omega_I \equiv 1, \omega_J \equiv 0) = \det(G_{L,I}, H_{L,J}, \mathrm{Id}_{L,K'}), \tag{A.6}$$

$$P_G(\omega_{I'} \equiv 1, \omega_J \equiv 0) = \det(G_{L,I'}, H_{L,J}, \mathrm{Id}_{L,K}). \tag{A.7}$$

Using $\mathrm{Id}_{L,j} - G_{L,j} = H_{L,j}$ and additivity of the determinant in the j-th column, we calculate

$$\begin{aligned} &P_G(\omega_I \equiv 1, \omega_{J'} \equiv 0) = P_G(\omega_I \equiv 1, \omega_J \equiv 0) - P_G(\omega_{I'} \equiv 1, \omega_J \equiv 0) \\ &= \det(G_{L,I}, H_{L,J}, \mathrm{Id}_{L,K'}) - \det(G_{L,I'}, H_{L,J}, \mathrm{Id}_{L,K}) = \det(G_{L,I}, H_{L,J'}, \mathrm{Id}_{L,K}). \end{aligned} \tag{A.8}$$

This completes the inductive proof of (A.4).

Given any $\omega \in \{0,1\}^l$, (A.4) gives us the same determinant for $P_G(\overline{\omega})$ and for $P_H(\omega)$, using that $\overline{\omega}$ equals 1 precisely on $I(\overline{\omega}) = I(\omega)^c$, while ω equals 0 precisely on the same set:

$$P_G(\overline{\omega}) = \det(G_{L,I(\omega)^c}, H_{L,I(\omega)}) = P_H(\omega). \tag{A.9}$$

□

Here is an interpretation of Lemma A.2. Viewing locations without particles as "antiparticles" or holes, exchanging particles with holes corresponds to exchanging G with $\mathrm{Id} - G$.

Lemma A.3. *For any $k \le l$ in $\mathbb{N}$ and $M \in \overline{\mathcal{G}}_l$ with $\operatorname{rank} M = k$ and $\operatorname{rank}(\mathrm{Id} - M) = l - k$, for any $J \subseteq [l]$ with $|J| \ne k$, one has*

$$P_M(\omega_J \equiv 1, \omega_{[l]\setminus J} \equiv 0) = 0. \tag{A.10}$$

In other words, the signed measure P_M is supported on configurations with precisely k particles.

Proof. For $J \subseteq [l]$ with $|J| > k$, let $\mathcal{E}_J = \{\{\omega_I \equiv 1\} : J \subseteq I \subseteq [l]\}$. Since $\operatorname{rank} M = k$, the measure P_M vanishes on $\mathcal{E}_J$. The event

$$\{\omega_J \equiv 1, \omega_{[l]\setminus J} \equiv 0\} = \{\omega_J \equiv 1\} \setminus \Big(\bigcup_{I:\ J \subsetneq I \subseteq [l]} \{\omega_I \equiv 1\} \Big) \tag{A.11}$$

belongs to $\sigma(\mathcal{E}_J)$. Since $\mathcal{E}_J$ is closed under intersections, it follows that this event has P_M-measure zero. Consequently, P_M is supported on configurations with at most k particles.

By assumption, $\mathrm{Id} - M$ has rank $l - k$. Consequently, $P_{\mathrm{Id} - M}$ is supported on configurations with at most $l - k$ particles. For $\omega \in \{0,1\}^l$, Lemma A.2 states that $P_M(\overline{\omega}) = P_{\mathrm{Id} - M}(\omega)$. From this we conclude that P_M is supported on configurations with at most $l - k$ holes or equivalently with at least k particles. Thus, the claim of the lemma follows. □

Consider the orthogonal projector $M(G) = \Psi^*(G)\Psi(G) \in \mathbb{C}^{2k\times 2k}$ from (3.2). Recall that $\iota : \{0,1\}^{2k} \to \{0,1\}^k$ denotes the projection to the first k coordinates. Slightly more generally than in Section 3.1, we denote by $\iota[\mu]$ the image signed measure of any signed measure μ on $\{0,1\}^{2k}$ with respect to ι. The following lemma shows that P_G is recovered from $P_{M(G)}$ by ignoring all locations indexed by $k+1,\dots,2k$.

Lemma A.4. *For $G \in \overline{\mathcal{G}}_k$, we have $P_G = \iota[P_{M(G)}]$.*

Proof. We use the notation from line (3.1). From

$$M(G) = \Psi(G)^*\Psi(G), \qquad \Psi(G) = (\psi(G), \phi(G)),$$

and $G = \psi(G)^*\psi(G)$, we get $M(G)_{[k][k]} = G$. Consequently, we obtain for $I \subseteq [k]$:

$$\begin{aligned} &P_G(\{\omega \in \{0,1\}^k : \omega_I \equiv 1\}) = \det G_{I,I} = \det M(G)_{I,I} \\ &= P_{M(G)}(\{\omega \in \{0,1\}^{2k} : \omega_I \equiv 1\}) = \iota[P_{M(G)}](\{\omega \in \{0,1\}^k : \omega_I \equiv 1\}) \end{aligned} \tag{A.12}$$

Since the events $\{\omega_I \equiv 1\}$, $I \subseteq [k]$, form a $\cap$-stable generator of the power set of $\{0,1\}^k$, the claim follows. □

Proof of Fact 1.5. Let $G \in \overline{\mathcal{G}}_k$. It remains to show that the signed measure P_G defined in Lemma A.1 is a probability measure.

By Lemma 3.1, the matrices $M(G)$ and $\mathrm{Id} - M(G)$ have rank k. Thus, Lemma A.3 implies that the signed measure $P_{M(G)}$ is supported on configurations with precisely k particles.

Next, we consider the finite measure $\mu_{\Psi(G)}$ introduced in Definition 3.2. Just as $P_{M(G)}$, it is supported on configurations with precisely k particles at $2k$ locations. For $I \subseteq [2k]$ with $|I| = k$, we obtain

$$\begin{aligned} &P_{M(G)}(\omega_I \equiv 1, \omega_{[2k]\setminus I} \equiv 0) = P_{M(G)}(\omega_I \equiv 1) \\ &= \det M(G)_{I,I} = \mu_{\Psi(G)}(\omega_I \equiv 1, \omega_{[2k]\setminus I} \equiv 0). \end{aligned} \tag{A.13}$$

This implies $P_{M(G)} = \mu_{\Psi(G)}$. Since $\mu_{\Psi(G)}$ is a positive measure, $P_{M(G)}$ is also positive.

Using Lemma A.4, we get that P_G is also a positive measure. It is normalized because of $P_G(\{0,1\}^k) = P_G(\omega_\emptyset \equiv 1) = 1$ because the determinant of the empty matrix equals 1 by definition. This proves the claim. □

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